



Flutter and Divergence Instability of Axially-Moving Nanoplates Resting on a Viscoelastic Foundation

Jingbo Duan¹, Dapeng Zhang^{2,*} and Wenjie Wang^{3,*}

- ¹ Department of Engineering Mechanics, Shijiazhuang Tiedao University, Shijiazhuang 050043, China; duanjingbo@stdu.edu.cn
- ² College of Aerospace Science and Engineering, National University of Defense Technology, Changsha 410073, China
- ³ School of Engineering, University of Warwick, Coventry CV4 7AL, UK
- * Correspondence: zhangdapeng201812@163.com (D.Z.); Wenjie.Wang.1@warwick.ac.uk (W.W.); Tel.: +86-1850-325-6216 (D.Z.)

Received: 12 February 2019; Accepted: 4 March 2019; Published: 15 March 2019



Abstract: Moving nanosystems often rest on a medium exhibiting viscoelastic behavior in engineering applications. The moving velocity and viscoelastic parameters of the medium usually have an interacting impact on the mechanical properties of nanostructures. This paper investigates the dynamic stability of an axially-moving nanoplate resting on a viscoelastic foundation based on the nonlocal elasticity theory. Firstly, the governing partial equations subject to appropriate boundary conditions are derived through utilizing the Hamilton's principle with the axial velocity, viscoelastic foundation, nonlocal effect and biaxial loadings taken into consideration. Subsequently, the characteristic equation describing the dynamic characteristics is obtained by employing the Galerkin strip distributed transfer function method. Then, complex frequency curves for the nanoplate are displayed graphically and the effects of viscoelastic foundation parameters, small-scale parameters and axial forces on divergence instability and coupled-mode flutter are analyzed, which show that these parameters play a crucial role in affecting nanostructural instability. The presented results benefit the designation of axially-moving graphene nanosheets or other plate-like nanostructures resting on a viscoelastic foundation.

Keywords: axially-moving nanoplates; viscoelastic foundation; complex frequency; divergence instability; mode-couple flutter

1. Introduction

Nano structures have attracted a huge amount of attention from the scientific community due to their extraordinary mechanical, electrical, thermal and other physical/chemical properties. These novel properties can be for nanoelectromechanical systems (NEMS), nanoresonators, nanosensors and nanogenerators [1,2], which are always modeled as nanobeams [3], nanorods [4], nanoribbons [5], nanoplates [6], nanoshells [7], etc to investigate the mechanical properties and other physical properties. Currently, there are three main kinds of approaches available for the study of nanomechanics, which are experimental techniques [8], molecular dynamic (MD) simulations [9] and modified continuum mechanics theories [10,11], respectively. However, nano scale experiments are exceedingly difficult and MD simulations are costly in terms of computation. As a remedy, continuum mechanics theories are a valid selection for nanomechanics. The nonlocal elasticity theory established by Eringen [12,13] is one of the typical continuum mechanics theories. It is assumed that the stress at a reference point is affected not only by the strain at that point but also by the strains at every point in the rest of the domain. Therefore, based on the nonlocal elasticity theory, a couple of researchers have applied their works to the bending, vibration and stability of nanostructures [14–16].



Moving nanosystems including NEMS, nanoresonators, nanosensors and nanogenerators are actually common in engineering applications, which can be modeled as moving continuum. Many researchers have studied the dynamical behaviors of such moving nanosystems including the transverse vibration characteristics, dynamical stabilities and dynamic response. Yang et al. [17] studied dynamic stability in transverse parametric vibrations of an axially accelerating tensioned beam of a Timoshenko model on simple supports. The Galerkin method is applied to discretize the governing equation and the averaging method is done to analyze the instability phenomena. Kiani [18] studied the dynamic instability of an axially moving single-walled carbon nanotube (SWCNT) with simply supported ends by using nonlocal Rayleigh beam theory. The obtained results show that small-scale parameters would result in the occurrence of both divergence and flutter instabilities at lower levels of speed. Mokhtari et al. [19] presented a time/wave domain analysis for an axially moving pre-stressed nanobeam by a wavelet-based spectral element (WSE) method, which is validated to be accurate and efficient. The effects of the moving nanobeam property including velocity, pretention and nonlocal parameter on vibration and wave characteristics and dispersion curves were investigated. Also, the instability of moving nanobeams was studied considering divergence and flutter. Wang et al. [20] investigated the dynamic behavior of axially moving nanobeams based on the nonlocal strain gradient theory considering the geometrical nonlinearity. The roles of small-scale parameters on the flutter critical velocity and stability were explained. Li [21] presented the transverse vibration behaviors of axially moving nanobeams considering the thermal effects and strain gradient. Rezaee and Lotfan [22] studied the nonlocal nonlinear vibrations of a viscoelastic Rayleigh nanobeam with small fluctuation in the axial speed by applying the nonlocal theory. Oveissi [23] investigated the longitudinal and transverse wave propagation of stationary and axially moving carbon nanotubes (CNTs) with conveying nano-fluid. Recently, Liu et al. [24] studied the transverse free vibration and stability of an axially moving nanoplate. The effects of dimensionless small-scale parameters, axial speed and boundary conditions on the natural frequencies in sub-critical region were discussed via the method of complex mode. The effects of small-scale parameters on divergence and flutter in super-critical region were investigated by the Galerkin method. Li et al. [25] investigated the transverse vibrations and steady-state responses of axially moving viscoelastic piezoelectric nanostructures containing the thermal effect and the instable behaviors of axially non-uniformly moving viscoelastic piezoelectric nanoplates.

On the other hand, the nanosystems are often embedded in a medium or resting on a foundation in many nanotechnology applications. In order to characterize the surrounding effects, Pradhan and Kumar [26] studied the vibration of orthotropic single layered graphene sheets embedded in Winkler-type and Pasternak-type foundations with the differential quadrature method employed to solve the governing differential equations for various boundary conditions. Jung et al. [27] developed a model for sigmoid functionally graded material (S-FGM) nanoplates embedded in a Pasternak elastic medium based on a modified couple stress theory and presented an analytical solution for buckling analysis of S-FGM nanoplates. Radic et al. [28] presented a buckling analysis of double-orthotropic nanoplates based on nonlocal elasticity theory with an assumption that two nanoplates are bonded by an internal elastic medium and surrounded by external elastic foundation. Asemi and Farajpour [29] proposed the thermo-electro-mechanical vibration characteristics of a piezoelectric-nanoplate system (PNPS) subjected to a non-uniform voltage distribution and embedded in a Pasternak elastic medium. The problem was solved by using the differential quadrature method (DQM) and results showed that the natural frequencies are quite sensitive to the non-uniform and nonlocal parameters. Asemi et al. [30] also investigated the effect of initial stress on the vibration responses for a double-piezoelectric-nanoplate system based on the nonlocal Kirchhoff plate theory, where the Pasternak-type elastic foundation model was used to account for the effect of shearing between the two nanoplates. Mohamed et al. [31] studied the vibration of a nonlocal Euler–Bernoulli beam embedded in a Pasternak elastic foundation. An efficient sixth-order finite difference discretization was adopted for various boundary conditions and the effects of nonlocal parameter, slenderness ratios

and boundary conditions on the dynamic characteristics of the beam were discussed. Arani and Jalaei [32] presented transient behavior of simply-supported orthotropic single-layered graphene sheet (SLGS) resting on orthotropic visco-Pasternak foundation subjected to dynamic loads. Researches show that the structural and foundation damping coefficients are effective parameters on the transient response, particularly for large damping coefficients. Gharib et al. [33] used the micromechanics method to consider the small-scale effect in the structural equations of a piezoelectric polymeric nanocomposite panel embedded in the Pasternak-type elastic substrate. Biloueiet et al. [34] studied the nonlinear buckling of straight concrete columns armed with single-walled carbon nanotubes (SWCNTs) resting on an elastic foundation. The column was modeled with Euler-Bernoulli beam theory. The characteristics of the equivalent composite were determined using the Mori–Tanaka model. The foundation around the column was simulated with spring and shear layer. Jamalpoor et al. [35] dealt with the free vibration and biaxial buckling of double-magneto-electro-elastic nanoplate systems (DMEENPS) subjected to initial external electric and magnetic potentials with the assumption that the two nanoplates are bonded with each other using a visco-Pasternak medium and are also limited to the external elastic substrate. Zenkour and Arefi [36] investigated the transient thermos-electro-mechanical vibration and bending analysis of a functionally graded piezoelectric nanosheet resting on visco-Pasternak foundation. The effects of some important parameters were studied on the fundamental frequencies of the system and the maximum deflection of the sheet under various thermal and electrical loadings. They also investigated the free vibration responses of a sandwich nanoplate resting on visco-Pasternak foundation based on the nonlocal Kirchhoff theory and Hamilton's principle [37]. Kolahchi et al. [38] examined the dynamic buckling for a sandwich nanoplate with the surrounding medium simulated by a visco-orthotropic Pasternak foundation model based on visco-nonlocal-refined Zigzag theories. Zhang et al. [39] studied the thermo-electro-mechanical vibration responses for a rectangular piezoelectric nanoplate resting on viscoelastic foundations by using the Galerkin strip distributed transfer function method, which enables one to obtain the semi-analytical solutions of natural frequencies for piezoelectric nanoplates with arbitrary boundary conditions. Liu et al. [40] investigated the size-dependent vibrational behaviors of functionally graded (FG) magneto-electro-viscoelastic nanobeams in the presence of porosities based on the nonlocal Timoshenko beam theory in conjunction with a visco-Pasternak foundation model. Liu and Lv [41] established a theoretical framework based on interval analysis method to study the wave-dispersion behavior of CNTs embedded in a Pasternak-type elastic medium with uncertain material properties. In their work, the material properties of CNTs were considered as uncertain-but-bounded variables, which is different from probabilistic analysis method.

As reviewed above, a lot of studies on the dynamic behaviors for nanosystems with axially-moving velocity or those resting on a foundation have been separately reported in recent years. However, the literature review shows that few studies are concerned with the dynamic instability of axially-moving nanoplates resting on a viscoelastic foundation for nanomechanical systems. As well known, in engineering practices, nanoresonators, nanosensors and nanogenerators are often embedded in a moving medium, which often exhibit viscoelastic behaviors. The objective of this study is to explore dynamical instability of an axially-moving nanoplate resting on a viscoelastic foundation. With the framework of the thin plate theory, we get the governing partial differential equations according to Hamilton's principle. After examining the developed mechanics model, the effects of foundation properties, axial moving velocity, nonlocal parameter and biaxial force on the divergent instability and coupled-mode flutter of nanoplates are shown by graphs and tables in detail, which could benefit the designation of moving graphene nanosheets or other plate-like nanostructures resting on a viscoelastic medium.

2. Basic Equations

A rectangular nanoplate resting on a viscoelastic foundation is shown in Figure 1. The length, width and uniform thickness of the nanoplate are denoted as l_a , l_b and h, respectively. The viscoelastic

foundation is represented by a visco-Pasternak foundation model, whose Winkler's modulus parameter is k_w , Pasternak's modulus parameter is k_G and the damping parameter is c_t . The axially-moving velocity of the nanoplate is v. The nanoplate is subjected to a biaxial force N_{Px} and N_{Py} along with xand y directions according to the rectangular coordinate system *oxyz* shown in Figure 1.



Figure 1. The axially-moving nanoplate resting on a viscoelastic foundation.

Based on the nonlocal elasticity theory [8], the basic constitutive equations for the nanoplate can be written as:

$$\begin{pmatrix} 1 - (e_0 a)^2 \nabla^2 \end{pmatrix} \sigma_{ij} = c_{ijkl} \varepsilon_{kl}, \sigma_{ij,j} = \rho \ddot{u}_i, \varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}),$$

$$(1)$$

where u_i , σ_{ij} and ε_{ij} are the components of nonlocal displacement, stress and strain vector, respectively. c_{ijkl} and ρ are the elastic constants and mass density of the nanoplate, respectively. Moreover, e_0a is the nonlocal parameter and ∇^2 is the Laplacian operator, respectively.

For simplicity, the displacement fields of the nanoplate can be obtained by neglecting the deformation of the middle surface, given by:

$$\begin{cases}
 u(x, y, z, t) = -z \frac{\partial w}{\partial x} \\
 \nu(x, y, z, t) = -z \frac{\partial w}{\partial y} \\
 w(x, y, z, t) = w(x, y, t)
 \end{cases},$$
(2)

where u, v and w are, respectively, the displacement components in the x, y and z directions. Based on the strain-displacement equations of thin plate theory, the nonzero strain components are given as:

$$\begin{cases} \varepsilon_{xx} = -z \frac{\partial^2 w}{\partial x^2} \\ \varepsilon_{yy} = -z \frac{\partial^2 w}{\partial y^2} \\ \gamma_{xy} = -2z \frac{\partial^2 w}{\partial x \partial y} \end{cases}$$
(3)

Based on Equations (1) and (3) and the classical plane stress relation, the nonlocal stress is thus derived as the following form:

$$\left(1 - (e_0 a)^2 \nabla^2\right) \left\{ \begin{array}{c} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{array} \right\} = \frac{E}{1 - \mu^2} \left[\begin{array}{cc} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1 + \mu}{2} \end{array} \right] \left\{ \begin{array}{c} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{array} \right\},$$
(4)

where *E* and μ denote elastic modulus and Poisson's ratio, respectively.

Further, the relation of the bending moment and stress in classical elasticity theory is still valid in the nonlocal stress. Then, the nonlocal bending moment can be presented by corresponding classical counterparts as: $\left(a_{1}=\left(\frac{2^{2}m}{2}-\frac{2^{2}m}{2}\right)\right)$

$$\left(1 - (e_0 a)^2 \nabla^2\right) \left\{ \begin{array}{c} M_{xx} \\ M_{yy} \\ M_{xy} \end{array} \right\} = \left\{ \begin{array}{c} -D\left(\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2}\right) \\ -D\left(\frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2}\right) \\ -D(1 - \mu)\frac{\partial^2 w}{\partial x \partial y} \end{array} \right\},$$
(5)

where $D = \frac{Eh^3}{12(1-\mu^2)}$.

Subsequently, the governing equations and boundary conditions for transverse vibration of the axially-moving nanoplate resting on a viscoelastic foundation can be derived utilizing the variational approach. The strain energy Π_U of the nanoplate can be stated as:

$$\Pi_{U} = \frac{1}{2} \int_{A} \int_{-h/2}^{h/2} \left(\sigma_{xx} \varepsilon_{xx} + \sigma_{xy} \gamma_{xy} + \sigma_{yy} \varepsilon_{yy} \right) dz dA, \tag{6}$$

where *A* is the area of the mid-plane of the nanoplate.

The kinetic energy Π_K of the moving nanoplate can be obtained as:

$$\Pi_{K} = \frac{1}{2} \int_{A} \rho h \left[\left(\frac{\partial w}{\partial t} + v \frac{\partial w}{\partial x} \right)^{2} + v^{2} \right] \mathrm{d}A.$$
(7)

Moreover, the work Π_W done by the external forces is obtained as:

$$\Pi_{W} = \frac{1}{2} \int_{A} \left(-N_{Q}w + N_{Px} \left(\frac{\partial w}{\partial x} \right)^{2} + N_{Py} \left(\frac{\partial w}{\partial y} \right)^{2} \right) \mathrm{d}A, \tag{8}$$

where N_{Px} and N_{Py} are respectively the normal forces in the *x* and *y* directions. N_Q is the reaction of the viscoelastic foundation, which can be obtained from:

$$N_Q = k_w w - k_G \nabla^2 w + c_t \frac{\partial w}{\partial t}.$$
(9)

Substituting Equations (6)–(9) into the Hamilton's equation

$$\delta \int_0^t (\Pi_U + \Pi_W - \Pi_K) dt = 0.$$
 (10)

Then, integrating by parts and setting the coefficient δw to zero, the governing equations of the axially-moving nanoplate resting on a viscoelastic foundation can be derived as:

$$\frac{\partial^2 M_{xx}}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_{yy}}{\partial y^2} - \left(k_w w - k_G \nabla^2 w + c_t \frac{\partial w}{\partial t} \right) - \left(N_{Px} \frac{\partial^2 w}{\partial x^2} + N_{Py} \frac{\partial^2 w}{\partial y^2} \right) \\ = \rho h \left(v^2 \frac{\partial^2 w}{\partial x^2} + 2v \frac{\partial^2 w}{\partial t \partial x} + \frac{\partial^2 w}{\partial t^2} \right).$$
(11)

The corresponding boundary conditions can also be obtained as follows:

$$w = 0 \text{ or } \left[\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} - N_{Px} \frac{\partial w}{\partial x} - \rho h \left(v \frac{\partial w}{\partial t} + v^2 \frac{\partial w}{\partial x} \right) \right] n_x + \left[\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} - N_{Py} \frac{\partial w}{\partial y} \right] n_y = 0,$$

$$\frac{\partial w}{\partial x} = 0 \text{ or } M_{xx} n_x + M_{xy} n_y = 0,$$

$$\frac{\partial w}{\partial y} = 0 \text{ or } M_{xy} n_x + M_{yy} n_y = 0,$$
(12)

where (n_x, n_y) are denoted as the direction cosines of the outward unit normal to the boundaries of the mid-plane.

Using the Galerkin strip distributed transfer function method (GSDTFM), the equivalent integral of the governing Equation (11) along with *y* direction can be stated as:

$$\int_{0}^{l} H \left[\frac{\partial^{2} M_{xx}}{\partial x^{2}} + 2 \frac{\partial^{2} M_{xy}}{\partial x \partial y} + \frac{\partial^{2} M_{yy}}{\partial y^{2}} - \left(k_{w} w - k_{G} \nabla^{2} w + c_{t} \frac{\partial w}{\partial t} \right) - \left(N_{Px} \frac{\partial^{2} w}{\partial x^{2}} + N_{Py} \frac{\partial^{2} w}{\partial y^{2}} \right) - \rho h \left(v^{2} \frac{\partial^{2} w}{\partial x^{2}} + 2v \frac{\partial^{2} w}{\partial t \partial x} + \frac{\partial^{2} w}{\partial t^{2}} \right) \right] dy = \mathbf{0},$$
(13)

where **H** is an admissible function.

Further, integrating by parts and considering the nonlocal bending moment equations shown in Equation (5), the weak form of the governing equations for the axially-moving nanoplate resting on a viscoelastic foundation can be derived as:

$$\int_{0}^{l} H\left(-D\frac{\partial^{4}w}{\partial x^{4}}-\mu D\frac{\partial^{4}w}{\partial x^{2}\partial y^{2}}\right) dy - 2\int_{0}^{l} \frac{\partial H}{\partial y}\left(-D(1-\mu)\frac{\partial^{3}w}{\partial x^{2}\partial y}\right) dy \\
+\int_{0}^{l} \frac{\partial^{2}H}{\partial y^{2}}\left(-\mu D\frac{\partial^{2}w}{\partial x^{2}}-D\frac{\partial^{2}w}{\partial y^{2}}\right) dy - \int_{0}^{l} H\left(1-(e_{0}a)^{2}\nabla^{2}\right)\left(k_{w}w-k_{G}\nabla^{2}w+c_{t}\frac{\partial w}{\partial t}\right) dy \\
-\int_{0}^{l} H\left(1-(e_{0}a)^{2}\nabla^{2}\right)\left(N_{Px}\frac{\partial^{2}w}{\partial x^{2}}+N_{Py}\frac{\partial^{2}w}{\partial y^{2}}\right) dy \\
-\int_{0}^{l} H\left(1-(e_{0}a)^{2}\nabla^{2}\right)\rho h\left(v^{2}\frac{\partial^{2}w}{\partial x^{2}}+2v\frac{\partial^{2}w}{\partial t^{2}}+\frac{\partial^{2}w}{\partial t^{2}}\right) dy = \mathbf{0}.$$
(14)

For the sake of convenience and generality, the following dimensionless quantities are defined as:

$$X = \frac{x}{l_a}, Y = \frac{y}{l_b}, W = \frac{w}{h}, \lambda = \frac{l_a}{l_b}, \alpha = \frac{e_{0a}}{l_a}, T = \frac{t}{l_a^2} \sqrt{\frac{D}{\rho h}}, C = l_a v \sqrt{\frac{\rho h}{D}},$$

$$K_1 = N_{Px} \frac{l_a^2}{D}, K_2 = N_{Py} \frac{l_a^2}{D}, K_w = \frac{l_a^4}{D} k_w, K_G = k_G \frac{l_a^2}{D}, L = \frac{l}{l_b}, C_t = \frac{l_a^2}{D} \sqrt{\frac{D}{\rho h}} c_t.$$
(15)

Using these dimensionless terms, the governing Equation (14) can be rewritten as:

$$\int_{0}^{L} \overline{\mathbf{H}} \left(-\frac{\partial^{4}W}{\partial X^{4}} - \mu\lambda^{2} \frac{\partial^{4}W}{\partial X^{2} \partial Y^{2}} \right) dY - 2 \int_{0}^{L} \frac{\partial \overline{\mathbf{H}}}{\partial Y} \left(-\lambda^{2} (1-\mu) \frac{\partial^{3}W}{\partial X^{2} \partial Y} \right) dY + \int_{0}^{L} \frac{\partial^{2} \overline{\mathbf{H}}}{\partial Y^{2}} \left(-\mu\lambda^{2} \frac{\partial^{2}W}{\partial X^{2}} - \lambda^{4} \frac{\partial^{2}W}{\partial Y^{2}} \right) dY
- \int_{0}^{L} \overline{\mathbf{H}} \left(1 - \alpha^{2} \left(\frac{\partial^{2}}{\partial X^{2}} + \lambda^{2} \frac{\partial^{2}}{\partial Y^{2}} \right) \right) \left(K_{w}W - K_{G} \left(\frac{\partial^{2}}{\partial X^{2}} + \lambda^{2} \frac{\partial^{2}}{\partial Y^{2}} \right) W + C_{t} \frac{\partial W}{\partial T} \right) dY,
- \int_{0}^{L} \overline{\mathbf{H}} \left(1 - \alpha^{2} \left(\frac{\partial^{2}}{\partial X^{2}} + \lambda^{2} \frac{\partial^{2}}{\partial Y^{2}} \right) \right) K_{1} \frac{\partial^{2}W}{\partial X^{2}} dY + \int_{0}^{L} \overline{\mathbf{H}} \left(1 - \alpha^{2} \left(\frac{\partial^{2}}{\partial X^{2}} + \lambda^{2} \frac{\partial^{2}}{\partial Y^{2}} \right) \right) K_{2} \lambda^{2} \frac{\partial^{2}W}{\partial Y^{2}} dY
- \int_{0}^{L} \overline{\mathbf{H}} \left(1 - \alpha^{2} \left(\frac{\partial^{2}}{\partial X^{2}} + \lambda^{2} \frac{\partial^{2}}{\partial Y^{2}} \right) \right) \left(C^{2} \frac{\partial^{2}W}{\partial X^{2}} + 2C \frac{\partial^{2}W}{\partial T \partial X} + \frac{\partial^{2}W}{\partial T^{2}} \right) dY = \mathbf{0}.$$
(16)

According to Equation (12), boundary conditions for the axially-moving nanoplate resting on a viscoelastic foundation can be obtained. For simplicity, in the following analysis, we only consider the simply supported boundary constraints for the nanoplate, which are given as follows:

$$\begin{split} W(X,Y)|_{X=0,1} &= \left. \frac{\partial W(X,Y)}{\partial X} \right|_{X=0,1} = 0, \\ W(X,Y)|_{Y=0,1} &= \left. \frac{\partial W(X,Y)}{\partial Y} \right|_{Y=0,1} = 0. \end{split}$$
(17)

3. Solution Method

In this section, the GSDTFM was adopted to investigate the dynamic instability of the axially-moving nanoplate resting on a viscoelastic foundation. Firstly, the nanoplate was divided into

a couple of strip elements along the *y* direction and there are *NE* strip elements and (*NE* + 1) nodal lines for the nanoplate in all, as shown in Figure 2. The length of the *j*th strip element is l_a and its width is l_e . The transverse displacement w(x, y, t) of the *j*th strip element can be interpolated by:

$$w(x, y, t) = \mathbf{N}(y)\delta_e(x, t), \tag{18}$$

where $\mathbf{N}(y)$ is the shape function and $\delta_e(x, t)$ is the unknown displacement of the e^{th} strip element. The expression of $\mathbf{N}(y)$ and $\delta_e(x, t)$ are given in literature [39], which are not herein presented for the sake of brevity.



Figure 2. Strip elements for a rectangular nanoplate.

According to the dimensionless quantities defined in Equation (15), Equation (18) can be rewritten as:

$$W(X,Y,T) = \overline{\mathbf{N}}(Y)\overline{\delta}_{\ell}(X,T),$$
(19)

where

$$\overline{\mathbf{N}}(Y) = \begin{bmatrix} 1 - 3\frac{Y^2}{L^2} + 2\frac{Y^3}{L^3} & l_b \left(Y - 2\frac{Y^2}{L} + \frac{Y^3}{L^2}\right) & 3\frac{Y^2}{L^2} - 2\frac{Y^3}{L^3} & l_b \left(-\frac{Y^2}{L} + \frac{Y^3}{L^2}\right) \end{bmatrix}$$

$$\overline{\delta}_e(X,T) = \begin{bmatrix} W_j & \Theta_j & W_{j+1} & \Theta_{j+1} \end{bmatrix}^T, \{\Theta_j, \Theta_{j+1}\} = \frac{1}{h} \{\theta_j, \theta_{j+1}\}, L = \frac{l_e}{l_b}.$$
(20)

Substituting Equation (19) into Equation (16) with $\overline{\mathbf{H}} = \overline{\mathbf{N}}^{\mathrm{T}}$ results in the following:

$$\overline{\mathbf{k}}_{e}^{4} \frac{\partial^{4} \overline{\delta}_{e}}{\partial X^{4}} + \overline{\mathbf{k}}_{e}^{2} \frac{\partial^{2} \overline{\delta}_{e}}{\partial X^{2}} + \overline{\mathbf{k}}_{e}^{0} \overline{\delta}_{e} + \overline{\mathbf{k}}_{e,1}^{1} \frac{\partial^{2} \overline{\delta}_{e}}{\partial X \partial T} + \overline{\mathbf{m}}_{e} \frac{\partial^{2} \overline{\delta}_{e}}{\partial T^{2}} + \overline{\mathbf{k}}_{e,2}^{2} \frac{\partial^{4} \overline{\delta}_{e}}{\partial X^{2} \partial T^{2}} + \overline{\mathbf{k}}_{e,1}^{3} \frac{\partial^{4} \overline{\delta}_{e}}{\partial X^{3} \partial T} = \mathbf{0}, \tag{21}$$

where

$$\begin{split} \overline{\mathbf{K}}_{e}^{4} &= \left[-1 - \alpha^{2} K_{1} + \alpha^{2} C^{2} + \alpha^{2} K_{G}\right] \int_{0}^{L} \overline{\mathbf{N}}^{\mathrm{T}} \overline{\mathbf{N}} dY, \\ \overline{\mathbf{K}}_{e}^{2} &= \left[-\mu \lambda^{2} + 2\alpha^{2} K_{G} \lambda^{2} - \alpha^{2} \lambda^{2} K_{1} - \alpha^{2} \lambda^{2} K_{2} + \alpha^{2} \lambda^{2} C^{2}\right] \int_{0}^{L} \overline{\mathbf{N}}^{\mathrm{T}} \frac{\partial^{2} \overline{\mathbf{N}}}{\partial Y^{2}} dY \\ &+ \left[2\lambda^{2}(1-\mu)\right] \int_{0}^{L} \frac{\partial \overline{\lambda}^{\mathrm{T}}}{\partial Y} \frac{\partial \overline{\lambda}}{\partial Y} dY + \left[-K_{G} - \alpha^{2} K_{w} + K_{1} - C^{2}\right] \int_{0}^{L} \overline{\mathbf{N}}^{\mathrm{T}} \overline{\mathbf{N}} dY - \mu \lambda^{2} \int_{0}^{L} \frac{\partial^{2} \overline{\lambda}^{\mathrm{T}}}{\partial Y^{2}} \overline{\mathbf{N}} dY, \\ \overline{\mathbf{K}}_{e}^{0} &= \left[-\lambda^{2} K_{G} - \lambda^{2} \alpha^{2} K_{w} + \lambda^{2} K_{2}\right] \int_{0}^{L} \overline{\mathbf{N}}^{\mathrm{T}} \frac{\partial^{2} \overline{\mathbf{N}}}{\partial Y^{2}} dY + K_{w} \int_{0}^{L} \overline{\mathbf{N}}^{\mathrm{T}} \overline{\mathbf{N}} dY \\ &+ \left[\lambda^{4} \alpha^{2} K_{G} - \alpha^{2} \lambda^{4} K_{2}\right] \int_{0}^{L} \overline{\mathbf{N}}^{\mathrm{T}} \frac{\partial^{4} \overline{\mathbf{N}}}{\partial Y^{4}} dY - \lambda^{4} \int_{0}^{L} \frac{\partial^{2} \overline{\lambda}^{\mathrm{T}}}{\partial Y^{2}} \frac{\partial^{2} \overline{\mathbf{N}}}{\partial Y^{2}} dY, \\ \overline{\mathbf{K}}_{e,1}^{1} &= -2C \int_{0}^{L} \overline{\mathbf{N}}^{\mathrm{T}} \overline{\mathbf{N}} dY + 2C \alpha^{2} \lambda^{2} \int_{0}^{L} \overline{\mathbf{N}}^{\mathrm{T}} \frac{\partial^{2} \overline{\mathbf{N}}}{\partial Y^{2}} dY, \\ \overline{\mathbf{K}}_{e,1}^{3} &= 2C \alpha^{2} \int_{0}^{L} \overline{\mathbf{N}}^{\mathrm{T}} \overline{\mathbf{N}} dY, \\ \overline{\mathbf{m}}_{e}^{3} &= -\int_{0}^{L} \overline{\mathbf{N}}^{\mathrm{T}} \overline{\mathbf{N}} d\xi + \alpha^{2} \lambda^{2} \int_{0}^{L} \overline{\mathbf{N}}^{\mathrm{T}} \frac{\partial^{2} \overline{\mathbf{N}}}{\partial Y^{2}} dY, \end{split}$$
(22)

where $\overline{\mathbf{k}}_{e}^{4}$, $\overline{\mathbf{k}}_{e}^{2}$, $\overline{\mathbf{k}}_{e,1}^{0}$, $\overline{\mathbf{k}}_{e,1}^{1}$, $\overline{\mathbf{k}}_{e,2}^{3}$, $\overline{\mathbf{k}}_{e,2}^{2}$ denote stiffness matrices of the strip element and $\overline{\mathbf{m}}_{e}$ denotes mass matrice of the strip element.

To obtain the global dynamic equilibrium equations of the nanoplate, a global nodal displacement vector is defined as:

$$\overline{\delta}(X,T) = \begin{bmatrix} W_1 & \Theta_1 & W_2 & \Theta_2 & \cdots & W_{NE+1} & \Theta_{NE+1} \end{bmatrix}^T.$$
(23)

Like the finite element method, the contributions of each strip element are added together and the global dynamic equilibrium equations are obtained as:

$$\overline{\mathbf{K}}^{4} \frac{\partial^{4} \overline{\delta}}{\partial X^{4}} + \overline{\mathbf{K}}^{2} \frac{\partial^{2} \overline{\delta}}{\partial X^{2}} + \overline{\mathbf{K}}^{0} \overline{\delta} + \overline{\mathbf{K}}^{1}_{1} \frac{\partial^{2} \overline{\delta}}{\partial X \partial T} + \overline{\mathbf{M}} \frac{\partial^{2} \overline{\delta}}{\partial T^{2}} + \overline{\mathbf{K}}^{2}_{2} \frac{\partial^{4} \overline{\delta}}{\partial X^{2} \partial T^{2}} + \overline{\mathbf{K}}^{3}_{1} \frac{\partial^{4} \overline{\delta}}{\partial X^{3} \partial T} = \mathbf{0},$$
(24)

where $\overline{\mathbf{K}}^{(4)}, \overline{\mathbf{K}}^{(2)}, \overline{\mathbf{K}}^{(0)}, \overline{\mathbf{K}}^{(1)}_1, \overline{\mathbf{K}}^{(2)}_2, \overline{\mathbf{K}}^{(3)}_1$ are global stiffness matrices, and $\overline{\mathbf{M}}$ is a global mass matrix.

Further, dealing with the displacement boundary conditions imposed on nodal lines, the global dynamic equilibrium equations are rewritten as:

$$\widetilde{\mathbf{K}}^{4} \frac{\partial^{4} \overline{\delta}'_{w}}{\partial X^{4}} + \widetilde{\mathbf{K}}^{2} \frac{\partial^{2} \overline{\delta}'_{w}}{\partial X^{2}} + \widetilde{\mathbf{K}}^{0} \overline{\delta}'_{w} + \widetilde{\mathbf{K}}^{1} \frac{\partial^{2} \overline{\delta}'_{w}}{\partial X \partial T} + \widetilde{\mathbf{M}} \frac{\partial^{2} \overline{\delta}'_{w}}{\partial T^{2}} + \widetilde{\mathbf{K}}^{2} \frac{\partial^{4} \overline{\delta}'_{w}}{\partial X^{2} \partial T^{2}} + \widetilde{\mathbf{K}}^{3} \frac{\partial^{4} \overline{\delta}'_{w}}{\partial X^{3} \partial T} = 0,$$
(25)

where

$$\widetilde{\mathbf{K}}^{4} = \mathbf{T}_{1}^{\mathrm{T}} \overline{\mathbf{K}}^{4} \mathbf{T}_{1}, \widetilde{\mathbf{K}}^{2} = \mathbf{T}_{1}^{\mathrm{T}} \overline{\mathbf{K}}^{2} \mathbf{T}_{1}, \widetilde{\mathbf{K}}^{0} = \mathbf{T}_{1}^{\mathrm{T}} \overline{\mathbf{K}}^{0} \mathbf{T}_{1},$$

$$\widetilde{\mathbf{K}}_{1}^{1} = \mathbf{T}_{1}^{\mathrm{T}} \overline{\mathbf{K}}_{1}^{1} \mathbf{T}_{1}, \widetilde{\mathbf{K}}_{2}^{2} = \mathbf{T}_{1}^{\mathrm{T}} \overline{\mathbf{K}}_{2}^{2} \mathbf{T}_{1}, \widetilde{\mathbf{K}}_{1}^{3} = \mathbf{T}_{1}^{\mathrm{T}} \overline{\mathbf{K}}_{1}^{3} \mathbf{T}_{1}, \widetilde{\mathbf{M}} = \mathbf{T}_{1}^{\mathrm{T}} \overline{\mathbf{M}} \mathbf{T}_{1},$$
(26)

where T_1 are row transformation matrices containing 0 and 1, which is referred to literature [39].

Next, the Fourier transform of Equation (25) with respect to time are given by:

$$\widetilde{\mathbf{K}}^{4} \frac{\partial^{4} \widehat{\delta}'_{w}}{\partial X^{4}} + i\omega \widetilde{\mathbf{K}}_{1}^{3} \frac{\partial^{3} \widehat{\delta}'_{w}}{\partial X^{3}} + \left(\widetilde{\mathbf{K}}^{2} + i\omega \widetilde{\mathbf{K}}_{1}^{2} + (i\omega)^{2} \widetilde{\mathbf{K}}_{2}^{2}\right) \frac{\partial^{2} \widehat{\delta}'_{w}}{\partial X^{2}} + i\omega \widetilde{\mathbf{K}}_{1}^{1} \frac{\partial \widehat{\delta}'_{w}}{\partial X} + \left(\widetilde{\mathbf{K}}^{0} + (i\omega)^{2} \widetilde{\mathbf{M}}\right) \widehat{\delta}'_{w} = \mathbf{0}, \quad (27)$$

where $\hat{\delta}'_w$ is the Fourier transformation form of $\overline{\delta}'_w$, $i\omega$ is the Fourier transform parameter, ω is the angular frequency and $i = \sqrt{-1}$.

If denoting a status vector as:

$$\boldsymbol{\eta}(X,i\omega) = \begin{bmatrix} \hat{\delta}'^{\mathrm{T}}_{w} & \frac{\partial \hat{\delta}'^{\mathrm{T}}_{w}}{\partial X} & \frac{\partial^{2} \hat{\delta}'^{\mathrm{T}}_{w}}{\partial X^{2}} & \frac{\partial^{3} \hat{\delta}'^{\mathrm{T}}_{w}}{\partial X^{3}} \end{bmatrix}^{\mathrm{T}},$$
(28)

Equation (27) can be rewritten in a compact form in state space:

$$\frac{\partial \mathbf{\eta}(X, i\omega)}{\partial X} = \mathbf{F}(i\omega)\mathbf{\eta}(X, i\omega), \tag{29}$$

where

$$\mathbf{F}(i\omega) = \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \\ \mathbf{f}_{1} & \mathbf{f}_{2} & \mathbf{f}_{3} & \mathbf{f}_{4} \end{bmatrix}, \\ \mathbf{f}_{1} = -\left(\widetilde{\mathbf{K}}^{(4)}\right)^{-1} \left(\widetilde{\mathbf{K}}^{(0)} + (i\omega)^{2}\widetilde{\mathbf{M}}\right), \\ \mathbf{f}_{2} = -i\omega\left(\widetilde{\mathbf{K}}^{(4)}\right)^{-1}\widetilde{\mathbf{K}}_{1}^{(1)}, \\ \mathbf{f}_{3} = -\left(\widetilde{\mathbf{K}}^{(4)}\right)^{-1} \left(\widetilde{\mathbf{K}}^{(2)} + (i\omega)\widetilde{\mathbf{K}}_{1}^{(2)} + (i\omega)^{2}\widetilde{\mathbf{K}}_{2}^{(2)}\right), \\ \mathbf{f}_{4} = -(i\omega)^{2} \left(\widetilde{\mathbf{K}}^{(4)}\right)^{-1} \widetilde{\mathbf{K}}_{1}^{(3)}. \end{aligned}$$
(30)

Introducing the vector $\eta(X, i\omega)$ and performing the Fourier transform to the boundary conditions in Equation (17), we get:

$$\mathbf{M}_{h}(i\omega)\mathbf{\eta}(-0.5,i\omega) + \mathbf{N}_{h}(i\omega)\mathbf{\eta}(0.5,i\omega) = \mathbf{0},\tag{31}$$

where \mathbf{M}_b and \mathbf{N}_b are selection matrice of boundary condition at the left and right edges of nanoplate, respectively, as shown below.

$$\mathbf{M}_{b}(i\omega) = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \mathbf{N}_{b}(i\omega) = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \end{bmatrix}.$$
 (32)

According to the transfer function method, the solution of Equation (29) can be expressed as follows:

$$\eta(X, i\omega) = e^{\mathbf{F}(i\omega)X} \eta(0, i\omega).$$
(33)

Substituting Equation (33) into Equation (31) results in:

$$\left[\mathbf{M}_{b}(i\omega)\mathbf{e}^{-0.5\mathbf{F}(i\omega)} + \mathbf{N}_{b}(i\omega)\mathbf{e}^{0.5\mathbf{F}(i\omega)}\right]\boldsymbol{\eta}(0,i\omega) = \mathbf{0}.$$
(34)

For the dynamic analysis of the nanoplate, the existence of a nontrivial solution of the corresponding Equation (34) requires that the determinant of the coefficient matrix must vanish

$$\det\left[\mathbf{M}_{b}(\mathrm{i}\omega)\mathrm{e}^{-0.5\mathrm{F}(\mathrm{i}\omega)} + \mathbf{N}_{b}(\mathrm{i}\omega)\mathrm{e}^{0.5\mathrm{F}(\mathrm{i}\omega)}\right] = 0. \tag{35}$$

The dimensionless natural frequency ω of the axially-moving nanoplate resting on a viscoelastic foundation can be obtained by solving the above transcendental characteristic equation.

4. Numerical Investigation

4.1. Comparison and Validation

To make a comparison with the previous literature [24], we adopted the following parameters: $l_a = l_b = 10 \text{ nm}, h = 0.34 \text{ nm}, E = 1.06 \text{ TPa}, \mu = 0.25, \rho = 2250 \text{ kg/m} \text{ and } K_1 = K_2 = 80$ for the axially-moving nanoplate without a foundation. The material properties of the foundation are the Winkler's modulus $k_w = 0$, the Pasternak's modulus $k_G = 0$, and the damping $c_t = 0$. According to the literature [39], strip element number NE = 6 was selected for GSDTFM in all of the following numerical calculations. The numerical results of the present work were compared with the corresponding results in literature [24], as shown in Table 1. It can be observed from Table 1 that the dimensionless fundamental frequencies for the axially-moving nanoplate in this paper were in excellent agreement with those in [24], demonstrating the accuracy and efficiency of the proposed method for vibration analysis.

 Table 1. Dimensionless natural frequencies for the axially-moving nanoplate without a foundation.

	ω ₁₁		ω ₁₂	
С	$\alpha = 0.0$	$\alpha = 0.005$	$\alpha = 0.0$	$\alpha = 0.005$
	Pressent/Ref. [24]	Pressent/Ref. [24]	Pressent/Ref. [24]	Pressent/Ref. [24]
0	44.3601/44.3709	44.3577/44.3687	79.8500/79.8941	79.9500/79.8753
2	43.3402/43.3435	43.3361/43.3409	78.5746/78.5923	78.5598/78.5714
4	40.2698/40.2712	40.2500/40.2674	74.6534/74.6768	74.6401/74.6494
6	35.1732/35.1804	35.1611/35.1745	68.1010/68.1130	68.0699/68.0746
8	28.0989/28.1030	28.0899/28.0940	58.8110/58.8225	58.7618/58.7690
10	19.0302/19.0359	19.0221/19.0228	46.6143/46.6159	46.5418/46.5434

For future comparisons with other researchers, the first dimensionless natural frequencies of the axially-moving nanoplate resting on a viscoelastic foundation before the dimensionless velocity C = 10 with small-parameter α = 0.01 and 0.10 are presented in Table 2. The material properties of the viscoelastic foundation were k_{w0} = 0.1 GPa/nm, k_{G0} = 0.25 GPa·nm and c_{t0} = 10⁻⁴ GPa·ns/nm, as referred to in the literature [39]. It can be seen from the tables that the dimensionless natural frequencies of the axially-moving nanoplate were complex numbers when it rested on a viscoelastic foundation.

Table 2. Dimensionless natural frequencies for the axially-moving nanoplate resting on a viscoelastic foundation ($\times 10^2$).

C	ω_{11}		ω_{12}	
	$\alpha = 0.01$	$\alpha = 0.10$	$\alpha = 0.01$	$\alpha = 0.10$
0	0.4860 + 0.0297i	0.4791 + 0.0273i	0.8345 + 0.0296i	0.7814 + 0.0219i
2	0.4761 + 0.0290i	0.4680 + 0.0265i	0.8248 + 0.0291i	0.7692 + 0.0214i
4	0.4467 + 0.0270i	0.4359 + 0.0244i	0.7968 + 0.0277i	0.7350 + 0.0202i
6	0.3994 + 0.0240i	0.3844 + 0.0214i	0.7532 + 0.0259i	0.6837 + 0.0186i
8	0.3345 + 0.0203i	0.3143 + 0.0177i	0.6961 + 0.0238i	0.6179 + 0.0169i
10	0.2509 + 0.0156i	0.2226 + 0.0129i	0.6263 + 0.0214i	0.5380 + 0.0151i

4.2. Flutter and Divergent Instability

The flutter instability refers to a dynamic process of structures transiting from stability to instability with a nonvanishing frequency, whereas the divergence instability refers to a structure passing from stability to instability at a zero frequency. In the following text, the mode-couple flutter and divergence instability of an axially-moving nanoplate resting on a visoelastic foundation with simply supported boundary condition are investigated. The small parameter of the nanoplate was $\alpha = 0.001$ and other parameters were the same as Section 4.1.

For comparison, the first four dimensionless natural frequencies ω_{11} , ω_{12} , ω_{21} and ω_{22} of the axially-moving nanoplate without a foundation with respect to moving speeds are plotted in Figure 3. As shown in the figure, the dimensionless frequencies were real numbers at the dimensionless speed C = 0. Along with the increase of axially moving speed, the first four dimensionless natural frequencies changed but remained real numbers before C = 12.8. This showed that the nanoplate was stable before C = 12.8. At the speed of C = 12.8, the real part of ω_{11} decreased to zero and its imaginary part had two branches. This indicated the nanoplate began divergence instability on its first order mode. C = 12.8 was a threshold critical value of ω_{11} , denoted as the divergence speed C_{div1}. Moreover, ω_{11} of the nanoplate was unstable in the interval [12.8, 14.1], but ω_{12} , ω_{21} and ω_{22} still kept stable. When the axially moving speed further increased to C = 16.2, ω_{11} and ω_{21} of the nanoplate coupled with each other into a pair of complex conjugate frequency. This means that the nanoplate exhibited a coupled-mode flutter instability. C = 16.2 was denoted as the flutter speed C_{flu1} . When C = 17.9, denoted as the divergence speed C_{div2} , the real part of ω_{12} decreased to zero with its imaginary part becoming two branches, representing the nanoplate becoming divergent instable again on its second order mode. The nanoplate stayed unstable again during the interval [17.9, 25.5]. At the speed of C = 26.2, denoted as the flutter speed C_{flu2} , the nanoplate exhibited coupled-mode flutter instability again with ω_{12} and ω_{22} merging together.

In contrast with Figure 3, Figure 4 presents the first four order dimensionless natural frequencies ω_{11} , ω_{12} , ω_{21} and ω_{22} of the axially-moving nanoplate resting on a viscoelastic foundation with respect to moving speeds. As can be seen, the dimensionless frequencies only had a real component at the speed of C = 0, but the values were slightly larger than those without a foundation. This implies that the rigidity of the nanoplate was enhanced due to the viscoelastic foundation. When the moving speed C > 0, the first four dimensionless frequencies ω_{11} , ω_{12} , ω_{21} and ω_{22} became complex numbers, which were significantly different from vibrations of the nanoplate without a foundation. Also, the divergence

instability type of the axially moving nanoplate resting on a viscoelastic foundation was similar to that without a foundation. However, the divergence speeds for ω_{11} and ω_{21} ($C_{div1} = 13.3$ and $C_{div2} = 18.3$) were larger than those without a foundation ($C_{div1} = 12.8$ and $C_{div2} = 17.9$). This means the divergence speed was more obviously affected by the viscoelastic foundation. Particularly, it is worth pointing out that the phenomenon of mode-couple flutter instabilities of ω_{11} coupling ω_{21} and ω_{12} coupling ω_{22} disappeared when the moving nanoplate rested on a viscoelastic foundation.



Figure 3. Dimensionless complex frequencies vs axially moving speed for the nanoplate without a foundation.



Figure 4. Dimensionless complex frequencies vs axially moving speed for the nanoplate resting on a viscoelastic foundation.

From the Figures 3 and 4, it was found that the viscoelastic foundation had a significant impact on the dynamic instability of the nanoplate. So, the effects of viscoelastic foundation model parameters including the Winkler's modulus k_w , the Pasternak's modulus k_G , and the damping c_t on the dimensionless complex frequencies of the axially-moving nanoplate are respectively investigated in the following text.

Figure 5 shows the change of dimensionless complex frequencies for the axially-moving nanoplate resting on a foundation with the Winkler's modulus $k_w = 0$, k_{w0} , $2k_{w0}$ (the Pasternak's modulus $k_G = 0$ and the damping $c_t = 0$), respectively. Note that the instability type of the axially-moving nanoplate along with the increase of the Winkler's modulus k_w included divergence instability and mode-couple

flutter instability. For divergence instability, as the increase of the Winkler's modulus k_w , divergence instability zeros for ω_{11} and ω_{12} were enlarged and the corresponding divergence speeds C_{div1} and C_{div2} were increased. For mode-couple flutter instability, flutter speeds C_{flu1} and C_{flu2} were also increased by raising the k_w . This indicates that the divergence instability and flutter instability is delayed when the k_w increases. Besides, the dimensionless natural frequencies ω_{11} and ω_{12} increased before their divergence instability along with the increase of the k_w , whereas ω_{21} and ω_{22} increased before their flutter instability. Figure 6 gives the vibration of the divergence speeds C_{div1} and C_{div2} and the flutter speeds C_{flu1} and C_{flu2} along with the increase of k_w . The changes of flutter speeds were dominant over divergence speeds, which indicates that the flutter instability of the nanoplate is more sensitive to the Winkler's modulus k_w .



Figure 5. Effect of the Winkler's modulus k_w on dimensionless complex frequencies of the axially-moving nanoplate.



Figure 6. Effect of the Winkler's modulus k_w on the flutter and divergence speed of the axially-moving nanoplate.

Figure 7 presents the change of dimensionless complex frequencies for the axially-moving nanoplate resting on a foundation with the Pasternak's modulus $k_G = 0$, k_{G0} , $2k_{G0}$ (the Winkler's modulus $k_w = 0$ and the damping $c_t = 0$), respectively. Similarly to the Pasternak's modulus parameter k_w , the instability type of the axially-moving nanoplate with $k_G = 0$, k_{G0} , $2k_{G0}$ still included divergence instability and flutter instability. However, for divergence instability, the instability zeros for ω_{11} and ω_{12} had a slightly change with the increase of the Pasternak's modulus k_G , although

the corresponding divergence speeds C_{div1} and C_{div2} were still increased. For mode-couple flutter instability, flutter speeds C_{flu1} and C_{flu2} were also increased with the increase of the Pasternak's modulus k_G . Also, the dimensionless natural frequencies ω_{11} and ω_{12} were increased before their divergence instability and ω_{21} and ω_{22} were also increased before their flutter instability with the increase of the Pasternak's modulus k_G . The vibrations of divergence speeds C_{div1} and C_{div2} and flutter speeds C_{flu1} and C_{flu2} along with the increase of the Pasternak's modulus k_G are plotted in Figure 8, illustrating that changes of the flutter speeds are no more dominant than those of the divergence speeds.



Figure 7. Effect of the Pasternak's modulus k_G on dimensionless complex frequencies of the axially-moving nanoplate.



Figure 8. Effect of the Pasternak's modulus k_G on the flutter and divergence speed of the axially-moving nanoplate.

Figure 9 illustrates the change of dimensionless complex frequencies for the axially-moving nanoplate resting on a foundation with the damping $c_t = 0$, c_{t0} , $2c_{t0}$ (the Winkler's modulus $k_w = 0$ and the Pasternak's modulus $k_G = 0$), respectively. Obviously, the instability type of the axially-moving nanoplate was altered with the appearance of the damping. When the damping $c_t = 0$, there was divergence instability in ω_{11} , ω_{12} and flutter instability in ω_{11} coupling ω_{21} and ω_{12} coupling ω_{22} . However, when the damping $c_t > 0$, the flutter instability disappeared and only divergence instability in ω_{11} and ω_{12} was left. Furthermore, the modes $\omega_{11} / \omega_{21}$ and $\omega_{12} / \omega_{22}$, which coupled

when the damping $c_t = 0$, separated with the appearance of the damping c_t . It was found that the damping c_t had almost no influences on the divergence speeds C_{div1}/C_{div2} and the real parts of the first two dimensionless complex frequencies $\text{Re}(\omega_{11})/\text{Re}(\omega_{12})$ before their divergence instability, but intensively affected the imaginary part of dimensionless complex frequencies.



Figure 9. Effect of the Pasternak's modulus k_G on dimensionless complex frequencies of the axially-moving nanoplate.

Figure 10 plots the change of dimensionless complex frequencies for the axially-moving nanoplate resting on a viscoelastic foundation with the small-parameter $\alpha = 0.001$, 0.050 and 0.100, respectively. As the small-parameter α increased, divergence instability zeros for ω_{11} and ω_{12} were enlarged but the corresponding divergence speeds C_{div1} and C_{div2} were decreased. The changes of divergence speeds C_{div1} and C_{div2} along with the increase of the small-parameter α are presented in Figure 11. It was seen that the divergence speeds C_{div1} and C_{div2} decreased significantly when $\alpha > 0.02$.

In Figure 12, the changes of dimensionless complex frequencies for the axially-moving nanoplate resting on a viscoelastic foundation with the biaxial force K = 40, 60 and 80 are presented. The biaxial force K clearly influenced the divergence instability zero and the divergence speed. Raising the biaxial force K, divergence instability zeros for ω_{11} and ω_{12} were slightly enlarged and the corresponding divergence speeds C_{div1} and C_{div2} were increased as shown in Figure 13.



Figure 10. Effect of the small-parameter α on dimensionless complex frequencies of the axially-moving nanoplate resting on a viscoelastic foundation.



Figure 11. Effect of the small-parameter α on the divergence speed of the axially-moving nanoplate.



Figure 12. Effect of the small-parameter α on dimensionless complex frequencies of the axially-moving nanoplate resting on a viscoelastic foundation.



Figure 13. Effect of the axis force *K* on the divergence speed of the axially-moving nanoplate resting on a viscoelastic foundation.

5. Conclusions

The dynamic instability of axially-moving nanoplates resting on a viscoelastic foundation was investigated based on the nonlocal elasticity theory. The viscoelastic foundation, nonlocal effect and biaxial loadings of the nanoplate are considered in the established model to capture the size-dependent mechanical property. The main conclusions are drawn as follow:

- (i) The viscoelastic foundation has a dominant impact on vibrations of the axially-moving nanoplate. The dimensionless natural frequencies of the axially-moving nanoplate resting on a viscoelastic foundation are complex numbers when the dimensionless moving speed C > 0.
- (ii) The Winkler's modulus k_w of the viscoelastic foundation significantly affects both the instability zone and the divergence speed. The Pasternak's modulus k_G of the viscoelastic foundation affects mainly the divergence speed. The damping c_t of the viscoelastic foundation has almost no influence on the instability zone or divergence speed.
- (iii) Mode-couple flutter disappears when the axially-moving nanoplate rests on a viscoelastic foundation. This change of instability types is mainly influenced by the damping c_t of the viscoelastic foundation.
- (iv) Small-parameter α affects both the divergence instability zero and the divergence speed of the axially-moving nanoplates resting on a viscoelastic foundation. Such influences become more substantial when $\alpha > 0.02$.
- (v) The divergence instability is also quite sensitive to the biaxial force, which influences the divergence instability zero and the corresponding divergence speed.

Author Contributions: Conceptualization, J.D., D.Z., W.W.; Methodology, J.D., D.Z.; Writing—Original Draft Preparation, J.D., D.Z.; Writing—Review & Editing, J.D., W.W.; Funding Acquisition, J.D.

Funding: This research was funded by the Natural Science Foundation of China (grant number 11702325 and 11302254) and the Natural Science Foundation of Hebei Province of China (grant number A2018210065).

Conflicts of Interest: The authors declare no conflict of interest.

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