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Capacity Optimization for RSMA-Based Multi-User System over Underwater Turbulence Channel

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Abstract: The underwater environment used for communication is harsh and complex, necessitating heightened standards for spectral efficiency and reliability in underwater wireless optical communication (UWOC) systems. The focus of this work is on the performance of multi-user UWOC systems operating in oblique channels of ocean turbulence downlink, where users are randomly distributed at a certain depth. A joint optimization scheme is proposed, which joints rate-splitting multiple access (RSMA) and power allocation so that the system's ergodic sum capacity is optimized to improve the transmission bandwidth. Furthermore, the probability density function (PDF) and cumulative distribution function (CDF) models for the received signal-to-noise ratio (SNR) of a multi-user multiple-input multiple-output (MIMO) system operating in the turbulent underwater oblique channels are established, accounting for the avalanche photodiode (APD) shot noise and solar radiation noise. Theoretical derivations are presented to quantify the ergodic capacity and outage probability of the multi-user system utilizing the RSMA technology. Subsequently, a numerical analysis is conducted to investigate the influence of the power allocation coefficient, RSMA, and the joint optimization algorithm on the performance of a two-user MIMO system leveraging RSMA. The simulation results show that our optimization scheme effectively reduces the outage probability, thereby achieving the maximum system sum rate and validating the practical feasibility and efficacy of the proposed scheme.

Keywords: turbulent oblique channel; rate-splitting multiple access; multi-user system; random distribution; power allocation



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1. Introduction

In underwater wireless sensor networks (UWSNs), the underwater acoustic communication faces significant limitations such as high energy consumption, prolonged delays, and substantial communication bandwidth requirements. Conversely, the advancements in high-brightness blue-green light-emitting diode (LED) and laser diode (LD) light source technology have enabled high-bandwidth underwater wireless optical communication (UWOC) to emerge as a viable alternative for short-distance underwater communication. UWOC not only offers high data transmission efficiency with low transmission delay, but also facilitate seamless communication between nodes in three-dimensional (3D) underwater environments, rendering them a highly appealing communication modality. Within the communication framework of these underwater nodes, there exists inevitably a multi-user communication system. Moreover, with the increase in equipment deployed underwater, multi-user technology in UWOC is garnering increasing attention from researchers. Meanwhile, the performance of multi-user UWOC systems may degrade due to interference caused by the concurrent scheduling of multiple optical links. Therefore, researchers have adopted different schemes to reduce interference and achieve reliable and efficient multi-user UWOC [1–9]. In [1], a rotation transform-aided transmission methodology has been

employed to mitigate interference from coexisting users and enhance system robustness. In [2], a zero-forcing precoding strategy in a 2×2 multiple-input multiple-output (MIMO) underwater visible-light communication system aiming to eradicate multi-user interference has been implemented. Power domain non-orthogonal multiple access (NOMA) is introduced for UWOC to improve the spectrum efficiency and sum rate in [3–6]. In [3], the authors experimentally demonstrated a NOMA with a polarization multiplexing UWOC system. In [4], a relay-aided NOMA optical wireless communication system is assessed in the underwater turbulence environment. In [5,6], full-duplex cooperative relay-aided NOMA UWOC systems have been studied. Since the NOMA is based on user grouping and ordering and enforces some users to fully decode and omit the interference generated by other users, its complexity increases significantly with the number of users.

Rate-splitting multiple access (RSMA) presents an alternative scheme for multi-user communications in UWOC networks. Furthermore, RSMA is shown to be a more general multiple-access scheme embracing space division multiple access (SDMA) and NOMA as special cases [10,11]. The main advantage of the RSMA scheme is that it relies on linearly precoded rate splitting with successive interference cancellation to decode part of the interference and treat the remaining part as noise. Based on its advantages, RSMA has also been integrated into multi-user UWOC, significantly mitigating multi-user interference and optimizing the system's aggregate throughput in [7–9]. In [7], the ergodic rate of an RSMA-based UWOC system is assessed with the log-normal weak oceanic turbulence. In [8,9], the performance of RSMA-based UWOC systems is investigated over the exponential generalized gamma (EGG) turbulence with pointing errors (PEs). However, the aforementioned efforts have focused primarily on horizontal oceanic turbulent channels. Furthermore, in underwater 3D wireless sensor networks where data traverse vertically or obliquely (from the deep to the shallow of the ocean or vice versa), the water depth emerges as a pivotal factor influencing system performance, warranting scrutiny. To our knowledge, there exists a dearth of studies examining RSMA schemes deployed across oblique or vertical channels over oceanic Gamma–Gamma turbulence with PEs.

In this paper, joint optimization combining RSMA and power allocation strategies for two-user systems randomly distributed at a certain depth in oblique channels of ocean turbulence downlink is delved into. Its primary objective is to evaluate the system's ergodic capacity and outage probability performance. By characterizing oceanic turbulent channels, commencing with the Gamma–Gamma distribution with PEs, the probability density function (PDF) and cumulative distribution function (CDF) of the received instantaneous signal-to-noise ratio (SNR) of the system are meticulously derived. Subsequently, analytical expressions for the ergodic capacity and outage probability specific to the RSMA implementation are formulated. To maximize the system sum rate while ensuring reliability and user fairness, a joint optimization algorithm combining RSMA and power allocation is proposed. Finally, to validate the feasibility of our proposed joint optimization scheme, a comprehensive numerical analysis focusing on the ergodic capacity and outage probability of RSMA systems operating in shallow underwater turbulent channels across diverse aquatic environments is undertaken. Our findings aim to provide valuable insights into the efficacy of combining RSMA and power allocation in enhancing the performance of UWOC systems across oblique or vertical channels over oceanic Gamma–Gamma turbulence with PEs.

The rest of this paper is organized as follows. In Sections 2 and 3, we discuss the system model and channel model for the multi-user RSMA system. In Section 4, the performance of the multi-user RSMA system in terms of the ergodic capacity and outage probability is analyzed, and the joint optimization of the RASM and power allocation algorithm is proposed to maximize the system's rate. In Section 5, we present the numerical results and discussions. Finally, the conclusion is given in Section 6.

2. System Model

The multi-user UWOC system consists of a transmitting node and N user nodes, and its communication scenario diagram is shown in Figure 1. The total optical power of the transmitting node is $P_{\max}$, θ_0 is the maximum divergence angle of the transmitting beam, and $\cos \theta_0 = 1/\sqrt{1 + (D/R)^2}$. The transmitting node is equipped with N_t vertically downward-oriented LDs, and N , $N_t \geq 2$. These independent users are all located underwater at the same depth D and randomly and uniformly distributed within the communication radius R of the transmitting node. Each user is equipped with a vertically upward-oriented avalanche photodiode (APD). The distance between the user node and the transmitting node is $l = \sqrt{D^2 + r^2}$, where the radial displacement r follows a uniform distribution, and the optical link length l also follows the same distribution, i.e., $f_l(l) = 2l/R^2$.

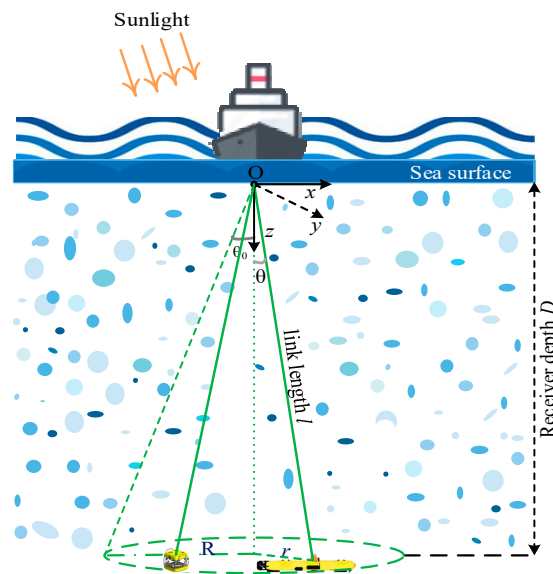


Figure 1. Communication scenario diagram of UWOC multi-user system.

The RSMA scheme is implemented in a multi-user UWOC system where one transmitting node with N_t LDs transmits N messages simultaneously to N single-APD users, and its system architecture is illustrated in Figure 2. At the transmitter, N messages $[u_1, \dots, u_i, \dots, u_N]^T$, $i \in \{1, \dots, N\}$, are divided into two parts: common messages $[u_{c1}, \dots, u_{ci}, \dots, u_{cN}]^T$ and private messages $[u_{p1}, \dots, u_{pi}, \dots, u_{pN}]^T$. The common messages $[u_{c1}, \dots, u_{ci}, \dots, u_{cN}]^T$ for N scheduled users are combined as u_c and encoded into a common stream s_c , which is decoded by each receiving user. In particular, the message u_i intended for user i is split into a private part u_{pi} and a common part u_{ci} . The private message u_{pi} of user i is encoded as the private stream s_i specific to user i and decoded by user i only. Therefore, for all user streams, $\mathbf{s} = [s_c, s_1, \dots, s_i, \dots, s_N]^T$, $s_c = a_c P_t$, and $s_i = a_{p,i} P_t$, where P_t represents the total optical power from the transmitting node, limited by $P_{\max}$. a_c and $a_{p,i}$ represent the power allocation coefficients allocated to the common and private messages, respectively, while they satisfy $a_c + \sum_{i=1}^N a_{p,i} = 1$. The transmitted vector $\mathbf{x}$ can be obtained after linearly precoding the $K + 1$ streams of $\mathbf{s}$, then completing the electric-to-optical (E/O) conversion, the driven LDs transmit the user data through an underwater wireless optical channel to the APD receiver.

At the receivers, the optical signal received by the APD of user i is then passed through an optical band-pass filter (OBPF) to complete the optical-to-electric (O/E) conversion and the electrical signal is outputted. Namely, the received signal of user i is denoted by [8]

$$y_i = \Re\eta_t\eta_r\mathbf{h}^T\mathbf{s} + n_i = \underbrace{\Re\eta_t\eta_r a_c P_t h_i s_c}_{\text{Common steam}} + \underbrace{\Re\eta_t\eta_r a_{p,i} P_t h_i s_i}_{\text{Private steam}} + \underbrace{\Re\eta_t\eta_r P_t h_i \sum_{j=1, j \neq i}^N a_{p,j} s_j}_{\text{interference}} + \underbrace{n_i}_{\text{AWGN}}, \quad (1)$$

where $\Re$ denotes the photodetector responsivity, and η_t and η_r , respectively, represent the E/O conversion efficiency of LD and the O/E conversion efficiency of APD. $\mathbf{h}$ represents the channel gain matrix, while h_i represents the channel gain between the transmitting node and user i . n_i is the additive white Gaussian noise (AWGN), which has a mean of zero and a variance of σ_n^2 . Since the receiving user node sets in this paper are all located at the same underwater depth, the received noise for each user is assumed to be the same. Since the dark current noise and thermal noise are relatively small, therefore, the APD shot noise and the solar radiation noise in the optical receiver are mainly considered in the UWOC links [4,12], i.e.,

$$\sigma_n^2 = \underbrace{2Bq\eta_r P_t}_{\text{shot noise}} + \underbrace{2\pi Bq\eta_r A_{rx} \theta_{FOV}^2 \Delta\lambda T_F E_{sol}}_{\text{solar noise}}, \quad (2)$$

where q is the electronic charge and B is the effective bandwidth of the receiver. The receiving aperture area of the receiver is $A_{rx} = \pi D_{rx}^2/4$, where D_{rx} represents the receiving aperture diameter, θ_{FOV} is the receiving field of view angle, $\Delta\lambda$ is the bandwidth of the band-pass filter at the front end of the APD photodetector, T_F is the transmittance, and E_{sol} represents the solar irradiance [12].

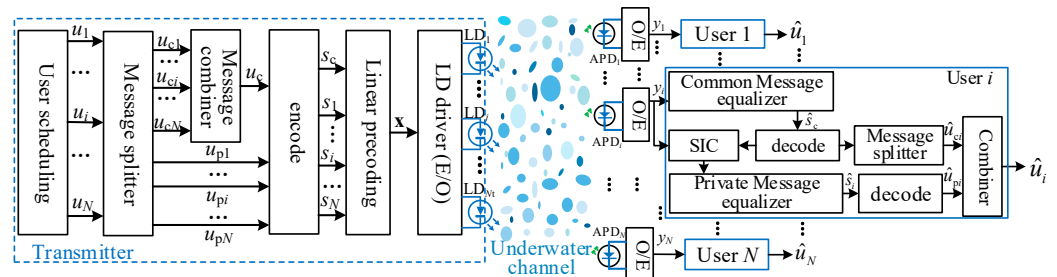


Figure 2. RSMA communication system structure diagram.

In the RSMA scheme, each user uses a two-step decoding method to extract the desired information from the received signal. The first step is to decode the common message, $\hat{s}_c$, treating all other messages as interference. The signal-to-interference-plus-noise-ratio (SINR) corresponding to the common message used to decode user i can be expressed as [9]

$$\gamma_{c,i} = \frac{a_c^2 \gamma_i}{1 + \gamma_i(1 - a_c)^2}, \gamma_i = \frac{(\Re\eta_t\eta_r P_t)^2 h_i^2}{\sigma_n^2} = \gamma_0 h_i^2, \quad (3)$$

where γ_i denotes the system's instantaneous SNR and $\gamma_0 = \frac{(\Re\eta_t\eta_r P_t)^2}{\sigma_n^2}$ is the average SNR.

In the second step, after the common message is successfully decoded, each user decodes its private message, $\hat{s}_i$, by subtracting the decoded common message from the received signal, while considering the private messages of all other users as interference. The SINR used to decode the received private message of user i is denoted by [9]

$$\gamma_{p,i} = \frac{a_{p,i}^2 \gamma_i}{1 + \gamma_i \sum_{j=1, j \neq i}^N a_{p,j}^2} = \frac{a_{p,i}^2 \gamma_i}{1 + B_p \gamma_i}, B_p = \sum_{j=1, j \neq i}^N a_{p,j}^2. \quad (4)$$

3. Underwater Channel Model

The underwater wireless optical channel is affected by various factors such as absorption, scattering, turbulence, and PEs.

3.1. Oceanic Turbulence Model

The Gamma–Gamma model can be used to characterize underwater ocean turbulence from the weak to the strong. The distribution of signal attenuation h_t due to oceanic turbulence is expressed as [13,14]

$$f_{h_t}(h_t) = \frac{2(\alpha\beta)^{(\alpha+\beta)/2}}{\Gamma(\alpha)\Gamma(\beta)} h_t^{\frac{\alpha+\beta}{2}-1} K_{\alpha-\beta}\left(2\sqrt{\alpha\beta}h_t\right), h_t \geq 0, \quad (5)$$

where $K_v(\cdot)$ is the modified Bessel function of the second kind with the order of v . $\Gamma(\cdot)$ denotes Gamma function. The turbulence parameters α and β that characterize the irradiance fluctuations are expressed as [14]

$$\alpha = \left[\exp\left(\frac{0.49\sigma_R^2}{(1 + 1.11\sigma_R^{12/5})^{7/6}}\right) - 1 \right]^{-1}, \quad \beta = \left[\exp\left(\frac{0.51\sigma_R^2}{(1 + 0.69\sigma_R^{12/5})^{5/6}}\right) - 1 \right]^{-1}, \quad (6)$$

where σ_R^2 is the Rytov variance for a plane wave. $\sigma_R^2 = 1.23C_n^2(l)k^{7/6}l^{11/6}$, where $C_n^2(l)$ indicates the oceanic index-of-refraction structure parameter [15], and $k = 2\pi/\lambda$ indicates the light wave number at the wavelength λ . κ denotes the amplitude of the spatial frequency. Thus, σ_R^2 can be represented as follows:

$$\sigma_R^2 = 19.68\pi^2 k^2 \int_0^D \int_0^\infty \kappa \Phi_n(\kappa, z) \left\{ 1 - \cos\left[\kappa^2 z(1 - z/D)/k\right] \right\} d\kappa dz, \quad (7)$$

where $\Phi_n(\kappa, z)$ denotes the spatial power spectrum of ocean turbulence, which can be applied to any depth of seawater [16], i.e.,

$$\Phi_n(\kappa, z) = \frac{C_0 \chi_n(z) \{1 + C_1 [\kappa \eta(z)]^{2/3}\}}{4\pi \kappa^{11/3} \varepsilon^{1/3}(z)} \times \frac{\omega^2(z) e^{-A_T(z)\delta(z)} - \omega(z)[1 + d_r(z)] e^{-A_{TS}(z)\delta(z)} + d_r(z) e^{-A_S(z)\delta(z)}}{\omega^2(z) - \omega(z)[1 + d_r(z)] + d_r(z)}, \quad (8)$$

where $C_0 = 0.72$, and $C_1 = 2.35$. $\varepsilon(z)$ indicates the turbulent kinetic energy dissipation rate in units of (m^2/s^3). $\chi_n(z)$ refers to the dissipation rate of the mean-squared refractive index. $\delta(z) = 1.5C_1^2[\kappa\eta(z)]^{4/3} + C_1^3[\kappa\eta(z)]^2$, where the term $\eta(z)$ denotes the Kolmogorov inner scale in units of (m), and $\eta(z) = 10^{-3}$ m. $A_T(z) = C_0 C_1^{-2}[P_T(z)]^{-1}$, $A_S(z) = C_0 C_1^{-2}[P_S(z)]^{-1}$, and $A_{TS}(z) = 0.5C_0 C_1^{-2}[P_{TS}(z)]^{-1}$, wherein the term $P_T(z)$ indicates the Planck number for temperature, $P_S(z)$ is the Schmidt number for salinity, and $[P_{TS}(z)]^{-1} = 0.5[P_T(z)]^{-1} + [P_S(z)]^{-1}$. $d_r(z)$ refers to the eddy diffusion rate of seawater, which can be approximated as a piecewise function of $\omega(z)$. Furthermore, the dimensionless parameter, $\omega(z)$, represents the relative intensity of temperature and salinity fluctuation [16,17]. Based on the above analysis, it is found that the ocean turbulence parameters α and β are only related to the depth difference between the receiving user node and the transmitting node.

3.2. Pointing Errors Model

The signal fading h_p that is caused by PEs is modeled as [18,19]

$$f_{h_p}(h_p) = \frac{\xi^2}{A_0^{\xi^2}} h_p^{\xi^2-1}, \quad 0 \leq h_p \leq A_0, \quad (9)$$

where the collected power at zero radial distance is denoted by $A_0 = [\text{erf}(v_0)]^2$, and $\text{erf}(\cdot)$ is the error function. $v_0 = \sqrt{\pi} R_{\text{rx}} / W_l$, R_{rx} denotes the radius of the receiving aperture on the receiving plane, and W_l is the beam waist radius at the distance l from the transmitter. Due to the effect of oceanic turbulence, W_l can be denoted by [20]

$$W_l^2 = W^2 + 8\pi^2 \left(\frac{l}{D} \right)^2 \int_0^D (D-z)^2 \int_0^\infty \kappa^3 \Phi_n(\kappa, z) d\kappa dz, \quad (10)$$

where $W^2 = W_0^2(\Lambda_0^2 + \Theta_0^2)$ is the beam radius at the variable distance l from the transmitter [21]. W_0 is the beam radius at the transmitter. Θ_0 and Λ_0 refer to a pair of non-dimensional quantities used as transmitter beam parameters. Θ_0 is called the curvature parameter and $\Theta_0 = 1 - l/F_0$, where F_0 is the radius of the curvature. Λ_0 is the Fresnel ratio at the transmitter plane and $\Lambda_0 = 2l/kW_0^2$. The curvature parameter $\Theta_0 = 1$ for the collimated beam. Furthermore, ξ is the ratio of the standard deviation of the equivalent beam radius W_{Leq} to the pointing error displacement on the receiving plane σ_s , signified by $\xi = W_{\text{Leq}}/2\sigma_s$. And the smaller the value of ξ , the more severe the pointing error. The equivalent beam width at the receiver is denoted by $W_{\text{Leq}}^2 = W_l^2 \left[\sqrt{\pi} \text{erf}(v_0) / (2v_0 e^{-v_0^2}) \right]$ [19]. Obviously, the impact of l on PEs is relatively small.

3.3. Attenuation Model

Signal attenuation due to absorption and scattering remains approximately constant during symbol transmission, meaning that it does not change with the instantaneous variation in the light intensity. It depends only on the optical link length l between the user node and the transmitter node, and on the average seawater attenuation coefficient c_{av} , that is, $h_a(l) = \exp(-c_{\text{av}}l)$, wherein $c_{\text{av}} = \frac{1}{D} \int_0^D c(\lambda, z) dz$, and $c(\lambda, z)$ denotes the attenuation coefficient for the light wavelength λ and the seawater depth variable z [22].

Based on the above analysis, and without a loss of generality, the effect of changes in user position on turbulence and PEs is neglected here, and only their effect on signal attenuation due to absorption and scattering is considered. The average signal attenuation $h_a^\#$ of any user channel following a random uniform distribution within the communication radius R is calculated by the following ([23], Eq. (3.351.7)):

$$\begin{aligned} h_a^\# &= \int_D^l h_a(l) f_l(l) dl \\ &= 2R^{-2} c_{\text{av}}^{-2} [\exp(-c_{\text{av}}D) - \exp(-c_{\text{av}}l)] + 2R^{-2} c_{\text{av}}^{-1} [D \exp(-c_{\text{av}}D) - l \exp(-c_{\text{av}}l)]. \end{aligned} \quad (11)$$

3.4. Joint Fading Channel Model

Assuming $h_a^\#$, $h_t(t)$, and $h_p(t)$ are independent of each other, the channel loss can be expressed as $h(t) = h_a^\# h_t(t) \cdot h_p(t)$. Therefore, the PDF of the joint fading channel coefficient h considering the effects of absorption, scattering, ocean turbulence, and PEs can be obtained as follows [24]:

$$f_h(h) = \frac{\alpha \beta \xi^2}{A_0 h_a^\# \Gamma(\alpha) \Gamma(\beta)} G_{1,3}^{3,0} \left(\frac{\alpha \beta h}{A_0 h_a^\#} \middle| \begin{matrix} \xi^2 \\ \xi^2 - 1, \alpha - 1, \beta - 1 \end{matrix} \right), \quad (12)$$

where $G_{p,q}^{m,n}[\cdot]$ is the Meijer G function [25]. According to the relation between the instantaneous SNR of γ_i and h_i for user i in (3), the PDF of γ_i can be obtained as follows:

$$f_{\gamma_i}(\gamma) = \frac{1}{2\gamma} \frac{\xi^2}{\Gamma(\alpha)\Gamma(\beta)} G_{1,3}^{3,0} \left(\frac{\alpha\beta}{A_0 h_a^\#} \sqrt{\frac{\gamma}{\gamma_0}} \middle| \frac{\xi^2 + 1}{\xi^2, \alpha, \beta} \right). \quad (13)$$

According to [25] (Eq. (07.34.21.0084.01)), the CDF of γ_i can be obtained as follows:

$$F_{\gamma_i}(\gamma) = \int_0^\gamma f_{\gamma_i}(\gamma) d\gamma = \frac{2^{\alpha+\beta-3} \xi^2}{\pi \Gamma(\alpha) \Gamma(\beta)} G_{3,7}^{6,1} \left[\left(\frac{\alpha\beta}{4A_0 h_a^\#} \right)^2 \frac{\gamma}{\gamma_0} \middle| \frac{1, \frac{\xi^2+1}{2}, \frac{\xi^2}{2}+1}{\frac{\xi^2}{2}, \frac{\xi^2+1}{2}, \frac{\alpha}{2}, \frac{\alpha+1}{2}, \frac{\beta}{2}, \frac{\beta+1}{2}, -\gamma} \right]. \quad (14)$$

4. Channel Simulation and System Performance Analysis

4.1. Channel Simulation

To accurately evaluate the multi-user channel within the communication radius of multi-beam coverage in this underwater communication scenario, channel simulations based on the theory in Section 2 will be performed. In order to integrate more closely with the real ocean channel, data from three observation nodes provided by the global ocean array for real-time geospatial oceanography (Argo) system were selected to demonstrate this. The information of the three observed nodes is given in Table 1.

Table 1. Selected Argo nodes in our simulation [26].

Water	Argo Node ID	Longitude	Latitude	Location
1	4902602	−64.587	72.253	Arctic ocean frigid zone
2	2902878	156.977	39.616	Pacific temperate zone
3	6990505	53.498	−12.681	East African coastal tropical zone

In each simulated water area, the system deploys two user nodes, each located underwater at a depth of 40 m, uniformly and randomly distributed in a circular area with a communication coverage radius of 30 m. The subsequent simulations yield PDF and CDF outcomes for the underwater optical channels, denoted as h_1 and h_2 , which are presented in Figure 3. Specifically, Figure 3a illustrates the link distances $(l_1, l_2) = (49.77 \text{ m}, 43.84 \text{ m})$ of user 1 and user 2, and the oceanic turbulence scintillation indices of the two user channels are 0.688 and 0.676, respectively. In Figure 3b, $(l_1, l_2) = (45.33 \text{ m}, 45.54 \text{ m})$, and their oceanic turbulence scintillation indices both are 0.008. In Figure 3c, $(l_1, l_2) = (40.98 \text{ m}, 42.79 \text{ m})$, and their oceanic turbulence scintillation indices are 0.031 and 0.027, respectively. As can be seen from Figure 3, it becomes evident that signal fading in water area 1 exerts the most pronounced effect among the three simulated waters. The red and blue dashed lines show that there is not much difference in signal fading between user 1 and user 2 at fixed user node positions within the communication coverage radius in the three water areas. Conversely, when user nodes are randomly positioned, their distribution is correlated with signal fading, which is enhanced compared to the fixed-node scenario. Furthermore, the disparity in signal fading between the two users within the communication radius is more pronounced, as depicted by the red and blue solid lines. Therefore, it is necessary to study the ergodic capacity and outage probability performance of a multi-user RSMA system with random distribution in the communication radius.

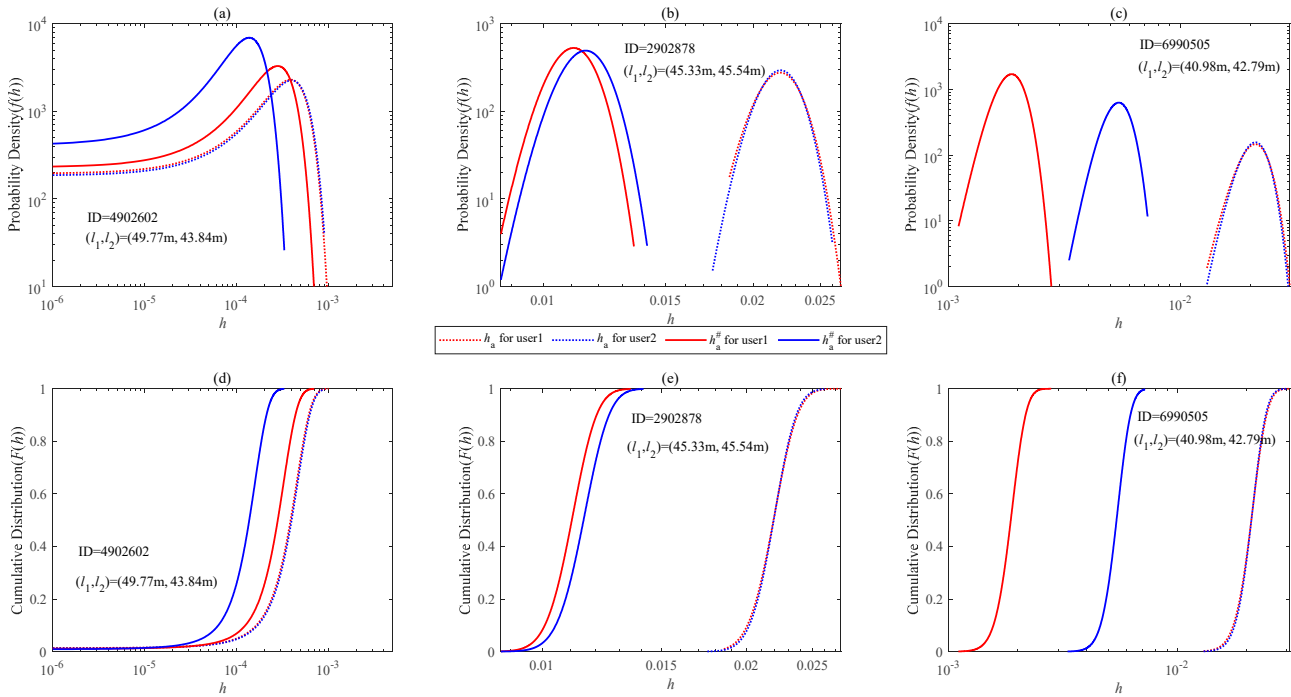


Figure 3. PDF and CDF of underwater channels h for two users in three different water areas of the scenario of Figure 1: (a) PDF of h in water area of ID = 4902602; (b) PDF of h in water area of ID = 2902878; (c) PDF of h in water area of ID = 6990505; (d) CDF of h in water area of ID = 4902602; (e) CDF of h in water area of ID = 2902878; (f) CDF of h in water area of ID = 6990505.

4.2. Ergodic Capacity

In the RSMA scheme, the common and private message capacities for user i can be expressed separately as follows [27]:

$$C_{c,i} = \frac{1}{\ln 2} \int_0^\infty \ln(1 + \gamma_{c,i}) f_{\gamma_i}(\gamma) d\gamma, \quad (15)$$

$$C_{p,i} = \frac{1}{\ln 2} \int_0^\infty \ln(1 + \gamma_{p,i}) f_{\gamma_i}(\gamma) d\gamma. \quad (16)$$

Hence, the ergodic sum capacity of the system can be derived by minimizing the common message capacity among all users, then optimizing and summing the capacity dedicated to private messages from each user. Mathematically, this ergodic sum capacity can be concisely expressed as follows [7]:

$$C_s = \min_{i \in \{1,2,\dots,N\}} C_{c,i} + \sum_{i=1}^N C_{p,i}. \quad (17)$$

Substituting (3), (4), and (13) into (15) and (16), the terms $\ln(1+\gamma_{c,i})$ and $\ln(1+\gamma_{p,i})$ can be approximated with the error bounded by $\zeta < 3.0 \times 10^{-8}$ utilizing the first eight terms of their Taylor expansion. According to [25] (Equation (07.34.21.0086.01)), the common and private message capacities for user i can be obtained as follows:

$$C_{c,i} \approx \frac{2^{\alpha+\beta-3}\zeta^2}{\pi \ln 2 \Gamma(\alpha) \Gamma(\beta)} \sum_{m=1}^8 \frac{(-1)^{m+1}}{\Gamma(m+1)} \left(\frac{a_c}{1-a_c} \right)^{2m} G_{3,7}^{7,1} \left[\left(\frac{\alpha\beta}{4A_0 h_a^\#} \right)^2 \frac{1}{(1-a_c)^2 \gamma_0} \middle| \begin{matrix} 1-m, \frac{\zeta^2+1}{2}, \frac{\zeta^2}{2}+1 \\ 0, \frac{\zeta^2}{2}, \frac{\zeta^2+1}{2}, \frac{\alpha}{2}, \frac{\alpha+1}{2}, \frac{\beta}{2}, \frac{\beta+1}{2} \end{matrix} \right], \quad (18)$$

$$C_{p,i} \approx \frac{2^{\alpha+\beta-3}\zeta^2}{\pi \ln 2 \Gamma(\alpha) \Gamma(\beta)} \sum_{m=1}^8 \frac{(-1)^{m+1}}{\Gamma(m+1)} \left(\frac{a_{p,i}^2}{B_p} \right)^m G_{3,7}^{7,1} \left[\left(\frac{\alpha\beta}{4A_0 h_a^{\#}} \right)^2 \frac{1}{\gamma_0 B_p} \left| \begin{matrix} 1-m, \frac{\zeta^2+1}{2}, \frac{\zeta^2}{2}+1 \\ 0, \frac{\zeta^2}{2}, \frac{\zeta^2+1}{2}, \frac{\alpha}{2}, \frac{\alpha+1}{2}, \frac{\beta}{2}, \frac{\beta+1}{2} \end{matrix} \right. \right]. \quad (19)$$

Substituting (18) and (19) into (17), the ergodic sum capacity of the system C_s can be obtained.

4.3. Outage Probability

The system may experience short-duration communication outages due to severe underwater channel fading. The outage capacity reflects the maximum capacity that the system can guarantee under certain reliability requirements, i.e., the maximum transmission rate of the system under acceptable outage probabilities. In the RSMA scheme, the maximum transmission rates for decoding common and private messages are denoted as R_{cth} and $R_{pth,i}$, respectively, and the corresponding SINR thresholds are $\gamma_{cth} = 2^{R_{cth}} - 1$ and $\gamma_{pth,i} = 2^{R_{pth,i}} - 1$, respectively. If the SINR used to decode common and private messages is lower than γ_{cth} and $\gamma_{pth,i}$, respectively, the link between the transmitting node and user i is interrupted. Then, the outage probability of user i is expressed as follows [8]:

$$P_{out,i} = 1 - \Pr \left\{ \gamma_{c,i} > \gamma_{cth}, \gamma_{p,i} > \gamma_{pth,i} \right\}. \quad (20)$$

By substituting (3) and (4) into (20), and utilizing (14), it can ultimately be obtained as

$$P_{out,i} = F_{\gamma}(\gamma_{th,i}) = \frac{2^{\alpha+\beta-3}\zeta^2}{\pi \Gamma(\alpha) \Gamma(\beta)} G_{3,7}^{6,1} \left[\left(\frac{\alpha\beta}{4A_0 h_a^{\#}} \right)^2 \frac{\gamma_{th,i}}{\gamma_0} \left| \begin{matrix} 1, \frac{\zeta^2+1}{2}, \frac{\zeta^2}{2}+1 \\ \frac{\zeta^2}{2}, \frac{\zeta^2+1}{2}, \frac{\alpha}{2}, \frac{\alpha+1}{2}, \frac{\beta}{2}, \frac{\beta+1}{2}, -\gamma_{th,i} \end{matrix} \right. \right], \quad (21)$$

where $\gamma_{th,i}$ denotes the SINR threshold for each user to correctly receive messages, $\gamma_{th,i} = \max \left\{ \frac{\gamma_{cth}}{a_c^2 - (1-a_c)^2 \gamma_{cth}}, \frac{\gamma_{pth,i}}{a_{p,i}^2 - B_p \gamma_{pth,i}} \right\}$, $\gamma_{cth} < \frac{a_c^2}{(1-a_c)^2}$, and $\gamma_{pth,i} < \frac{a_{p,i}^2}{B_p}$. The outage probability of the RSMA system can be expressed as follows [8]:

$$P_{out} = 1 - \prod_{i=1}^N (1 - P_{out,i}). \quad (22)$$

Finally, by substituting (21) into (22), the system outage probability can be obtained.

4.4. Joint Optimization of RASM and Power Allocation Algorithm

It is well known that RSMA provides user fairness enhancement by maximizing the worst-case rate between users to achieve maximum and minimum fairness. This section aims to maximize the system rate by optimizing the minimum rate for each user served by the transmitting node while ensuring the system's reliability, taking into account user fairness. Therefore, the proposed joint optimization (JO) algorithm considers two aspects. First, to guarantee the reliability of the system, it is necessary to satisfy the requirement that the private rate of each user is not less than the maximum transmission rate for decoding public and private messages. Secondly, the water filling (WF) algorithm is used to design the power allocation between user private messages [28]. Thus, based on the theoretical analysis in Sections 4.2 and 4.3, the optimization problem is described as follows:

$$\max R_s = \max \left(\min_{i \in \{1,2,\dots,N\}} R_{c,i} + \sum_{i=1}^N R_{p,i} \right), \quad (23a)$$

$$\text{s.t.} \begin{cases} P_i = \max\left(\mu(a_c, P_t) - \frac{1}{a_{p,i}}, 0\right) \\ \sum_{i=1}^N P_i = (1 - a_c)P_t, P_{\min} \leq P_t \leq P_{\max} \\ R_{p,i} \geq R_{\min} \end{cases} \quad (23b)$$

where μ is the water filling constant and a function of the three parameters N , a_c , and P_t . Its initial value is set as $\frac{1}{2}\left[(1 - a_c)P_t + \left(\frac{1}{a_{p,i}} - \frac{1}{a_{p,j}}\right)\right]$, $i, j \in \{1, 2, \dots, N\}$, $i \neq j$. $P_{\min}$ and $P_{\max}$, respectively, represent the minimum and maximum transmission powers allowed by the system, where $R_{\min} > \max(R_{\text{cth}}, R_{\text{pth},i})$, $i \in \{1, 2, \dots, N\}$.

Under a sum power constraint in (23b), the transmitting power control coefficient $a_{p,i}$ for user i is related to its channel gain h_i , is applied as follows:

$$a_{p,i} = \frac{h_i}{\|\mathbf{h}\|^2} \quad (24)$$

Meanwhile, in order to ensure user fairness and optimize the private rate of users, the following conditions must be satisfied:

$$\begin{cases} P_i = P_j = \frac{1}{2}a_c P_t, \text{ if } a_c P_t \leq \left|\frac{1}{a_{p,i}} - \frac{1}{a_{p,j}}\right|, \\ P_i = \frac{1}{2}\left(a_c P_t + \left|\frac{1}{a_{p,i}} - \frac{1}{a_{p,j}}\right|\right), P_j = \frac{1}{2}\left(a_c P_t - \left|\frac{1}{a_{p,i}} - \frac{1}{a_{p,j}}\right|\right), \text{ if } a_c P_t > \left|\frac{1}{a_{p,i}} - \frac{1}{a_{p,j}}\right|. \end{cases} \quad (25)$$

Obviously, (23a) is a convex optimization problem. It is suitable to use the Convex Optimization Toolbox (CVX) to solve this problem [29]. The pseudocode of the proposed JO algorithm considering RSMA and the WF power allocation is as follows. Here, its error tolerance is 10^{-5} (Algorithm 1).

Algorithm 1 JO algorithm:

Initialize: N and P_t , $P_{\min} \leq P_t \leq P_{\max}$;

- 1: According to (24), calculate $a_{p,i}, a_{p,j}$, $i, j \in \{1, \dots, N\}$, $i \neq j$;
 - 2: Satisfy $a_c + \sum_{i=1}^N a_{p,i} = 1$, obtain a_c ;
 - 3: Based on (25), obtain P_i, P_j ;
 - 4: Set P_{out} , Determine $R_{\min}$ and the SINR range for each user based on (22);
 - 5: According to $\gamma_{\text{cth}}, \gamma_{\text{pth},i}$, optimization variables of (23a) reduce to the power allocation using CVX tool until convergence, i.e., $a_{p,i}, a_c, R_{p,i}, R_c$.
-

5. Numerical Results and Discussions

In the following, numerical simulations and performance verification will be performed on the ergodic capacity and outage probability of the downlink of a UWOC system supporting RSMA for two users under a Gamma–Gamma turbulent oblique channel with PEs. The parameters for the system simulations are given in Table 2.

Table 2. System parameters in simulation.

Parameter Name	Symbol	Value
Minimum transmission power	$P_{\min}$	30 mW
Maximum transmission power	$P_{\max}$	200 mW
Electronic charge	q	1.602×10^{-19} C
OBPF bandwidth	$\Delta\lambda$	30 nm
Optical filter transmissivity	T_F	0.95
Half-angle view field	θ_{FOV}	30°
Receiver effective bandwidth	B	100 MHz
Excess noise factor	F_A	0.5

Table 2. Cont.

Parameter Name	Symbol	Value
Receiving aperture radius	R_{rx}	10 cm
Seawater depth at which user receiver is deployed	D	40 m
O/E conversion efficiency of APDs	η_r	0.6
E/O conversion efficiency of LDs	η_t	0.5
APD responsivity	$\mathfrak{R}$	0.8 A/W
Initial beam waist	W_0	5 cm
Communication coverage radius	R	30 m
Number of LDs	N_t	6
Number of APDs for each user	N_r	1
Optical wavelength	λ	532 nm

First, we compare and analyze the transmission rates (R_1 , R_2) of the two users at different system average SNRs, and the results are shown in Figure 4. When the power allocation coefficient ratio, $a_{p,1}/a_{p,2}$, of a private message for user 1 and user 2 changes from 0.1 to 10 in Figure 4, the transmission rates (R_1 , R_2) of the two users change with the increase in the system's average SNR. The data marked in Figure 4 are (R_1 , R_2) when $a_{p,1}/a_{p,2}$ is 1, respectively. Figure 5 demonstrates the changes in the system's ergodic sum rate while the system's average SNR increases. From Figures 4 and 5, both the transmission rates of two users and the system's ergodic sum rate increase as the average SNR of the system increases.

The transmission rates for these two users change as $a_{p,1}/a_{p,2}$ varies, as shown in Figure 6. According to the analysis in Figure 3, in the water area of ID = 6990505, the turbulence strengths are 0.031 and 0.027, respectively, in the channel where the two users are located, which are only slightly different. The changes in the transmission rate for these two users are similar with $a_{p,1}/a_{p,2}$ variations. Similarly, in the water area of ID = 4902602, the turbulence strengths are 0.688 and 0.676, respectively, in the channel where the two users are located, which are larger than the changes in the water area of ID = 4902602. Thus, by adjusting $a_{p,1}/a_{p,2}$, the transmission rate of two users in the RSMA system can be improved.

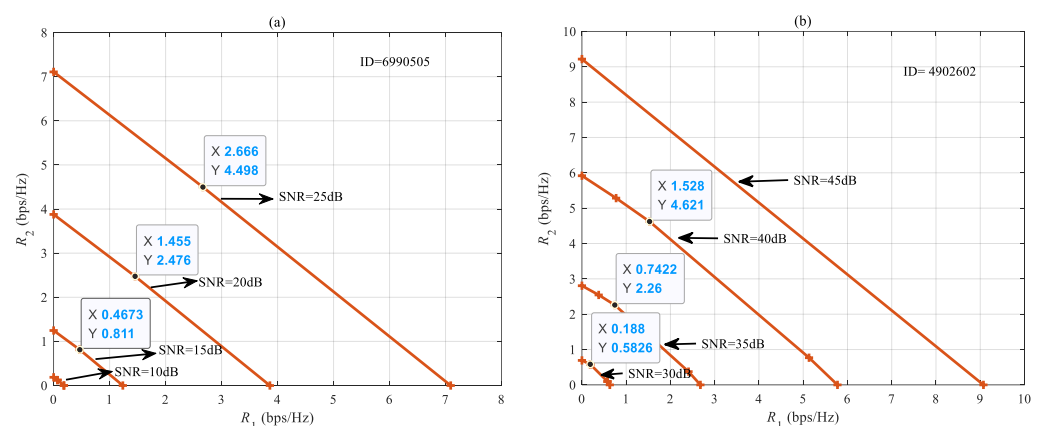


Figure 4. The transmission rate of user 1 versus that of user 2 in the different average SNRs: (a) water area of ID = 6990505; (b) water area of ID = 4902602.

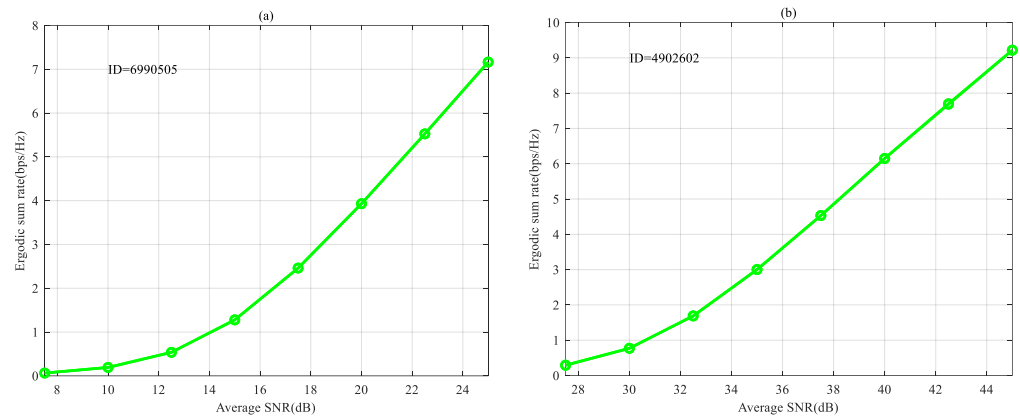


Figure 5. Ergodic sum rate in different average SNRs: (a) water area of ID = 6990505; (b) water area of ID = 4902602.

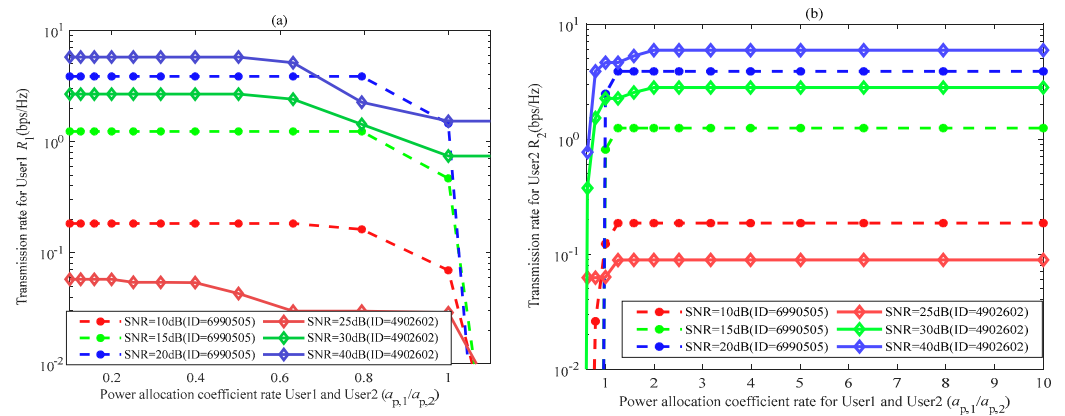


Figure 6. Transmission rate for user 1 and user 2 in different $a_{p,1}/a_{p,2}$: (a) R_1 for user 1; (b) R_2 for user 2.

For the water area of ID = 4902602, Figure 7 shows the outage probability of the RSMA system for different SINR thresholds. As shown in Figure 7, the system's outage probability increases with the increase in the SINR threshold set by each user. When the outage probability is set to 5×10^{-2} , the SINR thresholds of $\gamma_{th,1}$ and $\gamma_{th,2}$ for the water area of ID = 4902602 with an average SNR of 25 dB are 11 and 10 dB, respectively, as shown in Figure 7a. Similarly, Figure 7b shows the outage probability with an average SNR of 40 dB.

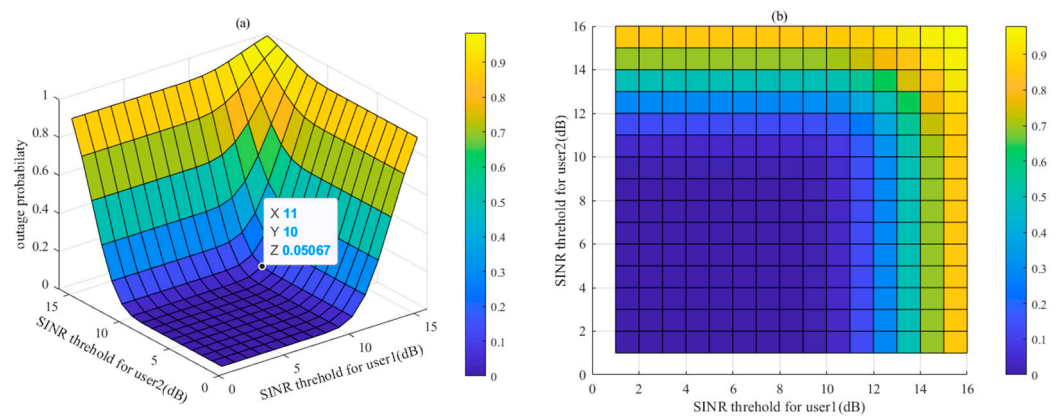


Figure 7. Outage probability in RSMA system for different SINR thresholds: (a) average SNR = 25 dB; (b) average SNR = 40 dB.

Finally, with the system outage probability $P_{\text{out}} = 5 \times 10^{-2}$, the equal power allocation (EPA) algorithm and the proposed JO algorithm in Section 4.3 are applied to analyze the impact of power allocation algorithms on the system's ergodic sum rate. The results are shown in Figure 8. The chosen water area ID in Figure 8 is 4902602. Figure 8a shows the ergodic sum rate of the RSMA system as a function of the average SNR, and Figure 8b shows the CPU time of the proposed JO algorithm, averaged over 30 random channels. As the average SNR increases, there is a slight increase in CPU running time, shown in Figure 8b. It can be seen that the JO algorithm of RSMA and the WF power allocation can effectively improve the system's ergodicity sum capacity by adaptively choosing reasonable power allocation coefficients.

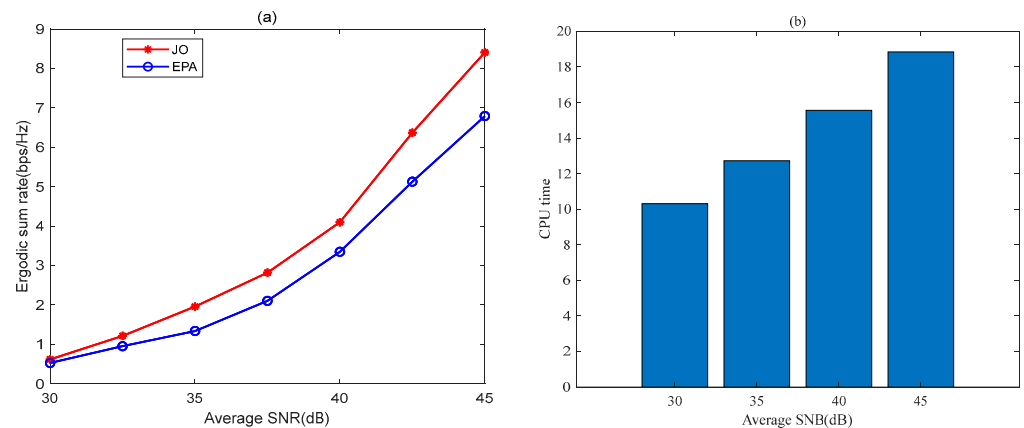


Figure 8. Ergodic sum rate of RSMA system under power allocation algorithm: (a) ergodic sum rate of RSMA system; (b) CPU time of proposed JO algorithm.

6. Conclusions

The UWOC link is complex and is usually subject to various uncertainties such as absorption, scattering, and turbulence in water. In order to accurately model underwater communication conditions, the Argo ocean monitoring data are employed and a two-user MIMO communication system with a joint optimization scheme of RASM and the power allocation algorithm under ocean turbulence with PEs is explored in this paper. It is found that the ergodic capacity of the RASM system is affected by the variation in the power allocation coefficient ratio of the private messages for users and the system's average SNR. The outage probability of the RSMA system will vary depending on the SINR threshold for each user and the system's average SNR. Our simulation results show that the joint optimization scheme of RASM and power allocation effectively improves the capacity of underwater wireless optical communication systems while ensuring system reliability. Further research can extend the analysis to N users at any depth in seawater and focus on exploring low-complexity algorithms to improve the performance and reliability of UWOC systems, ultimately achieving a robust UWOC in challenging underwater environments.

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