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A Novel Fractional-Order Grey Prediction Model and Its Modeling Error Analysis

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Abstract: Based on the grey prediction model GM(1,1), a novel fractional-order grey prediction model is proposed and its modeling error is systematically studied. In this paper, exponential data sequences are generated for numerical simulation. Via the numerical simulation method, the mean absolute percentage error (MAPE) of the fractional-order GM(1,1) with different values of order and development coefficient is compared to the GM(1,1) and the discrete GM(1,1). The error distribution of the sequences of exponential data is given. The GM(1,1) and the direct modeling GM(1,1) are both special cases of the fractional-order GM(1,1). The conclusion is helpful to further optimize the grey model using fractional-order operators and to expand the applicable bound of GM(1,1).

Keywords: fractional-order operator; grey prediction model; error analysis

1. Introduction

Grey system theory was developed to study uncertainty systems with small samples and poor information by Chinese scholar, Professor Deng Julong [1]. The grey system theory uses the generation and development method to extract valuable information of some known information in the unknown system, recognizes the correct description of the system's operational behavior and evolution law, and realizes the quantitative prediction of future changes [2,3]. This is the grey model (GM). The GM(1,1) is the basic model of grey prediction theory. Its modeling method has been very actively studied and a wealth of research results has been obtained. The representative research modeling results of GM(1,1) include GM(1,1) with zero-setting modeling [4], GM(1,1) with extended step-by-step optimum direct modeling [5], GM(1,1) with step-by-step optimum grey derivative background values [6], GM(1,1) based on optimum grey derivative [7], GM(1,1) direct modeling with step-by-step optimizing grey derivative whiten values [8], modeling and application of metabolic GM(1,1) [9], discrete GM(1,1) and its modeling mechanism and optimization [10,11], the buffer operator method [12,13], multivariable grey forecasting with a dynamic background-value coefficient [14], unbiased grey model with a weakening buffer operator [15], self-adapting intelligent grey model [16], GM(1,1) optimization based on the background value and boundary value correction [17], GM(1,1) with standardized interval grey number [18], and others. The combined grey prediction model was used to forecast electricity consumption [19] and foreign tourists [20]. Salmeron proposed an autonomous FGCM-based system for surveillance asset coordination [21]. Chang used the grey silhouette coefficient to build a novel procedure for multimodel development [22]. Aydemir developed an EPQ model by degree of greyness approach [23]. Özdemir applied grey model to predict the product demand [24]. Ma studied the framework of grey machine learning [25]. For the applicable bound of the grey prediction model, Professor Liu Sifeng systematically studied several basic forms and applicable bound of the GM(1,1) [26]. In the literature [27–30], the grey prediction model, based on fractional-order accumulation operator, was studied. Meng Wei studied that the fractional-order accumulating generation operator and the fractional-order reducing generation operator satisfied the commutative and exponential laws [31,32]. The same order of accumulating generation operator and reducing generation operator satisfied the reciprocal law, which lays a theoretical foundation for the grey prediction model with fractional-order operators [33].

Extant studies focused on the optimization and applicable bounds of traditional grey prediction models and the modeling method of fractional-order operators. The error distribution and applicable bound of grey prediction models with fractional-order operators has not been studied. This paper mainly studies a novel fractional-order grey prediction model based on the fractional-order operators and applies numerical simulation to study the error distribution of GM(1,1) with different values of order and development coefficient. The mean absolute percentage error (MAPE) of the medium-length exponential sequences are compared with the GM(1,1) and the discrete GM(1,1). The simulation results can help optimize the modeling method of grey prediction model with fractional-order operators and expand the applicable bound of grey prediction model.

The rest of this paper is organized as follows. Fractional-order grey generation operators are presented in Section 2. In Section 3, the definition and modeling steps of fractional-order GM(1,1) is discussed. The experimental data of experimental sequences is generated in Section 4. The error distribution of experimental data is studied by the method of numerical simulation. In this section, the MAPE of the GM(1,1), the discrete GM(1,1), and the GM(1,1) with different values of order and development coefficient is calculated and compared. A graph of the error distribution is drawn and the applicable bound of fractional-order GM(1,1) is analyzed. Finally, a conclusion is drawn in the last section.

2. Fractional-Order Grey Prediction Model

Definition 1. *Assume that* $X^{(0)} = (x^{(0)}(1), x^{(0)}(2), \cdots, x^{(0)}(n))$ *is the sequence of raw data, where* $r \in \mathbf{R}^+$*, we call* $X^{(r)} = (x^{(r)}(1), x^{(r)}(2), \cdots, x^{(r)}(n))$ *is the* r *th-order accumulating generation sequence of* $X^{(0)} = (x^{(0)}(1), x^{(0)}(2), \cdots, x^{(0)}(n))$ *[33], where* $\Gamma(n)$ *is the gamma function and*

$$x^{(r)}(k) = \sum_{i=1}^{k} \frac{\Gamma(r+k-i)}{\Gamma(k-i+1)\Gamma(r)} x^{(0)}(i), \quad k = 1, 2, \cdots, n \tag{1}$$

Definition 2. *Assume that* $X^{(0)} = (x^{(0)}(1), x^{(0)}(2), \cdots, x^{(0)}(n))$ *is the sequence of raw data, where* $r \in \mathbf{R}^+$*, we call* $X^{(-r)} = (x^{(-r)}(1), x^{(-r)}(2), \cdots, x^{(-r)}(n))$ *the* r *th-order reducing generation sequence of* $X^{(0)} = (x^{(0)}(1), x^{(0)}(2), \cdots, x^{(0)}(n))$ *[33], where*

$$x^{(-r)}(k) = \sum_{i=0}^{k-1} (-1)^i \frac{\Gamma(r+1)}{\Gamma(i+1)\Gamma(r-i+1)} x^{(0)}(k-i), \quad k = 1, 2, \cdots, n. \tag{2}$$

Theorem 1. *Assume that* $X^{(0)} = (x^{(0)}(1), x^{(0)}(2), \cdots, x^{(0)}(n))$ *is a sequence of raw data,* $X^{(p)}$ *is the* p *th-order accumulating generation sequence of the sequence of* $X^{(0)}$*, where* $p \in \mathbf{R}^+$*, and* $X^{(-q)}$ *is the* q *th-order reducing generation sequence of the sequence of* $X^{(0)}$*, where* $q \in \mathbf{R}^+$*. It follows that* $(X^{(p)})^{(-q)}$ *is the* q *th-order reducing generation sequence of the sequence of* $(X^{(p)})$*, and* $(X^{(-q)})^{(p)}$ *is the* p *th-order accumulating generation sequence of* $X^{(-q)}$*. The following holds true [33]:*

(i) If $p - q > 0$*,* $X^{(p-q)}$ *is the* $(p-q)$ *th-order accumulating generation sequence of the sequence of* $X^{(0)}$*.*

(ii) If $p - q < 0$*,* $X^{(p-q)}$ *is the* $(q-p)$ *th-order reducing generation sequence of the sequence of* $X^{(0)}$*.*

(iii) The fractional-order accumulating generation operator and the fractional-order reducing generation operator satisfy the commutative and exponential laws.

$$X^{(p-q)} = (X^{(p)})^{(-q)} = (X^{(-q)})^{(p)} \tag{3}$$

Definition 3. *Assume that* $X^{(r)} = (x^{(r)}(1), x^{(r)}(2), \cdots, x^{(r)}(n))$ *is defined as in Definition 1, and* $X^{(-r)} = (x^{(-r)}(1), x^{(-r)}(2), \cdots, x^{(-r)}(n))$ *is defined as in Definition 2.*

Thus, $Z^{(r)} = (z^{(r)}(2), z^{(r)}(3), \cdots, z^{(r)}(n))$ *, where*

$$z^{(r)}(k) = \frac{x^{(r)}(k) + x^{(r)}(k-1)}{2}, \quad k = 2, 3, \cdots, n, \tag{4}$$

we call

$$x^{(r-1)}(k) + az^{(r)}(k) = b \tag{5}$$

is the fractional-order GM(1,1). In particular, consider the following.

(i) If $r = 1$ *,* $x^{(r-1)}(k) + az^{(r)}(k) = b$ *is the GM(1,1).*

$$x^{(0)}(k) + az^{(1)}(k) = b\,.$$

(ii) If $r = 0$ *,* $x^{(r-1)}(k) + az^{(r)}(k) = b$ *is the direct modeling GM(1,1).*

$$x^{(-1)}(k) + az^{(0)}(k) = b\,.$$

Theorem 2. *Assume that the fractional-order GM(1,1) is defined as in Definition 3. Then, the parameter vector of* $x^{(r-1)}(k) + az^{(r)}(k) = b$ *,* $\hat{\mathbf{a}} = [a, b]^{\mathrm{T}}$ *, can be calculated by the least-squares method.*

$$\hat{\mathbf{a}} = (\mathbf{B}^{\mathrm{T}}\mathbf{B})^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{Y}\,. \tag{6}$$

It follows that

$$\mathbf{Y} = \begin{bmatrix} x^{(r-1)}(2) \\ x^{(r-1)}(3) \\ \vdots \\ x^{(r-1)}(n) \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -z^{(r)}(2) & 1 \\ -z^{(r)}(3) & 1 \\ \vdots & \vdots \\ -z^{(r)}(n) & 1 \end{bmatrix}, \tag{7}$$

where

$$x^{(r-1)}(k) = (x^{(r)})^{(-1)}(k) = x^{(r)}(k) - x^{(r)}(k-1)$$

$$= \sum_{i=1}^{k} \frac{\Gamma(r+k-i)}{\Gamma(k-i+1)\Gamma(r)} x^{(0)}(i) - \sum_{i=1}^{k-1} \frac{\Gamma(r+k-1-i)}{\Gamma(k-i)\Gamma(r)} x^{(0)}(i), \quad k = 2, 3, \cdots, n, \quad k-i \geq 1$$

$$z^{(r)}(k) = \frac{x^{(r)}(k) + x^{(r)}(k-1)}{2}$$

$$= \frac{\sum_{i=1}^{k} \frac{\Gamma(r+k-i)}{\Gamma(k-i+1)\Gamma(r)} x^{(0)}(i) + \sum_{i=1}^{k-1} \frac{\Gamma(r+k-i)}{\Gamma(k-i+1)\Gamma(r)} x^{(0)}(i)}{2}, \quad k = 2, 3, \cdots, n\,.$$

So, it follows that

$$\mathbf{Y}=\begin{bmatrix}\sum_{i=1}^{2}\frac{\Gamma(r+2-i)}{\Gamma(2-i+1)\Gamma(r)}x^{(0)}(i)-\sum_{i=1}^{1}\frac{\Gamma(r+2-1-i)}{\Gamma(2-i)\Gamma(r)}x^{(0)}(i)\\ \sum_{i=1}^{3}\frac{\Gamma(r+3-i)}{\Gamma(3-i+1)\Gamma(r)}x^{(0)}(i)-\sum_{i=1}^{2}\frac{\Gamma(r+3-1-i)}{\Gamma(3-i)\Gamma(r)}x^{(0)}(i)\\ \vdots\\ \sum_{i=1}^{n}\frac{\Gamma(r+n-i)}{\Gamma(n-i+1)\Gamma(r)}x^{(0)}(i)-\sum_{i=1}^{n-1}\frac{\Gamma(r+n-1-i)}{\Gamma(n-i)\Gamma(r)}x^{(0)}(i)\end{bmatrix}$$

$$=\begin{bmatrix}(r-1)x^{(0)}(1)+x^{(0)}(2)\\ \frac{r(r\text{-}1)}{2}x^{(0)}(1)+(r-1)x^{(0)}(2)+x^{(0)}(3)+\\ \vdots\\ \sum_{i=1}^{n}\frac{\Gamma(r+n-i)}{\Gamma(n-i+1)\Gamma(r)}x^{(0)}(i)-\sum_{i=1}^{n-1}\frac{\Gamma(r+n-1-i)}{\Gamma(n-i)\Gamma(r)}x^{(0)}(i)\end{bmatrix}$$

$$\mathbf{B}=\begin{bmatrix}-\frac{x^{(r)}(1)+x^{(r)}(2)}{2} & 1\\ -\frac{x^{(r)}(2)+x^{(r)}(3)}{2} & 1\\ \vdots & \vdots\\ -\frac{x^{(r)}(n-1)+x^{(r)}(n)}{2} & 1\end{bmatrix}=\begin{bmatrix}-\frac{1}{2}\left[(r+1)x^{(0)}(1)+x^{(0)}(2)\right] & 1\\ -\frac{1}{2}\left[\frac{r(r+3)}{2}x^{(0)}(1)+(r+1)x^{(0)}(2)+x^{(0)}(3)\right] & 1\\ \vdots & \vdots\\ -\frac{1}{2}\left[\sum_{i=1}^{n}\frac{\Gamma(r+n-i)}{\Gamma(n-i+1)\Gamma(r)}x^{(0)}(i)+\sum_{i=1}^{n-1}\frac{\Gamma(r+n-i)}{\Gamma(n-i+1)\Gamma(r)}x^{(0)}(i)\right] & 1\end{bmatrix}$$

Definition 4. *Assume that* $x^{(r-1)}(k)$ *and* $z^{(r)}(k)$ *are defined as in Theorem 1. Thus,*

$$\frac{dx^{(r)}}{dt}+ax^{(r)}=b \tag{8}$$

is called a whitenization (i.e., image) equation of the grey differential equation.

$$x^{(r-1)}(k)+az^{(r)}(k)=b\ .$$

Theorem 3. *Assume that* $\mathbf{B}$, $\mathbf{Y}$, *and* $\hat{\mathbf{a}}$ *are the same as in Theorem 2. If*

$$\hat{\mathbf{a}}=[a,b]^{\mathrm{T}}=(\mathbf{B}^{\mathrm{T}}\mathbf{B})^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{Y}\ ,$$

then the following is true.

(i) The solution (i.e., time response function) of the whitenization function of the fractional-order GM(1,1),

$$\frac{dx^{(r)}}{dt}+ax^{(r)}=b$$

is given by

$$\hat{x}^{(r)}(t)=(x^{(r)}(1)-\frac{b}{a})e^{-at}+\frac{b}{a}\ . \tag{9}$$

(ii) The time response sequence of fractional-order GM(1,1),

$$x^{(r-1)}(k)+az^{(r)}(k)=b\ ,$$

is given by

$$\hat{x}^{(r)}(k)=(x^{(0)}(1)-\frac{b}{a})e^{-a(k-1)}+\frac{b}{a},\ \ k=2,3,\cdots,n\ . \tag{10}$$

(iii) Let $x^{(1)}(0) = x^{(0)}(1)$. *Then, the restored values of* $x^{(0)}(k)$ *can be given by*

$$\hat{x}^{(0)}(k) = (\hat{x}^{(r)})^{(-r)}(k) = \begin{cases} x^{(0)}(1) & k = 1 \\ \sum_{i=0}^{k-1} (-1)^i \dfrac{\Gamma(r+1)}{\Gamma(i+1)\Gamma(r-i+1)} \hat{x}^{(r)}(k-i) & k = 2, 3, \cdots, n \end{cases} \quad (11)$$

The overall modeling steps of the fractional-order GM(1,1) are shown in Figure 1. The MATLAB program code for the fractional-order GM(1,1) is shown in the Appendix A.

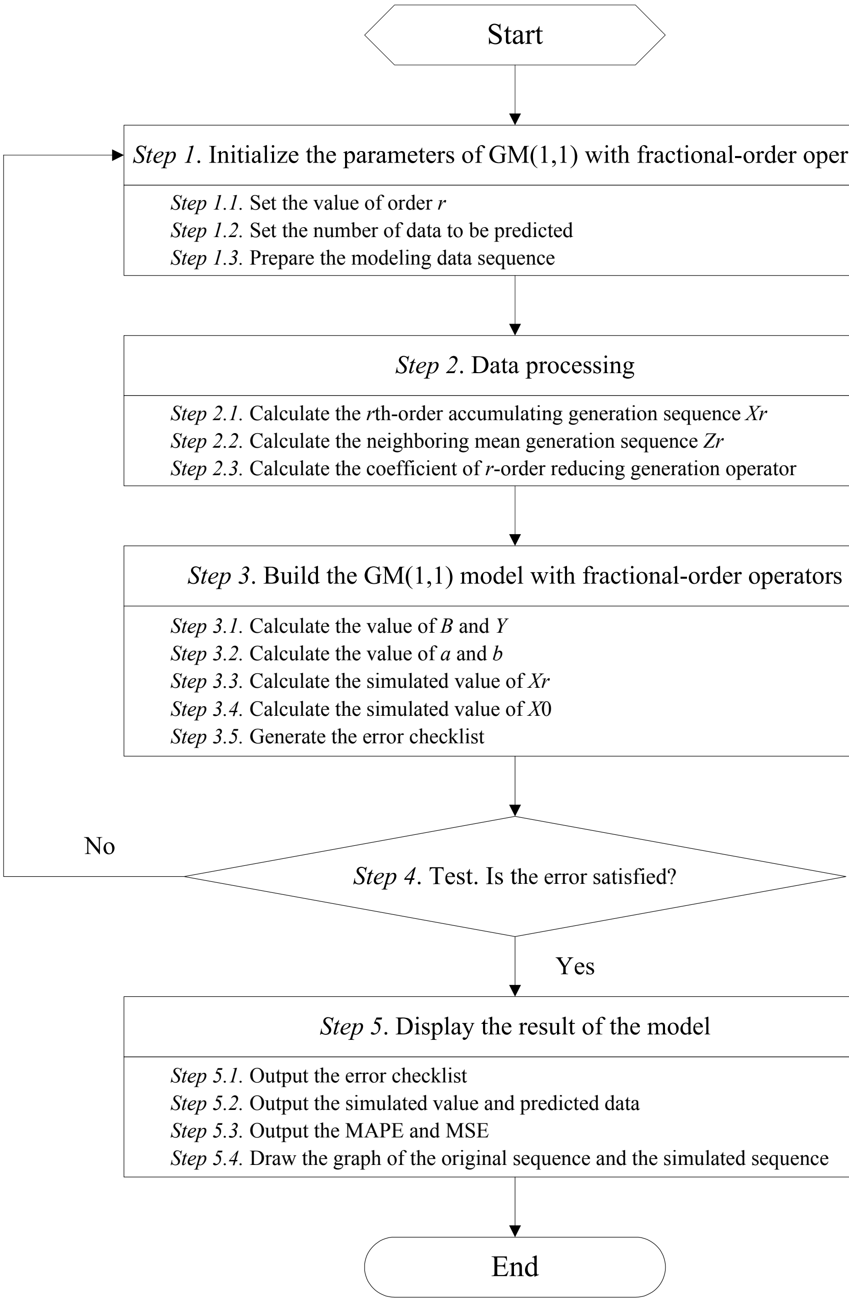


Figure 1. The modeling steps of the fractional-order GM(1,1).

3. Modeling Error Analysis

3.1. Data Preparation

The accumulating generation sequence of economic data satisfies the approximate exponential law and is suitable for the application of grey prediction model GM (1,1) [2]. Economic forecasting is an important application field of GM(1,1). French economist C. Juglar proposed that the economy has a cyclical fluctuation of nine to 10 years. This is generally called the "Juglar Cycle". Let the sequence length, $n = 9$, and set the development coefficient, $-a$ = −0.5, −0.4, −0.3, −0.2, −0.1, −0.05, −0.04, −0.03, −0.02, −0.01, 0.01, 0.02, 0.03, 0.04, 0.05, 0.1, 0.15, 0.2, 0.25, 0.3, 0.35, 0.4, 0.45, 0.5, 0.55, 0.6, 0.65, 0.7, 0.8, 0.9, 1. A total of $N = 31$ sequences of experimental data are generated. The sequences of exponential data are generated by Equation (12).

$$x_i^{(0)}(k) = e^{-ak} + \frac{b}{n}\sum_{j=1}^{n} e^{-aj}\ ,\quad i = 1, 2, \cdots, N\ ,\quad k = 1, 2, \cdots, n\,. \tag{12}$$

where the $\frac{1}{n}\sum_{j=1}^{n} e^{-aj}$ is the means of homogeneous exponential sequences, and $b = 1$ is the nonhomogeneous terms. We can get the data sequence as shown below.

$$-a = -0.5\,,\quad X_1^{(0)} = (x_1^{(0)}(1), x_1^{(0)}(2), x_1^{(0)}(3), x_1^{(0)}(4), x_1^{(0)}(5), x_1^{(0)}(6), x_1^{(0)}(7), x_1^{(0)}(8), x_1^{(0)}(9))$$

$$= (0.776, 0.537, 0.393, 0.305, 0.251, 0.219, 0.200, 0.188, 0.180)$$

……

$$-a = 0.01\,,\quad X_{11}^{(0)} = (x_{11}^{(0)}(1), x_{11}^{(0)}(2), x_{11}^{(0)}(3), x_{11}^{(0)}(4), x_{11}^{(0)}(5), x_{11}^{(0)}(6), x_{11}^{(0)}(7), x_{11}^{(0)}(8), x_{11}^{(0)}(9))$$

$$= (2.062, 2.072, 2.082, 2.092, 2.103, 2.114, 2.124, 2.135, 2.146)$$

……

$$-a = 1\,,\quad X_{31}^{(0)} = (x_{31}^{(0)}(1), x_{31}^{(0)}(2), x_{31}^{(0)}(3), x_{31}^{(0)}(4), x_{31}^{(0)}(5), x_{31}^{(0)}(6), x_{31}^{(0)}(7), x_{31}^{(0)}(8), x_{31}^{(0)}(9))$$

$$= (1426.9, 1431.5, 1444.2, 1478.7, 1572.6, 1827.6, 2520.8, 4405.1, 9527.2)$$

All the experimental data sequences are shown in Table 1. The sequences with smaller negative values of development coefficient $-a$ have sequences of smaller value and lower growth rate.

Table 1. The experimental data sequences.

Seq.	$-a$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$
$X_1^{(0)}$	−0.5	0.776	0.537	0.393	0.305	0.251	0.219	0.200	0.188	0.180
$X_2^{(0)}$	−0.4	0.890	0.669	0.521	0.422	0.355	0.310	0.281	0.261	0.247
$X_3^{(0)}$	−0.3	1.037	0.845	0.703	0.597	0.519	0.462	0.419	0.387	0.363
$X_4^{(0)}$	−0.2	1.238	1.089	0.968	0.868	0.787	0.720	0.665	0.621	0.584
$X_5^{(0)}$	−0.1	1.532	1.446	1.368	1.297	1.234	1.176	1.124	1.076	1.034
$X_6^{(0)}$	−0.05	1.737	1.690	1.646	1.604	1.564	1.526	1.490	1.456	1.423
$X_7^{(0)}$	−0.04	1.784	1.746	1.710	1.675	1.642	1.610	1.579	1.549	1.521
$X_8^{(0)}$	−0.03	1.834	1.805	1.777	1.750	1.724	1.699	1.674	1.650	1.627
$X_9^{(0)}$	−0.02	1.886	1.867	1.848	1.829	1.811	1.793	1.775	1.758	1.741
$X_{10}^{(0)}$	−0.01	1.942	1.932	1.922	1.912	1.903	1.893	1.884	1.875	1.866
$X_{11}^{(0)}$	0.01	2.062	2.072	2.082	2.092	2.103	2.114	2.124	2.135	2.146
$X_{12}^{(0)}$	0.02	2.127	2.148	2.169	2.190	2.212	2.234	2.257	2.280	2.304
$X_{13}^{(0)}$	0.03	2.196	2.227	2.260	2.293	2.327	2.363	2.399	2.437	2.475
$X_{14}^{(0)}$	0.04	2.269	2.311	2.355	2.401	2.449	2.499	2.551	2.605	2.661

$X_{15}^{(0)}$	0.05	2.346	2.400	2.457	2.516	2.579	2.645	2.714	2.787	2.863
$X_{16}^{(0)}$	0.1	2.809	2.926	3.054	3.196	3.353	3.526	3.718	3.930	4.164
$X_{17}^{(0)}$	0.15	3.441	3.629	3.848	4.101	4.396	4.739	5.137	5.599	6.137
$X_{18}^{(0)}$	0.2	4.317	4.587	4.917	5.321	5.814	6.415	7.150	8.048	9.145
$X_{19}^{(0)}$	0.25	5.548	5.912	6.381	6.982	7.754	8.745	10.018	11.653	13.751
$X_{20}^{(0)}$	0.3	7.300	7.772	8.410	9.270	10.432	12.000	14.116	16.973	20.830
$X_{21}^{(0)}$	0.35	9.823	10.418	11.262	12.459	14.159	16.570	19.992	24.849	31.740
$X_{22}^{(0)}$	0.4	13.489	14.223	15.318	16.951	19.387	23.021	28.442	36.530	48.596
$X_{23}^{(0)}$	0.45	18.861	19.752	21.150	23.342	26.780	32.172	40.629	53.891	74.690
$X_{24}^{(0)}$	0.5	26.786	27.856	29.619	32.526	37.320	45.223	58.253	79.736	115.15
$X_{25}^{(0)}$	0.55	38.549	39.820	42.023	45.841	52.459	63.929	83.809	118.27	177.99
$X_{26}^{(0)}$	0.6	56.100	57.598	60.328	65.301	74.364	90.876	120.96	175.79	275.68
$X_{27}^{(0)}$	0.65	82.405	84.159	87.519	93.954	106.28	129.89	175.12	261.76	427.72
$X_{28}^{(0)}$	0.7	121.99	124.03	128.14	136.42	153.09	186.66	254.26	390.40	664.55
$X_{29}^{(0)}$	0.8	272.29	275.01	281.08	294.59	324.66	391.57	540.49	871.91	1609.5
$X_{30}^{(0)}$	0.9	619.11	622.70	631.53	653.25	706.67	838.06	1161.2	1956.1	3911.1
$X_{31}^{(0)}$	1	1426.9	1431.5	1444.2	1478.7	1572.6	1827.6	2520.8	4405.1	9527.2

3.2. Results and Discussion

The GM(1,1), the discrete GM(1,1), and the fractional-order GM(1,1) are established by each sequence of exponential data, where the value of order $r =$ 0, 0.01, 0.05, 0.1, 0.2, 0.5, 0.7, 0.9, 1, 1.5, 1.8. The MAPE of the GM(1,1), the discrete GM(1,1), and the fractional-order GM(1,1) for each sequence is calculated. MAPE removed less than 100% is shown in Table 2.

Table 2. Mean absolute percentage error (MAPE) of the experimental data sequences for the three models.

Seq.	$-a$	Discrete GM	GM	Fractional-Order GM(1,1)										
				0	**0.01**	**0.05**	**0.1**	**0.2**	**0.5**	**0.7**	**0.9**	**1.0**	**1.5**	**1.8**
$X_{1}^{(0)}$	−0.5	10.6	10.5	1.0	1.0	1.2	2.0	3.8	4.6	7.8	10.4	10.5	2.8	19.5
$X_{2}^{(0)}$	−0.4	7.3	7.2	0.6	0.6	1.1	1.8	3.8	1.3	6.0	7.4	7.2	4.9	22.5
$X_{3}^{(0)}$	−0.3	4.3	4.3	0.3	0.4	0.9	1.8	3.7	1.1	4.3	4.8	4.3	7.9	24.9
$X_{4}^{(0)}$	−0.2	2.0	2.0	0.1	0.2	0.9	1.7	15.5	1.7	3.0	2.8	2.0	10.0	25.9
$X_{5}^{(0)}$	−0.1	0.5	0.5	0.0	0.2	0.8	9.8	1.2	2.0	2.3	1.4	0.5	10.8	25.3
$X_{6}^{(0)}$	−0.05	0.1	0.1	0.0	0.2	5.8	0.9	0.5	2.1	2.1	1.1	0.1	10.7	24.5
$X_{7}^{(0)}$	−0.04	0.1	0.1	0.0	0.2	7.7	0.4	0.7	2.1	2.0	1.0	0.1	10.7	24.3
$X_{8}^{(0)}$	−0.03	0.0	0.0	0.0	0.2	1.1	0.1	0.8	2.1	2.0	1.0	0.0	10.7	24.1
$X_{9}^{(0)}$	−0.02	0.0	0.0	0.0	0.2	0.2	0.3	1.0	2.1	2.0	1.0	0.0	10.6	23.8
$X_{10}^{(0)}$	−0.01	0.0	0.0	0.0	1.3	0.2	0.5	1.1	2.1	2.0	1.0	0.0	10.5	23.6
$X_{11}^{(0)}$	0.01	0.0	0.0	0.0	0.1	0.4	0.7	1.3	2.2	2.0	1.0	0.0	10.4	23.1
$X_{12}^{(0)}$	0.02	0.0	0.0	0.0	0.1	0.5	0.8	1.4	2.3	2.0	1.0	0.0	10.3	22.8
$X_{13}^{(0)}$	0.03	0.0	0.0	0.0	0.1	0.6	0.9	1.5	2.3	2.1	1.0	0.0	10.1	22.6
$X_{14}^{(0)}$	0.04	0.1	0.1	0.0	0.1	0.6	1.0	1.6	2.4	2.1	1.0	0.1	10.0	22.3
$X_{15}^{(0)}$	0.05	0.1	0.1	0.0	0.1	0.6	1.0	1.6	2.4	2.2	1.0	0.1	9.9	22.0
$X_{16}^{(0)}$	0.1	0.5	0.5	0.0	0.2	0.7	1.2	1.9	2.9	2.5	1.4	0.5	9.0	20.4
$X_{17}^{(0)}$	0.15	1.1	1.1	0.1	0.2	0.7	1.3	2.2	3.3	3.0	2.0	1.1	7.9	18.6
$X_{18}^{(0)}$	0.2	2.0	2.0	0.2	0.5	1.2	1.5	2.5	3.8	3.7	2.8	2.0	6.5	16.7

$X_{19}^{(0)}$	0.25	3.1	3.0	0.3	0.8	2.1	2.8	3.0	4.6	4.5	3.7	3.0	5.0	14.6
$X_{20}^{(0)}$	0.3	4.4	4.3	0.6	1.3	3.5	4.9	5.3	5.4	5.7	5.0	4.3	3.8	12.5
$X_{21}^{(0)}$	0.35	6.0	6.0	0.9	2.1	5.6	8.0	9.2	7.0	6.9	6.5	6.0	3.4	10.3
$X_{22}^{(0)}$	0.4	7.7	7.9	1.3	3.1	8.6	12.6	15.1	10.9	9.0	8.3	7.9	3.5	8.8
$X_{23}^{(0)}$	0.45	9.7	10.0	1.8	4.5	12.9	19.2	23.7	17.5	12.6	10.6	10.0	5.7	8.2
$X_{24}^{(0)}$	0.5	12.0	12.8	2.3	6.3	18.8	28.7	36.1	27.0	19.2	13.7	12.8	8.1	7.7
$X_{25}^{(0)}$	0.55	15.1	16.2	3.0	8.7	27.1	41.9	53.7	40.3	28.1	19.4	16.2	10.6	7.8
$X_{26}^{(0)}$	0.6	18.8	22.3	3.8	11.9	38.5	60.5	78.6	58.7	40.1	27.1	22.3	13.1	9.5
$X_{27}^{(0)}$	0.65	25.1	30.0	4.6	16.2	54.2	86.2	-	83.6	55.9	36.8	30.0	15.6	11.9
$X_{28}^{(0)}$	0.7	32.6	39.3	5.5	21.8	75.7	-	-	-	76.2	48.8	39.3	18.0	14.2
$X_{29}^{(0)}$	0.8	51.5	63.4	7.5	38.8	-	-	-	-	-	81.2	63.4	23.4	18.6
$X_{30}^{(0)}$	0.9	75.5	96.0	9.6	68.3	-	-	-	-	-	-	96.0	28.6	22.3
$X_{31}^{(0)}$	1	-	-	11.6	-	-	-	-	-	-	-	-	33.7	25.5

Furthermore, the prediction results of the GM(1,1) with fractional-order operators in this example are screened. Excluding the result of MAPE greater than 40%, the sequence of $-a \in (-0.5, 0.5)$ is retained. We generate the MAPE distribution plot of the fractional-order GM(1,1) with respect to the different combinations of values of development coefficient and order, as shown in Figure 2. When $-a = 0$ and $r = 1$, there is a singular point. Thus, the exponential function cannot be used to simulate a constant sequence.

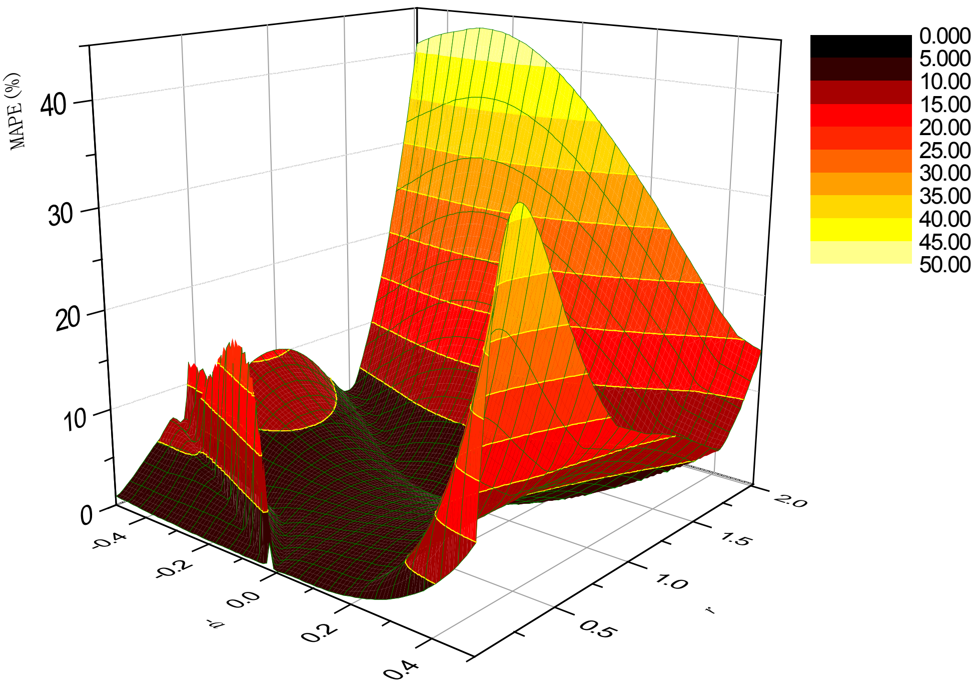


Figure 2. Error distribution of GM(1,1) with fractional-order operators.

From Table 2, we draw the following conclusions.

(i) If $r = 0$, direct modeling using raw data sequences has minimal MAPE.

(ii) If $r \neq 0$ and $-a$ is negative, the experimental data sequence is an exponential attenuation sequence. Both the discrete GM(1,1) and the GM(1,1) have larger MAPE. The smaller the value of $-a$, the faster the attenuation speed, and the larger the simulation error of the discrete GM(1,1) and GM(1,1). The GM(1,1) with fractional-order operators has higher simulation accuracy when $r \in (0, 0.5)$.

(iii) If $r \neq 0$ and $-a \in (-0.2, 0.2)$, the discrete GM(1,1) and the GM(1,1) have almost the same high simulation accuracy. The GM(1,1) with fractional-order operators has higher simulation accuracy when $r = 1$.

(iv) If $r \neq 0$ and $-a > 0.5$, the simulation accuracy of the discrete GM(1,1) and the GM(1,1) is reduced, and the GM(1,1) with fractional-order operators with larger values of order has higher simulation accuracy.

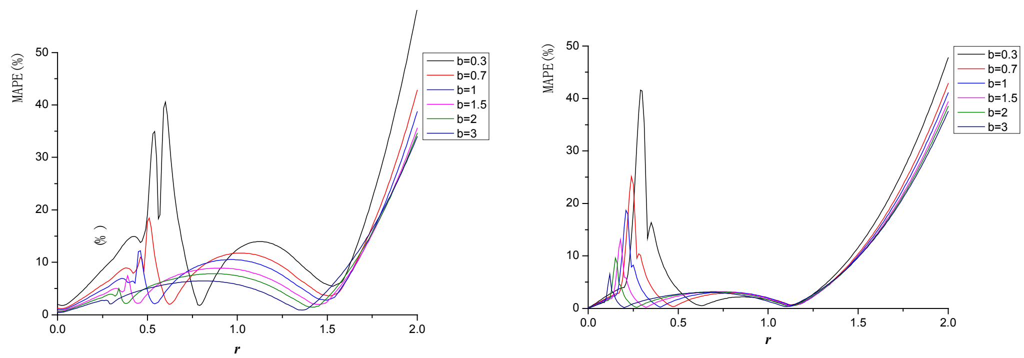
The value of nonhomogeneous coefficient b in Equation (12) affects the accuracy of modeling fitting.

When $b = 0$, it is a homogeneous exponential sequence. As the value of b increases, the nonhomogeneous exponential sequence is reflected in the upward shift of the homogeneous exponential sequence. Taking $b = 0.3$, 0.7, 1, 1.5, 2, and 3 as examples, under the different value of $-a$, the MAPE of the nonhomogeneous coefficient and order is shown in Figure 3.

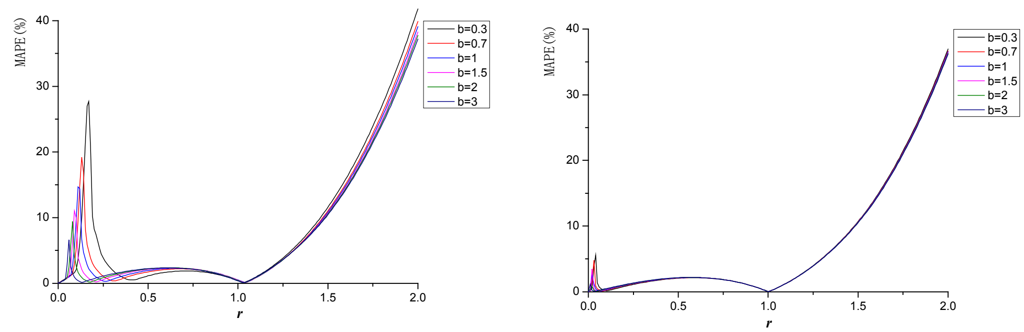
(i) If $-a < -0.1$, under different values of b, there are two minimum points of MAPE where the value of order in $0 < r < 1$ and $r > 1$. Furthermore, the larger the value of b, the larger the corresponding value of order is.

(ii) If $-0.02 < -a < 0.02$, the MAPE is the smallest when $r = 0$. The value of r increases firstly and then decreases. When $r = 1$, another minimum point of MAPE is obtained. When $r > 1$, the MAPE increases rapidly with the value of r.

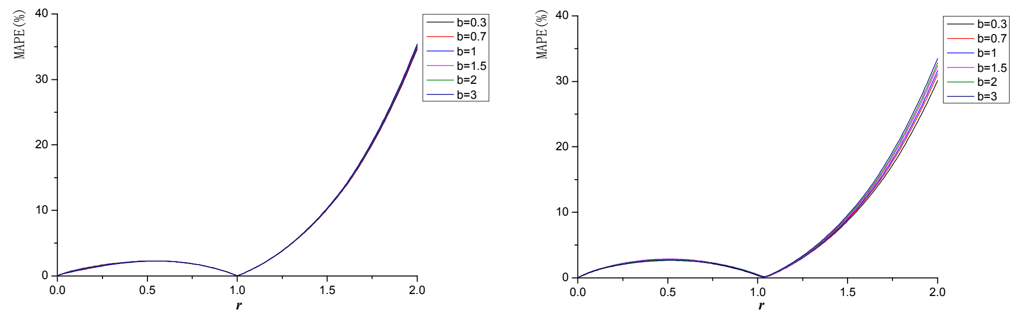
(iii) If $-a > 0.5$, the MAPE of GM(1,1) with optimal fractional-order operators is greater than 5%, indicating that these data are not suitable for the GM(1,1) with fractional-order operators.



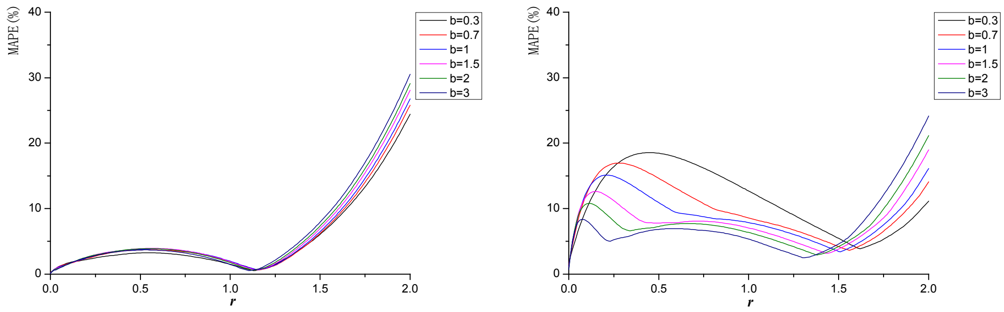
(**a**) $-a = -0.5$ (**b**) $-a = -0.2$



(**c**) $-a = -0.1$ (**d**) $-a = -0.02$



(**e**) $-a = 0.02$ (**f**) $-a = 0.1$



(**g**) $-a = 0.2$ (**h**) $-a = 0.4$

Figure 3. The MAPE of fractional-order GM(1,1) with changed $-b$ and r.

4. Conclusions

This paper presented the modeling method for the fractional-order GM(1,1). Experimental data sequences were generated, and the fitting MAPE distribution graph of the fractional-order GM(1,1) with different values of order and development coefficient was generated via numerical simulation. MAPE of the GM(1,1), the discrete GM(1,1), and fractional-order GM(1,1) was compared. The research shows that the GM(1,1) is a special case of the fractional-order GM(1,1) where $r = 1$. The fractional-order GM(1,1) is better than the classic grey prediction models, GM(1,1), and discrete GM(1,1). The fitting accuracy effectively expands the bound of grey model applications. The advantage of this method is that it can obtain better fitting accuracy by combining the optimal order algorithm, but the optimal algorithm will increase the computational complexity of model solving. This model only considers the equidistant sequence, and the nonequidistant sequence is also an application scenario worth studying.

Author Contributions: W.M. proposed the modeling method of the fractional-order prediction model; W.M. and B.Z. performed the simulations and the tests; and W.M., B.Z., and S.L. verified the simulation and test results and supervised the whole research procedure of this paper.

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Conflicts of Interest: The authors declare no conflicts of interest.

Appendix A

```
% This is the MATLAB program code for the fractional-order GM(1,1).
clear all;
% X0 is the original data sequence.
X0 = [8.21 9.52 10.51 12.72 14.84 17.89 21.22 26.79];
% r is the order of fractional-order GM(1,1). If r = 1, the model is GM(1,1).
r = 1;
n = numel(X0);
% Xr is the r-order accumulating generation sequence of the X0. If r = 0, the model is direct modeling GM(1,1).
if r == 0
    Xr = X0;
```

```
else
    for k = 1:n
        tmp = 0;
        for i = 1:k
            cc2(k,i) = gamma(r + k − i)/(gamma(k − i + 1)*gamma(r));
            tmp = tmp + cc2(k,i)*X0(i);
        end
        Xr(k) = tmp;
    end
end
% Zr is the neighboring mean production sequence of Xr.
for I = 2:n;
    Zr(i − 1) = (Xr(i) + Xr(i − 1))/2;
end;
% Xr_1 is the first order reducing generation sequence of Xr.
Xr_1(1) = X0(1);
for k = 2:n;
    Xr_1(k) = Xr(k) − Xr(k − 1);
end;
% Calculate the coefficient of fractional-order reducing generation operator.
for k = 1:n;
    for i = 0:k-1;
        if k – i ≥ 1
            cc1(k,i + 1) = (−1)^i*gamma(r+1)/(gamma(i + 1)*gamma(r – i + 1));
        else;
            cc1(k,i + 1) = 0;
        end;
    end;
end;
% Calculate the value of B and Y.
B = ones(n − 1,2);
Y = ones(n − 1,1);
for i = 1:n − 1;
    Y(i,1) = Xr_1(i + 1);
    B(i,1) = −Zr(i);
end;
% Calculate the value of a and b.
E = inv(B′*B)*B′*Y;
a = E(1);
b = E(2);
% Calculate the simulated value of Xr.
XrF(1) = X0(1);
for k = 2:n
    XrF(k) = (X0(1)-b/a)*exp(-a*(k-1))+b/a;
end
% Calculate the simulated data of X0.
if r == 0
    X0F = XrF;
else
    for k = 1:n
        tmp = 0;
        for i = 1:k
```

```
            tmp = tmp + XrF(k + 1 − i)*cc1(k,i);
        end
        X0F(k) = tmp;
    end
end
% Generate the MAPE checklist.
A = zeros(n,5);
A(1,1) = 1;
A(1,2) = X0(1);
for k = 2:n;
    A(k,1) = k;
    A(k,2) = X0(k);
    A(k,3) = X0F(k);
    A(k,4) = A(k,2)-A(k,3);
    A(k,5) = 100*abs(A(k,4))/A(k,2);
    A(k,6) = A(k,4)^2;
end;
% Calculate the Mean Absolute Percentage Error (MAPE) and Mean Squared Error
of the Model.
MAPE = mean(A(2:n,5));
MSE = mean(A(2:n,6));
clc;
% Output the original data sequence X0.
disp('The original sequence X0 is:');
disp(X0);
% Output the error checklist.
disp('The error checklist is:');
disp('      No.   Original data   Simulated data      Simulation error    APE ');
disp(A(:,1:5));
disp(['The MAPE(Mean Absolute Percentage Error) is: ',num2str(MAPE)]);
disp(['The MSE(Mean Squared Error) is: ',num2str(MSE)]);
% Draw the graph of the original sequence X0 and the simulated sequence X0F.
k = 1:1:n;
plot(k,X0,'+-',k,X0F,'*-');
legend('Original data','Simulated data',0);
```

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