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# Analysis of a Fractional-Order COVID-19 Epidemic Model with Lockdown

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**Abstract:** The outbreak of the coronavirus disease (COVID-19) has caused a lot of disruptions around the world. In an attempt to control the spread of the disease among the population, several measures such as lockdown, and mask mandates, amongst others, were implemented by many governments in their countries. To understand the effectiveness of these measures in controlling the disease, several mathematical models have been proposed in the literature. In this paper, we study a mathematical model of the coronavirus disease with lockdown by employing the Caputo fractional-order derivative. We establish the existence and uniqueness of the solution to the model. We also study the local and global stability of the disease-free equilibrium and endemic equilibrium solutions. By using the residual power series method, we obtain a fractional power series approximation of the analytic solution. Finally, to show the accuracy of the theoretical results, we provide some numerical and graphical results.

**Keywords:** COVID-19 epidemic model; lockdown; fractional-order differential equations; stability of equilibrium solutions; fractional power series



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## 1. Introduction and Preliminaries

The study of mathematical models of infectious diseases has attracted the attention of many researchers since they provide a better understanding of their evaluations, existence, stability, and control [1–6]. Most of the mathematical models of infectious diseases are composed of a system of integer-order differential equations. However, in the last few decades, fractional-order differential equation has been used in the modeling of biological phenomena because they provide a greater degree of accuracy than the integer-order models [7–13]. The theory of fractional calculus, which involves differentiation and integration of non-integer orders, is as old as the classical calculus of integer orders. However, this theory has gained considerable attention from many researchers in recent years due to the numerous applications found in the sciences, engineering, economics, control theory, and finance, amongst others. The increasing interest in using fractional calculus in the modeling of real-world phenomena is due to its various properties which are not found in classical calculus. Unlike the integer-order derivatives, which are local in nature, most of the fractional-order derivatives (for example, the Caputo and Riemann–Liouville fractional order derivatives [14,15]) are non-local and possess the memory effects which make it more superior because in many situations the future state of the model depends not only upon the current state but also on the previous history. For this realistic property, the usage of fractional-order systems is becoming popular to model the behavior of real systems in various fields of science and engineering. Many researchers have therefore expanded the integer-order models to fractional-order models via various mathematical techniques. Recently, several authors have used fractional-order models to understand the dynamics of the COVID-19 epidemic to achieve the best strategies to stop the spread of the disease, chose a better effective immunization program, allocate scarce resources to control or prevent infections and also predict the future course of the outbreak [16–27].

Motivated by the above works and the high interest in finding the best solution to control the COVID-19 pandemic, we study a fractional-order model to understand the dynamics of the COVID-19 infections when the population is under lockdown. Most of the COVID-19 models in the literature, to the best of our knowledge, do not include the impact of the preventive measures adopted by many governments around the world to control the virus. Lockdown was one of the preventive measures and that makes our model quite unique and interesting.

Several definitions of fractional derivatives and integrals have been provided in the literature. For the purpose of this work, we present the definitions and some properties of the Riemann–Liouville fractional integral, Caputo fractional derivative and other related fractional-order derivatives. For more information about the Riemann–Liouville fractional integral and the Caputo derivative, we refer the interested reader to [14,15] and the references therein.

**Definition 1** ([14]). *The Riemann–Liouville fractional integral of order  $\alpha > 0$  of a function  $f$  is given by*

$$J_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(z)}{(t-z)^{1-\alpha}} dz, \tag{1}$$

where  $\Gamma$  denotes the Gamma function defined by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \quad x > 0.$$

**Definition 2** ([14]). *The Caputo fractional derivative of order  $\alpha > 0$  is given by*

$$D_a^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(z)}{(t-z)^{\alpha-n+1}} dz, \quad t > a, n-1 < \alpha \leq n, n \in \mathbb{N}. \tag{2}$$

The Caputo derivative satisfies the following properties:

1.  $D_a^\alpha (b_1 f(t) + b_2 g(t)) = b_1 D_a^\alpha f(t) + b_2 D_a^\alpha g(t)$ ,  $b_1, b_2 \in \mathbb{R}$ .
2.  $D_a^\alpha (J_a^\alpha f(t)) = f(t)$ .
3.  $J_a^\alpha (D_a^\alpha f(t)) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (t-a)^k$ .
4.  $D_a^\alpha (c) = 0$ ,  $c \in \mathbb{R}$ .
5.  $D_a^\alpha (t-a)^\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)} (t-a)^{\lambda-\alpha}$ , for  $\lambda > n-1$ .
6.  $D_a^\alpha (t-a)^k = 0$ , for  $k = 0, 1, 2, \dots, n-1$ .

Most fractional-order differential equations do not have a closed-form solution, and thus approximation and numerical methods are extensively used. One way to find an approximated solution of a system of fractional-order differential equations is by using the technique of fractional power series. The following is the definition of the fractional power series.

**Definition 3** ([28]). *The fractional power series about  $t = t_0$  is defined as*

$$\sum_{m=0}^\infty c_m (t-t_0)^{m\alpha} = c_0 + c_1 (t-t_0)^\alpha + c_2 (t-t_0)^{2\alpha} + \dots, \tag{3}$$

where  $n-1 < \alpha \leq n$  for some  $n \in \mathbb{N}$  and  $c_m$  for  $m = 0, 1, 2, \dots$  are the coefficients of the power series.

The following result hold for the Caputo derivative.

**Theorem 1** ([28]). Suppose the fractional power series representation of  $f$  is of the form

$$f(t) = \sum_{m=0}^{\infty} c_m(t - t_0)^{m\alpha}.$$

If  $D_a^{m\alpha} f(t)$  for  $m = 0, 1, 2, 3 \dots$  are continuous on the interval  $(t_0, t_0 + \rho)$ , then  $c_m = \frac{D_a^{m\alpha} f(t_0)}{\Gamma(1 + m\alpha)}$ , where  $D_a^{m\alpha} = D_a^\alpha D_a^\alpha \dots D_a^\alpha$  ( $m$ -times) and  $\rho$  is the radius of convergence.

**Remark 1.** We note that the kernel function in the definition of the Caputo derivative  $(t - z)^{n-1-\alpha}$  has a singularity at  $z = t$ . Recently, some new fractional derivatives with non singular kernels were introduced in the literature and we present them here for the readers' reference.

In 2015, Caputo and Fabrizio [29] proposed the following generalization of the Caputo derivative as follows:

**Definition 4.** The Caputo–Fabrizio fractional-order derivative (CF) of order  $\alpha$  is defined as follows:

$${}^{CF}D_a^\alpha f(t) = \frac{M(\alpha)}{(1 - \alpha)} \int_a^t f'(z) \exp\left\{-\frac{\alpha(t - z)}{1 - \alpha}\right\} dz, \quad 0 < \alpha \leq 1 \tag{4}$$

where  $M(\alpha)$  is a normalization function such that  $M(0) = M(1) = 1$ . The Caputo–Fractional derivative of a constant function is zero, but unlike the Caputo derivative in (2), the kernel does not have a singularity at  $z = t$ .

The Caputo–Fabrizio fractional-order had also been generalized by Atangana and Baleanu in 2016 as follows:

**Definition 5** ([30]). Let  $f \in H^1(a, b)$ ,  $a < b$  and  $\alpha \in [0, 1]$ . The Atangana–Baleanu fractional-order derivative in the Caputo sense is given by

$${}^{ABC}D_a^\alpha f(t) = \frac{B(\alpha)}{(1 - \alpha)} \int_a^t f'(z) E_\alpha\left\{-\frac{\alpha(t - z)}{1 - \alpha}\right\} dz \tag{5}$$

where  $B(\alpha)$  is a normalization function such that  $B(0) = B(1) = 1$ ,  $E_\alpha(\cdot)$  is the one parameter Mittag–Leffler function (see Remark 3) and  $H^1(a, b)$  is the Sobolev space of order one defined as follows:

$$H^1(a, b) = \{u \in L^2(a, b) : u' \in L^2(a, b)\}.$$

We refer the interested reader to [29,30] and the references therein for more information about these fractional derivatives.

**Remark 2.** It is worth pointing out that recently, Hattaf [31] also introduced some new generalizations of the Atangana–Baleanu fractional-order derivatives by introducing a weight function and thus giving rise to the definition of several fractional-order derivatives with non-singular kernels.

Next, we present the definition of the two-parameter Mittag-Leffler function, which will be utilized later in the paper.

**Definition 6** ([28,32]). The two-parameter Mittag-Leffler function denoted by  $E_{\alpha,\beta}$ , is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \operatorname{Re}(\alpha), \operatorname{Re}(\beta) > 0, \beta \in \mathbb{C}. \tag{6}$$

**Remark 3.** When  $\beta = 1$  in (6), then we have the one parameter Mittag-Leffler function which is denoted by  $E_\alpha(\cdot)$ . That is,  $E_\alpha(z) = E_{\alpha,1}(z)$

## 2. Model Formulation

In order to understand the impact of lockdown in preventing the spread of COVID-19 infection, Baba et al. [33] proposed the following model governed by a system of nonlinear ordinary differential equations:

$$\begin{aligned}\frac{dS}{dt} &= \Lambda - \beta SI - \lambda_1 SL - dS + \gamma_1 I + \gamma_2 I_L + \theta_1 S_L, \\ \frac{dS_L}{dt} &= \lambda_1 SL - dS_L - \theta_1 S_L, \\ \frac{dI}{dt} &= \beta SI - \gamma_1 I - \alpha_1 I - dI - \lambda_2 IL + \theta_2 I_L, \\ \frac{dI_L}{dt} &= \lambda_2 IL - dI_L - \theta_2 I_L - \gamma_2 I_L - \alpha_2 I_L, \\ \frac{dL}{dt} &= \mu I - \phi L,\end{aligned}\tag{7}$$

subjected to the initial conditions

$$S(0) = S_0 > 0, S_L(0) = S_L^0 > 0, I(0) = I^0 > 0, I_L(0) = I_L^0 > 0 \text{ and } L(0) = L_0 > 0.$$

In this model, the authors considered a population of total size  $N(t)$  at time  $t$  with a constant recruitment rate of  $\Lambda$ . The population is divided into four compartments denoted by  $S(t)$ ,  $S_L(t)$ ,  $I(t)$  and  $I_L(t)$ . The  $S(t)$  class denotes the susceptible individuals that are not under lockdown. The group  $S_L(t)$  contains those individuals who are susceptible and are under lockdown. The group who are infected and are not under lockdown are represented by  $I(t)$ , while those individuals who are infected and are under lockdown are denoted by  $I_L(t)$ . Finally, the cumulative density of the lockdown program is denoted by  $L(t)$ . The authors have studied the local stability of the equilibrium solutions in relation to the basic reproduction number.

In order to include the memory effects and the past history to get a better understanding of the dynamics of COVID-19 infections under lockdown, we reformulate the model (7) by using the Caputo fractional derivative as follows:

$$\begin{aligned}D_0^\alpha S(t) &= \Lambda - \beta SI - \lambda_1 SL - dS + \gamma_1 I + \gamma_2 I_L + \theta_1 S_L \\ D_0^\alpha S_L(t) &= \lambda_1 SL - dS_L - \theta_1 S_L \\ D_0^\alpha I(t) &= \beta SI - \gamma_1 I - \alpha_1 I - dI - \lambda_2 IL + \theta_2 I_L \\ D_0^\alpha I_L(t) &= \lambda_2 IL - dI_L - \theta_2 I_L - \gamma_2 I_L - \alpha_2 I_L \\ D_0^\alpha L(t) &= \mu I - \phi L,\end{aligned}\tag{8}$$

where  $D_0^\alpha$  denotes the Caputo derivative for  $0 < \alpha \leq 1$ ; under the initial conditions

$$S(0) = S_0 > 0, S_L(0) = S_L^0 > 0, I(0) = I^0 > 0, I_L(0) = I_L^0 > 0 \text{ and } L(0) = L_0 > 0.$$

We explore some interesting results of the fractional-order COVID-19 model (8), description of the parameters shown in Table 1. In particular, after proving the existence of a unique positive global solution of the model, we study the local and global stability of the various equilibrium points of the fractional-order model using the comparison theory of fractional differential inequality and fractional La-Salle invariance principle. Moreover, we apply the method of residual power series technique to approximate the solution of the

fractional-order system (8). Finally, we provide numerical simulations to illustrate some of the theoretical results and error analysis to show how good the approximated solution is.

**Table 1.** Description of the parameters used in the COVID-19 epidemic model (8).

Parameters	Description of the Parameters
$\Lambda$	Recruitment rate
$\beta$	Infection contact rate
$\lambda_1$	Imposition of lockdown on susceptible group
$\lambda_2$	Imposition of lockdown on infected group
$\gamma_1$	Recovery rate of the infected group
$\gamma_2$	Recovery rate of the infected group under lockdown
$\alpha_1$	Death rate of the infected group
$\alpha_2$	Death rate of the infected group under lockdown
$d$	Natural death rates
$\theta_1$	Rate of transfer of susceptible lockdown individuals to susceptible class
$\theta_2$	Rate of transfer of susceptible lockdown individuals to infected class
$\mu$	rate of implementation of the lockdown program
$\phi$	rate of depletion of the lockdown program

**Remark 4.** In [17], the authors also discussed a slightly different extension of the model in (7) to fractional calculus where they include the order of the fractional derivative as powers on the parameters. They established the existence and uniqueness of solutions to their model using the Schauder and Banach fixed point theorems.

Denote the set  $\Omega$  as follows,

$$\Omega = \left\{ (S, S_L, I, I_L, L) \in \mathbb{R}_+^5 : S + S_L + I + I_L \leq \frac{\Lambda}{d} \text{ and } L \leq \frac{\mu\Lambda}{\phi d} \right\}.$$

**Theorem 2.** For any  $t \geq 0$  and  $X(0) \in \Omega$  there exist a unique positive solution  $X(t) = (S(t), S_L(t), I(t), I_L(t), L(t))$  of system (8). Moreover, the set  $\Omega$  attracts all solutions of system (8), and thus it is positively invariant.

**Proof.** Denote the vector field associated with system (8) by  $f(X(t))$ . Then  $f(X(t))$  can be written as  $f(X(t)) = A + (S(t)B + C + I(t)D)X(t)$ , where

$$A = \begin{pmatrix} \Lambda \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 & -\beta & 0 & -\lambda_1 \\ 0 & 0 & 0 & 0 & \lambda_1 \\ 0 & 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\lambda_2 \\ 0 & 0 & 0 & 0 & \lambda_2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$D = \begin{pmatrix} -d & \theta_1 & \gamma_1 & \gamma_2 & 0 \\ 0 & -d - \theta_1 & 0 & 0 & 0 \\ 0 & 0 & -\gamma - 1 - \alpha_1 - d & \theta_2 & 0 \\ 0 & 0 & 0 & -d - \theta - 2 - \gamma_2 - \alpha_2 & 0 \\ 0 & 0 & \mu & 0 & -\phi \end{pmatrix}.$$

Clearly,  $f(X(t))$  and  $\frac{\partial f(X)}{\partial X}$  are continuous for all  $X(t) \in \Omega$ . Moreover,

$$\|f(X(t))\| \leq \|A\| + \|(S(t)B + I(t)C + D)X\| \leq k_1 + k_2\|X\|,$$

where  $k_1 = \|A\|$  and  $k_2 = \max\{1, \frac{\Lambda}{d}\}\|B + C + D\|$  are two positive constants. Thus by Theorem (3.1) and remark (3.2) of [34], the system (8) has a unique solution.

Now, to prove that  $\Omega$  is positively invariant, let  $N(t) = S(t) + S_L(t) + I(t) + I_L(t)$ . Then adding the first four equations of system (8), we have

$$D_0^\alpha N(t) = \Lambda - dN(t) - \alpha_1 I - \alpha_2 I_L.$$

Taking the Laplace transform on both sides yields

$$\mathcal{L}(D_0^\alpha N(t)) = \frac{\Lambda}{s} - d\mathcal{L}(N(t)) - \alpha_1 \mathcal{L}(I(t)) - \alpha_2 \mathcal{L}(I_L(t)).$$

Simplifying this equation, we have the following inequality

$$\mathcal{L}(N(t)) \leq \frac{\Lambda s^{-1}}{s^\alpha + d} + \frac{s^{\alpha-1} N(0)}{s^\alpha + d}.$$

Taking the inverse Laplace transform and using the fact that

$$\mathcal{L}^{-1} \left[ \frac{s^{-(\alpha-\beta)}}{s^\beta - a} \right] = t^{\alpha-1} E_{\beta,\alpha}(at^\beta), \quad \alpha, \beta > 0, s^\alpha > |a|,$$

where  $E_{\alpha,\beta}(\cdot)$  is the Mittag-Leffler function defined in (6), we have

$$\begin{aligned} N(t) &\leq \Lambda t^\alpha E_{\alpha,\alpha+1}(-dt^\alpha) + N(0) E_{\alpha,1}(-dt^\alpha) \\ &= \Lambda t^\alpha E_{\alpha,\alpha+1}(-dt^\alpha) + N(0) \left( -dt^\alpha E_{\alpha,\alpha+1}(-dt^\alpha) + \frac{1}{\Gamma(1)} \right) \\ &\leq \Lambda t^\alpha E_{\alpha,\alpha+1}(-dt^\alpha) + \frac{\Lambda}{d} \left( -dt^\alpha E_{\alpha,\alpha+1}(-dt^\alpha) + \frac{1}{\Gamma(1)} \right) \\ &= \frac{\Lambda}{d\Gamma(1)} = \frac{\Lambda}{d}. \end{aligned} \tag{9}$$

Consequently, inequality (9) and the last equation of system (8), implies that  $D_0^\alpha L(t) \leq \frac{\mu\Lambda}{d} - \phi L(t)$ . Similar to the above discussion, taking the Laplace transform and using the Mittag-Leffler function, it follows that  $L(t) \leq \frac{\mu\Lambda}{\phi d}$  for any  $t > 0$ .

In conclusion, for any  $X(0) \in \Omega$  we have  $N(t) \leq \frac{\Lambda}{d}$  and  $L(t) \leq \frac{\mu\Lambda}{\phi d}$  for any  $t \geq 0$ , and thus  $\Omega$  is positively invariant.  $\square$

### 3. Equilibrium Points and Basic Reproduction Number

One way to see what will happen to the population eventually is to explore when the system is at equilibrium. By setting

$$D_0^\alpha S(t) = D_0^\alpha S_L(t) = D_0^\alpha I(t) = D_0^\alpha I_L(t) = D_0^\alpha L(t) = 0$$

we get the following equilibrium points

$$E_0 = \left( \frac{\Lambda}{d}, 0, 0, 0, 0 \right), \quad E_1 = (S_1, 0, I_1, 0, 0),$$

and

$$E_2 = (S^*, S_L^*, I^*, I_L^*, L^*),$$

where

$$S_1 = \frac{\gamma_1 + \alpha_1 + d}{\beta}, \quad I_1 = \frac{\Lambda\beta - d(\gamma_1 + \alpha_1 + d)}{\beta(\alpha_1 + d)} \tag{10}$$

$$S^* = \frac{\gamma_1 + \alpha_1 + d}{\beta} + \frac{\lambda_2 \mu (d + \gamma_2 + \alpha_2) I^*}{\beta \phi (d + \theta_2 + \gamma_2 + \alpha_2)}, \quad S_L^* = \frac{\lambda \mu S^* I^*}{\phi (d + \theta_1)} \tag{11}$$

$$I_L^* = \frac{\lambda_2 \mu (I^*)^2}{\phi (d + \theta_2 + \alpha_2 + \gamma_2)}, \quad L^* = \frac{\mu I^*}{\phi}$$

and  $I^* = \max\{\tilde{I}, 0\}$ , where  $\tilde{I}$  is the solution of the equation  $k_1 \tilde{I}^2 + k_2 \tilde{I} + k_3 = 0$ , where

$$k_1 = -\frac{\mu(\lambda_2 - \theta_2)}{\phi} \left[ 1 - \frac{\gamma_2}{\alpha_2 + \gamma_2 + d} - \frac{\theta_1 \lambda_1}{\beta \phi (d + \theta_1)} + \frac{\mu \lambda_1}{\beta \phi} \right]$$

$$k_2 = -(\gamma_1 + \alpha_1 + d) \left[ 1 + \frac{\lambda_1 \mu}{\beta \phi} - \frac{\gamma_1}{(\alpha_1 + \gamma_1 + d)(d + \theta_1) \beta \phi} + \frac{\mu d (\lambda_2 - \theta_2)}{(\alpha_1 + \gamma_1 + d) \beta \phi} \right] \quad \text{and}$$

$$k_3 = \Lambda - \frac{d(\gamma_1 + \alpha_1 + d)}{\beta}.$$

The disease-free equilibrium  $E_0$  is the case when the pathogen has suffered extinction and, in the long run, everyone in the population is susceptible. The endemic equilibrium  $E_1$  is the state where the disease cannot be totally eradicated and remains in the population without a lockdown, while  $E_2$  is the endemic equilibrium point in the presence of lockdown. The basic reproduction number of the model, denoted by  $\mathcal{R}_0$ , is a constant that is used to approximate the expected number of cases directly generated by one case in a population where all individuals are susceptible to infection. One way to calculate  $\mathcal{R}_0$  is using the next generation matrix approach [35]. The rate at which secondary infections are produced is given by

$$F(E_0) := \left( \frac{\partial F_i}{\partial x_j} \right) (E_0) = \begin{bmatrix} \frac{\beta \Lambda}{d} & \theta_2 \\ 0 & 0 \end{bmatrix},$$

where  $i, j \in \{1, 2\}$  and  $x_j \in \{I, I_L\}$  for  $j = 1, 2$ .

Similarly, the transfer of infection from compartment  $i$  to  $j$  is given by

$$V(E_0) := \left( \frac{\partial V_i}{\partial x_j} \right) (E_0) = \begin{bmatrix} \gamma_1 + \alpha_1 + d & 0 \\ 0 & d + \theta_2 + \gamma_2 + \alpha_2 \end{bmatrix},$$

where  $i, j \in \{1, 2\}$  and  $x_j \in \{I, I_L\}$  for  $j = 1, 2$ .

Now define the next generation matrix  $G$  as

$$G = FV^{-1} = \begin{bmatrix} \frac{\beta \Lambda}{d(\gamma_1 + \alpha_1 + d)} & \frac{\theta_2}{d + \theta_2 + \gamma_2 + \alpha_2} \\ 0 & 0 \end{bmatrix}.$$

Hence  $\mathcal{R}_0$  is the dominant eigenvalue of  $G$  and it is given by

$$\mathcal{R}_0 = \frac{\beta \Lambda}{d(\gamma_1 + \alpha_1 + d)}. \tag{12}$$

#### 4. Local and Global Asymptotic Stability of the Disease-Free and Endemic Equilibrium Points

We use the following theorem to prove the local asymptotic stability of the disease-free equilibrium point  $E_0$ .

**Theorem 3** ([36]). *Given a fractional-order system of differential equation*

$$D_0^\alpha x(t) = f(x), \quad 0 < \alpha \leq 1. \tag{13}$$

*Let  $x_0$  be an equilibrium point of the given system, and let  $A = D(f(x_0))$  be the Jacobian matrix of  $f$  evaluated at  $x_0$ . Then  $x_0$  is locally asymptotically stable if and only if  $|\arg(\lambda_i)| > \frac{\alpha \pi}{2}$ , for all eigenvalues  $\lambda_i$  of the matrix  $A$ .*

**Theorem 4.** *The disease-free equilibrium point  $E_0$  is locally asymptotically stable if  $\mathcal{R}_0 < 1$ .*

**Proof.** After calculating the associated Jacobian matrix of system (8), it can be shown that the characteristic equation of the Jacobian matrix satisfies

$$(\lambda + d)(\lambda + \phi)(\lambda + d + \theta_1)(\lambda + d + \theta_2 + \gamma_2 + \alpha_2) \left( \lambda + \left( -\frac{\beta\Lambda}{d} + \gamma_1 + \alpha_1 + d \right) \right) = 0.$$

Note that  $\mathcal{R}_0 < 1$  iff  $-\frac{\beta\Lambda}{d} + \gamma_1 + \alpha_1 + d > 0$ . As a result, all the eigenvalues are negative real numbers, and hence  $|\arg(\lambda)| = \pi > \frac{\alpha\pi}{2}$  for any  $0 < \alpha \leq 1$ . Thus by Theorem 3 the disease-free equilibrium point  $E_0$  is locally asymptotically stable.  $\square$

**Definition 7** ([32,37]). An equilibrium point  $x^*$  of system (8) is Mittag-Leffler stable if

$$\|x(t) - x^*\| \leq \{m(x_0 - x^*)E_\alpha(-\lambda t^\alpha)\}^b,$$

where  $\|\cdot\|$  is any norm on  $\mathbb{R}^3$ ,  $\lambda > 0, b > 0, m(0) = 0$  and  $m(x) \geq 0$ , where  $m(x)$  is locally Lipschitz on  $\Omega$  with Lipschitz constant  $m_0$ , and  $E_\alpha(\cdot)$  is a one-parameter Mittag-Leffler function which can be defined in terms of the two-parameter Mittag-Leffler function as  $E_\alpha(\cdot) = E_{\alpha,1}(\cdot)$ .

**Remark 5.** Since Mittag-Leffler stability implies global asymptotic stability (see Remark 4.4 in [38]), it is sufficient to prove the disease-free equilibrium point  $E_0$  is Mittag-Leffler stable on  $\Omega$ .

Note that a similar result is provided in Definition 5 [39] for a generalized Hattaf fractional (GHF) derivative which encloses the popular forms fractional derivatives with non-singular kernels.

**Theorem 5.** If  $\mathcal{R}_0 < 1$ , then the disease-free equilibrium point  $E_0$  is Mittag-Leffler stable on  $\Omega$ .

**Proof.** Define the positive definite Lyapunov function

$$V(t) = c_1 \left( \frac{\Lambda}{d} - S(t) \right) + S_L(t) + I(t) + I_L(t) + c_2 L(t), \tag{14}$$

where  $c_1$  and  $c_2$  are two positive constants that will be determined later in the proof. By linearity of the Caputo fractional derivative, we have

$$\begin{aligned} D_0^\alpha V(t) &= -c_1 D_0^\alpha S(t) + D_0^\alpha (S_L(t) + I(t) + I_L(t)) + c_2 D_0^\alpha L(t) \\ &= -c_1 (\Lambda - \beta S(t)I(t) - \lambda_1 S(t)L(t) - dS(t) + \gamma_1 I(t) + \gamma_2 I_L(t) + \theta_1 S_L(t)) \\ &\quad + (\lambda_1 S(t)L(t) - dS_L(t) - \theta_1 S_L(t) + \beta S(t)I(t) - \gamma_1 I(t)) \\ &\quad - (\gamma_1 + \alpha_1 + d)I(t) - (d + \gamma_2 + \alpha_2)I_L(t) \\ &= -c_1 \left( \frac{\Lambda}{d} - S(t) \right) - (c_1 \theta + \theta_1 + d)S_L(t) - (c_1 \gamma_1 + \gamma_1 + \alpha_1 + d - c_2 \mu)I(t) \\ &\quad - (c_1 \gamma_2 + d + \gamma_2 + \alpha_2)I_L(t) - c_2 \phi L(t) + c_1 \beta S(t)I(t) + c_1 \lambda_1 S(t)L(t) \\ &\quad + \lambda_1 S(t)L(t) + \beta S(t)I(t). \end{aligned} \tag{15}$$

Now since

$$\begin{aligned} c_1 \beta S(t)I(t) + c_1 \lambda_1 S(t)L(t) + \lambda_1 S(t)L(t) + \beta S(t)I(t) &= (1 + c_1)(\beta S(t)I(t) + \lambda_1 S(t)L(t)) \\ &\leq (1 + c_1) \frac{\Lambda}{d} (\beta I(t) + \lambda_1 L(t)), \end{aligned}$$

then Equation (15) can be rewritten as

$$\begin{aligned} D_0^\alpha V(t) &\leq -c_1 \left( \frac{\Lambda}{d} - S(t) \right) - (c_1 \theta + \theta_1 + d)S_L(t) \\ &\quad - \left( \gamma_1 + \alpha_1 + d - c_2 \mu - \frac{\beta\Lambda}{d} (1 + c_1) \right) I(t) - (c_1 \gamma_2 + d + \gamma_2 + \alpha_2)I_L(t) \end{aligned}$$

$$- \left( c_2\phi - (c_1 + 1)\lambda_1 \frac{\Lambda}{d} \right) L(t). \tag{16}$$

If  $\mathcal{R}_0 = \frac{\beta\Lambda}{d(\gamma_1 + \alpha_1 + d)} < 1$  then we can choose  $c_1, c_2 > 0$  such that

$$\gamma_1 + \alpha_1 + d - c_2\mu - \frac{\beta\Lambda}{d}(1 + c_1) > 0, \quad c_2\phi - (c_1 + 1)\lambda_1 \frac{\Lambda}{d} > 0.$$

Thus the inequality in (16) becomes

$$D_0^\alpha V(t) \leq -c_3 V(t) \tag{17}$$

where

$$c_3 = \min \left\{ c_1, c_1\theta + \theta_1 + d, \gamma_1 + \alpha_1 + d - c_2\mu - \frac{\beta\Lambda}{d}(1 + c_1), c_1\gamma_2 + d + \gamma_2 + \alpha_2, c_1\phi - (c_1 + 1)\frac{\lambda_1\Lambda}{d} \right\} > 0.$$

Now taking the Laplace transform of inequality (17) results

$$\mathcal{L}V(t) \leq \frac{s^{\alpha-1}}{s^\alpha + c_3} V(0). \tag{18}$$

Thus using (2), we have

$$V(t) \leq \mathcal{L}^{-1} \left( \frac{s^{\alpha-1} V(0)}{s^\alpha + c_3} \right) = V(0) E_\alpha(-c_3 t^\alpha), \quad \forall t \geq 0. \tag{19}$$

Let  $\|x(t)\| = |S(t)| + |S_L(t)| + |I(t)| + |I_L(t)| + |L(t)|$  be the norm defined on the solution  $x(t) = (S(t), S_L(t), I(t), I_L(t), L(t))$  of system (8). Then from Equation (14) it follows that

$$c_4 \|x(t) - E_0\| \leq V(t) \leq c_5 \|x(t) - E_0\|, \quad \forall t \geq 0 \tag{20}$$

where  $c_4 = \min\{1, c_1, c_2\}$  and  $c_5 = \max\{1, c_1, c_2\}$ . As a result, from inequality (19) and (20) it follows that  $\|x(t) - E_0\| \leq M(x_0 - E_0) E_\alpha(-c_3 t^\alpha)$ , where  $M(S, S_L, I, I_L, L) := \frac{c_4}{c_5}(S(t) + S_L(t) + I(t) + I_L(t) + L(t))$ . In conclusion, the disease-free equilibrium point is Mittag-Leffler stable, and thus it is globally asymptotically stable.  $\square$

**Remark 6.** *The proof of the local stability of the endemic equilibrium point  $E_1$  is similar to Theorem 4. Thus we will provide proof of the global stability of  $E_1$  by constructing a Lyapunov function.*

**Theorem 6 ([40]).** *Let  $x_0 \in \Gamma$  be an equilibrium point for the non-autonomous fractional-order system  $D_0^\alpha x(t) = f(t, x)$ . Also let  $L : [0, \infty) \times \Gamma \rightarrow \mathbb{R}$  be a continuously differentiable function such that*

$$W_1(x) \leq L(t, x(t)) \leq W_2(x)$$

and

$$D_0^\alpha L(t, x(t)) \leq -W_3(x)$$

for all  $\alpha \in (0, 1)$  and all  $x \in \Gamma$ , where  $W_1, W_2, W_3$  are continuous positive definite functions on  $\Gamma$ . Then the equilibrium point  $x_0$  is globally asymptotically stable.

Now using Theorem 6 we will show that the endemic equilibrium point  $E_1$  of system (8) is globally asymptotically stable under some conditions.

**Theorem 7.** *If  $\mathcal{R}_0 > 1$ , then  $E_1$  is globally asymptotically stable in  $\Omega$ .*

**Proof.** Consider the Lyapunov function

$$V(S, S_L, I, I_L, L) = L_1(S) + L_2(I) + L_3(S_L, I_L, L), \tag{21}$$

where

$$L_1 = S(t) - S_1 - S_1 \ln\left(\frac{S(t)}{S_1}\right), \quad L_2 = I(t) - I_1 - I_1 \ln\left(\frac{I(t)}{I_1}\right) \quad \text{and} \\ L_3 = S_L(t) + I_L(t) + kL(t),$$

and  $k$  is a positive constant to be determined later in the proof. Using Lemma 3.1 and Remark 3.1 of [41] we have

$$D_0^\alpha L_1 = D_0^\alpha \left[ S(t) - S_1 - S_1 \ln\left(\frac{S(t)}{S_1}\right) \right] \leq \left( 1 - \frac{S_1}{S(t)} \right) D_0^\alpha S(t) \tag{22}$$

$$D_0^\alpha L_2 = D_0^\alpha \left[ I(t) - I_1 - I_1 \ln\left(\frac{I(t)}{I_1}\right) \right] \leq \left( 1 - \frac{I_1}{I(t)} \right) D_0^\alpha I(t), \tag{23}$$

and by the linearity property of the fractional Caputo derivative, it follows that

$$D_0^\alpha L_3(t) = D_0^\alpha S_L(t) + D_0^\alpha I_L(t) + kD_0^\alpha L(t). \tag{24}$$

Now using Equations (10), (22)–(24) and the fact that

$$D_0^\alpha V = D_0^\alpha L_1 + D_0^\alpha L_2 + D_0^\alpha L_3,$$

we have

$$D_0^\alpha V \leq \left( 1 - \frac{S_1}{S(t)} \right) D_0^\alpha S(t) + \left( 1 - \frac{I_1}{I(t)} \right) D_0^\alpha I(t) + D_0^\alpha S_L(t) + D_0^\alpha I_L(t) + kD_0^\alpha L(t) \\ \leq \frac{-d - \beta I_1}{S} (S - S_1)^2 + \left( \gamma_1 + \alpha_1 + d - \frac{\beta \Lambda}{d} \right) (I - I_1)^2 \\ + (S_1 \lambda_1 - k\phi)L - dS_L. \tag{25}$$

Choose any  $k > \frac{S_1 \lambda_2}{\phi}$  and note that if  $\mathcal{R}_0 > 1$  then  $\gamma_1 + \alpha_1 + d - \frac{\beta \Lambda}{d} < 0$ . Thus if  $\mathcal{R}_0 > 1$  then  $D_0^\alpha V \leq 0$  and also  $D_0^\alpha V = 0$  if and only if  $S = S_1, I = I_1$  and  $S_L = I_L = L = 0$ . Therefore the largest compact invariant set in

$$E = \{(S, S_L, I, I_L, L) \in \Omega : D_0^\alpha V = 0\}$$

is the singleton set containing the endemic equilibrium  $E_1$ . Thus by Theorem 6, we conclude that the endemic equilibrium is globally asymptotically stable in  $\Omega$ .  $\square$

Since the proof of the local stability of the endemic equilibrium point with lockdown  $E_2$  is similar to the proof of (4), we will provide the proof of the global stability of the endemic equilibrium point with lockdown  $E_2$ .

**Theorem 8.** *If  $\mu \lambda_2 I^* - \mu \theta_2 I^* - \alpha_2 \phi L^* - \phi^2 < 0$ , then the endemic equilibrium point with lockdown,  $E_2$ , is globally asymptotically stable in  $\Omega$ .*

**Proof.** We start by defining a Lyapunov function

$$V = (S - S^* - S^* \ln(S)) + (S_L - S_L^* - S_L^* \ln(S_L)) + (I - I^* - I^* \ln(I)) \\ (I_L - I_L^* - I_L^* \ln(I_L)) + (L - L^* - L^* \ln(L)),$$

where  $E_2 = (S^*, S_L^*, I^*, I_L^*, L^*)$  is the endemic equilibrium point with lockdown given in Equation (11). Note that  $V(E_2) = 0$  and using Lemma 3.1 and Remark 3.1 of [41] we have the following

$$\begin{aligned} D_0^\alpha V &\leq \left(1 - \frac{S^*}{S}\right) D_0^\alpha S(t) + \left(1 - \frac{S_L^*}{S_L}\right) D_0^\alpha S_L(t) + \left(1 - \frac{I^*}{I}\right) D_0^\alpha I(t) \\ &\quad + \left(1 - \frac{I_L^*}{I_L}\right) D_0^\alpha I_L(t) + \left(1 - \frac{L^*}{L}\right) D_0^\alpha L(t) \\ &\leq (S - S^*) \left(\frac{\Lambda}{S} \beta I - d\right) + S^* \lambda_1 L - d(S_L - S_L^*) - \frac{S_L^* \lambda_1 S L}{S_L} + S_L^* \theta_1 \\ &\quad + (I - I^*) (\beta S - \alpha_1 - d) + I^* \gamma_1 + \lambda_2 I^* L - \frac{\theta_2 I^* I_L}{I} \\ &\quad - (I_L - I_L^*) (d + \alpha_2) - I_L^* \left(\frac{\lambda_2 I L}{I_L} - \theta_2 - \gamma_2\right) + (L - L^*) \left(\frac{\mu I}{L} - \phi\right) \end{aligned}$$

Using Equation (11) we have the following inequality

$$\begin{aligned} D_0^\alpha V &\leq \left(-\beta I^* - \frac{\lambda_1 \mu I^*}{\phi} - d\right) (S - S^*)^2 - (d + \theta) (S_L - S_L^*)^2 \\ &\quad + (\mu \lambda_2 I^* - \mu \theta_2 I^* - \alpha_2 \phi L^* - \phi^2) (I - I^*)^2 \\ &\quad - (d + \alpha_2) I_L^2 + (d + \alpha_2) I_L^* I_L - (\lambda_2 I L + d + \alpha_2 + \theta_2 + \gamma_2) (I_L^*)^2 \\ &\quad - \phi^2 L^2 + (\mu I + \mu I^*) \phi L + (\mu^2 I I^* - I^2 \mu^2 - \mu^2 (I^*)^2). \end{aligned}$$

Let  $f(I_L) := -(d + \alpha_2) I_L^2 + (d + \alpha_2) I_L^* I_L - (\lambda_2 I L + d + \alpha_2 + \theta_2 + \gamma_2) (I_L^*)^2$ . Then it follows that

$$((d + \alpha_2) I_L^*)^2 - 4((d + \alpha_2)) ((\lambda_2 I L + d + \alpha_2 + \theta_2 + \gamma_2) (I_L^*)^2) < 0.$$

Thus it follows that  $f(I_L) < 0$ . Similarly, it can be shown that if  $g(L) := -\phi^2 L^2 + (\mu I + \mu I^*) \phi L + (\mu^2 I I^* - I^2 \mu^2 - \mu^2 (I^*)^2)$ , then  $g(L) < 0$  since

$$((\mu I + \mu I^*) \phi)^2 - 4(\phi^2) (\mu^2 I I^* - I^2 \mu^2 - \mu^2 (I^*)^2) < 0$$

and the fact that  $(\mu^2 I I^* - I^2 \mu^2 - \mu^2 (I^*)^2) < 0$ . Thus under the given assumption that  $\mu \lambda_2 I^* - \mu \theta_2 I^* - \alpha_2 \phi L^* - \phi^2 < 0$  it can be concluded that  $D_0^\alpha V \leq 0$  and  $D_0^\alpha V = 0$  if and only if  $S = S^*, S_L = S_L^*, I = I^*$ . This result, together with the last two equations of (8), will imply that  $I_L = I_L^*$  and  $L = L^*$ . As a result, the endemic equilibrium point  $E_2$  is globally asymptotically stable.  $\square$

### 5. Approximate Solution

In this section, we present a solution of the fractional-order model by using the residual power series method, which consists of expressing the solution as fractional power series expanded about the initial point  $t = t_0$ .

Consider the following fractional-order model:

$$\begin{aligned} D_0^\alpha S(t) &= \Lambda - \beta SI - \lambda_1 SL - dS + \gamma_1 I + \gamma_2 I_L + \theta_1 S_L \\ D_0^\alpha S_L(t) &= \lambda_1 SL - hS_L \\ D_0^\alpha I(t) &= \beta SI - pI - \lambda_2 IL + \theta_2 I_L \\ D_0^\alpha I_L(t) &= \lambda_2 IL - rI_L \\ D_0^\alpha L(t) &= \mu I - \phi L, \end{aligned} \tag{26}$$

where  $D_0^\alpha$  denotes the Caputo derivative for  $0 < \alpha \leq 1$  with  $h = d + \theta_1, p = \gamma_1 + \alpha_1 + d$  and  $r = d + \theta_2 + \gamma_2 + \alpha_2$ ; under the initial conditions

$$S(0) = S_0 > 0, S_L(0) = S_L^0 > 0, I(0) = I^0 > 0, I_L(0) = I_L^0 > 0 \quad \text{and} \quad L(0) = L_0 > 0.$$

Similar to the procedure in [42], we do the following steps in order to obtain a fractional power series solution for the nonlinear fractional-order model in (26).

**Step 1:** Suppose that the fractional power series of  $S(t), S_L(t), I(t), I_L(t)$  and  $L(t)$  around  $t = 0$  are as follows:

$$\begin{aligned} S(t) &= \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(1+k\alpha)} t^{k\alpha}, \\ S_L(t) &= \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(1+k\alpha)} t^{k\alpha}, \\ I(t) &= \sum_{k=0}^{\infty} \frac{c_k}{\Gamma(1+k\alpha)} t^{k\alpha}, \\ I_L(t) &= \sum_{k=0}^{\infty} \frac{d_k}{\Gamma(1+k\alpha)} t^{k\alpha}, \\ L(t) &= \sum_{k=0}^{\infty} \frac{e_k}{\Gamma(1+k\alpha)} t^{k\alpha}, \end{aligned} \tag{27}$$

where  $0 \leq t < \eta$  for some  $\eta > 0$ . Now, we let  $S_n(t), S_{L,n}(t), I_n(t), I_{L,n}(t)$  and  $L_n(t)$  denote the  $n$ -th truncated power series approximation of  $S(t), S_L(t), I(t), I_L(t)$  and  $L(t)$ , respectively. That is,

$$\begin{aligned} S_n(t) &= \sum_{k=0}^n \frac{a_k}{\Gamma(1+k\alpha)} t^{k\alpha}, \\ S_{L,n}(t) &= \sum_{k=0}^n \frac{b_k}{\Gamma(1+k\alpha)} t^{k\alpha}, \\ I_n(t) &= \sum_{k=0}^n \frac{c_k}{\Gamma(1+k\alpha)} t^{k\alpha}, \\ I_{L,n}(t) &= \sum_{k=0}^n \frac{d_k}{\Gamma(1+k\alpha)} t^{k\alpha}, \\ L_n(t) &= \sum_{k=0}^n \frac{e_k}{\Gamma(1+k\alpha)} t^{k\alpha}. \end{aligned} \tag{28}$$

**Step 2:** Next, we define the residual functions for the model in (26) as follows:

$$\begin{aligned} Res_S(t) &= D_0^\alpha S(t) - \Lambda + \beta S(t)I(t) + \lambda_1 S(t)L(t) + dS(t) - \gamma_1 I(t) \\ &\quad - \gamma_2 I_L(t) - \theta_1 S_L(t) \\ Res_{S_L}(t) &= D_0^\alpha S_L(t) - \lambda_1 S(t)L(t) + hS_L(t) \\ Res_I(t) &= D_0^\alpha I(t) - \beta S(t)I(t) + pI(t) + \lambda_2 I(t)L(t) - \theta_2 I_L(t) \\ Res_{I_L}(t) &= D_0^\alpha I_L(t) - \lambda_2 I(t)L(t) + rI_L(t) \\ Res_L(t) &= D_0^\alpha L(t) - \mu I(t) + \phi L(t). \end{aligned} \tag{29}$$

Hence, the  $n$ -th residual functions for  $S(t), S_L(t), I(t), I_L(t)$  and  $L(t)$ , respectively, are as follows:

$$\begin{aligned} Res_{S_n}(t) &= D_0^\alpha S_n(t) - \Lambda + \beta S_n(t)I_n(t) + \lambda_1 S_n(t)L_n(t) + dS_n(t) - \gamma_1 I_n(t) \\ &\quad - \gamma_2 I_{L,n}(t) - \theta_1 S_{L,n}(t) \end{aligned}$$

$$\begin{aligned}
 Res_{S_{L,n}}(t) &= D_0^\alpha S_{L,n}(t) - \lambda_1 S_n(t)L_n(t) + hS_{L,n}(t) \\
 Res_{I_n}(t) &= D_0^\alpha I_n(t) - \beta S_n(t)I_n(t) + pI_n(t) + \lambda_2 I_n(t)L_n(t) - \theta_2 I_{L,n}(t) \\
 Res_{I_{L,n}}(t) &= D_0^\alpha I_{L,n}(t) - \lambda_2 I_n(t)L_n(t) + rI_{L,n}(t) \\
 Res_{L_n}(t) &= D_0^\alpha L_n(t) - \mu I_n(t) + \phi L_n(t).
 \end{aligned}
 \tag{30}$$

Now, we observe that

$$Res_{S_n}(t) = Res_{S_{L,n}}(t) = Res_{I_n}(t) = Res_{I_{L,n}}(t) = Res_{L_n}(t) = 0, \text{ and}$$

$$\lim_{n \rightarrow \infty} Res_{S_n}(t) = Res_S(t), \lim_{n \rightarrow \infty} Res_{S_{L,n}}(t) = Res_{S_L}(t), \lim_{n \rightarrow \infty} Res_{I_n}(t) = Res_I(t),$$

$$\lim_{n \rightarrow \infty} Res_{I_{L,n}}(t) = Res_{I_L}(t) \text{ and } \lim_{n \rightarrow \infty} Res_{L_n}(t) = Res_L(t) \text{ for all } t \geq 0.$$

Since the Caputo derivative of any constant is zero, it is straightforward to see that for  $k = 1, \dots, n$

$$\begin{aligned}
 D_0^{(k-1)\alpha} Res_S(0) &= D_0^{(k-1)\alpha} Res_{S_n}(0), \quad D_0^{(k-1)\alpha} Res_{S_L}(0) = D_0^{(k-1)\alpha} Res_{S_{L,n}}(0) \\
 D_0^{(k-1)\alpha} Res_I(0) &= D_0^{(k-1)\alpha} Res_{I_n}(0), \quad D_0^{(k-1)\alpha} Res_{I_L}(0) = D_0^{(k-1)\alpha} Res_{I_{L,n}}(0) \\
 \text{and } D_0^{(k-1)\alpha} Res_L(0) &= D_0^{(k-1)\alpha} Res_{L_n}(0).
 \end{aligned}$$

**Step 3:** To obtain the coefficients  $a_k, b_k, c_k, d_k$  and  $e_k$ , for  $k = 1, \dots, n$ , we substitute the  $n$ -th truncated series of  $S(t), S_L(t), I(t), I_L(t)$  and  $L(t)$  into (30) and then apply the Caputo fractional derivative operator  $D_0^{(n-1)\alpha}$  on  $S_n(t), S_{L,n}(t), I_n(t), I_{L,n}(t)$  and  $L_n(t)$ , and evaluate the result at  $t = 0$ . We obtain the following algebraic system of equations.

$$\begin{aligned}
 D_0^{(n-1)\alpha} Res_{S_n}(0) &= 0, \\
 D_0^{(n-1)\alpha} Res_{S_{L,n}}(0) &= 0, \\
 D_0^{(n-1)\alpha} Res_{I_n}(0) &= 0 \\
 D_0^{(n-1)\alpha} Res_{I_{L,n}}(0) &= 0, \\
 D_0^{(n-1)\alpha} Res_{L_n}(0) &= 0.
 \end{aligned}
 \tag{31}$$

for  $n = 1, 2, 3, \dots$ .

**Step 4:** We solve the algebraic system (31) for the values of  $a_k, b_k, c_k, d_k, e_k$  for  $k = 1, 2, 3, \dots, n$  to get the  $n$ -th residual power series approximate solution of the system in (26).

**Step 5:** We repeat the procedure to obtain a sufficient number of coefficients. Higher accuracy for the solution can be achieved by evaluating more terms in the series solution.

By following the steps outlined in above, we derive a recursive formula for the coefficients as follows:

First, we note that the coefficients  $a_0, b_0, c_0, d_0$  and  $e_0$  are given by the initial conditions. That is,

$$a_0 = S_0, \quad b_0 = S_L^0, \quad c_0 = I^0, \quad d_0 = I_L^0 \text{ and } e_0 = L_0.$$

The first truncated power series approximations will have the forms:

$$\begin{aligned}
 S_1(t) &= a_0 + \frac{a_1}{\Gamma(1 + \alpha)} t^\alpha, \\
 S_{L,1}(t) &= b_0 + \frac{b_1}{\Gamma(1 + \alpha)} t^\alpha,
 \end{aligned}$$

$$\begin{aligned}
 I_1(t) &= c_0 + \frac{c_1}{\Gamma(1+\alpha)}t^\alpha, \\
 I_{L,1}(t) &= d_0 + \frac{d_1}{\Gamma(1+\alpha)}t^\alpha, \text{ and} \\
 L_1(t) &= e_0 + \frac{e_1}{\Gamma(1+\alpha)}t^\alpha.
 \end{aligned}$$

Thus, the first residual functions are:

$$\begin{aligned}
 Res_{S_1}(t) &= D_0^\alpha \left( a_0 + \frac{a_1}{\Gamma(1+\alpha)}t^\alpha \right) - \Lambda + \beta \left( a_0 + \frac{a_1}{\Gamma(1+\alpha)}t^\alpha \right) \left( c_0 + \frac{c_1}{\Gamma(1+\alpha)}t^\alpha \right) \\
 &\quad + \lambda_1 \left( a_0 + \frac{a_1}{\Gamma(1+\alpha)}t^\alpha \right) \left( e_0 + \frac{e_1}{\Gamma(1+\alpha)}t^\alpha \right) \\
 &\quad + d \left( a_0 + \frac{a_1}{\Gamma(1+\alpha)}t^\alpha \right) - \gamma_1 \left( c_0 + \frac{c_1}{\Gamma(1+\alpha)}t^\alpha \right) \\
 &\quad - \gamma_2 \left( d_0 + \frac{d_1}{\Gamma(1+\alpha)}t^\alpha \right) - \theta_1 \left( b_0 + \frac{b_1}{\Gamma(1+\alpha)}t^\alpha \right), \\
 Res_{S_{L,1}}(t) &= D_0^\alpha \left( b_0 + \frac{b_1}{\Gamma(1+\alpha)}t^\alpha \right) - \lambda_1 \left( a_0 + \frac{a_1}{\Gamma(1+\alpha)}t^\alpha \right) \left( e_0 + \frac{e_1}{\Gamma(1+\alpha)}t^\alpha \right) \\
 &\quad + h \left( b_0 + \frac{b_1}{\Gamma(1+\alpha)}t^\alpha \right), \\
 Res_{I_1}(t) &= D_0^\alpha \left( c_0 + \frac{c_1}{\Gamma(1+\alpha)}t^\alpha \right) - \beta \left( a_0 + \frac{a_1}{\Gamma(1+\alpha)}t^\alpha \right) \left( c_0 + \frac{c_1}{\Gamma(1+\alpha)}t^\alpha \right) \\
 &\quad + p \left( c_0 + \frac{c_1}{\Gamma(1+\alpha)}t^\alpha \right) + \lambda_2 \left( c_0 + \frac{c_1}{\Gamma(1+\alpha)}t^\alpha \right) \left( e_0 + \frac{e_1}{\Gamma(1+\alpha)}t^\alpha \right) \\
 &\quad - \theta_2 \left( d_0 + \frac{d_1}{\Gamma(1+\alpha)}t^\alpha \right), \\
 Res_{I_{L,1}}(t) &= D_0^\alpha \left( d_0 + \frac{d_1}{\Gamma(1+\alpha)}t^\alpha \right) - \lambda_2 \left( c_0 + \frac{c_1}{\Gamma(1+\alpha)}t^\alpha \right) \left( e_0 + \frac{e_1}{\Gamma(1+\alpha)}t^\alpha \right) \\
 &\quad + r \left( d_0 + \frac{d_1}{\Gamma(1+\alpha)}t^\alpha \right) \text{ and} \\
 Res_{L_1}(t) &= D_0^\alpha \left( e_0 + \frac{e_1}{\Gamma(1+\alpha)}t^\alpha \right) - \mu \left( c_0 + \frac{c_1}{\Gamma(1+\alpha)}t^\alpha \right) + \phi \left( e_0 + \frac{e_1}{\Gamma(1+\alpha)}t^\alpha \right).
 \end{aligned}$$

Evaluating  $Res_{S_1}(t), Res_{S_{L,1}}(t), Res_{I_1}(t), Res_{I_{L,1}}(t)$  and  $Res_{L_1}(t)$  at  $t = 0$ , we have

$$\begin{aligned}
 Res_{S_1}(0) &= a_1 - \Lambda + \beta a_0 c_0 + \lambda_1 a_0 e_0 + d a_0 - \gamma_1 c_0 - \gamma_2 d_0 - \theta_1 b_0, \\
 Res_{S_{L,1}}(0) &= b_1 - \lambda_1 a_0 e_0 + h b_0, \\
 Res_{I_1}(0) &= c_1 - \beta a_0 c_0 + p c_0 + \lambda_2 c_0 e_0 - \theta_2 d_0, \\
 Res_{I_{L,1}}(0) &= d_1 - \lambda_2 c_0 e_0 + r d_0, \\
 Res_{L_1}(0) &= e_1 - \mu c_0 + \phi e_0.
 \end{aligned}$$

By solving the equations

$$Res_{S_1}(0) = 0, Res_{S_{L,1}}(0) = 0, Res_{I_1}(0) = 0, Res_{I_{L,1}}(0) = 0 \text{ and } Res_{L_1}(0) = 0,$$

we have that

$$\begin{aligned}
 a_1 &= \Lambda - \beta a_0 c_0 - \lambda_1 a_0 e_0 - d a_0 + \gamma_1 c_0 + \gamma_2 d_0 + \theta_1 b_0, \\
 b_1 &= \lambda_1 a_0 e_0 - h b_0, \\
 c_1 &= \beta a_0 c_0 - p c_0 - \lambda_2 c_0 e_0 + \theta_2 d_0, \\
 d_1 &= \lambda_2 c_0 e_0 - r d_0,
 \end{aligned}$$

$$e_1 = \mu c_0 - \phi e_0.$$

Now, the second truncated power series approximations will have the forms:

$$\begin{aligned} S_2(t) &= a_0 + \frac{a_1}{\Gamma(1+\alpha)}t^\alpha + \frac{a_2}{\Gamma(1+2\alpha)}t^{2\alpha}, \\ S_{L,2}(t) &= b_0 + \frac{b_1}{\Gamma(1+\alpha)}t^\alpha + \frac{b_2}{\Gamma(1+2\alpha)}t^{2\alpha}, \\ I_2(t) &= c_0 + \frac{c_1}{\Gamma(1+\alpha)}t^\alpha + \frac{c_2}{\Gamma(1+2\alpha)}t^{2\alpha}, \\ I_{L,2}(t) &= d_0 + \frac{d_1}{\Gamma(1+\alpha)}t^\alpha + \frac{d_2}{\Gamma(1+2\alpha)}t^{2\alpha}, \text{ and} \\ L_2(t) &= e_0 + \frac{e_1}{\Gamma(1+\alpha)}t^\alpha + \frac{e_2}{\Gamma(1+2\alpha)}t^{2\alpha}. \end{aligned}$$

So, the second residual functions are:

$$\begin{aligned} Res_{S_2}(t) &= D_0^\alpha \left( a_0 + \frac{a_1}{\Gamma(1+\alpha)}t^\alpha + \frac{a_2}{\Gamma(1+2\alpha)}t^{2\alpha} \right) - \Lambda \\ &\quad + \beta \left( a_0 + \frac{a_1}{\Gamma(1+\alpha)}t^\alpha + \frac{a_2}{\Gamma(1+2\alpha)}t^{2\alpha} \right) \left( c_0 + \frac{c_1}{\Gamma(1+\alpha)}t^\alpha + \frac{c_2}{\Gamma(1+2\alpha)}t^{2\alpha} \right) \\ &\quad + \lambda_1 \left( a_0 + \frac{a_1}{\Gamma(1+\alpha)}t^\alpha + \frac{a_2}{\Gamma(1+2\alpha)}t^{2\alpha} \right) \left( e_0 + \frac{e_1}{\Gamma(1+\alpha)}t^\alpha + \frac{e_2}{\Gamma(1+2\alpha)}t^{2\alpha} \right) \\ &\quad + d \left( a_0 + \frac{a_1}{\Gamma(1+\alpha)}t^\alpha + \frac{a_2}{\Gamma(1+2\alpha)}t^{2\alpha} \right) \\ &\quad - \gamma_1 \left( c_0 + \frac{c_1}{\Gamma(1+\alpha)}t^\alpha + \frac{c_2}{\Gamma(1+2\alpha)}t^{2\alpha} \right) \\ &\quad - \gamma_2 \left( d_0 + \frac{d_1}{\Gamma(1+\alpha)}t^\alpha + \frac{d_2}{\Gamma(1+2\alpha)}t^{2\alpha} \right) - \theta_1 \left( b_0 + \frac{b_1}{\Gamma(1+\alpha)}t^\alpha + \frac{b_2}{\Gamma(1+2\alpha)}t^{2\alpha} \right), \\ Res_{S_{L,2}}(t) &= D_0^\alpha \left( b_0 + \frac{b_1}{\Gamma(1+\alpha)}t^\alpha + \frac{b_2}{\Gamma(1+2\alpha)}t^{2\alpha} \right) \\ &\quad - \lambda_1 \left( a_0 + \frac{a_1}{\Gamma(1+\alpha)}t^\alpha + \frac{a_2}{\Gamma(1+2\alpha)}t^{2\alpha} \right) \left( e_0 + \frac{e_1}{\Gamma(1+\alpha)}t^\alpha + \frac{e_2}{\Gamma(1+2\alpha)}t^{2\alpha} \right) \\ &\quad + h \left( b_0 + \frac{b_1}{\Gamma(1+\alpha)}t^\alpha + \frac{b_2}{\Gamma(1+2\alpha)}t^{2\alpha} \right), \\ Res_{I_2}(t) &= D_0^\alpha \left( c_0 + \frac{c_1}{\Gamma(1+\alpha)}t^\alpha + \frac{c_2}{\Gamma(1+2\alpha)}t^{2\alpha} \right) \\ &\quad - \beta \left( a_0 + \frac{a_1}{\Gamma(1+\alpha)}t^\alpha + \frac{a_2}{\Gamma(1+2\alpha)}t^{2\alpha} \right) \left( c_0 + \frac{c_1}{\Gamma(1+\alpha)}t^\alpha + \frac{c_2}{\Gamma(1+2\alpha)}t^{2\alpha} \right) \\ &\quad + p \left( c_0 + \frac{c_1}{\Gamma(1+\alpha)}t^\alpha + \frac{c_2}{\Gamma(1+2\alpha)}t^{2\alpha} \right) \\ &\quad + \lambda_2 \left( c_0 + \frac{c_1}{\Gamma(1+\alpha)}t^\alpha + \frac{c_2}{\Gamma(1+2\alpha)}t^{2\alpha} \right) \left( e_0 + \frac{e_1}{\Gamma(1+\alpha)}t^\alpha + \frac{e_2}{\Gamma(1+2\alpha)}t^{2\alpha} \right) \\ &\quad - \theta_2 \left( d_0 + \frac{d_1}{\Gamma(1+\alpha)}t^\alpha + \frac{d_2}{\Gamma(1+2\alpha)}t^{2\alpha} \right), \\ Res_{I_{L,2}}(t) &= D_0^\alpha \left( d_0 + \frac{d_1}{\Gamma(1+\alpha)}t^\alpha + \frac{d_2}{\Gamma(1+2\alpha)}t^{2\alpha} \right) \\ &\quad - \lambda_2 \left( c_0 + \frac{c_1}{\Gamma(1+\alpha)}t^\alpha + \frac{c_2}{\Gamma(1+2\alpha)}t^{2\alpha} \right) \left( e_0 + \frac{e_1}{\Gamma(1+\alpha)}t^\alpha + \frac{e_2}{\Gamma(1+2\alpha)}t^{2\alpha} \right) \\ &\quad + r \left( d_0 + \frac{d_1}{\Gamma(1+\alpha)}t^\alpha + \frac{d_2}{\Gamma(1+2\alpha)}t^{2\alpha} \right) \text{ and} \end{aligned}$$

$$Res_{L_2}(t) = D_0^\alpha \left( e_0 + \frac{e_1}{\Gamma(1+\alpha)} t^\alpha + \frac{e_2}{\Gamma(1+2\alpha)} t^{2\alpha} \right) - \mu \left( c_0 + \frac{c_1}{\Gamma(1+\alpha)} t^\alpha + \frac{c_2}{\Gamma(1+2\alpha)} t^{2\alpha} \right) + \phi \left( e_0 + \frac{e_1}{\Gamma(1+\alpha)} t^\alpha + \frac{e_2}{\Gamma(1+2\alpha)} t^{2\alpha} \right)$$

We apply the operator  $D_0^\alpha$  to  $Res_{S_2}(t), Res_{S_{L_2}}(t), Res_{I_2}(t), Res_{I_{L_2}}(t)$  and  $Res_{L_2}(t)$  and then evaluate the result at  $t = 0$  to get

$$\begin{aligned} D_0^\alpha Res_{S_2}(0) &= a_2 + \beta[a_0c_1 + a_1c_0] + \lambda_1[a_0e_1 + a_1e_0] + da_1 - \gamma_1c_1 - \gamma_2d_1 - \theta_1b_1, \\ D_0^\alpha Res_{S_{L_2}}(0) &= b_2 - \lambda_1[a_0e_1 + a_1e_0] + hb_1, \\ D_0^\alpha Res_{I_2}(0) &= c_2 - \beta[a_0c_1 + a_1c_0] + pc_1 + \lambda_2[c_0e_1 + c_1e_0] - \theta_2d_1, \\ D_0^\alpha Res_{I_{L_2}}(0) &= d_2 - \lambda_2[c_0e_1 + c_1e_0] + rd_1, \\ D_0^\alpha Res_{L_2}(0) &= e_2 - \mu c_1 + \phi e_1. \end{aligned}$$

By solving the equations

$$D_0^\alpha Res_{S_2}(0) = 0, \quad D_0^\alpha Res_{S_{L_2}}(0) = 0, \quad D_0^\alpha Res_{I_2}(0) = 0, \quad D_0^\alpha Res_{I_{L_2}}(0) = 0 \text{ and } D_0^\alpha Res_{L_2}(0) = 0,$$

we have

$$\begin{aligned} a_2 &= -\beta[a_0c_1 + a_1c_0] - \lambda_1[a_0e_1 + a_1e_0] - da_1 + \gamma_1c_1 + \gamma_2d_1 + \theta_1b_1, \\ b_2 &= \lambda_1[a_0e_1 + a_1e_0] - hb_1, \\ c_2 &= \beta[a_0c_1 + a_1c_0] - pc_1 - \lambda_2[c_0e_1 + c_1e_0] + \theta_2d_1, \\ d_2 &= \lambda_2[c_0e_1 + c_1e_0] - rd_1, \\ e_2 &= \mu c_1 - \phi e_1. \end{aligned}$$

Similarly, the third truncated power series approximations are given by

$$\begin{aligned} S_2(t) &= a_0 + \frac{a_1}{\Gamma(1+\alpha)} t^\alpha + \frac{a_2}{\Gamma(1+2\alpha)} t^{2\alpha} + \frac{a_3}{\Gamma(1+3\alpha)} t^{3\alpha}, \\ S_{L_2}(t) &= b_0 + \frac{b_1}{\Gamma(1+\alpha)} t^\alpha + \frac{b_2}{\Gamma(1+2\alpha)} t^{2\alpha} + \frac{b_3}{\Gamma(1+3\alpha)} t^{3\alpha}, \\ I_2(t) &= c_0 + \frac{c_1}{\Gamma(1+\alpha)} t^\alpha + \frac{c_2}{\Gamma(1+2\alpha)} t^{2\alpha} + \frac{c_3}{\Gamma(1+3\alpha)} t^{3\alpha}, \\ I_{L_2}(t) &= d_0 + \frac{d_1}{\Gamma(1+\alpha)} t^\alpha + \frac{d_2}{\Gamma(1+2\alpha)} t^{2\alpha} + \frac{d_3}{\Gamma(1+3\alpha)} t^{3\alpha}, \text{ and} \\ L_2(t) &= e_0 + \frac{e_1}{\Gamma(1+\alpha)} t^\alpha + \frac{e_2}{\Gamma(1+2\alpha)} t^{2\alpha} + \frac{e_3}{\Gamma(1+3\alpha)} t^{3\alpha}. \end{aligned}$$

So, the third residual functions are:

$$\begin{aligned} Res_{S_3}(t) &= D_0^\alpha \left( a_0 + \frac{a_1}{\Gamma(1+\alpha)} t^\alpha + \frac{a_2}{\Gamma(1+2\alpha)} t^{2\alpha} + \frac{a_3}{\Gamma(1+3\alpha)} t^{3\alpha} \right) - \Lambda \\ &+ \beta \left( a_0 + \frac{a_1}{\Gamma(1+\alpha)} t^\alpha + \frac{a_2}{\Gamma(1+2\alpha)} t^{2\alpha} + \frac{a_3}{\Gamma(1+3\alpha)} t^{3\alpha} \right) \\ &\times \left( c_0 + \frac{c_1}{\Gamma(1+\alpha)} t^\alpha + \frac{c_2}{\Gamma(1+2\alpha)} t^{2\alpha} + \frac{c_3}{\Gamma(1+3\alpha)} t^{3\alpha} \right) \\ &+ \lambda_1 \left( a_0 + \frac{a_1}{\Gamma(1+\alpha)} t^\alpha + \frac{a_2}{\Gamma(1+2\alpha)} t^{2\alpha} + \frac{a_3}{\Gamma(1+3\alpha)} t^{3\alpha} \right) \\ &\times \left( e_0 + \frac{e_1}{\Gamma(1+\alpha)} t^\alpha + \frac{e_2}{\Gamma(1+2\alpha)} t^{2\alpha} + \frac{e_3}{\Gamma(1+3\alpha)} t^{3\alpha} \right) \\ &+ d \left( a_0 + \frac{a_1}{\Gamma(1+\alpha)} t^\alpha + \frac{a_2}{\Gamma(1+2\alpha)} t^{2\alpha} + \frac{a_3}{\Gamma(1+3\alpha)} t^{3\alpha} \right) \end{aligned}$$

$$\begin{aligned}
 & -\gamma_1 \left( c_0 + \frac{c_1}{\Gamma(1+\alpha)} t^\alpha + \frac{c_2}{\Gamma(1+2\alpha)} t^{2\alpha} + \frac{c_3}{\Gamma(1+3\alpha)} t^{3\alpha} \right) \\
 & -\gamma_2 \left( d_0 + \frac{d_1}{\Gamma(1+\alpha)} t^\alpha + \frac{d_2}{\Gamma(1+2\alpha)} t^{2\alpha} + \frac{d_3}{\Gamma(1+3\alpha)} t^{3\alpha} \right) \\
 & -\theta_1 \left( 3 + \frac{b_1}{\Gamma(1+\alpha)} t^\alpha + \frac{b_2}{\Gamma(1+2\alpha)} t^{2\alpha} + \frac{b_3}{\Gamma(1+3\alpha)} t^{3\alpha} \right), \\
 Res_{S_{L,3}}(t) &= D_0^\alpha \left( b_0 + \frac{b_1}{\Gamma(1+\alpha)} t^\alpha + \frac{b_2}{\Gamma(1+2\alpha)} t^{2\alpha} + \frac{b_3}{\Gamma(1+3\alpha)} t^{3\alpha} \right) \\
 & -\lambda_1 \left( a_0 + \frac{a_1}{\Gamma(1+\alpha)} t^\alpha + \frac{a_2}{\Gamma(1+2\alpha)} t^{2\alpha} + \frac{a_3}{\Gamma(1+3\alpha)} t^{3\alpha} \right) \\
 & \times \left( e_0 + \frac{e_1}{\Gamma(1+\alpha)} t^\alpha + \frac{e_2}{\Gamma(1+2\alpha)} t^{2\alpha} + \frac{e_3}{\Gamma(1+3\alpha)} t^{3\alpha} \right) \\
 & + h \left( b_0 + \frac{b_1}{\Gamma(1+\alpha)} t^\alpha + \frac{b_2}{\Gamma(1+2\alpha)} t^{2\alpha} + \frac{b_3}{\Gamma(1+3\alpha)} t^{3\alpha} \right), \\
 Res_{I_3}(t) &= D_0^\alpha \left( c_0 + \frac{c_1}{\Gamma(1+\alpha)} t^\alpha + \frac{c_2}{\Gamma(1+2\alpha)} t^{2\alpha} + \frac{c_3}{\Gamma(1+3\alpha)} t^{3\alpha} \right) \\
 & -\beta \left( a_0 + \frac{a_1}{\Gamma(1+\alpha)} t^\alpha + \frac{a_2}{\Gamma(1+2\alpha)} t^{2\alpha} + \frac{a_3}{\Gamma(1+3\alpha)} t^{3\alpha} \right) \\
 & \times \left( c_0 + \frac{c_1}{\Gamma(1+\alpha)} t^\alpha + \frac{c_2}{\Gamma(1+2\alpha)} t^{2\alpha} + \frac{c_3}{\Gamma(1+3\alpha)} t^{3\alpha} \right) \\
 & + p \left( c_0 + \frac{c_1}{\Gamma(1+\alpha)} t^\alpha + \frac{c_2}{\Gamma(1+2\alpha)} t^{2\alpha} + \frac{c_3}{\Gamma(1+3\alpha)} t^{3\alpha} \right) \\
 & + \lambda_2 \left( c_0 + \frac{c_1}{\Gamma(1+\alpha)} t^\alpha + \frac{c_2}{\Gamma(1+2\alpha)} t^{2\alpha} + \frac{c_3}{\Gamma(1+3\alpha)} t^{3\alpha} \right) \\
 & \times \left( e_0 + \frac{e_1}{\Gamma(1+\alpha)} t^\alpha + \frac{e_2}{\Gamma(1+2\alpha)} t^{2\alpha} + \frac{e_3}{\Gamma(1+3\alpha)} t^{3\alpha} \right) \\
 & -\theta_2 \left( d_0 + \frac{d_1}{\Gamma(1+\alpha)} t^\alpha + \frac{d_2}{\Gamma(1+2\alpha)} t^{2\alpha} + \frac{d_3}{\Gamma(1+3\alpha)} t^{3\alpha} \right), \\
 Res_{I_{L,3}}(t) &= D_0^\alpha \left( d_0 + \frac{d_1}{\Gamma(1+\alpha)} t^\alpha + \frac{d_2}{\Gamma(1+2\alpha)} t^{2\alpha} + \frac{d_3}{\Gamma(1+3\alpha)} t^{3\alpha} \right) \\
 & -\lambda_2 \left( c_0 + \frac{c_1}{\Gamma(1+\alpha)} t^\alpha + \frac{c_2}{\Gamma(1+2\alpha)} t^{2\alpha} + \frac{c_3}{\Gamma(1+3\alpha)} t^{3\alpha} \right) \\
 & \times \left( e_0 + \frac{e_1}{\Gamma(1+\alpha)} t^\alpha + \frac{e_2}{\Gamma(1+2\alpha)} t^{2\alpha} + \frac{e_3}{\Gamma(1+3\alpha)} t^{3\alpha} \right) \\
 & + r \left( d_0 + \frac{d_1}{\Gamma(1+\alpha)} t^\alpha + \frac{d_2}{\Gamma(1+2\alpha)} t^{2\alpha} + \frac{d_3}{\Gamma(1+3\alpha)} t^{3\alpha} \right) \text{ and} \\
 Res_{L_3}(t) &= D_0^\alpha \left( e_0 + \frac{e_1}{\Gamma(1+\alpha)} t^\alpha + \frac{e_2}{\Gamma(1+2\alpha)} t^{2\alpha} + \frac{e_3}{\Gamma(1+3\alpha)} t^{3\alpha} \right) \\
 & -\mu \left( c_0 + \frac{c_1}{\Gamma(1+\alpha)} t^\alpha + \frac{c_2}{\Gamma(1+2\alpha)} t^{2\alpha} + \frac{c_3}{\Gamma(1+3\alpha)} t^{3\alpha} \right) \\
 & + \phi \left( e_0 + \frac{e_1}{\Gamma(1+\alpha)} t^\alpha + \frac{e_2}{\Gamma(1+2\alpha)} t^{2\alpha} + \frac{e_3}{\Gamma(1+3\alpha)} t^{3\alpha} \right)
 \end{aligned}$$

We apply the operator  $D_0^{2\alpha}$  to  $Res_{S_3}(t), Res_{S_{L,3}}(t), Res_{I_3}(t), Res_{I_{L,3}}(t)$  and  $Res_{L_3}(t)$  and then evaluate the result at  $t = 0$  to get

$$\begin{aligned}
 D_0^{2\alpha} Res_{S_3}(0) &= a_3 + \beta \left[ a_0 c_2 + \frac{a_1 c_1 \Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2} + a_2 c_0 \right] + \lambda_1 \left[ a_0 e_2 + \frac{a_1 e_1 \Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2} + a_2 e_0 \right] \\
 &+ d a_2 - \gamma_1 c_2 - \gamma_2 d_2 - \theta_1 b_2,
 \end{aligned}$$

$$\begin{aligned}
 D_0^{2\alpha} Res_{S_{L,3}}(0) &= b_3 - \lambda_1 \left[ a_0 e_2 + \frac{a_1 e_1 \Gamma(1 + 2\alpha)}{\Gamma(1 + \alpha)^2} + a_2 e_0 \right] + h b_2, \\
 D_0^{2\alpha} Res_{I_3}(0) &= c_3 - \beta \left[ a_0 c_2 + \frac{a_1 c_1 \Gamma(1 + 2\alpha)}{\Gamma(1 + \alpha)^2} + a_2 c_0 \right] + p c_2 \\
 &\quad + \lambda_2 \left[ c_0 e_2 + \frac{c_1 e_1 \Gamma(1 + 2\alpha)}{\Gamma(1 + \alpha)^2} + c_2 e_0 \right] - \theta_2 d_2, \\
 D_0^{2\alpha} Res_{I_{L,3}}(0) &= d_3 - \lambda_2 \left[ c_0 e_2 + \frac{c_1 e_1 \Gamma(1 + 2\alpha)}{\Gamma(1 + \alpha)^2} + c_2 e_0 \right] + r d_2, \\
 D_0^{2\alpha} Res_{L_3}(0) &= e_3 - \mu c_2 + \phi e_2.
 \end{aligned}$$

By solving the equations

$$D_0^\alpha Res_{S_3}(0) = 0, \quad D_0^\alpha Res_{S_{L,3}}(0) = 0, \quad D_0^\alpha Res_{I_3}(0) = 0, \quad D_0^\alpha Res_{I_{L,3}}(0) = 0 \text{ and } D_0^\alpha Res_{L_3}(0) = 0,$$

we have

$$\begin{aligned}
 a_3 &= -\beta \left[ a_0 c_2 + \frac{a_1 c_1 \Gamma(1 + 2\alpha)}{\Gamma(1 + \alpha)^2} + a_2 c_0 \right] - \lambda_1 \left[ a_0 e_2 + \frac{a_1 e_1 \Gamma(1 + 2\alpha)}{\Gamma(1 + \alpha)^2} + a_2 e_0 \right] \\
 &\quad - d a_2 + \gamma_1 c_2 + \gamma_2 d_2 + \theta_1 b_2, \\
 b_3 &= \lambda_1 \left[ a_0 e_2 + \frac{a_1 e_1 \Gamma(1 + 2\alpha)}{\Gamma(1 + \alpha)^2} + a_2 e_0 \right] - h b_2, \\
 c_3 &= \beta \left[ a_0 c_2 + \frac{a_1 c_1 \Gamma(1 + 2\alpha)}{\Gamma(1 + \alpha)^2} + a_2 c_0 \right] - \lambda_2 \left[ c_0 e_2 + \frac{c_1 e_1 \Gamma(1 + 2\alpha)}{\Gamma(1 + \alpha)^2} + c_2 e_0 \right] \\
 &\quad - p c_2 + \theta_2 d_2, \\
 d_3 &= \lambda_2 \left[ c_0 e_2 + \frac{c_1 e_1 \Gamma(1 + 2\alpha)}{\Gamma(1 + \alpha)^2} + c_2 e_0 \right] - r d_2, \\
 e_3 &= \mu c_2 - \phi e_2.
 \end{aligned}$$

By continuing in this direction, we deduce that the coefficients  $a_n, b_n, c_n, d_n$  and  $e_n$  for  $n \geq 2$  are given by the recursive formula

$$\begin{aligned}
 a_n &= -\beta \Gamma(1 + (n - 1)\alpha) \sum_{i=0}^{n-1} \frac{c_{n-1-i} a_i}{\Gamma(1 + i\alpha) \Gamma(1 + (n - 1 - i)\alpha)} \\
 &\quad - \lambda_1 \Gamma(1 + (n - 1)\alpha) \sum_{i=0}^{n-1} \frac{e_{n-1-i} a_i}{\Gamma(1 + i\alpha) \Gamma(1 + (n - 1 - i)\alpha)} \\
 &\quad - d a_{n-1} + \gamma_1 c_{n-1} + \gamma_2 d_{n-1} + \theta_1 b_{n-1}, \\
 b_n &= \lambda_1 \Gamma(1 + (n - 1)\alpha) \sum_{i=0}^{n-1} \frac{e_{n-1-i} a_i}{\Gamma(1 + i\alpha) \Gamma(1 + (n - 1 - i)\alpha)} - h b_{n-1}, \\
 c_n &= \beta \Gamma(1 + (n - 1)\alpha) \sum_{i=0}^{n-1} \frac{c_{n-1-i} a_i}{\Gamma(1 + i\alpha) \Gamma(1 + (n - 1 - i)\alpha)} \\
 &\quad - \lambda_2 \Gamma(1 + (n - 1)\alpha) \sum_{i=0}^{n-1} \frac{e_{n-1-i} c_i}{\Gamma(1 + i\alpha) \Gamma(1 + (n - 1 - i)\alpha)} \\
 &\quad - p c_{n-1} + \theta_2 d_{n-1}, \\
 d_n &= \lambda_2 \Gamma(1 + (n - 1)\alpha) \sum_{i=0}^{n-1} \frac{e_{n-1-i} c_i}{\Gamma(1 + i\alpha) \Gamma(1 + (n - 1 - i)\alpha)} - r d_{n-1}, \\
 e_n &= \mu c_{n-1} - \phi e_{n-1}.
 \end{aligned} \tag{32}$$

**Remark 7.** It is worth noting that these recursive formulas for the coefficients will ensure that we obtain higher-order approximate solutions as compared with similar results in the literature with

a smaller order of the approximate solutions. Thus, due to this recursive nature of the coefficients, we can calculate  $a_n, b_n, c_n, d_n$  and  $e_n$  for any large  $n$  values if necessary. For the purpose of the numerical simulation given in Section 6, we used  $n = 25$ . That is, the coefficients  $a_n, b_n, c_n, d_n, e_n$  for  $n = 1, 2, 3, \dots, 25$  are computed and the approximated fractional power series solutions of system (8) are plotted in section 6, numerical simulation and examples.

### 6. Numerical Simulations and Examples

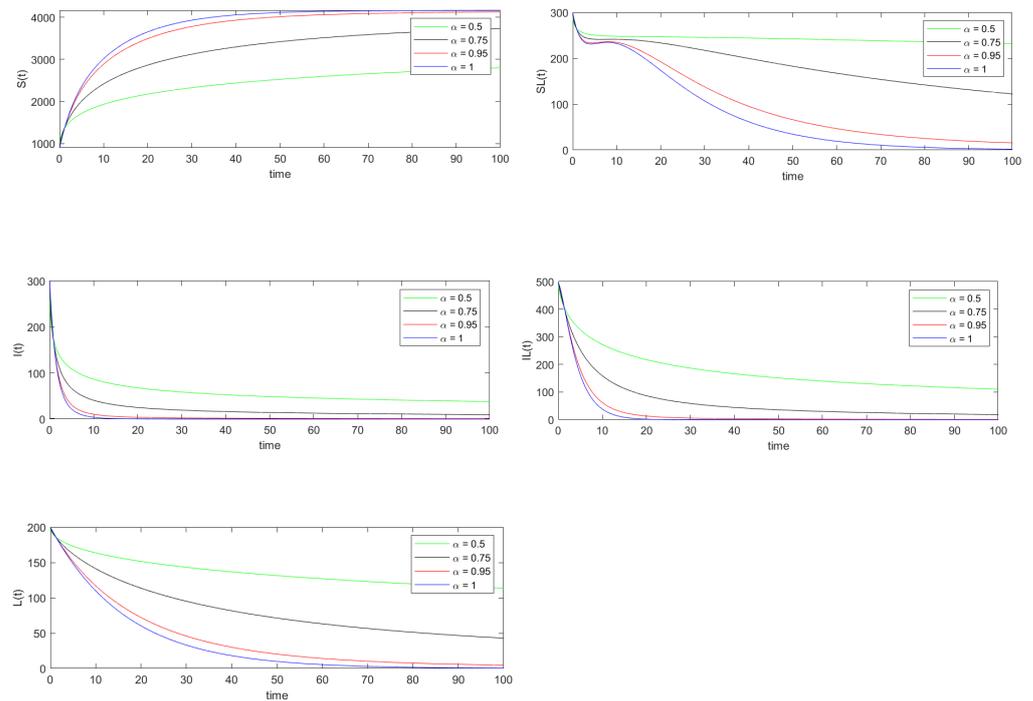
The following table provides the description of the different parameters used in the COVID-19 model. The parameter values under column “data set 1” yield a basic reproduction number  $\mathcal{R}_0 = 0.2373$ . Similarly, if the parameters under the column “data set 2” are used, then we have  $\mathcal{R}_0 = 4.5707$ . In Section 6.1, we simulate both the exact and approximated solutions of system (8) using the values under “data set 1” of Table 2 for various  $\alpha$  values ( $\alpha = 0.5, 0.75, 0.95$  and  $1$ ). In Section 6.2, we simulate the exact and approximated solutions of system (8) using the parameter values under “data set 2” of table Table 2. Moreover, in both cases ( $\mathcal{R}_0 < 1$  and  $\mathcal{R}_0 > 1$ ) the relative errors, due to approximating the exact solution of system (8) by the residual power series method, are computed for different  $\alpha$  values as indicated in Tables 3 and 4.

**Table 2.** Input parameter values used to simulate the trajectories of the solution of the model as shown in Figures 1–4. The parameters under the column data set 1 will result  $\mathcal{R}_0 = 0.2373 < 1$ , and the parameters under data set 2 yield  $\mathcal{R}_0 = 4.5707 > 1$ .

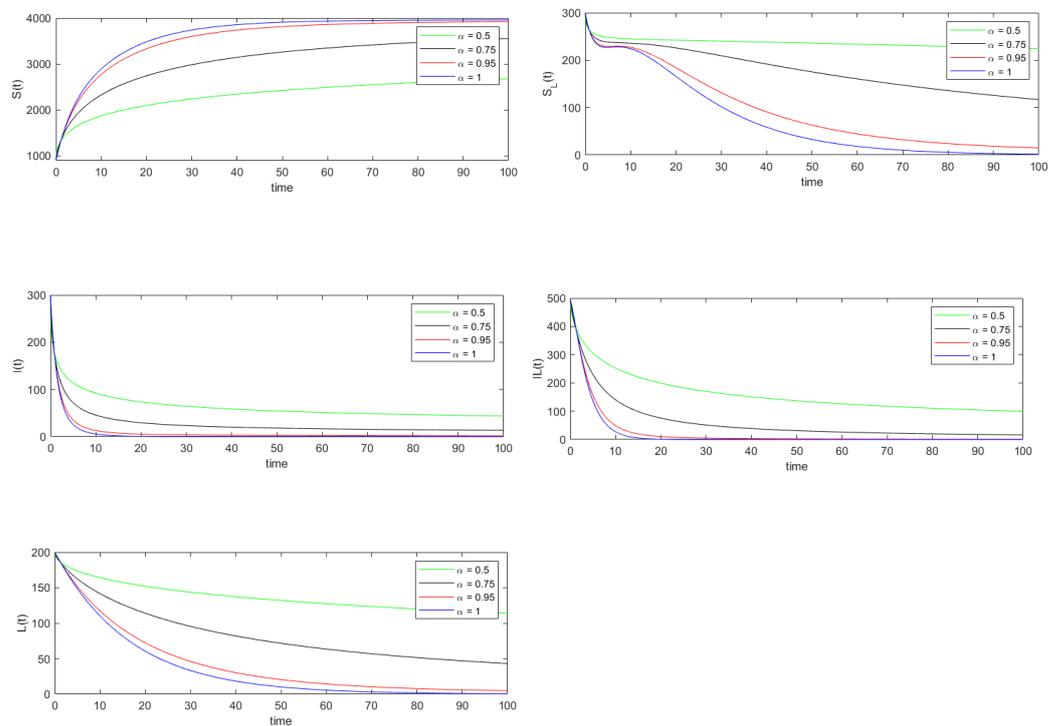
Parameters	Description of the Parameters	Data Set 1	Data Set 2	Reference
$\Lambda$	Recruitment rate	400	400	[38,43]
$\beta$	Infection contact rate	$1.7 \times 10^{-5}$	$1.8 \times 10^{-4}$	[44]
$\lambda_1$	Imposition of lockdown on susceptible group	$2 \times 10^{-4}$	$2 \times 10^{-5}$	[38,43]
$\lambda_2$	Imposition of lockdown on infected group	0.002	0.002	[38,43]
$\gamma_1$	Recovery rate of the infected group	0.16979	0.16979	[38,43]
$\gamma_2$	Recovery rate of the infected group under lockdown	0.16979	0.16979	[38,43]
$\alpha_1$	Death rate of the infected group	0.03275	0.03275	[44]
$\alpha_2$	Death rate of the infected group under lockdown	0.03275	0.03275	[44]
$d$	Natural death rates	0.096	0.06	[38,43]
$\theta_1$	Rate of transfer of susceptible lockdown individuals to susceptible class	0.2	0.52	[38,43]
$\theta_2$	Rate of transfer of susceptible lockdown individuals to infected class	0.02	0.2	[38,43]
$\mu$	rate of implementation of the lockdown program	$5 \times 10^{-4}$	$5 \times 10^{-5}$	[38,43]
$\phi$	rate of depletion of the lockdown program	0.06	0.06	[38,43]

#### 6.1. Exact and Approximated Solutions of System (8) When $\mathcal{R}_0 < 1$

Using data set 1, the basic reproduction number  $\mathcal{R}_0$  is calculated to be 0.2373. Thus by Theorems 4 and 5, the disease-free equilibrium point is both locally and globally asymptotically stable. The disease-free equilibrium point is calculated to be  $E_0 = (4167, 0, 0, 0, 0)$ . The simulations in Figures 1 and 2 support the fact that the trajectories of system (8) converges to this equilibrium point  $E_0 = (4167, 0, 0, 0, 0)$ .



**Figure 1.** Numerical simulation of the trajectories of the solution of system (8) when  $\mathcal{R}_0 < 1$ .



**Figure 2.** Approximated solution of system (8) using the residual power series method when  $\mathcal{R}_0 < 1$ . Note that the values of the parameters used in this figure are the same as the one in Figure 1.

In Figures 1 and 2 the exact and approximated solutions of system (8) are plotted for  $0 \leq t \leq 100$ , respectively. In order to show the accuracy of the residual power series method for approximating the solution of system (8), we have calculated the relative error of each state variable for various  $\alpha$  values. A time step size  $\Delta t = 2^{-6}$  is considered and

for each state variables  $x(t) \in \{S(t), S_L(t), I(t), I_L(t), L(t)\}$  the values  $x(i\Delta t)$  are calculated where  $0 \leq i \leq 6400$ . The relative error for the state variable  $x(t)$  is computed using

$$Rel_{x(t)} = \max_{0 \leq i \leq 6400} \left\{ \frac{|Exact(x(i\Delta t)) - Appr(x(i\Delta t))|}{Exact(x(i\Delta t))} \right\}, \tag{33}$$

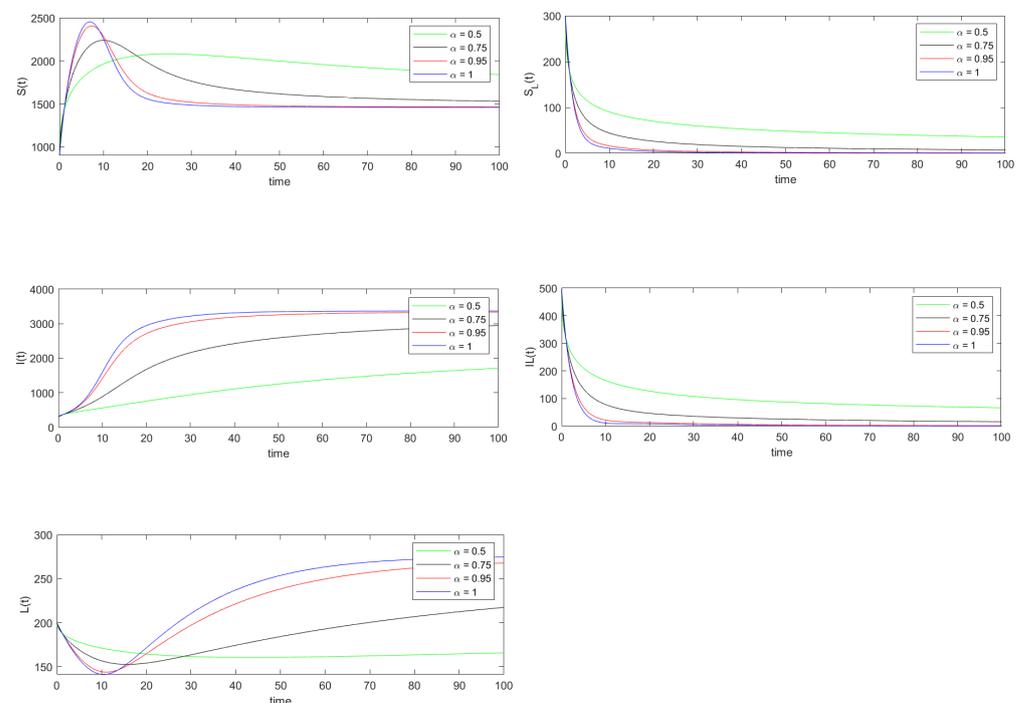
where  $Exact(x(i\Delta t))$  and  $Appr(x(i\Delta t))$  represent the exact and approximated values of the variable  $x(t)$  at  $t = i\Delta t$  for  $0 \leq i \leq 6400$ . As can be seen from Table 3, the relative error of each state variable decreases as the value of  $\alpha$  increases from 0.5 to 1.

**Table 3.** Relative error of the state variables for different values of  $\alpha$  when  $\mathcal{R}_0 < 1$ .

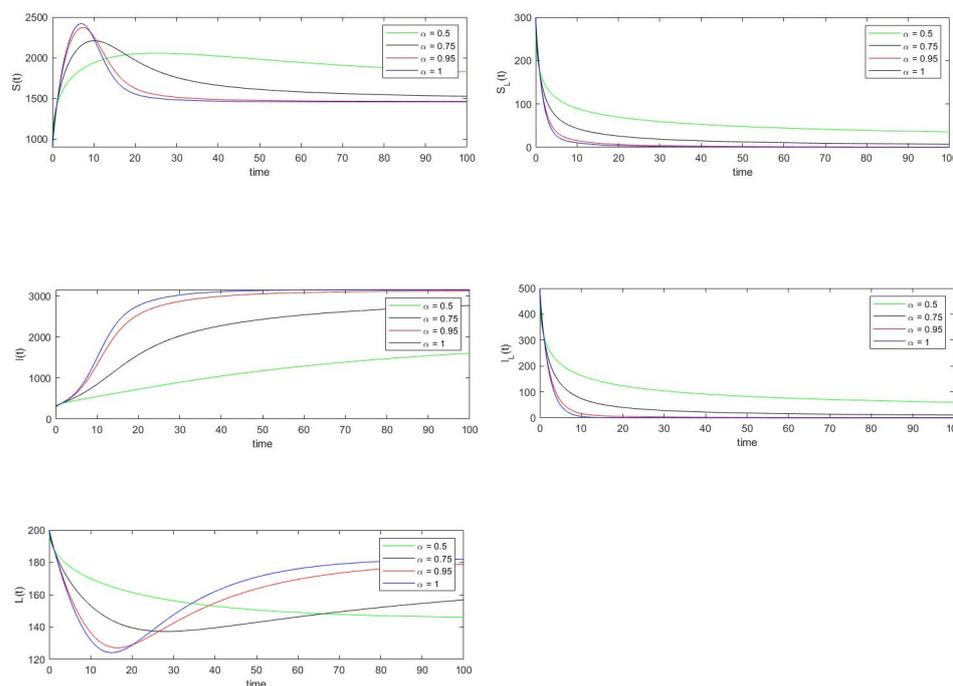
	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 0.95$	$\alpha = 1$
$S$	$3.876 \times 10^{-6}$	$3.143 \times 10^{-6}$	$2.025 \times 10^{-6}$	$1.202 \times 10^{-6}$
$S_L$	$4.302 \times 10^{-6}$	$2.710 \times 10^{-6}$	$7.094 \times 10^{-7}$	$4.851 \times 10^{-7}$
$I$	$6.813 \times 10^{-7}$	$2.421 \times 10^{-7}$	$8.059 \times 10^{-8}$	$3.140 \times 10^{-8}$
$I_L$	$1.783 \times 10^{-6}$	$0.448 \times 10^{-6}$	$2.220 \times 10^{-7}$	$1.504 \times 10^{-7}$
$L$	$1.570 \times 10^{-6}$	$2.097 \times 10^{-6}$	$1.313 \times 10^{-6}$	$9.177 \times 10^{-7}$

### 6.2. Exact and Approximated Solutions of System (8) When $\mathcal{R}_0 > 1$

The basic reproduction number is calculated using the parameter values under data set 2. The value of  $\mathcal{R}_0$  is found to be 4.5707, and also the condition  $\lambda_2 < \theta_2$  is satisfied. Thus by Remark 6 and Theorem 7 the endemic equilibrium point  $E_1 = (1458, 0, 3370, 0, 0)$ , which is calculated using Equation (10), is both locally and globally asymptotically stable. Note that both Figures 3 and 4 show that the trajectories of the solution of system (8) converges to the equilibrium point  $E_1$ .



**Figure 3.** Numerical simulation of the trajectories of the solution of system (8) when  $\mathcal{R}_0 > 1$ .



**Figure 4.** Approximated solution of system (8) using the residual power series method when  $\mathcal{R}_0 > 1$ . The values of the parameters used in this figure are the same as in Figure 3.

Similar to the previous discussion, the relative errors of the state variables are calculated for various  $\alpha$  values using Equation (33).

**Table 4.** Relative error of the state variables for different values of  $\alpha$  when  $\mathcal{R}_0 > 1$ .

	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 0.95$	$\alpha = 1$
$S$	$4.013 \times 10^{-6}$	$1.571 \times 10^{-6}$	$3.014 \times 10^{-7}$	$2.624 \times 10^{-7}$
$S_L$	$8.201 \times 10^{-5}$	$2.930 \times 10^{-5}$	$5.466 \times 10^{-6}$	$2.181 \times 10^{-6}$
$I$	$1.099 \times 10^{-6}$	$8.205 \times 10^{-7}$	$3.272 \times 10^{-7}$	$6.142 \times 10^{-8}$
$I_L$	$4.240 \times 10^{-6}$	$3.279 \times 10^{-6}$	$1.142 \times 10^{-6}$	$8.529 \times 10^{-7}$
$L$	$5.014 \times 10^{-5}$	$1.792 \times 10^{-5}$	$7.263 \times 10^{-6}$	$1.817 \times 10^{-6}$

### 7. Conclusions and Discussion

In this paper, a fractional-order COVID-19 epidemic model with lockdown is proposed and analyzed. In particular, some of the most important epidemiological constants, such as the equilibrium points ( $E_0, E_1$  and  $E_2$ ) and the basic reproduction number  $\mathcal{R}_0$ , are calculated. The existence and uniqueness of a positive global solution of the fractional-order system (8) is established. Moreover, by implementing various techniques such as the comparison theory of fractional-order differential equations, Mittag-Leffler stability (Theorem 5), and fractional La-Salle invariance principle (Theorem 6), the local and global stabilities of the equilibrium points are discussed. Additionally, the method of residual power series is used to approximate the solution of system (8). As can be seen in Equation (32), the coefficients  $a_n, b_n, c_n, d_n$  and  $e_n$  have recursive nature. That is, for any  $n \geq 1$ , each coefficients  $a_n, b_n, c_n, d_n$  and  $e_n$  can be determined from the previous ones  $a_{n-1}, b_{n-1}, c_{n-1}, d_{n-1}$  and  $e_{n-1}$ . Thus we can calculate as many coefficients as we need in order to get a very good approximation of the solution of system (8). Moreover, it is very interesting to note that as the value of  $\alpha$  increases from 0.5 to 1, the trajectories of the solution of the fractional-order system are converging to the case where  $\alpha = 1$ . That is, the solutions of the fractional-order

differential equation converge to the solutions of the system of the ordinary differential equation as the order  $\alpha$  increases towards 1. In Section 5, numerical simulations and examples are presented to show the stability of the equilibrium points when  $\mathcal{R}_0 < 1$  and  $\mathcal{R}_0 > 1$ . The exact solutions of system (8) for various  $\alpha$  values are shown in Figures 1 and 3. Similarly, the approximated solutions of system (8) using the residual power series method are included in Figures 2 and 4. Finally, in order to see the accuracy of the approximated solutions, we provided the relative errors for different  $\alpha$  values as shown in Tables 3 and 4. The model that is studied in this paper can be used to predict and understand how COVID-19 infection spreads in the presence of a lockdown. In addition to understanding the dynamics of the infection, it is very important to determine the relative importance of the various parameters used in the model. One such technique is the study of the sensitivity of the parameters in relation to the basic reproduction number  $\mathcal{R}_0$ , and the two endemic equilibrium points  $E_1$  and  $E_2$ , which are considered to be the most important values in the study of deterministic epidemic models. One such approach is to calculate the normalized sensitivity index (NSI), denoted by  $SI_\kappa$ , where  $\kappa$  is the given parameter. For example, if  $\kappa$  is one of the parameters in  $\mathcal{R}_0$ , then the NSI of  $\kappa$  is defined as  $SI_\kappa = \frac{\kappa}{\mathcal{R}_0} \frac{\partial \mathcal{R}_0}{\partial \kappa}$ . Using this result, we have calculated the NSI of the parameters  $\beta$  and  $\gamma_1$ , which are directly affected by the infection. As a result, we have  $SI_\beta = 1$ , and  $SI_{\gamma_1} = \frac{-\gamma_1}{\gamma_1 + \alpha_1 + d} < 0$ . Thus in order to decrease  $\mathcal{R}_0$ , the expected number of infections generated by one case, we have to either decrease the infection contact rate  $\beta$  or increase the recovery rate of the infected group  $\gamma_1$ . This further implies imposing some prevention actions, such as avoiding contact with people who are infected, wearing masks, and keeping a distance from an infectious person, will help to decrease the value of  $\mathcal{R}_0$ , which directly leads to mitigating the infection. By modifying this model, one can study other aspects of the infection, such as the impact of vaccination, the effect of population diffusion, and other important factors that determine the transmission and persistence of the infection.

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