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Linguistic Pythagorean Einstein Operators and Their Application to Decision Making

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Abstract: Linguistic Pythagorean fuzzy (LPF) set is an efficacious technique to comprehensively represent uncertain assessment information by combining the Pythagorean fuzzy numbers and linguistic variables. In this paper, we define several novel essential operations of LPF numbers based upon Einstein operations and discuss several relations between these operations. For solving the LPF numbers fusion problem, several LPF aggregation operators, including LPF Einstein weighted averaging (LPFEWA) operator, LPF Einstein weighted geometric (LPFEWG) operator and LPF Einstein hybrid operator, are propounded; the prominent characteristics of these operators are investigated as well. Furthermore, a multi-attribute group decision making (MAGDM) approach is presented on the basis of the developed operators under an LPF environment. Ultimately, two application cases are utilized to demonstrate the practicality and feasibility of the developed decision approach and the comparison analysis is provided to manifest the merits of it.

Keywords: multi-attribute group decision making; Linguistic Pythagorean fuzzy set; aggregation operator; Einstein operations

1. Introduction

Multi-attribute group decision-making is an important component of modern decision making (DM) science, which has been widely considered by multitudinous scholars and has achieved rich research achievements [1–6]. Due to the uncertainty and complexity of DM environments, it is arduous to portray the evaluation value of alternatives by precise numerical value. Therefore, Zadeh [7] originally propounded the fuzzy set (FS) theory to address this limitation. Since its introduction, a large number of achievements on FS have been acquired in many fields, including fuzzy control [8–10], decision analysis [11–15] and so on. However, one of the defects of FS is that it only reveals the membership degree (MD) of an element to a selected objective. Hence, as an effective extension of FS, Atanassov [16] introduced the notion of an intuitionistic fuzzy set (IFS), which can effectively deal with the defect of FS by adding a non-membership degree (NMD). Since its emergence, a lot of research results have been attained in theory and application. Xu [17] proposed several aggregation operators for aggregating an intuitionistic fuzzy number (IFN). Xia et al. [18] propounded some aggregation operators based upon Archimedean operations of IFN. Abdullah et al. [19] extended the DEMATEL method to interval-valued intuitionistic fuzzy setting to handle the problem of sustainable solid waste management. Zhang et al. [20] propounded an extended TODIM approach combined with Choquet integral to sort products with online reviews. Shen et al. [21] developed an extended TOPSIS approach with a novel distance measure to do with the credit risk evaluation issue. Liu et al. [22]

propounded several intuitionistic fuzzy Dombi Bonferroni mean operators, which can take into account the correlation of different attributes. Liu et al. [23] presented a dynamic intuitionistic fuzzy DM approach by combining evidential reasoning and a dynamic intuitionistic fuzzy geometric operator.

IFS theory has been applied in diverse DM approaches and practical applications. However, in several particular situations, the evaluation information provided by experts is disabled under the intuitionistic fuzzy context, that is, the sum of the membership degree and non-membership degree is bigger than 1. In order to surmount this defect, Yager [24,25] propounded the notion of the Pythagorean fuzzy set (PFS) by extending the FS and IFS whose square sum of MD and NMD is less than or equal to one. PFS is more effective and flexible for solving the vagueness and uncertainty issues than are FS and IFS. Since its introduction, increasingly investigators have paid their attention to research on PFS and have obtained many achievements. These achievements are summarized as follows: (1) The foundation theories such as operational laws [26,27], score function [28,29], information entropy measure [30–32], distance measure and similarity measure [33–37], and so forth; (2) The DM methodologies on PFS? Peng and Yang [38] proposed the MABAC approach combined with Choquet integral to resolve the MAGDM problem with Pythagorean fuzzy information. Liang et al. [39] propounded a novel approach by synthesizing the three-way decision theory and TOPSIS ideal solutions under a Pythagorean fuzzy environment. Khan et al. [40] presented an extended VIKOR method to tackle MAGDM, the problem of unknown attribute weight; (3) The aggregation operators aspect? Garg [41] propounded generalized Pythagorean fuzzy geometric aggregation operators on the basis of Einstein operations. Garg [42] proposed several Pythagorean fuzzy geometric operators based on the novel neutrality operation of Pythagorean fuzzy number (PFN). Li et al. [43] developed some novel interactive hybrid weighted operators for fusing Pythagorean fuzzy information. Zhu and Li [44] presented an MAGDM approach based on Muirhead Mean operators under a Pythagorean fuzzy environment. Khan et al. [45] developed an MADM approach based on a new ranking methodology with Pythagorean trapezoidal uncertain linguistic fuzzy information. More research results on the theories and techniques can be found in References [46–51].

The theory discussed above can only deal with uncertainty from a quantitative point of view. However, in actual problems, several attribute values are qualitative in essence and cannot describe an accurate numerical value. Under these circumstances, it is more convenient to express the preference information of DMs with linguistic variables. For this issue, Zadeh [52–54] propounded the notion of a linguistic variable and utilized it to express assessment information in the form of natural linguistics. For instance, when we evaluate the performance of a computer, we often use natural linguistics such as “excellent”, “very good”, “good” to provide evaluation information instead of a numerical value. Hence, a linguistic variable is more in line with human cognitive thinking to provide vague and uncertain assessment information. For reducing the information loss in the computation procedure, Xu [55] further propounded the notion of a continuous linguistic term set (LTS). Zhang et al. [56] originally proposed the conception of a linguistic intuitionistic fuzzy set (LIFS) by combining the linguistic method and IFS. Chen et al. [57] presented several aggregation operators of linguistic intuitionistic fuzzy number (LIFN) to construct a DM methodology. Garg [58] proposed a novel DM methodology by combining possibility degree measures and linguistic intuitionistic fuzzy Einstein aggregation operators. Garg [59] defined the concept of a linguistic interval-valued Atanassov intuitionistic fuzzy set and introduced its basic theory and aggregation operators. However, in such special cases, if an expert shall provide his or her preference information in the form of LIFN, denoted as (s_4, s_5) , in which s_t denotes the linguistic term and $t \in [0, 8]$. It is obvious that the LIFS fails to address this situation because of $4 + 5 \not\leq 8$. Therefore, in this setting, the above-mentioned approaches cannot resolve its validly. To overcome this shortcoming, Garg [60] firstly brought forward the concept of a linguistic Pythagorean fuzzy set (LPFS) by synthesizing the theory of PFS with the concept of linguistic terms to model qualitative assessment information. The element in LPFS is called the linguistic Pythagorean fuzzy number (LPFN), which is composed of linguistic MD and NMD that are denoted as the linguistic term. Aiming at the mentioned particular case, we can resolve it easily

under the LPF environment because of $4^2 + 5^2 \leq 8$. Furthermore, from the difference of IFN and PFN, we can find that an LIFN is also an LPFN, while an LPFN may not be an LIFN, which means that the LPFN has a wider scope than LIFN to express evaluation information for DMs. Later on, Li et al. [61] proposed the partitioned Bonferroni mean aggregation operators to fuse LPF information. Li et al. [62] presented a novel TOPSIS approach by combining a correlation coefficient and an entropy measure to resolve decision issues.

Aggregation operators are one of the hot topics in the field of decision support systems to aggregate the assessment information and to sort alternatives. It is known that most aggregation operators are propounded on the basis of the algebraic T-norm and S-norm but the algebraic operations lack flexibility and robustness. As a particular Archimedean T-norm and S-norm [63], Einstein T-norm and S-norm not only has the same characteristics of smooth approximation as the algebraic but also has stronger flexibility and robustness than algebraic T-norm and S-norm. So, aggregation operators have been widely applied to generate operational laws for fuzzy numbers. On the basis of the aforementioned, we can find that (1) The LPFN can express uncertainty information more generally than LIFN and PFN; (2) The traditional operational laws of LPFN lack flexibility and robustness during the procedure of information integration; (3) The Einstein T-norm and S-norm can not only generate operational laws but can also improve the flexibility and smoothness more than Algebraic operations. Accordingly, it is meaningful to utilize Einstein T-norm and S-norm to model the intersection and union on LPFN to propound novel aggregation operators. The main research objective and contributions of this article are:

1. To extend the Einstein T-norm and T-conorm to LPFS and propose novel operational laws of LPFNs to improve the flexibility and robustness of the proposed approach;
2. To propose several LPF Einstein operators such as LPF Einstein averaging operators, LPF Einstein geometry operators, LPF Einstein hybrid operators and discuss several related properties of these operators;
3. To present a novel DM method based on the proposed operators to solve MAGDM problems in practical situations;
4. To provide an application example to illustrate the validity of the presented approach and give a comparative analysis to show its advantages.

The organization of the paper is described below. Section 2 briefly introduces several fundamental concepts. Section 3 defines the novel operations of LPFN based on Einstein operation. Section 4 brings forward several LPFE operators and proves their related properties. We utilize those presented operators to construct an MAGDM approach to solve MAGDM problems in Section 5. Consequently, several examples are given to illustrate the validity of the proposed method and show its advantages by comparing it with other approaches in Section 6. Several conclusions are given at the end.

2. Preliminaries

This section reviews several foundation knowledge of LPFS and Einstein operation.

2.1. Linguistic Pythagorean Fuzzy Set

Garg [60] originally propounded LPFS by synthesizing PFS and LTS. The definition of LPFS is depicted below:

Definition 1 ([60]). Assume that $\mathcal{X}$ is a fixed set and $\tilde{\mathcal{S}} = \{s_\alpha | \alpha \in [0, t]\}$ with a nonnegative integer t be a continue LTS. A LPFS $\mathcal{A}$ on $\mathcal{X}$ is defined as:

$$\mathcal{A} = \{(x, s_\alpha(x), s_\beta(x)) | x \in X\}. \quad (1)$$

where $s_\alpha(x)$ and $s_\beta(x)$ denote linguistic MD and linguistic NMD of the element x to $\mathcal{A}$. Each pair $(s_\alpha(x), s_\beta(x))$ is simplified as (s_α, s_β) , which stand for an LPFN and meets that $\alpha, \beta \in [0, t]$ and $\alpha^2 + \beta^2 \leq t^2$. For any $x \in \mathcal{X}$, $\Pi_{\mathcal{A}} = s \sqrt{t^2 - \alpha^2 - \beta^2}$ is named linguistic hesitancy degree of x to $\mathcal{A}$.

Garg [60] also presented the score function with an accuracy function to build the comparison approach of LPFNs.

Definition 2 ([60]). Let $\mathcal{A} = (s_{\alpha_1}, s_{\beta_1})$ be a LPFN with $s_{\alpha_1}, s_{\beta_1} \in \tilde{\mathcal{S}}$. The score function $\mathcal{T}$ and accuracy function $\mathcal{H}$ of $\mathcal{A}$ are defined as:

$$\mathcal{T}(\mathcal{A}) = s \sqrt{\frac{t^2 + \alpha_1^2 - \beta_1^2}{2}}, \quad (2)$$

$$\mathcal{H}(\mathcal{A}) = s \sqrt{\alpha_1^2 + \beta_1^2}. \quad (3)$$

For comparing two LPFNs $\mathcal{A}$ and $\mathcal{B}$, the comparison method is given as:

1. if $\mathcal{T}(\mathcal{A}) > \mathcal{T}(\mathcal{B})$, then $\mathcal{A} \succ \mathcal{B}$;
2. if $\mathcal{T}(\mathcal{A}) = \mathcal{T}(\mathcal{B})$, then
 - if $\mathcal{H}(\mathcal{A}) < \mathcal{H}(\mathcal{B})$, then $\mathcal{A} \prec \mathcal{B}$;
 - if $\mathcal{H}(\mathcal{A}) = \mathcal{H}(\mathcal{B})$, then $\mathcal{A} \sim \mathcal{B}$.

2.2. Einstein T-Norm and S-Norm

Definition 3 ([63]). For arbitrary two real numbers $\rho, q \in [0, 1]$, the Einstein sums S_Ξ and Einstein product T_Ξ are defined as follows:

$$S_\Xi(\rho, q) = \rho \oplus q = \frac{\rho + q}{1 + \rho q}, \quad (4)$$

$$T_\Xi(\rho, q) = \rho \otimes q = \frac{\rho q}{1 + (1 - \rho)(1 - q)}. \quad (5)$$

3. Einstein Operations of LPFNs

Under this section, we redefine the operational laws of LPFNs based upon the Einstein operations and discuss its properties in detail.

Definition 4. Let $\mathcal{A} = (s_{\alpha_1}, s_{\beta_1})$ and $\mathcal{B} = (s_{\alpha_2}, s_{\beta_2})$ be two LPFNs and $\lambda > 0$, then the LPFE operations are defined as

1. $\mathcal{A} \oplus \mathcal{B} = \left(s \sqrt{\frac{t^2(\alpha_1^2 + \alpha_2^2)}{t^4 + \alpha_1^2 \alpha_2^2}}, s \sqrt{\frac{\beta_1 \beta_2}{\sqrt{t^4 + (t^2 - \beta_1^2)(t^2 - \beta_2^2)}}} \right);$
2. $\mathcal{A} \otimes \mathcal{B} = \left(s \sqrt{\frac{\alpha_1 \alpha_2}{\sqrt{t^4 + (t^2 - \alpha_1^2)(t^2 - \alpha_2^2)}}}, s \sqrt{\frac{t^2(\beta_1^2 + \beta_2^2)}{t^4 + \beta_1^2 \beta_2^2}} \right);$
3. $\lambda \mathcal{A} = \left(s \sqrt{\frac{(t^2 + \alpha_1^2)^\lambda - (t^2 - \alpha_1^2)^\lambda}{(t^2 + \alpha_1^2)^\lambda + (t^2 - \alpha_1^2)^\lambda}}, s \sqrt{\frac{\sqrt{2} \beta_1^\lambda}{\sqrt{(2t^2 - \beta_1^2)^\lambda + (\beta_1^2)^\lambda}}} \right);$
4. $\mathcal{A}^\lambda = \left(s \sqrt{\frac{\sqrt{2} \alpha_1^\lambda}{\sqrt{(2t^2 - \alpha_1^2)^\lambda + (\alpha_1^2)^\lambda}}}, s \sqrt{\frac{(t^2 + \beta_1^2)^\lambda - (t^2 - \beta_1^2)^\lambda}{(t^2 + \beta_1^2)^\lambda + (t^2 - \beta_1^2)^\lambda}} \right).$

Theorem 1. Let $\mathcal{A}_i = (s_{\alpha_i}, s_{\beta_i})$, $(i = 1, 2)$ be two LPFNs, then $\mathcal{A}_1 \oplus \mathcal{A}_2$, $\mathcal{A}_1 \otimes \mathcal{A}_2$, $\lambda \mathcal{A}_1$ and $\mathcal{A}_1^\lambda$ are also LPFNs, respectively.

Proof. Due to $\mathcal{A}_i = (s_{\alpha_i}, s_{\beta_i})$ ($i = 1, 2$) be two LPFNs, then we get $\alpha_i, \beta_i \in [0, t]$, $\alpha_i/t, \beta_i/t \in [0, 1]$ and $\alpha_i^2 + \beta_i^2 \in [0, t^2]$. Since $(\alpha_1^2/t^2 - 1)(\alpha_2^2/t^2 - 1) \geq 0$, then, $(\alpha_1^2/t^2)(\alpha_2^2/t^2) - (\alpha_1^2/t^2 + \alpha_2^2/t^2) + 1 \geq 0$, so, $(\alpha_1^2/t^2 + \alpha_2^2/t^2)/((\alpha_1^2/t^2)(\alpha_2^2/t^2) + 1) \leq 1$. Similarly, since $1 + (1 - \beta_1^2/t^2)(1 - \beta_2^2/t^2) \geq 1$, $(\beta_1/t \cdot \beta_2/t) \leq 1$, then $((\beta_1/t)(\beta_2/t))/(1 + (1 - \beta_1^2/t^2)(1 - \beta_2^2/t^2)) \leq 1$, hence, $(\beta_1\beta_2)/(\sqrt{t^4 + (t^4 - \beta_1^2)(t^4 - \beta_2^2)}) \leq 1$. Accordingly, we summarize that $t\sqrt{\frac{t^2(\alpha_1^2 + \alpha_2^2)}{t^4 + \alpha_1^2\alpha_2^2}} \in [0, t]$ and $t\frac{\beta_1\beta_2}{\sqrt{t^4 + (t^2 - \beta_1^2)(t^2 - \beta_2^2)}} \in [0, t]$. Furthermore

$$\begin{aligned} & \left(t\sqrt{\frac{t^2(\alpha_1^2 + \alpha_2^2)}{t^4 + \alpha_1^2\alpha_2^2}} \right)^2 + \left(t\frac{\beta_1\beta_2}{\sqrt{t^4 + (t^2 - \beta_1^2)(t^2 - \beta_2^2)}} \right)^2 \\ & \leq t^2 \left(\frac{t^2(\alpha_1^2 + \alpha_2^2)}{t^4 + \alpha_1^2\alpha_2^2} + \frac{(t^2 - \alpha_1^2)(t^2 - \alpha_2^2)}{t^4 + \alpha_1^2\alpha_2^2} \right) \\ & = t^2 \left(\frac{t^2(\alpha_1^2 + \alpha_2^2) + (t^2 - \alpha_1^2)(t^2 - \alpha_2^2)}{t^4 + \alpha_1^2\alpha_2^2} \right) \\ & = t^2. \end{aligned}$$

Hence, we can obtain that $\mathcal{A}_1 \oplus \mathcal{A}_2$ is a LPFN.

Furthermore, for $\lambda \mathcal{A}$, we have

$$\sqrt{\frac{(t^2 + \alpha_1^2)^\lambda - (t^2 - \alpha_1^2)^\lambda}{(t^2 + \alpha_1^2)^\lambda + (t^2 - \alpha_1^2)^\lambda}} = \sqrt{1 - \frac{2(t^2 - \alpha_1^2)^\lambda}{(t^2 + \alpha_1^2)^\lambda + (t^2 - \alpha_1^2)^\lambda}} \leq \sqrt{1 - (t^2 - \alpha_1^2)^\lambda} \leq 1,$$

and

$$t^2 + \alpha_1^2 \geq t^2 - \alpha_1^2 \Rightarrow (t^2 + \alpha_1^2)^\lambda - (t^2 - \alpha_1^2)^\lambda \geq 0 \Rightarrow \sqrt{\frac{(t^2 + \alpha_1^2)^\lambda - (t^2 - \alpha_1^2)^\lambda}{(t^2 + \alpha_1^2)^\lambda + (t^2 - \alpha_1^2)^\lambda}} \geq 0.$$

Accordingly,

$$0 \leq t\sqrt{\frac{(t^2 + \alpha_1^2)^\lambda - (t^2 - \alpha_1^2)^\lambda}{(t^2 + \alpha_1^2)^\lambda + (t^2 - \alpha_1^2)^\lambda}} \leq t.$$

Similarly,

$$\sqrt{\frac{2(t^2 - \alpha_1^2)^\lambda}{(t^2 + \alpha_1^2)^\lambda + (t^2 - \alpha_1^2)^\lambda}} \leq (t^2 - \alpha_1^2)^\lambda \leq 1, (\beta_1^2)^\lambda \geq 0 \Rightarrow \sqrt{\frac{(\beta_1^2)^\lambda}{(t^2 + \alpha_1^2)^\lambda + (t^2 - \alpha_1^2)^\lambda}} \geq 0.$$

Thus,

$$0 \leq t\frac{\sqrt{2}\beta_1^\lambda}{\sqrt{(2t^2 - \beta_1^2)^\lambda + (\beta_1^2)^\lambda}} \leq t.$$

Moreover,

$$\begin{aligned} & \left(t \sqrt{\frac{(t^2 + \alpha_1^2)^\lambda - (t^2 - \alpha_1^2)^\lambda}{(t^2 + \alpha_1^2)^\lambda + (t^2 - \alpha_1^2)^\lambda}} \right)^2 + \left(t \frac{\sqrt{2}\beta_1^\lambda}{\sqrt{(2t^2 - \beta_1^2)^\lambda + (\beta_1^2)^\lambda}} \right)^2 \\ & \leq t^2 \left(1 - \frac{2(t^2 - \alpha_1^2)^\lambda}{(t^2 + \alpha_1^2)^\lambda + (t^2 - \alpha_1^2)^\lambda} + \frac{2(t^2 - \alpha_1^2)^\lambda}{(t^2 + \alpha_1^2)^\lambda + (t^2 - \alpha_1^2)^\lambda} \right) = t^2. \end{aligned}$$

Hence, we can obtain that $\lambda\mathcal{A}_1$ is a LPFN.

Similarly, we can get $\mathcal{A}_1 \otimes \mathcal{A}_2$ and $\mathcal{A}_1^\lambda$ are also LPFNs. $\square$

Theorem 2. Let $\mathcal{A} = (s_{\alpha_1}, s_{\beta_1})$ and $\mathcal{B} = (s_{\alpha_2}, s_{\beta_2})$ be two LPFNs and $\lambda, \lambda_1, \lambda_2 > 0$ be three real numbers, then we have

1. $\mathcal{A} \oplus \mathcal{B} = \mathcal{B} \oplus \mathcal{A}$;
2. $\mathcal{A} \otimes \mathcal{B} = \mathcal{B} \otimes \mathcal{A}$;
3. $\lambda(\mathcal{A} \oplus \mathcal{B}) = \lambda\mathcal{A} \oplus \lambda\mathcal{B}$;
4. $(\mathcal{A} \otimes \mathcal{B})^\lambda = \mathcal{A}^\lambda \otimes \mathcal{B}^\lambda$;
5. $(\lambda_1 \oplus \lambda_2)\mathcal{A} = \lambda_1\mathcal{A} \oplus \lambda_2\mathcal{A}$;
6. $\mathcal{A}^{\lambda_1} \otimes \mathcal{A}^{\lambda_2} = \mathcal{A}^{\lambda_1 + \lambda_2}$.

Proof. We only prove (1), (3) and (5).

(1)

$$\mathcal{A} \oplus \mathcal{B} = \left(s_{t \sqrt{\frac{i^2(\alpha_1^2 + \alpha_2^2)}{i^4 + \alpha_1^2 \alpha_2^2}}}, s_{t \frac{\beta_1 \beta_2}{\sqrt{i^4 + (i^2 - \beta_1^2)(i^2 - \beta_2^2)}}} \right) = \left(s_{t \sqrt{\frac{i^2(\alpha_2^2 + \alpha_1^2)}{i^4 + \alpha_2^2 \alpha_1^2}}}, s_{t \frac{\beta_2 \beta_1}{\sqrt{i^4 + (i^2 - \beta_2^2)(i^2 - \beta_1^2)}}} \right) = \mathcal{B} \oplus \mathcal{A}.$$

(3) Since

$$\mathcal{A} \oplus \mathcal{B} = \left(s_{t \sqrt{\frac{i^2(\alpha_1^2 + \alpha_2^2)}{i^4 + \alpha_1^2 \alpha_2^2}}}, s_{t \frac{\beta_1 \beta_2}{\sqrt{i^4 + (i^2 - \beta_1^2)(i^2 - \beta_2^2)}}} \right) = \left(s_{t \sqrt{\frac{(i^2 + \alpha_1^2) \cdot (i^2 + \alpha_2^2) - (i^2 - \alpha_1^2) \cdot (i^2 - \alpha_2^2)}{(i^2 + \alpha_1^2) \cdot (i^2 + \alpha_2^2) + (i^2 - \alpha_1^2) \cdot (i^2 - \alpha_2^2)}}}, s_{t \frac{\sqrt{2}\beta_1 \beta_2}{\sqrt{\beta_1^2 \beta_2^2 + (2i^2 - \beta_1^2) \cdot (2i^2 - \beta_2^2)}}} \right).$$

Let $a = (t^2 + \alpha_1^2)(t^2 + \alpha_2^2)$, $b = (t^2 - \alpha_1^2)(t^2 - \alpha_2^2)$, $c = \beta_1^2 \cdot \beta_2^2$, $d = (2t^2 - \beta_1^2) \cdot (2t^2 - \beta_2^2)$, then

$$\mathcal{A} \oplus \mathcal{B} = \left(s_{t \sqrt{\frac{a-b}{a+b}}}, s_{t \sqrt{\frac{2c}{d+c}}} \right).$$

By the laws of LPFNs, we can get

$$\begin{aligned} \lambda(\mathcal{A} \oplus \mathcal{B}) &= \lambda \left(s_{t \sqrt{\frac{a-b}{a+b}}}, s_{t \sqrt{\frac{2c}{d+c}}} \right) \\ &= \left(s_{t \sqrt{\frac{(i^2 + \frac{i^2(a-b)}{a+b})^\lambda - (i^2 - \frac{i^2(a-b)}{a+b})^\lambda}{(i^2 + \frac{i^2(a-b)}{a+b})^\lambda + (i^2 - \frac{i^2(a-b)}{a+b})^\lambda}}}, s_{t \frac{(\sqrt{2}(t \sqrt{\frac{2c}{d+c}})^\lambda)}{\sqrt{(2i^2 - \frac{2ct^2}{d+c})^\lambda + (\frac{2ct^2}{d+c})^\lambda}}} \right) = \left(s_{t \sqrt{\frac{a^\lambda - b^\lambda}{a^\lambda + b^\lambda}}}, s_{t \sqrt{\frac{2c^\lambda}{c^\lambda + d^\lambda}}} \right) \\ &= \left(s_{t \sqrt{\frac{(i^2 + \alpha_1^2)^\lambda \cdot (i^2 + \alpha_2^2)^\lambda - (i^2 - \alpha_1^2)^\lambda \cdot (i^2 - \alpha_2^2)^\lambda}{(i^2 + \alpha_1^2)^\lambda \cdot (i^2 + \alpha_2^2)^\lambda + (i^2 - \alpha_1^2)^\lambda \cdot (i^2 - \alpha_2^2)^\lambda}}}, s_{t \frac{\sqrt{2}\beta_1^\lambda \cdot \beta_2^\lambda}{\sqrt{(\beta_1^2)^\lambda \cdot (\beta_2^2)^\lambda + (2i^2 - \beta_1^2)^\lambda \cdot (2i^2 - \beta_2^2)^\lambda}}} \right). \end{aligned}$$

Furthermore,

$$\lambda \mathcal{A} = \left(s_{t\sqrt{\frac{(t^2+\alpha_1^2)^\lambda - (t^2-\alpha_1^2)^\lambda}{(t^2+\alpha_1^2)^\lambda + (t^2-\alpha_1^2)^\lambda}}}, s_{t\sqrt{\frac{2c_1}{d_1+c_1}}} \right) = \left(s_{t\sqrt{\frac{a_1-b_1}{a_1+b_1}}}, s_{t\sqrt{\frac{2c_1}{d_1+c_1}}} \right)$$

and

$$\lambda \mathcal{B} = \left(s_{t\sqrt{\frac{(t^2+\alpha_2^2)^\lambda - (t^2-\alpha_2^2)^\lambda}{(t^2+\alpha_2^2)^\lambda + (t^2-\alpha_2^2)^\lambda}}}, s_{t\sqrt{\frac{2c_2}{d_2+c_2}}} \right) = \left(s_{t\sqrt{\frac{a_2-b_2}{a_2+b_2}}}, s_{t\sqrt{\frac{2c_2}{d_2+c_2}}} \right),$$

then

$$\begin{aligned} \lambda \mathcal{A} \oplus \lambda \mathcal{B} &= \left(s_{t\sqrt{\frac{a_1-b_1}{a_1+b_1}}}, s_{t\sqrt{\frac{2c_1}{d_1+c_1}}} \right) \oplus \left(s_{t\sqrt{\frac{a_2-b_2}{a_2+b_2}}}, s_{t\sqrt{\frac{2c_2}{d_2+c_2}}} \right) = \left(s_{t\sqrt{\frac{a_1 a_2 - b_1 b_2}{a_1 a_2 + b_1 b_2}}}, s_{t\sqrt{\frac{2c_1 c_2}{c_1 d_2 + d_1 c_2}}} \right) \\ &= \left(s_{t\sqrt{\frac{(t^2+\alpha_1^2)^\lambda \cdot (t^2+\alpha_2^2)^\lambda - (t^2-\alpha_1^2)^\lambda \cdot (t^2-\alpha_2^2)^\lambda}{(t^2+\alpha_1^2)^\lambda \cdot (t^2+\alpha_2^2)^\lambda + (t^2-\alpha_1^2)^\lambda \cdot (t^2-\alpha_2^2)^\lambda}}}, s_{t\sqrt{\frac{\sqrt{2}\beta_1^\lambda \cdot \beta_2^\lambda}{\sqrt{(\beta_1^2)^\lambda \cdot (\beta_2^2)^\lambda + (2t^2 - \beta_1^2)^\lambda \cdot (2t^2 - \beta_2^2)^\lambda}}}} \right), \end{aligned}$$

where $a_1 = (t^2 + \alpha_1^2)^\lambda, b_1 = (t^2 - \alpha_1^2)^\lambda, c_1 = (\beta_1^2)^\lambda, d_1 = (2t^2 - (\beta_1^2)^\lambda), a_2 = (t^2 + \alpha_2^2)^\lambda, b_2 = (t^2 - \alpha_2^2)^\lambda, c_2 = (\beta_2^2)^\lambda, d_2 = (2t^2 - (\beta_2^2)^\lambda)$.

Hence, we can obtain $\lambda(\mathcal{A} \oplus \mathcal{B}) = \lambda \mathcal{A} \oplus \lambda \mathcal{B}$.

(5) Due to

$$\lambda_1 \mathcal{A} = \left(s_{t\sqrt{\frac{(t^2+\alpha_1^2)^{\lambda_1} - (t^2-\alpha_1^2)^{\lambda_1}}{(t^2+\alpha_1^2)^{\lambda_1} + (t^2-\alpha_1^2)^{\lambda_1}}}}, s_{t\sqrt{\frac{2p_1}{p_1+q_1}}} \right) = \left(s_{t\sqrt{\frac{m_1-n_1}{m_1+n_1}}}, s_{t\sqrt{\frac{2p_1}{p_1+q_1}}} \right)$$

and

$$\lambda_2 \mathcal{A} = \left(s_{t\sqrt{\frac{(t^2+\alpha_1^2)^{\lambda_2} - (t^2-\alpha_1^2)^{\lambda_2}}{(t^2+\alpha_1^2)^{\lambda_2} + (t^2-\alpha_1^2)^{\lambda_2}}}}, s_{t\sqrt{\frac{2p_2}{p_2+q_2}}} \right) = \left(s_{t\sqrt{\frac{m_2-n_2}{m_2+n_2}}}, s_{t\sqrt{\frac{2p_2}{p_2+q_2}}} \right),$$

where $m_i = (t^2 + \alpha_1^2)^{\lambda_i}, n_i = (t^2 - \alpha_1^2)^{\lambda_i}, p_i = (\beta_1^2)^{\lambda_i}, q_i = (2t^2 - (\beta_1^2)^{\lambda_i}),$ for $i = 1, 2$.

So, we have

$$\begin{aligned} \lambda_1 \cdot \mathcal{A} \oplus \lambda_2 \cdot \mathcal{A} &= \left(s_{t\sqrt{\frac{m_1-n_1}{m_1+n_1}}}, s_{t\sqrt{\frac{2p_1}{p_1+q_1}}} \right) \oplus \left(s_{t\sqrt{\frac{m_2-n_2}{m_2+n_2}}}, s_{t\sqrt{\frac{2p_2}{p_2+q_2}}} \right) = \left(s_{t\sqrt{\frac{m_1 m_2 - n_1 n_2}{m_1 m_2 + n_1 n_2}}}, s_{t\sqrt{\frac{2p_1 p_2}{p_1 p_2 + q_1 q_2}}} \right) \\ &= \left(s_{t\sqrt{\frac{(t^2+\alpha_1^2)^{\lambda_1+\lambda_2} - (t^2-\alpha_1^2)^{\lambda_1+\lambda_2}}{(t^2+\alpha_1^2)^{\lambda_1+\lambda_2} + (t^2-\alpha_1^2)^{\lambda_1+\lambda_2}}}}, s_{t\sqrt{\frac{\sqrt{2}\beta_1^{\lambda_1+\lambda_2}}{\sqrt{(2t^2-\beta_1^2)^{\lambda_1+\lambda_2} + (\beta_1^2)^{\lambda_1+\lambda_2}}}}} \right) \\ &= (\lambda_1 + \lambda_2) \mathcal{A}. \end{aligned}$$

Hence, $(\lambda_1 \oplus \lambda_2) \mathcal{A} = \lambda_1 \cdot \mathcal{A} \oplus \lambda_2 \cdot \mathcal{A}$. $\square$

4. LPFE Aggregation Operators

In this part, we shall develop several novel aggregation operators for fusing LPF information based on the novel operational laws.

4.1. LPFE Averaging Operators

In this section, we will propose several average operators of LPFN including an LPFE weighted average operator, an LPFE order weighted average (LPFEOWA) operator and an LPFE hybrid average (LPFEHA) operator.

4.1.1. LPFEWA Operator

Definition 5. Suppose that $\kappa_i = (s_{\alpha_i}, s_{\beta_i})$ ($i = 1, 2, \dots, g$) be a set of LPFNs. Then the LPFEWA operator is defined as

$$LPFEWA(\kappa_1, \kappa_2, \dots, \kappa_g) = \tau_1 \kappa_1 \oplus \tau_2 \kappa_2 \oplus \dots \oplus \tau_g \kappa_g,$$

where $\tau = (\tau_1, \tau_2, \dots, \tau_g)^T$ be the weight vector with $\tau_i \in [0, 1]$ and $\sum_{i=1}^g \tau_i = 1$.

Theorem 3. For a collection LPFNs $\kappa_i = (s_{\alpha_i}, s_{\beta_i})$ ($i = 1, 2, \dots, g$), then the fusion value generated by LPFEWA operator is also a LPFN and

$$LPFEWA(\kappa_1, \kappa_2, \dots, \kappa_g) = \left(s \sqrt{\frac{\prod_{i=1}^g (t^2 + \alpha_i^2)^{\tau_i} - \prod_{i=1}^g (t^2 - \alpha_i^2)^{\tau_i}}{\prod_{i=1}^g (t^2 + \alpha_i^2)^{\tau_i} + \prod_{i=1}^g (t^2 - \alpha_i^2)^{\tau_i}}}, s \frac{\sqrt{2} \prod_{i=1}^g \beta_i^{\tau_i}}{\sqrt{\prod_{i=1}^g (2t^2 - \beta_i^2)^{\tau_i} + \prod_{i=1}^g (\beta_i^2)^{\tau_i}}} \right), \quad (6)$$

where $\tau = (\tau_1, \tau_2, \dots, \tau_g)^T$ be the weight vector with $\tau_i \in [0, 1]$ and $\sum_{i=1}^g \tau_i = 1$.

Proof. When $g = 2$

$$LPFEWA(\kappa_1, \kappa_2) = \tau_1 \kappa_1 \oplus \tau_2 \kappa_2.$$

According to Definition 4, we get

$$\tau_1 \kappa_1 = \left(s \sqrt{\frac{(t^2 + \alpha_1^2)^{\tau_1} - (t^2 - \alpha_1^2)^{\tau_1}}{(t^2 + \alpha_1^2)^{\tau_1} + (t^2 - \alpha_1^2)^{\tau_1}}}, s \frac{\sqrt{2} \beta_1^{\tau_1}}{\sqrt{(2t^2 - \beta_1^2)^{\tau_1} + (\beta_1^2)^{\tau_1}}} \right), \tau_2 \kappa_2 = \left(s \sqrt{\frac{(t^2 + \alpha_2^2)^{\tau_2} - (t^2 - \alpha_2^2)^{\tau_2}}{(t^2 + \alpha_2^2)^{\tau_2} + (t^2 - \alpha_2^2)^{\tau_2}}}, s \frac{\sqrt{2} \beta_2^{\tau_2}}{\sqrt{(2t^2 - \beta_2^2)^{\tau_2} + (\beta_2^2)^{\tau_2}}} \right).$$

Then

$$\begin{aligned} LPFEWA(\kappa_1, \kappa_2) &= \tau_1 \kappa_1 \oplus \tau_2 \kappa_2 \\ &= \left(s \sqrt{\frac{(t^2 + \alpha_1^2)^{\tau_1} - (t^2 - \alpha_1^2)^{\tau_1} + (t^2 + \alpha_2^2)^{\tau_2} - (t^2 - \alpha_2^2)^{\tau_2}}{(t^2 + \alpha_1^2)^{\tau_1} + (t^2 - \alpha_1^2)^{\tau_1} + (t^2 + \alpha_2^2)^{\tau_2} + (t^2 - \alpha_2^2)^{\tau_2}}}, s \frac{(\frac{\sqrt{2} \beta_1^{\tau_1}}{\sqrt{(2t^2 - \beta_1^2)^{\tau_1} + (\beta_1^2)^{\tau_1}}})(\frac{\sqrt{2} \beta_2^{\tau_2}}{\sqrt{(2t^2 - \beta_2^2)^{\tau_2} + (\beta_2^2)^{\tau_2}}})}{1 + (\frac{(t^2 + \alpha_1^2)^{\tau_1} - (t^2 - \alpha_1^2)^{\tau_1}}{(t^2 + \alpha_1^2)^{\tau_1} + (t^2 - \alpha_1^2)^{\tau_1}})(\frac{(t^2 + \alpha_2^2)^{\tau_2} - (t^2 - \alpha_2^2)^{\tau_2}}{(t^2 + \alpha_2^2)^{\tau_2} + (t^2 - \alpha_2^2)^{\tau_2}})} \right) \\ &= \left(s \sqrt{\frac{((t^2 + \alpha_1^2)^{\tau_1} - (t^2 - \alpha_1^2)^{\tau_2}) - ((t^2 - \alpha_1^2)^{\tau_1} - (t^2 - \alpha_2^2)^{\tau_2})}{((t^2 + \alpha_1^2)^{\tau_1} - (t^2 - \alpha_1^2)^{\tau_2}) + ((t^2 - \alpha_1^2)^{\tau_1} - (t^2 - \alpha_2^2)^{\tau_2})}}, s \frac{\sqrt{2} (\beta_1^{\tau_1} \cdot \beta_2^{\tau_2})}{\sqrt{(2t^2 - \beta_1^2)^{\tau_1} \cdot (2t^2 - \beta_2^2)^{\tau_2} + (\beta_1^2)^{\tau_1} \cdot (\beta_2^2)^{\tau_2}}} \right). \end{aligned}$$

Hence, the result is valid for $g = 2$.

When the consequence is valid for $g = k$, we have

$$LPFEWA(A_1, A_2, \dots, A_k) = \left(s \sqrt{\frac{\prod_{i=1}^k (t^2 + \alpha_i^2)^{\tau_i} - \prod_{i=1}^k (t^2 - \alpha_i^2)^{\tau_i}}{\prod_{i=1}^k (t^2 + \alpha_i^2)^{\tau_i} + \prod_{i=1}^k (t^2 - \alpha_i^2)^{\tau_i}}}, s \frac{\sqrt{2} \prod_{i=1}^k \beta_i^{\tau_i}}{\sqrt{\prod_{i=1}^k (2t^2 - \beta_i^2)^{\tau_i} + \prod_{i=1}^k (\beta_i^2)^{\tau_i}}} \right).$$

Now, when $g = k + 1$, one has

$$\begin{aligned} LPFEWA(A_1, A_2, \dots, A_{k+1}) &= LPFEWA(A_1, A_2, \dots, A_k) \oplus \tau_{k+1} A_{k+1} \\ &= \left(s \sqrt{\frac{\prod_{i=1}^k (t^2 + \alpha_i^2)^{\tau_i} - \prod_{i=1}^k (t^2 - \alpha_i^2)^{\tau_i}}{\prod_{i=1}^k (t^2 + \alpha_i^2)^{\tau_i} + \prod_{i=1}^k (t^2 - \alpha_i^2)^{\tau_i}}}, s \frac{\sqrt{2} \prod_{i=1}^k \beta_i^{\tau_i}}{\sqrt{\prod_{i=1}^k (2t^2 - \beta_i^2)^{\tau_i} + \prod_{i=1}^k (\beta_i^2)^{\tau_i}}} \right) \\ &\oplus \left(s \sqrt{\frac{(t^2 + \alpha_{k+1}^2)^{\tau_{k+1}} - (t^2 - \alpha_{k+1}^2)^{\tau_{k+1}}}{(t^2 + \alpha_{k+1}^2)^{\tau_{k+1}} + (t^2 - \alpha_{k+1}^2)^{\tau_{k+1}}}}, s \frac{\sqrt{2} \beta_{k+1}^{\tau_{k+1}}}{\sqrt{(2t^2 - \beta_{k+1}^2)^{\tau_{k+1}} + (\beta_{k+1}^2)^{\tau_{k+1}}}} \right) \\ &= \left(s \sqrt{\frac{\prod_{i=1}^{k+1} (t^2 + \alpha_i^2)^{\tau_i} - \prod_{i=1}^{k+1} (t^2 - \alpha_i^2)^{\tau_i}}{\prod_{i=1}^{k+1} (t^2 + \alpha_i^2)^{\tau_i} + \prod_{i=1}^{k+1} (t^2 - \alpha_i^2)^{\tau_i}}}, s \frac{\sqrt{2} \prod_{i=1}^{k+1} \beta_i^{\tau_i}}{\sqrt{\prod_{i=1}^{k+1} (2t^2 - \beta_i^2)^{\tau_i} + \prod_{i=1}^{k+1} (\beta_i^2)^{\tau_i}}} \right). \end{aligned}$$

Hence, Equation (6) holds for any g , that is, the proof of Theorem 3 is finished. $\square$

Based on Theorem 3, we can easily deduce the following properties.

Theorem 4. Let $\kappa_i = (s_{\alpha_i}, s_{\beta_i}), \iota_i = (s_{\alpha'_i}, s_{\beta'_i}) (i = 1, 2, \dots, g)$ be two collections of LPFNs and $\tau = (\tau_1, \tau_2, \dots, \tau_g)^T$ is the weight vector with $\tau_i \in [0, 1]$ and $\sum_{i=1}^g \tau_i = 1$. We can deduce the following properties:
T1 (Idempotency): If $\kappa_i = (s_{\alpha_i}, s_{\beta_i}) = (s_{\alpha}, s_{\beta})$ for all i , then

$$LPFEWA(\kappa_1, \kappa_2, \dots, \kappa_g) = (s_{\alpha}, s_{\beta}).$$

T2 (Monotonicity): If $\kappa_i \leq \iota_i$, that is, $s_{\alpha_i} \leq s_{\alpha'_i}$ and $s_{\beta_i} \geq s_{\beta'_i}$, then we have

$$LPFEWA(\kappa_1, \kappa_2, \dots, \kappa_g) \leq LPFEWA(\iota_1, \iota_2, \dots, \iota_g).$$

T3 (Boundedness): Suppose $\kappa^- = \min(\kappa_1, \kappa_2, \dots, \kappa_g), \kappa^+ = \max(\kappa_1, \kappa_2, \dots, \kappa_g)$, then

$$\kappa^- \leq LPFEWA(\kappa_1, \kappa_2, \dots, \kappa_g) \leq \kappa^+.$$

Proof. Since $\kappa_i = (s_{\alpha_i}, s_{\beta_i}), \iota_i = (s_{\alpha'_i}, s_{\beta'_i}) (i = 1, 2, \dots, g)$ be two sets of LPFNs. Then

(T1) when $\kappa_i = (s_{\alpha_i}, s_{\beta_i}) = (s_{\alpha}, s_{\beta})$ for each i , one has

$$\begin{aligned} LPFEWA(\kappa_1, \kappa_2, \dots, \kappa_g) &= \left(s \sqrt{\frac{\prod_{i=1}^g (t^2 + \alpha_i^2)^{\tau_i} - \prod_{i=1}^g (t^2 - \alpha_i^2)^{\tau_i}}{\prod_{i=1}^g (t^2 + \alpha_i^2)^{\tau_i} + \prod_{i=1}^g (t^2 - \alpha_i^2)^{\tau_i}}}, s \frac{\sqrt{2} \prod_{i=1}^g \beta_i^{\tau_i}}{\sqrt{\prod_{i=1}^g (2t^2 - \beta_i^2)^{\tau_i} + \prod_{i=1}^g (\beta_i^2)^{\tau_i}}} \right) \\ &= \left(s \sqrt{\frac{(t^2 + \alpha_i^2)^{\sum_{i=1}^g \tau_i} - (t^2 - \alpha_i^2)^{\sum_{i=1}^g \tau_i}}{(t^2 + \alpha_i^2)^{\sum_{i=1}^g \tau_i} + (t^2 - \alpha_i^2)^{\sum_{i=1}^g \tau_i}}}, s \frac{\sqrt{2} \beta_i^{\sum_{i=1}^g \tau_i}}{\sqrt{(2t^2 - \beta_i^2)^{\sum_{i=1}^g \tau_i} + (\beta_i^2)^{\sum_{i=1}^g \tau_i}}} \right) \\ &= \left(s_{t(\frac{\alpha_i}{t})}, s_{t(\frac{\beta_i}{t})} \right) = \kappa_i. \end{aligned}$$

(T2) Since $\kappa_i \leq \iota_i$ for all i , then $\tau_i \kappa_i \leq \tau_i \iota_i$. Accordingly, we can deduce $\oplus_{i=1}^g \tau_i \kappa_i \leq \oplus_{i=1}^g \tau_i \iota_i$. For $LPFEWA(\kappa_1, \kappa_2, \dots, \kappa_g) = \oplus_{i=1}^g \tau_i \kappa_i$ and $LPFEWA(\iota_1, \iota_2, \dots, \iota_g) = \oplus_{i=1}^g \tau_i \iota_i$, we can generate $LPFEWA(\kappa_1, \kappa_2, \dots, \kappa_g) \leq LPFEWA(\iota_1, \iota_2, \dots, \iota_g)$.

(T3) Since $\kappa^- = \min(\kappa_1, \kappa_2, \dots, \kappa_g)$, $\kappa^+ = \max(\kappa_1, \kappa_2, \dots, \kappa_g)$. By means of Theorem (T2), we have

$$LPFEWA(\kappa^-, \kappa^-, \dots, \kappa^-) \leq LPFEWA(\kappa_1, \kappa_2, \dots, \kappa_g) \leq LPFEWA(\kappa^+, \kappa^+, \dots, \kappa^+).$$

Furthermore, by means of Theorem (T1), we have

$$LPFEWA(\kappa^+, \kappa^+, \dots, \kappa^+) = \kappa^+, \quad LPFEWA(\kappa^-, \kappa^-, \dots, \kappa^-) = \kappa^-.$$

Accordingly, we can deduce $\kappa^- \leq LPFEWA(\kappa_1, \kappa_2, \dots, \kappa_g) \leq \kappa^+$. $\square$

4.1.2. LPFEOWA Operator

Definition 6. Suppose that $\kappa_i = (s_{\alpha_i}, s_{\beta_i}) (i = 1, 2, \dots, g)$ be a set of LPFNs. Then the LPFEOWA Operator is depicted as:

$$LPFEOWA(\kappa_1, \kappa_2, \dots, \kappa_g) = \tau_1 \kappa_{\sigma(1)} \oplus \tau_2 \kappa_{\sigma(2)} \oplus \dots \oplus \tau_g \kappa_{\sigma(g)},$$

where $(\sigma(1), \sigma(2), \dots, \sigma(g))$ is a permutation of $(i = 1, 2, \dots, g)$, such that $\kappa_{\sigma(i-1)} \geq \kappa_{\sigma(i)}$ for each i , $\tau = (\tau_1, \tau_2, \dots, \tau_g)^T$ is the weight vector with $\tau_i \in [0, 1]$ and $\sum_{i=1}^g \tau_i = 1$.

Similar to LPFWA operator, we can deduce the following theorems of LPEOWA operator.

Theorem 5. Let $\kappa_i = (s_{\alpha_i}, s_{\beta_i})$ be a set of LPFNs, then the fusion result by LPFWA operator is deduced as:

$$LPFEOWA(\kappa_1, \kappa_2, \dots, \kappa_g) = \left(s \sqrt{\frac{\prod_{i=1}^g (t^2 + \alpha_{\sigma(i)}^2)^{\tau_i} - \prod_{i=1}^g (t^2 - \alpha_{\sigma(i)}^2)^{\tau_i}}{\prod_{i=1}^g (t^2 + \alpha_{\sigma(i)}^2)^{\tau_i} + \prod_{i=1}^g (t^2 - \alpha_{\sigma(i)}^2)^{\tau_i}}}, s \frac{\sqrt{2} \prod_{i=1}^g (\beta_{\sigma(i)})^{\tau_i}}{\sqrt{\prod_{i=1}^g (2t^2 - \beta_{\sigma(i)}^2)^{\tau_i} + \prod_{i=1}^g (\beta_{\sigma(i)}^2)^{\tau_i}}} \right), \quad (7)$$

where $(\sigma(1), \sigma(2), \dots, \sigma(g))$ is a permutation of $(i = 1, 2, \dots, g)$, such that $\kappa_{\sigma(i-1)} \geq \kappa_{\sigma(i)}$ for each i , $\tau = (\tau_1, \tau_2, \dots, \tau_g)^T$ is the weight vector with $\tau_i \in [0, 1]$ and $\sum_{i=1}^g \tau_i = 1$. Apparently, if $w = (\frac{1}{g}, \frac{1}{g}, \dots, \frac{1}{g})$, the LPFEOWA operator will reduce to LPFWA operator.

Similar to the LPFEWA operator, it is easy to prove that the proposed operators satisfying the following properties.

Theorem 6. Let $\kappa_i = (s_{\alpha_i}, s_{\beta_i})$, $\iota_i = (s_{\alpha'_i}, s_{\beta'_i}) (i = 1, 2, \dots, g)$ be two collections of LPFNs and $\tau = (\tau_1, \tau_2, \dots, \tau_g)^T$ is the weight vector with $\tau_i \in [0, 1]$ and $\sum_{i=1}^g \tau_i = 1$. We can deduce the following properties:

T1 (Idempotency): If $\kappa_i = (s_{\alpha_i}, s_{\beta_i}) = (s_{\alpha}, s_{\beta})$ for all i , then

$$LPFEOWA(\kappa_1, \kappa_2, \dots, \kappa_g) = (s_{\alpha}, s_{\beta}).$$

T2 (Monotonicity): If $\kappa_i \leq \iota_i$ for all i , then

$$LPFEOWA(\kappa_1, \kappa_2, \dots, \kappa_g) \leq LPFEOWA(\iota_1, \iota_2, \dots, \iota_g).$$

T3 (Boundedness): Suppose $\kappa^- = \min(\kappa_1, \kappa_2, \dots, \kappa_g)$, $\kappa^+ = \max(\kappa_1, \kappa_2, \dots, \kappa_g)$, then

$$\kappa^- \leq LPFEOWA(\kappa_1, \kappa_2, \dots, \kappa_g) \leq \kappa^+.$$

T4 (Commutativity): Let $\iota_i = (s_{\alpha_i}, s_{\beta_i}) (i = 1, 2, \dots, g)$ is any permutation of $\kappa_i = (s_{\alpha_i}, s_{\beta_i})$. Then

$$LPFEOWA(\kappa_1, \kappa_2, \dots, \kappa_g) = LPFEOWA(\iota_1, \iota_2, \dots, \iota_g).$$

The proof is analogous to the Theorem 4, so we omit it here.

4.1.3. LPFEHA Operator

Definition 7. Let $\kappa_i = (s_{\alpha_i}, s_{\beta_i}) \in \Delta, (i = 1, 2, \dots, g)$. The LPFEHA operator is defined as

$$LPFEHA_{\omega, \tau}(\kappa_1, \kappa_2, \dots, \kappa_g) = \tau_1 \dot{\kappa}_{\sigma(1)} \oplus \tau_2 \dot{\kappa}_{\sigma(2)} \oplus \dots \oplus \tau_g \dot{\kappa}_{\sigma(g)} \quad (8)$$

$$= \left(S \sqrt{\frac{\prod_{i=1}^g (t^2 + (\dot{\kappa}_{\sigma(i)})^2)^{\tau_i} - \prod_{i=1}^g (t^2 - (\dot{\kappa}_{\sigma(i)})^2)^{\tau_i}}{\prod_{i=1}^g (t^2 + (\dot{\kappa}_{\sigma(i)})^2)^{\tau_i} + \prod_{i=1}^g (t^2 - (\dot{\kappa}_{\sigma(i)})^2)^{\tau_i}}}, S \frac{\sqrt{2} \prod_{i=1}^g (\dot{\beta}_{\sigma(i)})^{\tau_i}}{\sqrt{\prod_{i=1}^g (2t^2 - (\dot{\beta}_{\sigma(i)})^2)^{\tau_i} + \prod_{i=1}^g (\dot{\beta}_{\sigma(i)}^2)^{\tau_i}}} \right),$$

where $\tau = (\tau_1, \tau_2, \dots, \tau_g)^T$ is the weight vector with $\tau_i \in [0, 1]$ and $\sum_{i=1}^g \tau_i = 1$. and $\dot{\kappa}_i = g\omega_i \alpha_i (i = 1, 2, \dots, g)$, $(\dot{\kappa}_{\sigma(1)}, \dot{\kappa}_{\sigma(2)}, \dots, \dot{\kappa}_{\sigma(g)})$ is any permutation of a collection of the weighted LPFNs $(\dot{\kappa}_1, \dot{\kappa}_2, \dots, \dot{\kappa}_n)$ such that $\dot{\kappa}_{\sigma(i)} \geq \dot{\kappa}_{\sigma(i+1)} (i = 1, 2, \dots, n-1)$; $\omega = (\omega_1, \omega_2, \dots, \omega_g)^T$ is the weight vector of $\kappa_i (i = 1, 2, \dots, g)$, with $\omega_i \in [0, 1] (i = 1, 2, \dots, g)$, $\sum_{i=1}^g \omega_i = 1$, g is the balancing coefficient.

Theorem 7. The LPFEWA and LPFEOWA operators are a particular case of the LPFEHA operators.

Proof. (1) If $\tau = (\frac{1}{g}, \frac{1}{g}, \dots, \frac{1}{g})^T$. Then,

$$\begin{aligned} LPFEHA_{\omega, \tau}(\kappa_1, \kappa_2, \dots, \kappa_g) &= \tau_1 \dot{\kappa}_{\sigma(1)} \oplus \tau_2 \dot{\kappa}_{\sigma(2)} \oplus \dots \oplus \tau_g \dot{\kappa}_{\sigma(g)} \\ &= \frac{1}{g} (\dot{\kappa}_{\sigma(1)} \oplus \dot{\kappa}_{\sigma(2)} \oplus \dots \oplus \dot{\kappa}_{\sigma(g)}) \\ &= \omega_1 \kappa_1 \oplus \omega_2 \kappa_2 \oplus \dots \omega_g \kappa_g \\ &= LPFEWA_{\omega}(\kappa_1, \kappa_2, \dots, \kappa_g) \end{aligned}$$

(2) If $\omega = (\frac{1}{g}, \frac{1}{g}, \dots, \frac{1}{g})^T$ and $\dot{\kappa}_i = \kappa_i (i = 1, 2, \dots, g)$. Then

$$\begin{aligned} LPFEHA_{\tau, \omega}(\kappa_1, \kappa_2, \dots, \kappa_g) &= \tau_1 \dot{\kappa}_{\sigma(1)} \oplus \tau_2 \dot{\kappa}_{\sigma(2)} \oplus \dots \oplus \tau_g \dot{\kappa}_{\sigma(g)} \\ &= \tau_1 \kappa_{\sigma(1)} \oplus \tau_2 \kappa_{\sigma(2)} \oplus \dots \oplus \tau_g \kappa_{\sigma(g)} \\ &= LPFEOWA_{\tau}(\kappa_1, \kappa_2, \dots, \kappa_g) \end{aligned}$$

□

4.2. LPFE Geometric Operators

In this section, we will propose several geometric operators of LPFN including LPFE weighted geometric operator, LPFE order weighted geometric (LPFEOWG) operator and LPFE hybrid geometric (LPFEHG) operator.

4.2.1. LPFEWG Operator

Definition 8. Suppose that $\kappa_i = (s_{\alpha_i}, s_{\beta_i}) (i = 1, 2, \dots, g)$ be a set of LPFNs. Then the LPFEWA operator is defined as follows:

$$LPFEWG(\kappa_1, \kappa_2, \dots, \kappa_g) = \kappa_1^{\tau_1} \oplus \kappa_2^{\tau_2} \oplus \dots \oplus \kappa_g^{\tau_g},$$

where $\tau = (\tau_1, \tau_2, \dots, \tau_g)^T$ is the weight vector with $\tau_i \in [0, 1]$ and $\sum_{i=1}^g \tau_i = 1$.

Theorem 8. Assume that $\kappa_i = (s_{\alpha_i}, s_{\beta_i}) (i = 1, 2, \dots, g)$ be a set of LPFNs. The aggregated value of them using the LPFWG operator is also a LPFN and

$$LPFEWG(\kappa_1, \kappa_2, \dots, \kappa_g) = \left(s, t \frac{\sqrt{2} \prod_{i=1}^g \alpha_i^{\tau_i}}{\sqrt{\prod_{i=1}^g (2t^2 - \alpha_i^2)^{\tau_i} + \prod_{i=1}^g (\alpha_i^2)^{\tau_i}}} , s \sqrt{\frac{\prod_{i=1}^g (t^2 + \beta_i^2)^{\tau_i} - \prod_{i=1}^g (t^2 - \beta_i^2)^{\tau_i}}{\prod_{i=1}^g (t^2 + \beta_i^2)^{\tau_i} + \prod_{i=1}^g (t^2 - \beta_i^2)^{\tau_i}}} \right), \quad (9)$$

where $\tau = (\tau_1, \tau_2, \dots, \tau_g)^T$ is the weight vector with $\tau_i \in [0, 1]$ and $\sum_{i=1}^g \tau_i = 1$.

Proof. When $g = 2$, we have

$$LPFEWG(\kappa_1, \kappa_2) = \kappa_1^{\tau_1} \otimes \kappa_2^{\tau_2}.$$

Then

$$\kappa_1^{\tau_1} = \left(s, t \frac{\sqrt{2} \alpha_1^{\tau_1}}{\sqrt{(2t^2 - \alpha_1^2)^{\tau_1} + (\alpha_1^2)^{\tau_1}}} , s \sqrt{\frac{(t^2 + \beta_1^2)^{\tau_1} - (t^2 - \beta_1^2)^{\tau_1}}{(t^2 + \beta_1^2)^{\tau_1} + (t^2 - \beta_1^2)^{\tau_1}}} \right), \quad \kappa_2^{\tau_2} = \left(s, t \frac{\sqrt{2} \alpha_2^{\tau_2}}{\sqrt{(2t^2 - \alpha_2^2)^{\tau_2} + (\alpha_2^2)^{\tau_2}}} , s \sqrt{\frac{(t^2 + \beta_2^2)^{\tau_2} - (t^2 - \beta_2^2)^{\tau_2}}{(t^2 + \beta_2^2)^{\tau_2} + (t^2 - \beta_2^2)^{\tau_2}}} \right).$$

Hence,

$$\begin{aligned} LPFEWG(\kappa_1, \kappa_2) &= \kappa_1^{\tau_1} \oplus \kappa_2^{\tau_2} \\ &= \left(s, t \frac{\left(\frac{\sqrt{2} \alpha_1^{\tau_1}}{\sqrt{(2t^2 - \alpha_1^2)^{\tau_1} + (\alpha_1^2)^{\tau_1}}} \right) \left(\frac{\sqrt{2} \alpha_2^{\tau_2}}{\sqrt{(2t^2 - \alpha_2^2)^{\tau_2} + (\alpha_2^2)^{\tau_2}}} \right)}{\sqrt{1 + \left(\frac{(t^2 + \beta_1^2)^{\tau_1} - (t^2 - \beta_1^2)^{\tau_1}}{(t^2 + \beta_1^2)^{\tau_1} + (t^2 - \beta_1^2)^{\tau_1}} \right) \left(\frac{(t^2 + \beta_2^2)^{\tau_2} - (t^2 - \beta_2^2)^{\tau_2}}{(t^2 + \beta_2^2)^{\tau_2} + (t^2 - \beta_2^2)^{\tau_2}} \right)}} , s \sqrt{\frac{\left(\frac{(t^2 + \beta_1^2)^{\tau_1} - (t^2 - \beta_1^2)^{\tau_1}}{(t^2 + \beta_1^2)^{\tau_1} + (t^2 - \beta_1^2)^{\tau_1}} \right) \left(\frac{(t^2 + \beta_2^2)^{\tau_2} - (t^2 - \beta_2^2)^{\tau_2}}{(t^2 + \beta_2^2)^{\tau_2} + (t^2 - \beta_2^2)^{\tau_2}} \right)}{1 + \left(\frac{(t^2 + \beta_1^2)^{\tau_1} - (t^2 - \beta_1^2)^{\tau_1}}{(t^2 + \beta_1^2)^{\tau_1} + (t^2 - \beta_1^2)^{\tau_1}} \right) \left(\frac{(t^2 + \beta_2^2)^{\tau_2} - (t^2 - \beta_2^2)^{\tau_2}}{(t^2 + \beta_2^2)^{\tau_2} + (t^2 - \beta_2^2)^{\tau_2}} \right)}} \right) \\ &= \left(s, t \frac{\sqrt{2} (\alpha_1^{\tau_1} \cdot \alpha_2^{\tau_2})}{\sqrt{(2t^2 - \alpha_1^2)^{\tau_1} \cdot (2t^2 - \alpha_2^2)^{\tau_2} + (\alpha_1^2)^{\tau_1} \cdot (\alpha_2^2)^{\tau_2}}} , s \sqrt{\frac{((t^2 + \beta_1^2)^{\tau_1} \cdot ((t^2 + \beta_2^2)^{\tau_2}) - ((t^2 - \beta_1^2)^{\tau_1} \cdot ((t^2 - \beta_2^2)^{\tau_2}))}{((t^2 + \beta_1^2)^{\tau_1} \cdot ((t^2 + \beta_2^2)^{\tau_2}) + ((t^2 - \beta_1^2)^{\tau_1} \cdot ((t^2 - \beta_2^2)^{\tau_2}))}} \right) \end{aligned}$$

Hence, the result is valid for $g = 2$.

When the consequence is valid for $g = k$, we have

$$LPFEWG(\kappa_1, \kappa_2, \dots, \kappa_k) = \left(s, t \frac{\sqrt{2} \prod_{i=1}^k \alpha_i^{\tau_i}}{\sqrt{\prod_{i=1}^k (2t^2 - \alpha_i^2)^{\tau_i} + \prod_{i=1}^k (\alpha_i^2)^{\tau_i}}} , s \sqrt{\frac{\prod_{i=1}^k (t^2 + \beta_i^2)^{\tau_i} - \prod_{i=1}^k (t^2 - \beta_i^2)^{\tau_i}}{\prod_{i=1}^k (t^2 + \beta_i^2)^{\tau_i} + \prod_{i=1}^k (t^2 - \beta_i^2)^{\tau_i}}} \right).$$

Now, when $g = k + 1$, we have

$$\begin{aligned} LPFEWG(A_1, A_2, \dots, A_{k+1}) &= LPFEWG(A_1, A_2, \dots, A_k) \oplus A_{k+1}^{\tau_{k+1}} \\ &= \left(s, t \frac{\sqrt{2} \prod_{i=1}^k \alpha_i^{\tau_i}}{\sqrt{\prod_{i=1}^k (2t^2 - \alpha_i^2)^{\tau_i} + \prod_{i=1}^k (\alpha_i^2)^{\tau_i}}} , s \sqrt{\frac{\prod_{i=1}^k (t^2 + \beta_i^2)^{\tau_i} - \prod_{i=1}^k (t^2 - \beta_i^2)^{\tau_i}}{\prod_{i=1}^k (t^2 + \beta_i^2)^{\tau_i} + \prod_{i=1}^k (t^2 - \beta_i^2)^{\tau_i}}} \right) \\ &\otimes \left(s, t \frac{\sqrt{2} \alpha_{k+1}^{\tau_{k+1}}}{\sqrt{(2t^2 - \alpha_{k+1}^2)^{\tau_{k+1}} + (\alpha_{k+1}^2)^{\tau_{k+1}}}} , s \sqrt{\frac{(t^2 + \beta_{k+1}^2)^{\tau_{k+1}} - (t^2 - \beta_{k+1}^2)^{\tau_{k+1}}}{(t^2 + \beta_{k+1}^2)^{\tau_{k+1}} + (t^2 - \beta_{k+1}^2)^{\tau_{k+1}}}} \right) \\ &= \left(s, t \frac{\sqrt{2} \prod_{i=1}^{k+1} \alpha_i^{\tau_i}}{\sqrt{\prod_{i=1}^{k+1} (2t^2 - \alpha_i^2)^{\tau_i} + \prod_{i=1}^{k+1} (\alpha_i^2)^{\tau_i}}} , s \sqrt{\frac{\prod_{i=1}^{k+1} (t^2 + \beta_i^2)^{\tau_i} - \prod_{i=1}^{k+1} (t^2 - \beta_i^2)^{\tau_i}}{\prod_{i=1}^{k+1} (t^2 + \beta_i^2)^{\tau_i} + \prod_{i=1}^{k+1} (t^2 - \beta_i^2)^{\tau_i}}} \right). \end{aligned}$$

Hence, Equation (9) holds for any $g = k + 1$, the proof of Theorem 8 is completed. $\square$

Based on Theorem 8, it is easy to deduce the following properties of the LPFEWG operator.

Theorem 9. Let $\kappa_i = (s_{\alpha_i}, s_{\beta_i}), l_i = (s_{\alpha'_i}, s_{\beta'_i}) (i = 1, 2, \dots, g)$ be a collection of LPFNs and $\tau = (\tau_1, \tau_2, \dots, \tau_g)^T$ is the weight vector with $\tau_i \in [0, 1]$ and $\sum_{i=1}^g \tau_i = 1$. We can obtain that the following properties:

T1 (Idempotency): If $\kappa_i = (s_{\alpha_i}, s_{\beta_i}) = (s_{\alpha}, s_{\beta})$ for all i , then

$$LPFEWG(\kappa_1, \kappa_2, \dots, \kappa_g) = (s_{\alpha}, s_{\beta}).$$

T2 (Monotonicity): If $\kappa_i \leq l_i$ for all i , then

$$LPFEWG(\kappa_1, \kappa_2, \dots, \kappa_g) \leq LPFEWG(l_1, l_2, \dots, l_g).$$

T3 (Boundedness): Suppose $\kappa^- = \min(\kappa_1, \kappa_2, \dots, \kappa_g), \kappa^+ = \max(\kappa_1, \kappa_2, \dots, \kappa_g)$, then

$$\kappa^- \leq LPFEWG(\kappa_1, \kappa_2, \dots, \kappa_g) \leq \kappa^+.$$

The proof is analogous to Theorem 4.

4.2.2. LPFEOWG Operator

Definition 9. Suppose that $\kappa_i = (s_{\alpha_i}, s_{\beta_i}) (i = 1, 2, \dots, g)$ be a set of LPFNs. Then the LPFEWA operator is defined as follows:

$$LPFEOWG(\kappa_1, \kappa_2, \dots, \kappa_g) = \kappa_{\sigma(1)}^{\tau_1} \oplus \kappa_{\sigma(2)}^{\tau_2} \oplus \dots \oplus \kappa_{\sigma(g)}^{\tau_g},$$

where $(\sigma(1), \sigma(2), \dots, \sigma(g))$ is a permutation of $(i = 1, 2, \dots, g)$, such that $\kappa_{\sigma(i-1)} \geq \kappa_{\sigma(i)}$ for each i , $\tau = (\tau_1, \tau_2, \dots, \tau_g)^T$ is the weight vector with $\tau_i \in [0, 1]$ and $\sum_{i=1}^g \tau_i = 1$.

Theorem 10. Let $A = \{A_i = (s_{\alpha_i}, s_{\beta_i}) | i = 1, 2, \dots, n\}$, then the LPFEOWG operator is defined as follows:

$$LPFEOWG(\kappa_1, \kappa_2, \dots, \kappa_g) = \left(s \frac{\sqrt{2} \prod_{i=1}^g (s_{\sigma(i)})^{w_i}}{\sqrt{\prod_{i=1}^g (2t^2 - a_{\sigma(i)}^2)^{\tau_i} + \prod_{i=1}^g (a_{\sigma(i)}^2)^{\tau_i}}}, s \frac{\sqrt{\prod_{i=1}^g (t^2 + a_{\sigma(i)}^2)^{\tau_i} - \prod_{i=1}^g (t^2 - a_{\sigma(i)}^2)^{\tau_i}}}{\sqrt{\prod_{i=1}^g (t^2 + a_{\sigma(i)}^2)^{\tau_i} + \prod_{i=1}^g (t^2 - a_{\sigma(i)}^2)^{\tau_i}}} \right), \quad (10)$$

where $(\sigma(1), \sigma(2), \dots, \sigma(g))$ is a permutation of $(i = 1, 2, \dots, g)$, such that $\kappa_{\sigma(i-1)} \geq \kappa_{\sigma(i)}$ for each i , $\tau = (\tau_1, \tau_2, \dots, \tau_g)^T$ is the weight vector with $\tau_i \in [0, 1]$ and $\sum_{i=1}^g \tau_i = 1$. Apparently, if $w = (\frac{1}{g}, \frac{1}{g}, \dots, \frac{1}{g})$, the LPFEOWA operator will reduce to LPFWA operator.

The LPFEOWG operator also meets the theorems deduced in Theorem 4.

4.2.3. LPFEHG Operator

Definition 10. Let $\kappa_i = (s_{\alpha_i}, s_{\beta_i}) \in \Delta, (i = 1, 2, \dots, g)$. The LPFEHG operator is defined as

$$LPFEHG_{\omega, w}(\kappa_1, \kappa_2, \dots, \kappa_g) = \check{\kappa}_{\sigma(1)}^{\tau_1} \oplus \check{\kappa}_{\sigma(2)}^{\tau_2} \oplus \dots \oplus \check{\kappa}_{\sigma(g)}^{\tau_g} \quad (11)$$

$$= \left(s \frac{\sqrt{2} \prod_{i=1}^g (\check{\alpha}_{\sigma(i)})^{w_i}}{\sqrt{\prod_{i=1}^g (2t^2 - (\check{\alpha}_{\sigma(i)})^2)^{w_i} + \prod_{i=1}^g (\check{\alpha}_{\sigma(i)})^{2w_i}}}, s \frac{\sqrt{\prod_{i=1}^g (t^2 + (\check{\beta}_{\sigma(i)})^2)^{w_i} - \prod_{i=1}^g (t^2 - (\check{\beta}_{\sigma(i)})^2)^{w_i}}}{\sqrt{\prod_{i=1}^g (t^2 + (\check{\beta}_{\sigma(i)})^2)^{w_i} + \prod_{i=1}^g (t^2 - (\check{\beta}_{\sigma(i)})^2)^{w_i}}} \right),$$

where $\tau = (\tau_1, \tau_2, \dots, \tau_g)^T$ is the weight vector with $\tau_i \in [0, 1]$ and $\sum_{i=1}^g \tau_i = 1$. and $\kappa_i = g\omega_i\alpha_i$ ($i = 1, 2, \dots, g$), $(\kappa_{\sigma(1)}, \kappa_{\sigma(2)}, \dots, \kappa_{\sigma(g)})$ is any permutation of a collection of the weighted LPFNs $(\kappa_1, \kappa_2, \dots, \kappa_n)$ such that $\kappa_{\sigma(i)} \geq \kappa_{\sigma(i+1)}$ ($i = 1, 2, \dots, n-1$); $\omega = (\omega_1, \omega_2, \dots, \omega_g)^T$ is the weight vector of κ_i ($i = 1, 2, \dots, g$), with $\omega_i \in [0, 1]$ ($i = 1, 2, \dots, g$), $\sum_{i=1}^g \omega_i = 1$, g is the balancing coefficient.

Theorem 11. The LPFEWG and LPFEOWG operators are a special case of the LPFEHG operators.

Proof. (1) If $w = (\frac{1}{g}, \frac{1}{g}, \dots, \frac{1}{g})^T$. Then,

$$\begin{aligned} LPFEHA_{\tau, \omega}(\kappa_1, \kappa_2, \dots, \kappa_n) &= \check{\kappa}_{\sigma(1)}^{\tau_1} \oplus \check{\kappa}_{\sigma(2)}^{\tau_2} \oplus \dots \oplus \check{\kappa}_{\sigma(g)}^{\tau_g} \\ &= (\check{\kappa}_{\sigma(1)} \oplus \check{\kappa}_{\sigma(2)} \oplus \dots \oplus \check{\kappa}_{\sigma(g)})^{\frac{1}{g}} \\ &= \kappa_1^{\omega_1} \oplus \kappa_2^{\omega_2} \oplus \dots \oplus \kappa_n^{\omega_g} \\ &= LPFWG_{\omega}(\kappa_1, \kappa_2, \dots, \kappa_g) \end{aligned}$$

(2) If $\omega = (\frac{1}{g}, \frac{1}{g}, \dots, \frac{1}{g})^T$ and $\check{\kappa}_i = \kappa_i$ ($i = 1, 2, \dots, g$). Then,

$$\begin{aligned} LPFEHA_{\tau, \omega}(\kappa_1, \kappa_2, \dots, \kappa_n) &= \check{\kappa}_{\sigma(1)}^{\tau_1} \oplus \check{\kappa}_{\sigma(2)}^{\tau_2} \oplus \dots \oplus \check{\kappa}_{\sigma(g)}^{\tau_g} \\ &= \kappa_{\sigma(1)}^{\tau_1} \oplus \kappa_{\sigma(2)}^{\tau_2} \oplus \dots \oplus \kappa_{\sigma(g)}^{\tau_g} \\ &= LPFOWG_{\tau}(\kappa_1, \kappa_2, \dots, \kappa_g) \end{aligned}$$

□

5. The Developed Decision Making Approach

In this part, we will develop an MAGDM approach based on the presented LPFE operators to cope with DM issues with LPF information.

Let $Y = \{Y_1, Y_2, \dots, Y_m\}$ be a group of m alternatives and $\mathfrak{S} = \{\mathfrak{S}_1, \mathfrak{S}_2, \dots, \mathfrak{S}_n\}$ be a group of attributes whose weight vector $\tau = (\tau_1, \tau_2, \dots, \tau_n)^T$, with $\tau_j \in [0, 1]$ ($j = 1, 2, \dots, n$) and $\sum_{j=1}^n \tau_j = 1$. Assume that there is a collection of δ experts $Z = \{Z_1, Z_2, \dots, Z_{\delta}\}$ with weight vector $\lambda = \{\lambda_1, \lambda_2, \dots, \lambda_{\delta}\}^T$, with $\lambda_k \in [0, 1]$ ($k = 1, 2, \dots, \delta$) and $\sum_{k=1}^{\delta} \lambda_k = 1$. Expert Z_k provides the evaluation information on alternative Y_i with respect to the attribute $\mathfrak{S}_j$ in the form of LPFNs, $q_{ij}^k = (s_{\alpha_{ij}}^k, s_{\beta_{ij}}^k)$, where $s_{\alpha_{ij}}^k, s_{\beta_{ij}}^k \in \bar{S}$. Then the LPF decision matrices $Q^k = (q_{ij}^k)_{m \times n}$ are constructed. The decision process is described in the following (the procedure of the propounded MAGDM approach is shown in Figure 1).

Step 1: Normalize the decision matrices. Because there are two kinds of attribute, we need transform them to the same type attribute. That is, normalize the decision matrices Q^k into a normalization matrices $A^k = (a_{ij}^k)_{m \times n}$, where

$$a_{ij} = \begin{cases} (s_{\alpha_{ij}}^k, s_{\beta_{ij}}^k) & \text{for benefit attribute } \mathfrak{S}_j. \\ (s_{\beta_{ij}}^k, s_{\alpha_{ij}}^k) & \text{for cost attribute } \mathfrak{S}_j. \end{cases}$$

Step 2: Utilize the LPFOWA operator to fuse all the DMs' information.

$$q_{ij} = (s_{\alpha_{ij}}, s_{\beta_{ij}}) = LPFEOWA(q_{ij}^1, q_{ij}^2, \dots, q_{ij}^{\delta}),$$

or the LPFEOWG operator

$$q_{ij} = (s_{\alpha_{ij}}, s_{\beta_{ij}}) = LPFEOWG(q_{ij}^1, q_{ij}^2, \dots, q_{ij}^{\delta}).$$

Step 3: Utilize the LPFEWA operator to aggregate all q_{ij} for each alternative Y_i as follows.

$$q_i = \text{LPFEWA}(q_{i1}, q_{i2}, \dots, q_{in})$$

or the LPFEWG operators

$$q_i = \text{LPFEWG}(q_{i1}, q_{i2}, \dots, q_{in}).$$

Step 4: Calculating the score values $\mathcal{T}(Y_i)$ for each q_i .

Step 5: Rank all alternatives Y_i ($i = 1, 2, \dots, m$).

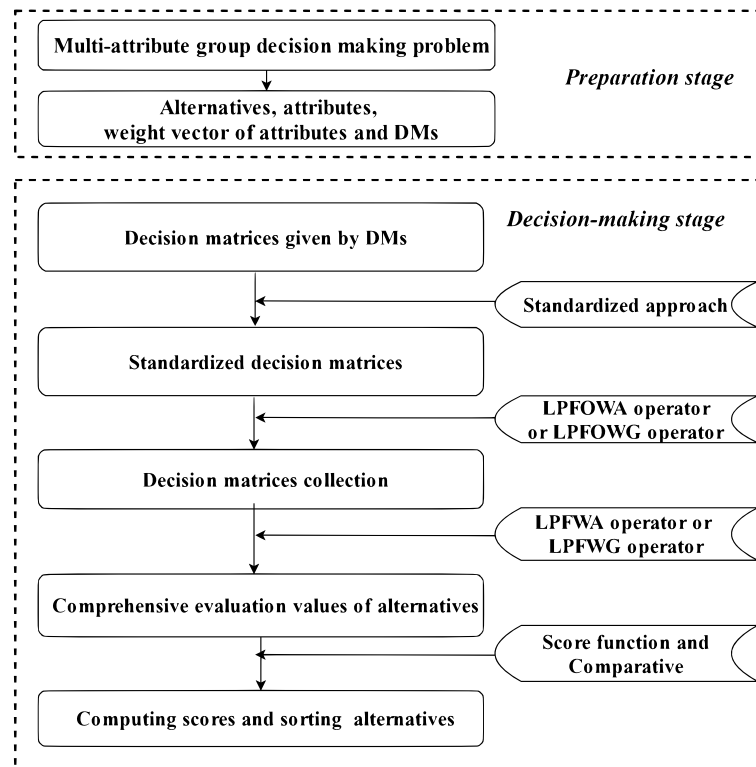


Figure 1. The decision process of the propounded MAGDM approach.

6. Numerical Example and Comparative Analysis

In this section, we utilize the proposed MAGDM approach to deal with practical problems and perform a comparative analysis to show its effectiveness and merits.

6.1. Numerical Example

Example 1. This practical example is cited and applied from Reference [57], which is about a company selecting a supplier. Assuming that $Y = \{Y_1, Y_2, Y_3, Y_4\}$ is a set of four optional suppliers as alternatives and the group of attributes $\mathfrak{S} = \{\mathfrak{S}_1, \mathfrak{S}_2, \mathfrak{S}_3, \mathfrak{S}_4, \mathfrak{S}_5\}$ and the weight of the attribute is $\tau = (0.25, 0.2, 0.15, 0.18, 0.22)^T$. The alternatives $Y = \{Y_1, Y_2, Y_3, Y_4\}$ are evaluated with respect to the attributes $\mathfrak{S} = \{\mathfrak{S}_1, \mathfrak{S}_2, \mathfrak{S}_3, \mathfrak{S}_4, \mathfrak{S}_5\}$ by DMs' using the LPFNs based on the linguistic term set $\mathcal{S} = \{s_0 = \text{extremely poor}, s_1 = \text{very poor}, s_2 = \text{poor}, s_3 = \text{slightly poor}, s_4 = \text{fair}, s_5 = \text{slightly good}, s_6 = \text{good}, s_7 = \text{very good}, s_8 = \text{extremely good}\}$. Assume that the weight vector of DMs' is $\lambda = (0.25, 0.3, 0.2, 0.25)^T$. Then the LPF decision matrices $A_k = (a_{ij}^k)_{4 \times 5}$ are listed in Tables 1–4.

Table 1. Decision matrix Q^1 by Z_1 .

	$\mathfrak{S}_1$	$\mathfrak{S}_2$	$\mathfrak{S}_3$	$\mathfrak{S}_4$	$\mathfrak{S}_5$
Y_1	(s_7, s_1)	(s_6, s_2)	(s_4, s_3)	(s_7, s_1)	(s_5, s_2)
Y_2	(s_6, s_2)	(s_5, s_2)	(s_6, s_1)	(s_6, s_2)	(s_7, s_1)
Y_3	(s_6, s_1)	(s_5, s_3)	(s_7, s_1)	(s_5, s_1)	(s_3, s_4)
Y_4	(s_5, s_2)	(s_7, s_1)	(s_4, s_3)	(s_6, s_1)	(s_4, s_4)

Table 2. Decision matrix Q^2 by Z_2 .

	$\mathfrak{S}_1$	$\mathfrak{S}_2$	$\mathfrak{S}_3$	$\mathfrak{S}_4$	$\mathfrak{S}_5$
Y_1	(s_7, s_1)	(s_4, s_4)	(s_6, s_2)	(s_5, s_1)	(s_3, s_5)
Y_2	(s_7, s_1)	(s_5, s_1)	(s_6, s_1)	(s_5, s_2)	(s_4, s_3)
Y_3	(s_6, s_2)	(s_6, s_1)	(s_7, s_1)	(s_5, s_3)	(s_4, s_4)
Y_4	(s_5, s_2)	(s_4, s_3)	(s_5, s_2)	(s_7, s_1)	(s_5, s_3)

Table 3. Decision matrix Q^3 by Z_3 .

	$\mathfrak{S}_1$	$\mathfrak{S}_2$	$\mathfrak{S}_3$	$\mathfrak{S}_4$	$\mathfrak{S}_5$
Y_1	(s_6, s_1)	(s_5, s_2)	(s_3, s_4)	(s_7, s_1)	(s_5, s_2)
Y_2	(s_7, s_1)	(s_6, s_2)	(s_7, s_1)	(s_6, s_2)	(s_5, s_1)
Y_3	(s_5, s_3)	(s_5, s_2)	(s_6, s_1)	(s_4, s_3)	(s_3, s_1)
Y_4	(s_6, s_2)	(s_7, s_1)	(s_5, s_1)	(s_5, s_2)	(s_5, s_4)

Table 4. Decision matrix Q^4 by Z_4 .

	$\mathfrak{S}_1$	$\mathfrak{S}_2$	$\mathfrak{S}_3$	$\mathfrak{S}_4$	$\mathfrak{S}_5$
Y_1	(s_5, s_3)	(s_4, s_4)	(s_7, s_1)	(s_5, s_1)	(s_4, s_2)
Y_2	(s_6, s_1)	(s_7, s_1)	(s_6, s_1)	(s_5, s_2)	(s_6, s_1)
Y_3	(s_5, s_2)	(s_3, s_4)	(s_6, s_2)	(s_3, s_3)	(s_5, s_2)
Y_4	(s_4, s_3)	(s_5, s_1)	(s_4, s_2)	(s_6, s_2)	(s_5, s_2)

Now, we use the proposed approach to cope with this practical problems.

Step 1: The normalization is omitted because all attributes are considered as the benefit attributes.

Step 2: Utilize the LPFWA operator to aggregate decision matrix $Q^1 - Q^4$, the aggregation result is shown in Table 5.

Step 3: Utilize the LPFEWA operator or LPFEWG operator to aggregate all q_{ij} for each alternative Y_i as follows.

$$q_1 = (s_{5.3862}, s_{2.6144}), \quad q_2 = (s_{5.6262}, s_{2.2769}), \quad q_3 = (s_{4.8177}, s_{2.7532}), \quad q_4 = (s_{5.1312}, s_{2.5480}).$$

Step 4: According to the score function to calculate the score value of each alternative Y_i ($i = 1, 2, 3, 4$);

$$\mathcal{T}(q_1) = 6.5641, \quad \mathcal{T}(q_2) = 6.7257, \quad \mathcal{T}(q_3) = 6.3099, \quad \mathcal{T}(q_4) = 6.4744.$$

Step 5: the order relation of alternative Y_i ($i = 1, 2, \dots, 4$) is obtained as follows: $Y_2 \succ Y_1 \succ Y_4 \succ Y_3$. Hence, the best alternative is Y_2 .

Table 5. Decision matrix by using LPFOWG operators.

	$\mathfrak{S}_1$	$\mathfrak{S}_2$	$\mathfrak{S}_3$	$\mathfrak{S}_4$	$\mathfrak{S}_5$
Y_1	($s_{6.4795}, s_{1.3229}$)	($s_{4.9191}, s_{2.9534}$)	($s_{5.5728}, s_{2.2836}$)	($s_{6.3484}, s_{1.1908}$)	($s_{4.4073}, s_{2.5479}$)
Y_2	($s_{6.6295}, s_{1.1908}$)	($s_{5.9726}, s_{1.4670}$)	($s_{6.3141}, s_{1.4670}$)	($s_{5.5985}, s_2$)	($s_{5.8224}, s_{1.3229}$)
Y_3	($s_{5.2885}, s_{1.8690}$)	($s_{4.9355}, s_{2.2836}$)	($s_{6.6295}, s_{1.1908}$)	($s_{4.4073}, s_{2.2962}$)	($s_{3.9091}, s_{2.5015}$)
Y_4	($s_{5.4217}, s_{2.2166}$)	($s_{6.2248}, s_{1.3229}$)	($s_{4.5895}, s_{1.8690}$)	($s_{6.1317}, s_{1.4670}$)	($s_{4.7789}, s_{3.1959}$)

Example 2. This example is cited from Reference [60]. The following example is that a multinational company is planning the strategic objectives of its financial strategy group for the next year and expanding it to other countries. To this end, the planning department after consultation is to provide four strategic options for selection, which is depicted as follows:

1. Y_1 : Expand to Asia;
2. Y_2 : Expand to African;
3. Y_3 : Expand to Northern American;
4. Y_4 : Expand to all three continent.

In order to evaluate the given alternatives, the company considers the following factors as the attributes of alternatives which are shown as follows:

1. $\mathfrak{S}_1$: Short term interests;
2. $\mathfrak{S}_2$: Medium-term interest;
3. $\mathfrak{S}_3$: Long-term interests;
4. $\mathfrak{S}_4$: Strategic risk.

Table 6. Decision matrix R^1 by expert Z_1 .

	$\mathfrak{S}_1$	$\mathfrak{S}_2$	$\mathfrak{S}_3$	$\mathfrak{S}_4$
Y_1	(s_6, s_1)	(s_3, s_1)	(s_3, s_3)	(s_1, s_6)
Y_2	(s_3, s_4)	(s_3, s_4)	(s_2, s_5)	(s_2, s_4)
Y_3	(s_1, s_3)	(s_2, s_3)	(s_3, s_2)	(s_6, s_1)
Y_4	(s_6, s_2)	(s_4, s_3)	(s_5, s_1)	(s_7, s_1)

Table 7. Decision matrix R^2 by expert Z_2 .

	$\mathfrak{S}_1$	$\mathfrak{S}_2$	$\mathfrak{S}_3$	$\mathfrak{S}_4$
Y_1	(s_3, s_2)	(s_3, s_1)	(s_3, s_4)	(s_2, s_3)
Y_2	(s_5, s_2)	(s_2, s_1)	(s_3, s_4)	(s_2, s_5)
Y_3	(s_2, s_3)	(s_2, s_3)	(s_1, s_2)	(s_3, s_3)
Y_4	(s_5, s_2)	(s_4, s_3)	(s_5, s_2)	(s_4, s_1)

Table 8. Decision matrix R^3 by expert Z_3 .

	$\mathfrak{S}_1$	$\mathfrak{S}_2$	$\mathfrak{S}_3$	$\mathfrak{S}_4$
Y_1	(s_3, s_3)	(s_3, s_5)	(s_6, s_1)	(s_2, s_6)
Y_2	(s_3, s_2)	(s_2, s_4)	(s_2, s_1)	(s_3, s_4)
Y_3	(s_6, s_1)	(s_2, s_5)	(s_3, s_4)	(s_1, s_3)
Y_4	(s_5, s_1)	(s_4, s_4)	(s_6, s_2)	(s_5, s_2)

In order to acquire the best alternative, the company invited three strategic decision experts Z_1, Z_2, Z_3 to provide their evaluation information for each alternative with respect to the given attribute by the linguistic term set $\mathcal{S} = \{s_0 = \text{extremely poor}, s_1 = \text{very poor}, s_2 = \text{poor}, s_3 = \text{slightly poor}, s_4 = \text{fair}, s_5 = \text{slightly good}, s_6 = \text{good}, s_7 = \text{very good}, s_8 = \text{extremely good}\}$. Then the LPF decision matrices R^1, R^2 and R^3 are listed in Tables 6–8. The associated weight vector of attributes is $\tau = (0.3, 0.1, 0.2, 0.4)$. Next, the best alternative is selected by utilizing the proposed approach and the detailed steps are described below:

Step 1: The normalization is omitted because all attributes are considered benefit attributes.

Step 2: Utilize the LPFOWA operator to aggregate decision matrix $R^1 - R^3$, the aggregation result is shown in Table 9.

Step 3: Utilize the LPFEWA operator or LPFEWG operator to aggregate all q_{ij} for each alternative Y_i as follows.

$$q_1 = (s_{2.6532}, s_{1.9591}), \quad q_2 = (s_{2.7984}, s_{2.5207}), \quad q_3 = (s_{2.4721}, s_{2.2021}), \quad q_4 = (s_{5.0590}, s_{1.6489}).$$

Step 4: According to the score function to compute the score value of each alternative q_i ($i = 1, 2, 3, 4$)

$$\mathcal{T}(q_1) = 5.7966, \mathcal{T}(q_2) = 5.7218, \mathcal{T}(q_3) = 5.7142, \mathcal{T}(q_4) = 6.5907.$$

Step 5: By the comparison laws defined in Definition 2, the order relation of alternative Y_i ($i = 1, 2, 3, 4$) is obtained as follows: $Y_4 \succ Y_1 \succ Y_2 \succ Y_3$. Hence, the best alternative is Y_4 .

Table 9. Decision matrix by using LPFOWG operators.

	$\mathfrak{S}_1$	$\mathfrak{S}_2$	$\mathfrak{S}_3$	$\mathfrak{S}_4$
Y_1	$(s_{3.6051}, s_{2.1203})$	$(s_{3.2221}, s_{2.6726})$	$(s_{3.6015}, s_{2.9691})$	$(s_{1.6928}, s_{5.5123})$
Y_2	$(s_{3.4175}, s_{2.6413})$	$(s_{2.2102}, s_{3.6176})$	$(s_{2.3137}, s_{2.6413})$	$(s_{2.7965}, s_{2.6594})$
Y_3	$(s_{2.2713}, s_{2.6594})$	$(s_{2.2102}, s_{3.6176})$	$(s_{2.3137}, s_{2.6413})$	$(s_{2.7965}, s_{2.6594})$
Y_4	$(s_{5.2362}, s_{1.8089})$	$(s_{3.7361}, s_{3.2754})$	$(s_{5.2362}, s_{1.8089})$	$(s_{5.2015}, s_{1.3153})$

6.2. Comparative Analysis

In order to verify the validity and practicability of the developed method and to analyze its merits in this article, we compare the proposed method with the existing approaches, including the method based on linguistic intuitionistic fuzzy weighted average operator proposed by Chen et al. [57], the method based on linguistic intuitionistic fuzzy Einstein weighted average (LIFEWA) operator proposed by Garg [58] and the method based on LPF weighted average operator proposed by Garg [60]. We utilize the proposed approach to resolve the application examples in the existing methods, the final sorting results are displayed in Table 10. From Table 10, it is apparent that the ranking results obtained by the proposed method are same as those of the existing methods. That proves the validity of the propounded approach in this article. In what follows, the advantages of the proposed method are demonstrated through a detailed comparison with above-mentioned methods.

Table 10. A comparison of preference order for different approaches.

	Ranking Orders in the Original Literature	Ranking Orders Based on Proposed Approach
Example in [57]	$Y_2 \succ Y_1 \succ Y_4 \succ Y_3$	$Y_2 \succ Y_1 \succ Y_4 \succ Y_3$
Example in [58]	$Y_2 \succ Y_1 \succ Y_4 \succ Y_3$	$Y_2 \succ Y_1 \succ Y_4 \succ Y_3$
Example in [60]	$Y_4 \succ Y_3 \succ Y_1 \succ Y_2$	$Y_4 \succ Y_3 \succ Y_1 \succ Y_2$

(1) Compared with the method based on LIFWA operator propounded by Chen et al. [57], although the LIFWA operator can be utilized to aggregate fuzzy and uncertain information, it lacks flexibility and robustness during the information fusion process because its operational laws are generated by Algebraic T-norm and S-norm. Besides, the LIFN can not address several special situations in real-life issues. For instance, if an expert provides his or her preference information in the form of LIFN denoted as (s_5, s_6) with $s_6, s_5 \in \mathcal{S}$, we can find that the LIFWA operator fails to resolve situation because of $5 + 6 \not\leq 8$. However, the approach based on the LPFEWA operator can overcome the above defects. The operational rules generated by Einstein T-norm and S-norm can improve the robustness and smoothness, the LPFS can address the special situation because of $5^2 + 6^2 \leq 8^2$. Accordingly, the presented approach in this paper is more general and flexible at resolving actual issues than LIFN.

(2) Compared with the method based on the LIFWEA operator propounded by Garg [58], the evaluation information of the approach is in the form of LPFN. We can find that the sorting result of alternatives in the Example is $Y_2 \succ Y_1 \succ Y_4 \succ Y_3$, which is the same as with the method in Reference [58]. However, if the decision maker Z_1 in Example 1 changes his preference information for alternative Y_1 with respect to $\mathfrak{S}_1$ and $\mathfrak{S}_2$ to (s_4, s_5) and (s_3, s_7) , respectively, we cannot acquire the score values and ranking relation of the alternative by utilizing the LIFWEA operator. For this limitation, the LPFEWA operator can efficiently overcome it and attain the final sorting of alternatives. Hence, the LPFS can provide more freedom for the evaluator to express their assessment information in practical problems.

(3) Compared with the method based on the LPFWA operator propounded by Garg [60], the LPFWA operator is obtained on the basis of Algebraic T-norm and S-norm. The proposed approach based upon the LPFEWA operator is generated through Einstein T-norm and S-norm, which has more robustness and smoothness than Algebraic operations.

To sum up, the developed approach fully demonstrates flexibility and effectiveness by combining LPFS and Einstein operation. It is more general than LIFS and PFS in solving practical problems. The significant characteristics of the proposed and other existing approaches are summarized in Table 11.

Table 11. Feature comparison for different methods.

Approaches	Whether Quantitative Description of Information	Whether Qualitative Description of Information	Describe a Wider Range of Information	Have Generalized Features
The method propounded by Xu in [17]	NO	NO	NO	NO
The method propounded by Xia et al. in [18]	NO	NO	NO	YES
The method propounded by Chen et al. in [57]	YES	YES	NO	YES
The method propounded by Garg in [60]	YES	YES	YES	NO
The propounded method in this paper	YES	YES	YES	YES

7. Conclusions

LPFS is the extension of PFS, where membership degree and nonmembership degree are indicated by the linguistic terms to process vague information from a qualitative point of view. In this article, we propose a multi-attribute group decision making approach for dealing with decision making problems. To begin with, we define the novel operational laws of LPFNs by utilizing the Einstein operation. Thereafter, we present several novel LPF Einstein averaging and geometric aggregation operators to fuse the diverse assessment information of alternatives. Some desirable properties of these aggregation operators are discussed in detail. Moreover, we utilize these aggregation operators to construct a novel decision making method to tackle practical issues with LPF information.

Two illustration examples are utilized to demonstrate the validity and practicality of the proposed method. A comparative analysis with existing methods is carried out to display the superiorities of the presented approach. In future research, we will further investigate the Einstein operations in other fuzzy environments such as linguistic single-valued neutrosophic sets [64], picture 2-tuple linguistic sets [65] and so on. Furthermore, we can apply the propounded aggregation operators to solve practical issues, such as information fusion, knowledge management and risk evaluation. At the same time, we will continue to consider the theory of linguistic Pythagorean fuzzy sets and its methodologies and applications in decision support systems, big data analysis and so on.

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