

Article

Turbulent Transport in a Stratified Shear Flow

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Abstract: Within the framework of the theory of unsteady turbulent flows in a stratified fluid, a new parameterization of the turbulent Prandtl number is proposed. The parameterization is included in the k - ϵ -closure and used within the three-dimensional model of thermohydrodynamics of an enclosed water body where density distribution includes pycnocline. This allows us to describe turbulence in a stratified shear flow without the restrictions associated with the gradient Richardson number and justify the choice of closure constants. Numerical experiments, where the downward penetration of turbulence was considered, confirm the advantage of the developed approach in describing the effects neglected in the classical closures.

Keywords: turbulence; numerical simulation; gradient Richardson number; turbulent Prandtl number

1. Introduction

The processes of turbulent mixing in the hydrosphere, associated, in particular, with meso- and submesoscale movements, play an important role in the dynamics of the ocean and inland waters. In particular, the instability of large-scale flows, including the breakdown of mesoscale eddies on topographic irregularities and transition to submesoscale processes, including internal waves and turbulent jets [1], and the resulting turbulent mixing, can lead to the formation of fine structure with areas of sharp gradients of hydrophysical quantities (temperature, salinity, flow velocity, etc.).

Currently, the most widely used and practically significant are three main approaches to modeling stratified turbulent flows. The first one is direct numerical simulation (DNS) of the Navier–Stokes equations, which may be used for a detailed study of turbulence over the entire range of its scales. The second approach, large eddy simulation (LES), uses a coarser grid and is based on the concept of “filtering” turbulence to explicitly resolve the largest scales of the flow. However, the relatively high requirements for computing resources limit the application of the above approaches for predicting processes of various scales in the environment.

The third approach to turbulence modeling is based on the ensemble-averaged description of random fields of hydrothermodynamic quantities in the Reynolds–Averaged Navier–Stokes (RANS) equations. In this approach, the mean flow is calculated using RANS equations, which include Reynolds stresses $\langle u_i' u_j' \rangle$ (u_i' denotes the fluctuations of the i -th velocity component, and the angular brackets denote ensemble averaging). For the Reynolds stresses, the transport equations can also be derived; in this case, they will include additional unknowns—the moments of thermohydrodynamic fields of higher order. In turbulence modeling, the problem of determining these stresses is known as the “closure problem”. One of the most common classes of RANS models used in modern oceanology and limnology refers to the two-parameter models that include transport equations for two parameters characterizing turbulence (they are commonly the turbulent kinetic energy k and another dimensional variable, for example, the rate of dissipation of turbulence



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energy ε , the product of k and the turbulence length scale L , or the frequency of turbulent pulsations, etc.) [2–5].

When simulating water bodies with seasonal temperature stratification, the presence of shear due to the wind stress (wind forcing), as well as strong turbulent mixing due to the breaking of surface waves, the so-called “ k - ε ” scheme is most often used in practical modeling [2,3]. In this scheme, a gradient approximation is applied to the turbulent stress: $-\langle u'_i u'_j \rangle = K_m (\frac{\partial \langle u_i \rangle}{\partial x_j} + \frac{\partial \langle u_j \rangle}{\partial x_i}) - \frac{1}{3} \delta_{ij} \langle u'^2 \rangle$, where the eddy viscosity K_m is calculated from the ratio $K_m = c_\mu k^2 / \varepsilon$. Under the assumption of locally isotropic turbulence, the linear scale of turbulence L is calculated from the formula $L = \frac{Ck^{3/2}}{\varepsilon}$; where c_μ and C are the coefficients which are usually determined empirically. The equations of the k - ε theory are as follows:

$$\begin{aligned} \frac{\partial k}{\partial t} &= \frac{\partial}{\partial z} \left(\frac{K_m}{\delta_k} + \nu \right) \frac{\partial k}{\partial z} + P + B - \varepsilon, \\ \frac{\partial \varepsilon}{\partial t} &= \frac{\partial}{\partial z} \left(\frac{K_m}{\delta_\varepsilon} + \nu \right) \frac{\partial \varepsilon}{\partial z} + \frac{\varepsilon}{k} (C_{1\varepsilon} \cdot P - C_{2\varepsilon} \cdot \varepsilon + C_{3\varepsilon} \cdot B), \\ K_m &= C_\varepsilon \frac{k^2}{\varepsilon}, \\ K_h &= Pr_T^{-1} \cdot K_m. \end{aligned} \quad (1)$$

Here, ν is the molecular viscosity, δ_k and δ_ε are turbulent Schmidt numbers, the function $P = C_\varepsilon \frac{k^2}{\varepsilon} \left[\left(\frac{\partial u}{\partial z} \right)^2 + \left(\frac{\partial v}{\partial z} \right)^2 \right]$ defines the generation of turbulent energy by the velocity shear, $B = -\frac{g}{\rho} C_{\varepsilon,T} \frac{k^2}{\varepsilon} \frac{\partial \rho}{\partial z}$ is responsible for turbulent energy generation and consumption due to buoyancy, $C_{1\varepsilon}$, $C_{2\varepsilon}$, $C_{3\varepsilon}$, C_ε , $C_{\varepsilon,T}$ are empirical constants [6], and K_h is the eddy thermal conductivity.

The corresponding approximate transport equation for ε is commonly obtained under numerous physical assumptions and includes four additional empirical constants. It should also be noted that one of the significant drawbacks of RANS models is the ensemble averaging method used in their derivation, which does not allow, for example, to identify physical mechanisms of turbulence generation, such as, for example, internal gravity waves. In addition, a disadvantage of traditional RANS models is the assumption that their empirical coefficients are assumed constant. In flows with stable stratification, where internal gravity waves and turbulence–wave interaction can play an important role, this assumption is not fulfilled.

Despite the popularity and, in many cases, the effectiveness of the k - ε scheme, it is often insufficient for the calculation of the dynamics of turbulent shear flows in a stratified fluid at high Richardson numbers. It is known that the values of the gradient Richardson number $Ri = -g \frac{\frac{dp}{dz}}{(dU_0/dz)^2}$ (where g is the gravity acceleration, ρ is density, and U_0 is velocity) in the upper mixed ocean layer (UML) including pycnocline, often significantly exceed the critical value $Ri = 1/4$, determined by the necessary condition $Ri < 1/4$ for the Miles–Howard hydrodynamic instability [7,8] obtained in the linear approximation. On the other hand, to maintain the turbulence that already exists, a less demanding condition [9] is required: $Ri < Pr_T$, where $Pr_T = K_m / K_h$ is the turbulent Prandtl number equal to the ratio of the coefficient of eddy viscosity K_m to the eddy diffusivity of scalar quantities, K_h . This condition is a general consequence of the turbulent energy balance equation, which means that the generation of turbulence energy due to the velocity shear exceeds its consumption due to the work of the buoyancy forces and dissipation. It is important to note that Pr_T is constant in the standard k - ε closure, and the terms describing the buoyancy forces are either included in the transport equation for k and ε (with certain additional assumptions), or the additional algebraic relations for turbulent stresses and fluxes of various scalar quantities, for example, heat, which is typical for temperature-stratified water bodies, are used. These modifications lead to a significant complication of schemes for closing the turbulent Reynolds stresses and to the appearance of additional empirical constants.

At the same time, measurements show that even at $Ri > Pr_T$, there is turbulence in the pycnocline, which has an intermittent nature, in the form of randomly located spots [10,11]. This turbulence provides an efficient transport of heat, salt, nutrients, and pollutants through pycnocline. Turbulent transport of this type is often much more significant than the effects of molecular thermal conductivity and diffusion. In general, the problem of the dependence of the small-scale turbulence on shear and stratification at large values of Ri remains an urgent and actively developing area of limnology and oceanology [12].

Over the past 20 years, new approaches based on spectral theories were developed [13,14] to describe the features of stratified turbulent flows [15,16]. These models can in some respects be an alternative to the traditional RANS models; they potentially allow one to more consistently describe non-stationary processes of turbulent-wave interaction than the RANS models. The QNSE (quasi-normal scale elimination) theory has been implemented in the atmospheric model WRF [17,18] and has been validated against the oceanic data both on meso- and submesoscales [19,20]. This theory also confirms the absence of a critical Richardson number [21], which is investigated in this paper using the more well-known RANS approach.

In this paper, we proposed a new approach for the analysis of stratified water layers within the $k-\varepsilon$ scheme, taking into account the $Pr_T(Ri)$ parameterization based on the original model [22] of unsteady turbulent flows in a stratified fluid. The authors of [22] used a sequential mathematical procedure based on an approximate solution of the equation for the probability distribution function $\tilde{f}(\mathbf{v}, \lambda, \mathbf{r}, t)$ ($\mathbf{v}$ is velocity, and λ is density at the given $\mathbf{r}$ and t) of the values of the hydrophysical fields and reduced the uncertainty of gradient semi-empirical RANS schemes at $Ri > 1$. Later, in the works of the Zilitinkevich's group [23,24], the theory of energy and flux-budget (EFB) turbulence closure was proposed. It is based on the balance equations for the energies and fluxes, and the relaxation equation for a turbulent time scale.

As noted above, one of the main results of the model [22] and the proposed parameterization based on it, is the conclusion that there is no critical value of Ri as a measure of flow laminarization in a stably stratified medium. Among other applications, this is of interest for describing mesoscale (horizontal scales 10–100 km) and submesoscale (10^0 – 10^5 m) movements in the hydrosphere [12]. For example, ocean studies at high latitudes indicate that the values of Ri range from 3 to 40 for the Gulf Stream [25] and 2 to 20 for the Florida current [26]. In this case, the shear of the mesoscale flow can interact with internal waves, causing increased mixing [27]. The geostrophically balanced flow of the upwelling-driven coastal jet off Oregon also has $Ri > 1$ [28]. Therefore, today, with the growing interest in meso- and submesoscale models of hydrosphere circulation, the problem of parameterization of turbulent mixing processes, including regions of large Ri , remains topical.

2. Turbulence Closure

Without dwelling on the details which are described in the work [22], here we briefly outline the main points of the model.

We began by introducing the variable probability distribution function $\tilde{f}(\mathbf{v}, \lambda, \mathbf{r}, t)$ for the fluid velocity $\mathbf{v}$ and density λ :

$$\tilde{f}(\mathbf{v}, \lambda, \mathbf{r}, t) = \delta(\mathbf{u} - \mathbf{v})\delta(\rho - \lambda), \quad (2)$$

where δ is Dirac delta-function, and the angular parentheses denote the ensemble averaging. Using this together with the Navier–Stokes equations for $\mathbf{u}$ and ρ , Ostrovsky and Troitskaya [22] obtained the expressions for the average fluxes of momentum, turbulent kinetic energy, density fluctuations, and mass. They are the same as those used in the common $k-\varepsilon$ theory, except for the last one, having the form:

$$\langle \rho u'_i \rangle = -LV \left(\frac{\partial \langle \rho \rangle}{\partial x_i} + g_i \frac{\langle \rho'^2 \rangle}{V^2 \rho_0} - \frac{\mathbf{g} \beta_i}{V^2 \rho_0} \right). \quad (3)$$

Here $V = \sqrt{\langle u^2 \rangle}$ is the characteristic scale of velocity, $\mathbf{g}$ is the gravity acceleration, and β_i are the components of the vector

$$\beta = \frac{1}{4\pi} \int d\mathbf{r}_1 \frac{\partial}{\partial \vec{r}} \frac{1}{|\vec{r} - \vec{r}_1|} \frac{\partial}{\partial z_1} \langle \rho'(\vec{r}, t) \rho'(\vec{r}_1, t) \rangle, \quad (4)$$

which characterizes the effect of pressure fluctuations due to random displacements of a particle in a stratified fluid.

The expression (3) for the mass flux includes the summand $g_i \frac{\langle \rho'^2 \rangle}{V^2 \rho_0} - \frac{\mathbf{g} \beta_i}{V^2 \rho_0}$, which, as shown below, leads to some significant differences from results obtained within the framework of known gradient models [1].

As shown in [22], the components of the vector β have the form $\beta_x = \beta_y = 0$, $\beta_z = \langle \rho'^2 \rangle \cdot R$ for a statistically homogeneous field of density fluctuations. Here R is the anisotropy parameter:

$$R = \begin{cases} 1, & L_z \ll L_r, \\ \approx \left(\frac{L_r}{L_z} \right)^2, & L_z \gg L_r, \end{cases} \quad (5)$$

where L_z and L_r are the vertical and horizontal scales of the density field correlation, respectively.

The physical meaning of the additional terms in (3) mentioned above is related to the dependence of the force acting upon a random displacement of a liquid particle in a stratified medium on the ratio of the characteristic scales L_z and L_r .

As a result, a closed model of a turbulent flow in a stratified fluid is obtained. It includes the equations for the mean values of velocity $\langle \mathbf{u} \rangle$, density $\langle \rho \rangle$, turbulent kinetic energy $k = 3V^2/2$, and variance of density pulsations $\langle \rho'^2 \rangle$:

$$\begin{aligned} \frac{\partial \langle u_i \rangle}{\partial t} + \langle u_j \rangle \frac{\partial \langle u_i \rangle}{\partial x_j} + \frac{1}{\rho_0} \frac{\partial \langle p \rangle}{\partial x_i} + g_i \frac{\langle \rho \rangle - \rho_0}{\rho_0} &= \frac{\partial}{\partial x_j} \left(L \sqrt{k} \left(\frac{\partial \langle u_i \rangle}{\partial x_j} + \frac{\partial \langle u_j \rangle}{\partial x_i} \right) \right), \\ \frac{\partial \langle \rho \rangle}{\partial t} + \langle u_i \rangle \frac{\partial \langle \rho \rangle}{\partial x_i} &= 2 \frac{\partial}{\partial x_i} L \sqrt{k} \left(\frac{\partial \langle \rho \rangle}{\partial x_i} + \frac{3}{2k\rho_0} (g_i \langle \rho'^2 \rangle + \mathbf{g} \beta_i) \right), \\ \frac{\partial k}{\partial t} + \langle u_i \rangle \frac{\partial k}{\partial x_i} - L \sqrt{k} \left(\frac{\partial \langle u_i \rangle}{\partial x_j} + \frac{\partial \langle u_j \rangle}{\partial x_i} \right)^2 - \frac{\mathbf{g}}{\rho_0} L \sqrt{k} \times \\ &\times \left(\frac{\partial \langle \rho \rangle}{\partial z} + \frac{3\mathbf{g}}{2k\rho_0} (\langle \rho'^2 \rangle + \beta_z) \right) + \frac{Ck^{3/2}}{L} = \frac{5}{3} \frac{\partial}{\partial x_i} \left(L \sqrt{k} \frac{\partial k}{\partial x_i} \right), \\ \frac{\partial \langle \rho'^2 \rangle}{\partial t} + \langle u_i \rangle \frac{\partial \langle \rho'^2 \rangle}{\partial x_i} - 2 \frac{\partial \langle \rho \rangle}{\partial x_i} L \sqrt{k} \left(\frac{\partial \langle \rho \rangle}{\partial x_i} + (g_i \langle \rho'^2 \rangle - \mathbf{g} \beta_i) \frac{3}{2k\rho_0} \right) + \\ &+ \frac{Dk^{1/2}}{L} \langle \rho'^2 \rangle = \frac{\partial}{\partial x_i} L \sqrt{k} \frac{\partial \langle \rho'^2 \rangle}{\partial x_i}. \end{aligned} \quad (6)$$

In an incompressible fluid considered here, $\nabla \cdot \mathbf{u} = 0$. This system includes mutual transformation between kinetic and potential energies of turbulent pulsations. Here (see also [22]), turbulence can be maintained by mean velocity shear (including that generated by internal waves) at any values of Ri . In particular, there is no turbulence “breakdown” phenomenon, in which, in certain phases of the wave, the velocity shear cannot maintain a nonzero level of turbulent energy.

Note that in the last two equations in (6) Kolmogorov’s hypotheses for the dissipation rate of turbulent energy ε and turbulent diffusion coefficient ε_D are used; namely, $\varepsilon = \frac{Ck^{3/2}}{L}$ and $\varepsilon_D = \frac{Dk^{1/2}}{L} \langle \rho'^2 \rangle$, where $C \sim D = 0.09$ are empirical constants and L is turbulent length scale. In the aforementioned $k - \varepsilon$ scheme, the dissipation rate is described with an empirical coefficient K_m in Equation (1) to account for non-equilibrium conditions.

First, we used the system (6) to consider the evolution of homogeneous turbulence in a current with constant shear. This simplified model allows us to demonstrate the exchange between kinetic and potential energies at any Richardson numbers and find the dependence of turbulent Prandtl number on Richardson number. The velocity shear V_{0z} is constant, and the

stratification corresponds to the constant Brunt–Väisälä frequency. The system (6) was reduced to two ordinary equations for kinetic k and potential $\Pi = \frac{\langle \rho'^2 \rangle g^2}{2N^2 \rho_0^2}$ energies of turbulence:

$$\begin{aligned} \frac{dk}{dt} &= V_{0z}^2 L \sqrt{k} - N^2 L \sqrt{k} \left(1 - \frac{3\Pi}{k} (1 - R) \right) - \frac{Ck^{3/2}}{L}, \\ \frac{d\Pi}{dt} &= N^2 L \sqrt{k} \left(1 - \frac{3\Pi}{k} (1 - R) \right) - \frac{Dk^{1/2}\Pi}{L}. \end{aligned} \quad (7)$$

First, we provided a stationary non-zero solution of Equation (7). It had a form:

$$\begin{aligned} k_{st} &= \frac{V_{0z}^2 L^2}{2C} f(Ri), \\ \Pi_{st} &= \frac{V_{0z}^2 L^2}{C} - k_{st}, \end{aligned} \quad (8)$$

where $f(Ri)$ is a function of the Richardson number, $f(Ri) = 1 - (4 - 3R)Ri + [1 + Ri^2(4 - 3R)^2 + (4 - 6R)Ri]^{\frac{1}{2}}$. The time of the establishment of the equilibrium had an order $t_1 \sim 1/V_{0z}\sqrt{C}$ (see also Figure 1).

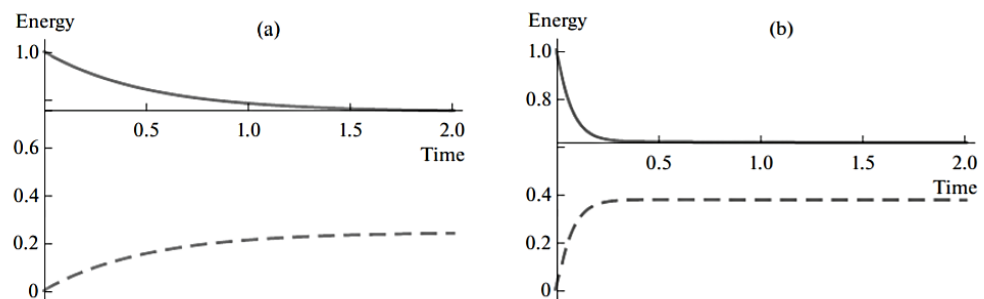


Figure 1. Dimensionless kinetic energy $\bar{k} = k/k_x$, $k_x = (V_{0z})^2 L^2 / C$ (the solid line) and potential energy $\bar{\Pi} = \Pi/\Pi_x$, $\Pi_x = (V_{0z})^2 L^2 / C$ (the dashed line) of turbulence as functions of dimensionless time $\bar{t} = t/t_x$, $t_x = 1/V_{0z}\sqrt{C}$ at different values of the Richardson number (a) 0.5 and (b) 5. The initial conditions and the anisotropy parameter $R = 0.5$ are the same in both cases.

Figure 1 shows the dependence of the dimensionless kinetic $\bar{k}(t)$ and the potential $\bar{\Pi}(t)$ energies of turbulence on time, corresponding to the model (6). For the chosen initial conditions, the kinetic energy of turbulent fluctuations decreases, whereas the potential energy of density fluctuations increases.

Note that if $Ri > 1$ (see Figure 1b), then in a short time $t_2 \sim \left(\frac{\sqrt{k}}{N^2 L} \right) = Ri^{-1} \ll t_1$, a quasi-stationary relation between the energies, $k = 3\Pi(1 - R)$, is established. The subsequent evolution to the stationary state (8) does not depend on the Richardson number, namely:

$$\frac{dk}{dt} = L\sqrt{k} \frac{3(1-R)}{4-3R} V_{0z}^2 - \frac{k^{\frac{3}{2}}(3(1-R)(C+D))}{(4-3R)L}.$$

An important parameter is the turbulent Prandtl number. In the stationary state, it is given by:

$$Pr_T = \frac{\langle u'_i u'_j \rangle \rho_{0z}}{\langle \rho' u'_i \rangle V_{0z}} = \left(1 - \frac{3\Pi}{k} (1 - R) \right)^{-1},$$

Using (8), it is easy to obtain an increasing dependence of the Prandtl number on the Richardson number:

$$Pr_T(Ri) = \frac{(4 - 3R)Ri + 1 + \left(((4 - 3R)Ri + 1)^2 - 4Ri \right)^{\frac{1}{2}}}{2}. \quad (9)$$

In particular, in the asymptotic case, $i \gg 1$, we have:

$$Pr_T(Ri) = (4 - 3R)Ri. \quad (10)$$

Therefore, the condition of turbulence sustainment in a stratified fluid, $Ri < Pr_T$, [13] is satisfied at any value of Ri ; i.e., there is no generation threshold for the Richardson number. Zilitinkevich [23,24] obtained similar expressions for the stationary values of $\Pi(Ri)$, $k(Ri)$, and $Pr_T(Ri)$ in the framework of the EFB model. The comparison of the above dependence (10) with the EFB model developed in these works is presented in Figure 2.

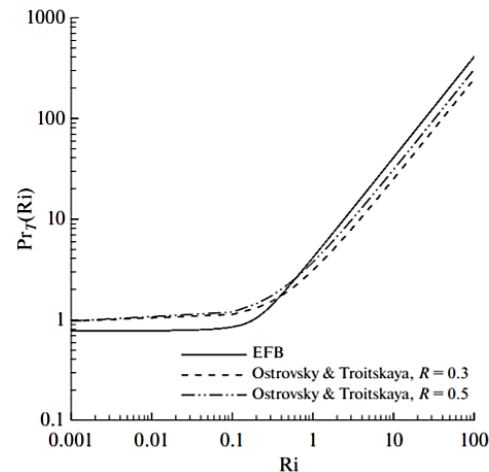


Figure 2. Dependence of the Prandtl number on the Richardson number from the EFB model [23,24] and the model by Ostrovsky and Troitskaya [22].

These parameterizations are equivalent to a constant. It can be seen that $Pr_T(Ri)$ is an indefinitely increasing function, and for large values of Ri it grows linearly.

Note that the expressions (8) for the stationary values have the same form as in the k - ε model, where the eddy viscosity coefficient is $K_m = c_\mu \sqrt{k}L$, where $k = \left(\frac{C_\varepsilon}{L}\right)^{2/3}$ so that the turbulence scale is $L = C \frac{k^{3/2}}{\varepsilon}$. Using here the traditional approximation for the diffusion rate of the scalar value (e.g., [29]), one can determine the dissipation rate of the potential energy of turbulent fluctuations ε_Π when the temporal scales of velocity fluctuations and the scalar value are equal:

$$\frac{\Pi}{\varepsilon_\Pi} \cdot \frac{\varepsilon}{k} = 1.$$

The expressions (8) (then will have) the form:

$$\begin{aligned} \frac{\varepsilon^2}{k^2} &= \frac{V_{0z}^2 C}{2} f(Ri), \\ \frac{\Pi}{k} &= \frac{2(1-f(Ri))}{f(Ri)}. \end{aligned} \quad (11)$$

Note also that from the expressions for the sources P and B of turbulent energy from (1) applied to the above example of homogeneous turbulence, in the stationary case we obtained:

$$C_{1\varepsilon} V_{0z}^2 - C_{3\varepsilon} N^2 \left(1 - \frac{3\Pi}{k}(1-R)\right) = \frac{C_{2\varepsilon}}{C} \frac{\varepsilon^2}{k^2}. \quad (12)$$

From this expression and the formulae (11) it follows that, as already mentioned, the parameters of the k - ε model are not independent but they are related by the following equation depending on Ri :

$$C_{1\varepsilon} - C_{3\varepsilon} Ri \left(1 - \frac{6(1-f(Ri))}{f(Ri)} (1-R) \right) = \frac{C_{2\varepsilon}}{C_2} f(Ri). \quad (13)$$

In particular, in the limit of $Ri \rightarrow \infty$, for the function $f(Ri)$ this relation takes the form:

$$C_{1\varepsilon}(4-3R) - C_{3\varepsilon} = 3C_{2\varepsilon}(1-R).$$

Thus, we supplemented the system (1) with the expression (9) to additionally take into account the contribution of shear and stratification, remove restrictions on the description of turbulence at large Ri , and justify the choice of constants. Hence, applying the theory developed in [22] to the description of turbulence in the k - ε model, the resulting equation can be written in the form:

$$\begin{aligned} \frac{\partial k}{\partial t} &= \frac{\partial}{\partial z} \left(\frac{K_m}{\delta_k} + \nu \right) \frac{\partial k}{\partial z} + P + B - \varepsilon, \\ \frac{\partial \varepsilon}{\partial t} &= \frac{\partial}{\partial z} \left(\frac{K_m}{\delta_\varepsilon} + \nu \right) \frac{\partial \varepsilon}{\partial z} + \frac{\varepsilon}{k} (C_{1\varepsilon} \cdot P - C_{2\varepsilon} \cdot \varepsilon + C_{3\varepsilon} \cdot B), \\ K_m &= C_\varepsilon \frac{k^2}{\varepsilon}, \\ K_h &= \left(\frac{(4-3R) Ri + 1 + ((4-3R) Ri + 1)^2 - 4Ri}{2} \right)^{-1} \cdot K_m. \end{aligned} \quad (14)$$

Here the constants in the second equation are related by (13). These equations are used for the specific calculations described in the next section.

3. Numerical Experiments

Using the developed modified turbulent closure, we considered here some realistic examples of the interaction between turbulence and shear flow. Based on the above theory, we studied the effect of the turbulent Prandtl number parameterization on the description of mixing in a stratified shear flow using the three-dimensional hydrostatic RANS model. The numerical code, developed at Moscow State University [30–32] unites different approaches (RANS, LES, DNS) to describe geophysical turbulent flows with high temporal and spatial resolution. The equations of the model are as follows:

$$\begin{aligned} \frac{\partial u}{\partial t} &= -A(u) + D_H(u, \chi_m) + D_z(u, K_m + \nu) - \mathbf{g} \frac{\partial \eta}{\partial x} - \frac{\mathbf{g}}{\rho_0} \frac{\partial}{\partial x} \int_z^\eta \rho dz' + f_{Cor} v, \\ \frac{\partial v}{\partial t} &= -A(v) + D_H(v, \chi_m) + D_z(v, K_m + \nu) - \mathbf{g} \frac{\partial \eta}{\partial y} - \frac{\mathbf{g}}{\rho_0} \frac{\partial}{\partial y} \int_z^\eta \rho dz' - f_{Cor} u, \\ \nabla \cdot \mathbf{u} &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \\ \frac{\partial T}{\partial t} &= -A(T) + D_H(T, \chi_h) + D_z(T, K_h + \lambda), \\ \frac{\partial S}{\partial t} &= -A(S) + D_H(S, \chi_h) + D_z(S, K_h + \lambda), \\ \rho &= \rho(T, S), \\ \frac{\partial \eta}{\partial t} &= w. \end{aligned} \quad (15)$$

Here $\mathbf{u} = (u, v, w)$ is the velocity vector, η is the free surface deviation from the equilibrium state, f_{Cor} is the Coriolis parameter, T is the temperature, S is the salinity, ρ is the density, K_m (χ_m) and K_h (χ_h) are coefficients of vertical (horizontal) turbulent viscosity and temperature conductivity, respectively, ν and λ are coefficients of molecular viscosity and temperature conductivity, respectively, and z is the vertical coordinate going from the water bottom $z = -H(x, y)$ to the surface. In addition, $A(q)$ is the advection operator, and $D_H(q, \lambda)$ and $D_z(q, K)$ are the operators defining horizontal and vertical diffusion with the coefficients λ and K , respectively. The coefficients K_m and K_h describing vertical turbulent mixing were calculated using the standard k - ε model (1).

To estimate the effect of turbulent Prandtl number parameterization on the description of turbulent mixing processes, two configurations of the model were used: an idealized reservoir with a rectangular cross-section, prescribed constant wind speed, neglecting the effects of short-wave radiation and Coriolis force, and a more specific configuration of the model corresponding to the Finnish lake, Kuivajärvi. In both cases, we compared the results obtained using the standard k - ϵ scheme (1) and the scheme with parameterization described above (14). In the standard scheme, the turbulent Prandtl number was set constant and equal to 1.25. Such a value agrees with estimates of Pr_{T_0} in the conditions of neutral stratification according to the data of laboratory experiments [33,34] and is commonly used in calculations of circulation in inland water bodies with neutral (or close to neutral) stratification (see, e.g., [35]). The values of constants in k -epsilon model are taken from the data given in [6]. In the second, modified scheme, the expressions (14) and (13) were used for the calculation of these constants.

The idealized setting used the following parameters: 10 m depth, surface temperature 20 °C with an initial gradient $\partial T/\partial z = 1.5$ °C/m, which corresponds to the Brent–Väisälä frequency (buoyancy frequency) $N = 4 \cdot 10^{-2} \text{ s}^{-1}$, constant momentum flux on the surface: $\tau = 10^{-2} \text{ N/m}^2$.

Figure 3 demonstrates that even in this idealized case, the description of the vertical distribution of turbulent kinetic energy is sensitive to the proposed modification.

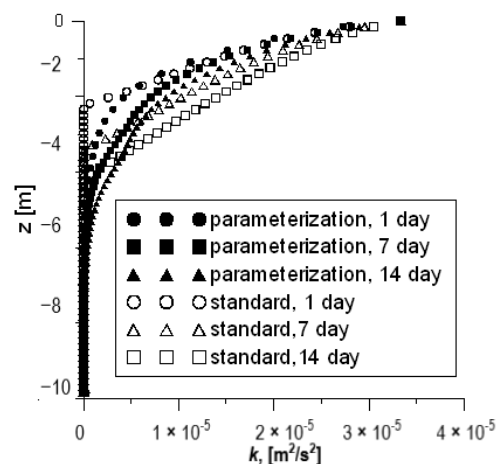


Figure 3. Profiles of the kinetic energy of turbulence in different days of calculation using the standard and modified schemes in the first numerical experiment.

In the case of a standard closure, the turbulence breakdown effect exists, and we see sharp transitions where the kinetic energy of turbulence becomes equal to zero. In the case of a modified closure the vertical distribution of turbulent kinetic energy is smooth within transient layer. This is the feature of the modified closure—it implies the existence of turbulence at $Ri > 1$.

For the example of Lake Kuivajärvi, the results of parameterization were investigated for a real water body. This small body of water is located in Southern Finland. The results of measurements made in 2013 [36,37] were used as atmospheric forcing data during the entire calculation time (components of wind speed, fluxes of short-wave radiation, sensible and latent heat), as well as a set of the initial conditions (for example, initial density vertical distribution). In the measurement area, the water depth was 12 m. A period of 24 h in June of 2013 was chosen when the intense heat flux resulted in large values of the Richardson number. The vertical distribution of the turbulence kinetic energy is shown in Figure 4, and the vertical distribution of the Richardson number values is shown in Figure 5.

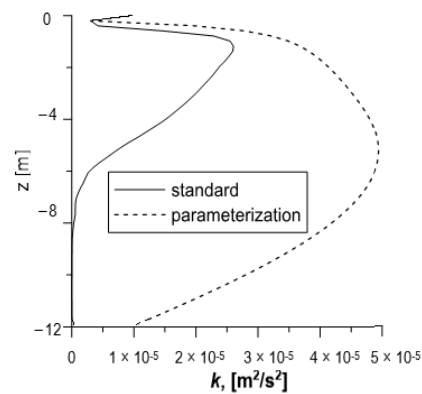


Figure 4. Profiles of the kinetic energy of turbulence using the standard and modified schemes in the lake.

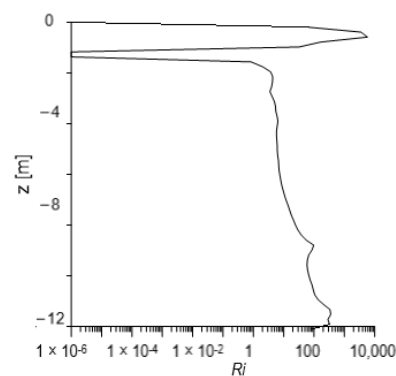


Figure 5. Profile of the Richardson number in the same numerical experiment.

Using our parameterization, we calculated the evolution of temperature distribution from May to October of 2013. Figure 6 shows a significant downward penetration of heat, which agrees with the observed data [38]. This confirms the advantages of the proposed approach compared to, for example, the Canuto stability functions [39] with the limited function of the Prandtl number used in [38].

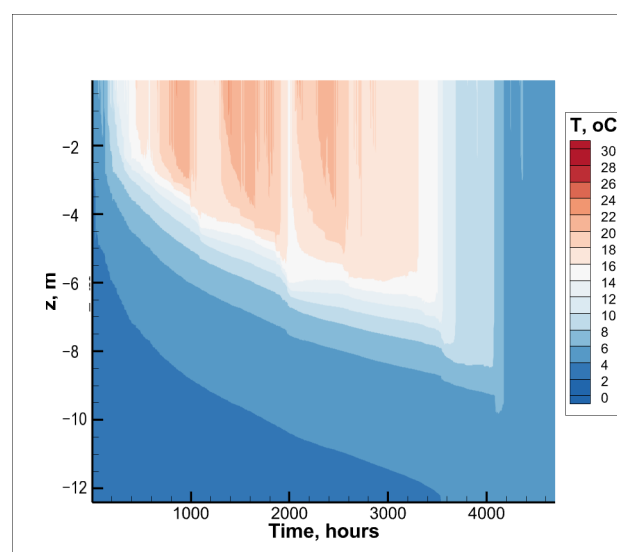


Figure 6. Time–depth distribution of temperature in Kuivajärvi Lake.

In the numerical experiment, the Richardson number varies significantly in the range from ~ 0.001 to ~ 100 – 1000 , reaching large values already at a shallow depth. In this case,

the kinetic energy varies smoothly over the entire mixing region. In the framework of the standard model (with the fixed Prandtl number), the turbulence below the thermocline is suppressed by viscosity and dissipation, whereas molecular diffusion is insufficient for transport through the thermocline. On the contrary, for the improved model, which takes into account an increase in Pr_T due to an increase in K_m at large Ri , a significant energy transfer through thermocline does exist. Significant penetration of scalar substances along the depth corresponds to the data observed in natural conditions [38].

4. Conclusions

In this paper, we proposed a parameterization of the turbulent Prandtl number based on a model that takes into account the mutual transformation of the kinetic and potential energies of turbulent pulsations. This parameterization is introduced into the formula for the coefficient of eddy viscosity in the $k-\varepsilon$ scheme to correctly take into account the stratification when calculating the thermohydrodynamics of stratified water objects. Besides, the obtained results confirm the choice of closure constants calculated as functions of the Richardson number. To test the chosen parametrization, two types of numerical experiments were carried out: a configuration of an idealized water body and the one with the use of field measurements carried out on Lake Kuivajärvi. For the calculations, a three-dimensional hydrostatic model was used. The calculation results allow us to conclude that the description of vertical mixing in stratified water bodies is sensitive to the parameterization of the turbulent Prandtl number.

The parameterization leads to the smoothing of all sharp changes in the vertical distributions of turbulent kinetic energy and thickness of the transient layer, and there is no turbulence “breakdown” phenomenon since turbulence is maintained at any values of the Richardson number. For the example of Lake Kuivajärvi, the fundamental role of parameterization in calculating the thermodynamics of a reservoir under the conditions of a formed thermocline was demonstrated. It was shown that due to an increase in the coefficient of turbulent diffusion, there is significant penetration of small-scale turbulence below the thermocline.

These results can have broader applications in numerical models of the atmosphere, ocean, and inland water bodies, in particular, when calculating seasonal and interannual dynamics, as well as for describing the processes of transfer of biochemical impurities, through a thermocline.

As for the application of the obtained results to the description of mesoscale and submesoscale processes, the interest in using the proposed approach is confirmed by studies of ocean currents and, in particular, by the presence of large values of the Richardson number in many regions of the World Ocean. At present, we plan to include the proposed parameterization in the oceanic block of the INM RAS climate model [40] for further application of the proposed approach in the framework of the parameterization of meso- and submesoscale processes in global weather and climate studies.

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