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Entanglement and Entropy of a Three-Qubit System Interacting with a Quantum Spin Environment

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Abstract: Throughout this paper, we describe an analytical solution for a three-qubit system characterized by a finite temperature within a thermodynamic limit influenced by a quantum spin environment. As applications to the presented solution, we investigate the effect of the temperature, the coupling constant ϵ_0 within the spin- qubit system and an external magnetic field on the three-particles residual entanglement N_{abc}, the concurrence $C(\rho)$, the information entropy $H(\sigma_Z)$ and the linear entropy $P_P(t)$. The results show an inverse relationship between the entanglement and entropy, where the degree of both is controlled by controlling the temperature T and the coupling constant ϵ_0.

Keywords: quantum spin; entropy; thermodynamic; concurrence

1. Introduction

Due to their significant role in quantum computation, the principles of open quantum schemes have received increasing consideration in the latest years. Usually, these theories examine the evolution concerning a quantum system in interaction with a specific nature which generally consists of bosons, fermions or localized spins [1–3]. The characteristics of the quantum can be obtained through the average of the degrees of freedom in the environment. Motivated by the progress in quantum information and comp, there has been a growing interest in mathematical and physical exploration of highly entangled states. This entanglement occurs when two or more particles are integrated and generated to the extent that their quantum states become inseparable and can not be depicted independently. Entangling between both near and remote pairs of qubits has yielded various applications in the quantum information processing, such as the entanglement generation in nanophotonic architectures [4–10]. Recently, there has been a rising curiosity in exploring the entanglement within the systems of quantum spin together with Heisenberg interactions [11–16]. Being both a simple and at the same time solid system, the Heisenberg model was used to visualize a quantum computer, and also to simulate nuclear and electronic spins [17,18], and quantum dots [19,20]. Moreover, it has demonstrated a very vital applicability in quantum state transfer [21]. However, the previous studies were not essentially discussing the relationship between the system and its surrounding environment as the pivotal point of interest, as any quantum system would naturally interact with the surrounding environment. What is the most crucial, thus, is not the interaction itself, but, practically speaking, the decoherence which that interaction leads to. Investigating the decoherence [22,23] process is undoubtedly fundamental for the theoretical basics. However, penetrating the process of decoherence within the quantum system is more important for building strategies of error correction, by which the collapse of quantum computers [24] can be detained. When interacting with an environment, the components of a composite quantum system, such as a multi-qubit quantum computer, are disentangled under the influence of the decoherence process. Qubits of a quantum computer may sometimes interact with substantially independent environments. Yet, in some cases, those environments are not completely independent, as the correlations within them are effectively significant.

Increasingly applied in quantum information theory and quantum computation [25], the dynamic behavior of an individual spin or multi-spin system's interaction with a spin bath has recently gained much attention [26–29]. A study of three-qubit quantum state has special significance because a three-qubit quantum state is the real start towards investigating multipartite entanglement [30]. Three-qubit quantum entanglement can also exhibit more complex entanglement structure than the two-qubit quantum entanglement [31]. However, in most cases of the non-Markovian process, it is quite tricky to get a specific solution to the time evolution of the reduced density matrix which is traced over by the surrounding environment. This casts more importance on our endeavor to study the case of the non-Markovian three-qubit quantum system by employing an innovative operator procedure to reach a specific solution to our system.

The problem we are introducing in this paper is a quantum system composed of three qubits characterized by a finite temperature within a thermodynamic limit influenced by a quantum spin environment. We obtained the situation of a two-qubit system through the tracing of a three-qubit case. The article [32] was studying the two-qubit state, and our paper has studied a three-qubit, so our paper is the extension of [32]. Through our case, we can reach the state of the two-qubit, and by changing some parameters we get the results obtained in [32]; i.e., the results are consistent with the results of reference [32]. Additionally, if we follow the same treatment, we can solve more atoms. We have calculated the entanglement and entropy as applications on our problem. We calculated the three-particle residual entanglement N_{abc} as a kind of entanglement on the case of a three-qubit system, while we calculated the concurrence $C(\rho)$ on a two-qubit quantum system. We calculated the information entropy $H(\sigma_Z)$ and the linear entropy $P_P(t)$ in both cases, two-qubit and three-qubit. The aim of these calculations was to establish a relationship between the entanglement and entropy, determined by the change of parameters.

This paper is ordered according to the following arrangement: The Hamiltonian system, along with the analytical solution of a three-qubit system characterized by a finite temperature within a thermodynamic limit influenced by a quantum spin environment, is discussed in Section 2. In Section 3, we discuss the three-particles residual entanglement N_{abc}, the concurrence $C(\rho)$, the information entropy $H(\sigma_Z)$ and the linear entropy $P_P(t)$. Finally, Section 4 presents the conclusion.

2. The Model

We study a quantum system composed of three qubits characterized by a finite temperature within a thermodynamic limit influenced by a quantum spin environment. The Hamiltonian we are studying is the extension of the Hamiltonian studied in the reference [32]. Here, we consider that the anisotropy and the magnetic field are not homogeneous, yet the spin environment remains without change. The Hamiltonian system can be written as:

$$
\begin{aligned}
H &= H_P + H_{PQ} + H_Q, && (1)\\
H_P &= \epsilon_0(S_{01}^z + S_{02}^z + S_{03}^z) + \omega(S_{01}^+ S_{02}^- + S_{01}^- S_{02}^+ + S_{02}^+ S_{03}^- + S_{02}^- S_{03}^+ && (2)\\
&\quad + S_{01}^+ S_{03}^- + S_{01}^- S_{03}^+),\\
H_Q &= \frac{q}{M}\sum_{r\neq j}^{M}(S_r^+ S_j^- + S_r^- S_j^+), && (3)\\
H_{PQ} &= \frac{q_0}{\sqrt{M}}[(S_{01}^+ + S_{02}^+ + S_{03}^+)\sum_{r=1}^{M} S_r^-] + \frac{q_0}{\sqrt{M}}[(S_{01}^- + S_{02}^- + S_{03}^-)\sum_{r=1}^{M} S_r^+], && (4)
\end{aligned}
$$

where H_P, H_Q and H_{PQ} are the system Hamiltonian, bath and system–bath interaction. S_{0r}^+ and S_{0r}^- ($r = 1,2,3$) denote the spin system operators' [33–36].

Through the substitution of the operator of the collective angular momentum, $J_\pm = \sum_{r=1}^{M} S_r^\pm$, the Hamiltonian of Equations (3) and (4) can be written as follows:

$$H_{PQ} = \frac{q_0}{\sqrt{M}}[(S_{01}^+ + S_{02}^+ + S_{03}^+)J_-] + \frac{q_0}{\sqrt{M}}[(S_{01}^- + S_{02}^- + S_{03}^-)J_+], \tag{5}$$

$$H_Q = \frac{q}{M}\sum_{r\neq j}^{M}(J_+J_- + J_-J_+) - q. \tag{6}$$

We apply the Holstein–Primakoff transformation in order to transform the infinite-dimensional Fock space of boson creation and annihilation operators to finite-dimensional subspaces of the spin operators [37].

The transformation of Holstein–Primakoff can be expressed as:

$$J_+ = b^\dagger(\sqrt{M - b^\dagger b}),\ J_- = (\sqrt{M - b^\dagger b})b, \text{ with } \left[b^\dagger, b\right] = 1. \tag{7}$$

The Hamiltonian in Equations (5) and (6) can be written as:

$$H_{PQ} = q_0[(S_{01}^+ + S_{02}^+ + S_{03}^+)(\sqrt{1 - \frac{b^\dagger b}{M}})b] + q_0[(S_{01}^- + S_{02}^- + S_{03}^-)b^\dagger(\sqrt{1 - \frac{b^\dagger b}{M}})], \tag{8}$$

$$H_Q = q[b^\dagger(1 - \frac{b^\dagger b}{M})b + (\sqrt{1 - \frac{b^\dagger b}{M}})bb^\dagger(\sqrt{1 - \frac{b^\dagger b}{M}})] - q. \tag{9}$$

If $M \longrightarrow \infty$ (when the system characterized by a finite temperature within a thermodynamic limit), H_{PQ} and H_Q can be written as follows:

$$
\begin{aligned}
H_{PQ} &= q_0[(S_{01}^+ + S_{02}^+ + S_{03}^+)b + (S_{01}^- + S_{02}^- + S_{03}^-)b^\dagger], && (10)\\
H_Q &= 2qb^\dagger b. && (11)
\end{aligned}
$$

Here, we find the approximation that $\frac{b^\dagger b}{M}$ approaches zero, due to the deficient energy of the elementary excitations according to the interconnection between the system and the bath. Accordingly, we deduce the time evolution of the density operator of the spin quantum system. Taking into consideration that the Hamiltonian is time-independent, we can present the density matrix as follows:

$$\rho(t) = \exp(-iHt)\rho(0)\exp(iHt). \tag{12}$$

We suppose that $\rho(0) = \rho_P(0) \otimes \rho_Q$, i.e., $\rho(0)$ is separable between the bath and system. The initial state of the spin system can be described by $\rho_P(0)$, and $\rho_Q = \frac{\exp(\frac{-H_Q}{KT})}{R}$ defines the spin environment, where $R = \frac{1}{1-\exp(\frac{-2q}{T})}$ is the separation function, T is the temperature and K is the Boltzmann constant ($K = 1$). The reduced density matrix of scheme is attained by deriving the trace over the spin system; i.e., $\rho_P(t) = Tr_Q(\rho(t))$.

By taking the initial state $|\psi\rangle = \cos\alpha\cos\beta\,|--+\rangle + \cos\alpha\sin\beta\,|-+-\rangle + \sin\alpha\,|+--\rangle$, $0 \leq \alpha \prec \pi$, $0 \leq \beta \prec \pi$, $\rho_P(0) = |\psi\rangle\langle\psi|$. Therefore, the reduced density matrix of the scheme can be written as follows:

$$\begin{aligned}
\rho_P(t) \quad = \quad & \frac{1}{R}\cos^2\alpha\cos^2\beta Tr_Q[\exp(-iHt)\,|--+\rangle\exp(\frac{-H_Q}{T})\,\langle+--|\exp(iHt)] + \\
& \frac{1}{R}\cos^2\alpha\sin^2\beta Tr_Q[\exp(-iHt)\,|-+-\rangle\exp(\frac{-H_Q}{T})\,\langle-+-|\exp(iHt)] + \\
& \frac{1}{R}\sin^2\alpha Tr_Q[\exp(-iHt)\,|+--\rangle\exp(\frac{-H_Q}{T})\,\langle--+|\exp(iHt)] +
\end{aligned}$$

$$\begin{aligned}
\{ \quad & \frac{1}{R}\cos^2\alpha\cos\beta\sin\beta Tr_Q[\exp(-iHt)\,|--+\rangle\exp(\frac{-H_Q}{T})\,\langle-+-|\exp(iHt)] + \\
& \frac{1}{R}\cos\alpha\cos\beta\sin\alpha Tr_Q[\exp(-iHt)\,|--+\rangle\exp(\frac{-H_Q}{T})\,\langle--+|\exp(iHt)] + \\
& \frac{1}{R}\cos\alpha\sin\beta\sin\alpha Tr_Q[\exp(-iHt)\,|-+-\rangle\exp(\frac{-H_Q}{T})\,\langle--+|\exp(iHt)] \\
& +h.c\}. \qquad (13)
\end{aligned}$$

To calculate $\exp(-iHt)\,|--+\rangle$, for example, we assume

$$\begin{aligned}
\exp(-iHt)\,|--+\rangle \quad = \quad & U\,|---\rangle + V\,|--+\rangle + X\,|-+-\rangle + Y\,|-++\rangle + G\,|+--\rangle + \\
& J\,|+-+\rangle + L\,|++-\rangle + F\,|+++\rangle\,, \qquad (14)
\end{aligned}$$

where U, V, X, Y, G, J, L and F are functions of operators b, $b^\dagger$, and time t. Here, we adopt the Schrödinger equation because it is considered to be a special case of the master equation in Lindblad form [38]. By applying the identity of Schrödinger equation

$$i\frac{d}{dt}(\exp(-iHt)\,|--+\rangle) = H(\exp(-iHt)\,|--+\rangle), \qquad (15)$$

and Equation (14), we get the following equations

$$
\begin{aligned}
i\frac{dU}{dt} &= (-\frac{3}{2}\epsilon_0 + 2qb^\dagger b)U + q_0 b^\dagger (V + X + G), \\
i\frac{dV}{dt} &= (-\frac{1}{2}\epsilon_0 + 2qb^\dagger b)V + q_0 bU + q_0 b^\dagger (Y + J) + \omega(X + G), \\
i\frac{dX}{dt} &= (-\frac{1}{2}\epsilon_0 + 2qb^\dagger b)X + q_0 bU + q_0 b^\dagger (Y + L) + \omega(V + G), \\
i\frac{dY}{dt} &= (\frac{1}{2}\epsilon_0 + 2qb^\dagger b)Y + q_0 b^\dagger F + q_0 b(V + X) + \omega(J + L), \\
i\frac{dG}{dt} &= (-\frac{1}{2}\epsilon_0 + 2qb^\dagger b)G + q_0 bU + q_0 b^\dagger (J + L) + \omega(V + X), \\
i\frac{dJ}{dt} &= (\frac{1}{2}\epsilon_0 + 2qb^\dagger b)J + q_0 b^\dagger F + q_0 b(V + G) + \omega(Y + L), \\
i\frac{dL}{dt} &= (\frac{1}{2}\epsilon_0 + 2qb^\dagger b)L + q_0 b^\dagger F + q_0 b(X + G) + \omega(Y + J), \\
i\frac{dF}{dt} &= (\frac{3}{2}\epsilon_0 + 2qb^\dagger b)F + q_0 b(Y + J + L),
\end{aligned} \tag{16}
$$

with the consideration that the initial conditions $U(0) = X(0) = ... = F(0) = 0$, and $V(0) = 1$, from Equation (14), the pervious differential equations, are unsolvable by any traditional methods for ordinary variables, because the differential equations are composed of non-commuting operator variables. So, we can use the following transformation:

$$
\begin{aligned}
U &= b^\dagger \exp(-i2q(b^\dagger b + 1)t)U_1, \;\; V = \exp(-i2q(b^\dagger b + 1)t)V_1, \\
X &= \exp(-i2q(b^\dagger b + 1)t)X_1, \;\; Y = b\exp(-i2q(b^\dagger b + 1)t)Y_1, \\
G &= \exp(-i2q(b^\dagger b + 1)t)G_1, \;\; J = b\exp(-i2q(b^\dagger b + 1)t)J_1, \\
L &= b\exp(-i2q(b^\dagger b + 1)t)L_1, \;\; F = bb\exp(-i2q(b^\dagger b + 1)t)F_1.
\end{aligned} \tag{17}
$$

Then the Equation (16) becomes, from Equation (17), as follows:

$$
\begin{aligned}
i\frac{dU_1}{dt} &= -\frac{3}{2}\epsilon_0 U_1 + q_0(V_1 + X_1 + G_1), \\
i\frac{dV_1}{dt} &= (-\frac{1}{2}\epsilon_0 - 2q)V_1 + q_0(n + 1)U_1 + nq_0(Y_1 + J_1) + \omega(X_1 + G_1), \\
i\frac{dX_1}{dt} &= (-\frac{1}{2}\epsilon_0 - 2q)X_1 + q_0(n + 1)U_1 + nq_0(Y_1 + L_1) + \omega(V_1 + G_1), \\
i\frac{dY_1}{dt} &= (\frac{1}{2}\epsilon_0 - 4q)Y_1 + q_0(n - 1)F_1 + q_0(V_1 + X_1) + \omega(J_1 + L_1), \\
i\frac{dG_1}{dt} &= (-\frac{1}{2}\epsilon_0 - 2q)G_1 + q_0(n + 1)U_1 + nq_0(J_1 + L_1) + \omega(V_1 + X_1), \\
i\frac{dJ_1}{dt} &= (\frac{1}{2}\epsilon_0 - 4q)J_1 + q_0(n - 1)F_1 + q_0(V_1 + G_1) + \omega(Y_1 + L_1), \\
i\frac{dL_1}{dt} &= (\frac{1}{2}\epsilon_0 - 4q)L_1 + q_0(n - 1)F_1 + q_0 b(X_1 + G_1) + \omega(Y_1 + J_1), \\
i\frac{dF_1}{dt} &= (\frac{3}{2}\epsilon_0 - 6q)F_1 + q_0(Y_1 + J_1 + L_1),
\end{aligned} \tag{18}
$$

with $n = b^\dagger b$ and the initial conditions $U_1(0) = X_1(0) = ... = F_1(0) = 0$, and $V_1(0) = 1$. Numerically we can solve the differential Equation (18). Therefore, we can obtain the variables U_1, V_1, X_1, Y_1, G_1, J_1, L_1 and F_1 as functions of n and t, which commute with each other. Hence, we can obtain U, V, X, Y, G, J, L and F, from Equation (17).

In the same way, we can calculate $\exp(-iHt)\left|-+-\right\rangle$ and $\exp(-iHt)\left|+--\right\rangle$, so we suppose

$$\begin{aligned}
\exp(-iHt)\left|-+-\right\rangle &= \chi\left|---\right\rangle+\Gamma\left|--+\right\rangle+\gamma\left|-+-\right\rangle+\delta\left|-++\right\rangle \\
&\quad +\lambda\left|+--\right\rangle+\mu\left|+-+\right\rangle+\nu\left|++-\right\rangle+\eta\left|+++\right\rangle, \\
\exp(-iHt)\left|+--\right\rangle &= \tilde{\chi}\left|---\right\rangle+\tilde{\Gamma}\left|--+\right\rangle+\tilde{\gamma}\left|-+-\right\rangle+\tilde{\delta}\left|-++\right\rangle \\
&\quad +\tilde{\lambda}\left|+--\right\rangle+\tilde{\mu}\left|+-+\right\rangle+\tilde{\nu}\left|++-\right\rangle+\tilde{\eta}\left|+++\right\rangle,
\end{aligned} \tag{19}$$

but here the initial conditions are $\chi_1(0)=\Gamma_1(0)=...=\eta_1(0)=0$, and $\gamma_1(0)=1$, $\tilde{\chi}_1(0)=\tilde{\Gamma}_1(0)=...=\tilde{\eta}_1(0)=0$, and $\lambda_1(0)=1$.

After the same transformation in Equation (17), we get eight differential equations—Equation (17), that can be solved numerically. Hence, we can obtain the variables χ_1, $\Gamma_1,...,$ η_1 and $\tilde{\chi}_1$, $\tilde{\Gamma}_1,...,$ $\tilde{\eta}_1$ as functions of n and t, which commute with each other. So, we can obtain χ, $\Gamma,...,$ η and $\tilde{\chi}$, $\tilde{\Gamma},...,$ $\tilde{\eta}$. Then ($\rho_P(t)=Tr_Q(\rho(t))$) can be expressed in the basis $\{\left|1\right\rangle=\left|---\right\rangle, \left|2\right\rangle=\left|--+\right\rangle, \left|3\right\rangle=\left|-+-\right\rangle, \left|4\right\rangle=\left|-++\right\rangle, \left|5\right\rangle=\left|+--\right\rangle, \left|6\right\rangle=\left|+-+\right\rangle, \left|7\right\rangle=\left|++-\right\rangle, \left|8\right\rangle=\left|+++\right\rangle\}$.

$$\begin{aligned}
\rho_{11} &= \frac{1}{Z}\sum_{n=0}^{\infty}(\cos^2\alpha\cos^2\beta U_1U_1^\dagger+\cos^2\alpha\sin^2\beta\chi_1\chi_1^\dagger+\sin^2\alpha\tilde{\chi}_1^\dagger\tilde{\chi}_1+ \\
&\quad \cos^2\alpha\cos\beta\sin\beta(U_1\chi_1^\dagger+\chi_1U_1^\dagger)+\cos\alpha\sin\alpha\cos\beta(U_1^\dagger\tilde{\chi}_1+\tilde{\chi}_1U_1^\dagger) \\
&\quad +\cos\alpha\sin\alpha\sin\beta(\chi_1^\dagger\tilde{\chi}_1+\tilde{\chi}_1\chi_1^\dagger))(n+1)\exp(\frac{-2qn}{T}), \\
\rho_{22} &= \frac{1}{Z}\sum_{n=0}^{\infty}(\cos^2\alpha\cos^2\beta V_1V_1^\dagger+\cos^2\alpha\sin^2\beta\Gamma_1\Gamma_1^\dagger+\sin^2\alpha\tilde{\Gamma}_1^\dagger\tilde{\Gamma}_1+ \\
&\quad \cos^2\alpha\cos\beta\sin\beta(V_1\Gamma_1^\dagger+\Gamma_1V_1^\dagger)+\cos\alpha\sin\alpha\cos\beta(V_1^\dagger\tilde{\Gamma}_1+\tilde{\Gamma}_1V_1^\dagger) \\
&\quad +\cos\alpha\sin\alpha\sin\beta(\Gamma_1\tilde{\Gamma}_1^\dagger+\tilde{\Gamma}_1\Gamma_1^\dagger))\exp(\frac{-2qn}{T}), \\
\rho_{33} &= \frac{1}{Z}\sum_{n=0}^{\infty}(\cos^2\alpha\cos^2\beta X_1X_1^\dagger+\cos^2\alpha\sin^2\beta\gamma_1\gamma_1^\dagger+\sin^2\alpha\tilde{\gamma}_1\tilde{\gamma}_1^\dagger+ \\
&\quad \cos^2\alpha\cos\beta\sin\beta(X_1X_1^\dagger+\gamma_1\gamma_1^\dagger)+\cos\alpha\sin\alpha\cos\beta(X_1\tilde{\gamma}_1^\dagger+\tilde{\gamma}_1X_1^\dagger) \\
&\quad +\cos\alpha\sin\alpha\sin\beta(\gamma_1\tilde{\gamma}_1^\dagger+\tilde{\gamma}_1\gamma_1^\dagger))\exp(\frac{-2qn}{T}), \\
\rho_{44} &= \frac{1}{Z}\sum_{n=0}^{\infty}(\cos^2\alpha\cos^2\beta Y_1Y_1^\dagger+\cos^2\alpha\sin^2\beta\delta_1\delta_1^\dagger+\sin^2\alpha\tilde{\delta}_1^\dagger\tilde{\delta}_1+ \\
&\quad \cos^2\alpha\cos\beta\sin\beta(Y_1\delta_1^\dagger+\delta_1Y_1^\dagger)+\cos\alpha\sin\alpha\cos\beta(Y_1\tilde{\delta}_1^\dagger+\tilde{\delta}_1Y_1^\dagger) \\
&\quad +\cos\alpha\sin\alpha\sin\beta(\delta_1\tilde{\delta}_1^\dagger+\tilde{\delta}_1\delta_1^\dagger))n\exp(\frac{-2qn}{T}), \\
\rho_{55} &= \frac{1}{Z}\sum_{n=0}^{\infty}(\cos^2\alpha\cos^2\beta G_1G_1^\dagger+\cos^2\alpha\sin^2\beta\lambda_1\lambda_1^\dagger+\sin^2\alpha\tilde{\lambda}_1\tilde{\lambda}_1^\dagger+ \\
&\quad \cos^2\alpha\cos\beta\sin\beta(G_1\lambda_1^\dagger+\lambda_1G_1^\dagger)+\cos\alpha\sin\alpha\cos\beta(G_1\tilde{\lambda}_1^\dagger+\tilde{\lambda}_1G_1^\dagger) \\
&\quad +\cos\alpha\sin\alpha\sin\beta(\lambda_1\tilde{\lambda}_1^\dagger+\tilde{\lambda}_1\lambda_1^\dagger))\exp(\frac{-2qn}{T}), \\
\rho_{66} &= \frac{1}{Z}\sum_{n=0}^{\infty}(\cos^2\alpha\cos^2\beta J_1J_1^\dagger+\cos^2\alpha\sin^2\beta\mu_1\mu_1^\dagger+\sin^2\alpha\tilde{\mu}_1\tilde{\mu}_1^\dagger+ \\
&\quad \cos^2\alpha\cos\beta\sin\beta(J_1\mu_1^\dagger+\mu_1J_1^\dagger)+\cos\alpha\sin\alpha\cos\beta(J_1\tilde{\mu}_1^\dagger+\tilde{\mu}_1J_1^\dagger) \\
&\quad +\cos\alpha\sin\alpha\sin\beta(\mu_1\tilde{\mu}_1^\dagger+\tilde{\mu}_1\mu_1^\dagger))n\exp(\frac{-2qn}{T}),
\end{aligned}$$

$$
\begin{aligned}
\rho_{77} \;=\;& \frac{1}{Z}\sum_{n=0}^{\infty}(\cos^2\alpha\cos^2\beta L_1L_1^{\dagger}+\cos^2\alpha\sin^2\beta\nu_1\nu_1^{\dagger}+\sin^2\alpha\tilde{\nu}_1\tilde{\nu}_1^{\dagger}+\\
& \cos^2\alpha\cos\beta\sin\beta(L_1\nu_1^{\dagger}+\nu_1L_1^{\dagger})+\cos\alpha\sin\alpha\cos\beta(L_1\tilde{\nu}_1^{\dagger}+\tilde{\nu}_1L_1^{\dagger})\\
& +\cos\alpha\sin\alpha\sin\beta(\nu_1\tilde{\nu}_1^{\dagger}+\tilde{\nu}_1\nu_1^{\dagger}))n\exp(\frac{-2qn}{T}),\\
\rho_{88} \;=\;& \frac{1}{Z}\sum_{n=0}^{\infty}(\cos^2\alpha\cos^2\beta F_1F_1^{\dagger}+\cos^2\alpha\sin^2\beta\eta_1\eta_1^{\dagger}+\sin^2\alpha\tilde{\eta}_1\tilde{\eta}_1^{\dagger}+\\
& \cos^2\alpha\cos\beta\sin\beta(F_1\eta_1^{\dagger}+\eta_1F_1^{\dagger})+\cos\alpha\sin\alpha\cos\beta(F_1\tilde{\eta}_1^{\dagger}+\tilde{\eta}_1F_1^{\dagger})\\
& +\cos\alpha\sin\alpha\sin\beta(\eta_1\tilde{\eta}_1^{\dagger}+\tilde{\eta}_1\eta_1^{\dagger}))n(n-1)\exp(\frac{-2qn}{T}),\\
\rho_{23} \;=\;& \frac{1}{Z}\sum_{n=0}^{\infty}(\cos^2\alpha\cos^2\beta V_1X_1^{\dagger}+\cos^2\alpha\sin^2\beta\Gamma_1\gamma_1^{\dagger}+\sin^2\alpha\tilde{\Gamma}_1\tilde{\gamma}_1^{\dagger}+\\
& \cos^2\alpha\cos\beta\sin\beta(V_1\gamma_1^{\dagger}+\Gamma_1X_1^{\dagger})+\cos\alpha\sin\alpha\cos\beta(V_1\tilde{\gamma}_1^{\dagger}+\tilde{\Gamma}_1X_1^{\dagger})\\
& +\cos\alpha\sin\alpha\sin\beta(\Gamma_1\tilde{\gamma}_1^{\dagger}+\tilde{\Gamma}_1\gamma_1^{\dagger}))\exp(\frac{-2qn}{T}),\\
\rho_{25} \;=\;& \frac{1}{Z}\sum_{n=0}^{\infty}(\cos^2\alpha\cos^2\beta V_1G_1^{\dagger}+\cos^2\alpha\sin^2\beta\Gamma_1\lambda_1^{\dagger}+\sin^2\alpha\tilde{\Gamma}_1\tilde{\lambda}_1^{\dagger}+\\
& \cos^2\alpha\cos\beta\sin\beta(V_1\lambda_1^{\dagger}+\Gamma_1G_1^{\dagger})+\cos\alpha\sin\alpha\cos\beta(V_1\tilde{\lambda}_1^{\dagger}+\tilde{\Gamma}_1G_1^{\dagger})\\
& +\cos\alpha\sin\alpha\sin\beta(\Gamma_1\tilde{\lambda}_1^{\dagger}+\tilde{\Gamma}_1\lambda_1^{\dagger}))\exp(\frac{-2qn}{T}),\\
\rho_{35} \;=\;& \frac{1}{Z}\sum_{n=0}^{\infty}(\cos^2\alpha\cos^2\beta X_1G_1^{\dagger}+\cos^2\alpha\sin^2\beta\gamma_1\lambda_1^{\dagger}+\sin^2\alpha\tilde{\gamma}_1\tilde{\lambda}_1^{\dagger}+\\
& \cos^2\alpha\cos\beta\sin\beta(X_1\lambda_1^{\dagger}+\gamma_1G_1^{\dagger})+\cos\alpha\sin\alpha\cos\beta(X_1\tilde{\lambda}_1^{\dagger}+\tilde{\gamma}_1G_1^{\dagger})\\
& +\cos\alpha\sin\alpha\sin\beta(\gamma_1\tilde{\lambda}_1^{\dagger}+\tilde{\gamma}_1\lambda_1^{\dagger}))\exp(\frac{-2qn}{T}),\\
\rho_{46} \;=\;& \frac{1}{Z}\sum_{n=0}^{\infty}(\cos^2\alpha\cos^2\beta Y_1J_1^{\dagger}+\cos^2\alpha\sin^2\beta\delta_1\mu_1^{\dagger}+\sin^2\alpha\tilde{\delta}_1\tilde{\mu}_1^{\dagger}+\\
& \cos^2\alpha\cos\beta\sin\beta(Y_1\mu_1^{\dagger}+\delta_1J_1^{\dagger})+\cos\alpha\sin\alpha\cos\beta(Y_1\tilde{\mu}_1^{\dagger}+\tilde{\delta}_1J_1^{\dagger})\\
& +\cos\alpha\sin\alpha\sin\beta(\delta_1\tilde{\mu}_1^{\dagger}+\tilde{\delta}_1\mu_1^{\dagger}))\exp(\frac{-2qn}{T}),\\
\rho_{47} \;=\;& \frac{1}{Z}\sum_{n=0}^{\infty}(\cos^2\alpha\cos^2\beta Y_1L_1^{\dagger}+\cos^2\alpha\sin^2\beta\delta_1\nu_1^{\dagger}+\sin^2\alpha\tilde{\delta}_1\tilde{\nu}_1^{\dagger}+\\
& \cos^2\alpha\cos\beta\sin\beta(Y_1\nu_1^{\dagger}+\delta_1L_1^{\dagger})+\cos\alpha\sin\alpha\cos\beta(Y_1\tilde{\nu}_1^{\dagger}+\tilde{P}_1L_1^{\dagger})\\
& +\cos\alpha\sin\alpha\sin\beta(\delta_1\tilde{\nu}_1^{\dagger}+\tilde{P}_1\nu_1^{\dagger}))\exp(\frac{-2qn}{T}),\\
\rho_{67} \;=\;& \frac{1}{Z}\sum_{n=0}^{\infty}(\cos^2\alpha\cos^2\beta J_1L_1^{\dagger}+\cos^2\alpha\sin^2\beta\mu_1\nu_1^{\dagger}+\sin^2\alpha\tilde{\mu}_1\tilde{\nu}_1^{\dagger}+\\
& \cos^2\alpha\cos\beta\sin\beta(J_1\nu_1^{\dagger}+\mu_1L_1^{\dagger})+\cos\alpha\sin\alpha\cos\beta(J_1\tilde{\nu}_1^{\dagger}+\tilde{\mu}_1L_1^{\dagger})\\
& +\cos\alpha\sin\alpha\sin\beta(\mu_1\tilde{\nu}_1^{\dagger}+\tilde{\mu}_1\nu_1^{\dagger}))\exp(\frac{-2qn}{T});
\end{aligned}
\tag{20}
$$

the unmentioned elements ρ_{ij} are equal zero.

Depending on the previous treatments, we calculate the three-particles' residual entanglement N_{abc}, the information entropy $H(\sigma_Z)$ and the linear entropy $P_P(t)$.

The ($\rho_{ij}(t) = Tr_k(\rho_P(t))$, i, j and k are equal a, b, c, $i \neq j \neq k$) and can be expressed on the basis $\{|1\rangle = |--\rangle, |2\rangle = |-+\rangle, |3\rangle = |+-\rangle, |4\rangle = |++\rangle\}$;

$$\begin{aligned}
\rho_{11} &= \frac{1}{Z}\sum_{n=0}^{\infty}\{\cos^2\alpha\cos^2\beta[U_1U_1^{\dagger}(n+1)+V_1V_1^{\dagger}]+\cos^2\alpha\sin^2\beta[\chi_1\chi_1^{\dagger}(n+1)\\
&\quad+\Gamma_1\Gamma_1^{\dagger}]+\sin^2\alpha[\tilde{\chi}_1\tilde{\chi}_1^{\dagger}(n+1)+\tilde{\Gamma}_1\tilde{\Gamma}_1^{\dagger}]+\{\cos^2\alpha\cos\beta\sin\beta\\
&\quad[U_1\chi_1^{\dagger}(n+1)+V_1\Gamma_1^{\dagger}]+\cos\alpha\sin\alpha\cos\beta[U_1\tilde{\chi}_1^{\dagger}(n+1)+V_1\tilde{\Gamma}_1^{\dagger}]\\
&\quad+\cos\alpha\sin\alpha\sin\beta[\chi_1\tilde{\chi}_1^{\dagger}(n+1)+\Gamma_1\tilde{\Gamma}_1^{\dagger}]+h.c\}\}\exp(\frac{-2qn}{T}),\\
\rho_{22} &= \frac{1}{Z}\sum_{n=0}^{\infty}\{\cos^2\alpha\cos^2\beta[X_1X_1^{\dagger}+Y_1Y_1^{\dagger}n]+\cos^2\alpha\sin^2\beta[\gamma_1\gamma_1^{\dagger}+\delta_1\delta_1^{\dagger}n]\\
&\quad+\sin^2\alpha[\tilde{\gamma}_1\tilde{\gamma}_1^{\dagger}+\tilde{\delta}_1\tilde{\delta}_1^{\dagger}n]+\{\cos^2\alpha\cos\beta\sin\beta[X_1\gamma_1^{\dagger}+Y_1\delta_1^{\dagger}n]+\\
&\quad\cos\alpha\sin\alpha\cos\beta[X_1\tilde{\gamma}_1^{\dagger}+Y_1\tilde{\delta}_1^{\dagger}n]+\cos\alpha\sin\alpha\sin\beta[\gamma_1\tilde{\gamma}_1^{\dagger}+\delta_1\tilde{\delta}_1^{\dagger}n]\\
&\quad+h.c\}\}\exp(\frac{-2qn}{T}),\\
\rho_{33} &= \frac{1}{Z}\sum_{n=0}^{\infty}\{\cos^2\alpha\cos^2\beta[G_1G_1^{\dagger}+J_1J_1^{\dagger}n]+\cos^2\alpha\sin^2\beta[\lambda_1\lambda_1^{\dagger}+\mu_1\mu_1^{\dagger}n]\\
&\quad+\sin^2\alpha[\tilde{\lambda}_1\tilde{\lambda}_1^{\dagger}+\tilde{\mu}_1\tilde{\mu}_1^{\dagger}n]+\{\cos^2\alpha\cos\beta\sin\beta[G_1\lambda_1^{\dagger}+J_1\mu_1^{\dagger}n]\\
&\quad+\cos\alpha\sin\alpha\cos\beta[G_1\tilde{\lambda}_1^{\dagger}+J_1\tilde{\mu}_1^{\dagger}n]+\cos\alpha\sin\alpha\sin\beta[\lambda_1\tilde{\lambda}_1^{\dagger}+\mu_1\tilde{\mu}_1^{\dagger}n]\\
&\quad+h.c\}\}\exp(\frac{-2qn}{T}),\\
\rho_{44} &= \frac{1}{Z}\sum_{n=0}^{\infty}\{\cos^2\alpha\cos^2\beta[L_1L_1^{\dagger}+F_1F_1^{\dagger}(n-1)]n+\cos^2\alpha\sin^2\beta[\nu_1\nu_1^{\dagger}\\
&\quad+\eta_1\eta_1^{\dagger}(n-1)]n+\sin^2\alpha[\tilde{\nu}_1\tilde{\nu}_1^{\dagger}+\tilde{\eta}_1\tilde{\eta}_1^{\dagger}(n-1)]n+\{\cos^2\alpha\cos\beta\sin\beta\\
&\quad[L_1\nu_1^{\dagger}+F_1\eta_1^{\dagger}(n-1)]n+\cos\alpha\sin\alpha\cos\beta[L_1\tilde{\nu}_1^{\dagger}+F_1\tilde{\eta}_1^{\dagger}(n-1)]n\\
&\quad+\cos\alpha\sin\alpha\sin\beta[\nu_1\tilde{\nu}_1^{\dagger}+\eta_1\tilde{\eta}_1^{\dagger}(n-1)]n+h.c\}\}\exp(\frac{-2qn}{T}),
\end{aligned}$$

$$\begin{aligned}
\rho_{23} &= \frac{1}{Z}\sum_{n=0}^{\infty}\{\cos^2\alpha\cos^2\beta[X_1G_1^{\dagger}+Y_1J_1^{\dagger}n]+\cos^2\alpha\sin^2\beta[\gamma_1\lambda_1^{\dagger}+\delta_1\mu_1^{\dagger}n]\\
&\quad+\sin^2\alpha[\tilde{\gamma}_1\tilde{\lambda}_1^{\dagger}+\tilde{\delta}_1\tilde{\mu}_1^{\dagger}n]+\{\cos^2\alpha\cos\beta\sin\beta[\gamma_1G_1^{\dagger}+\delta_1J_1^{\dagger}n]\\
&\quad+\cos\alpha\sin\alpha\cos\beta[\tilde{\gamma}_1G_1^{\dagger}+\tilde{\delta}_1J_1^{\dagger}n]+\cos\alpha\sin\alpha\sin\beta[\tilde{\gamma}_1\lambda_1^{\dagger}+\tilde{\delta}_1\mu_1^{\dagger}n]\\
&\quad+h.c\}\}\exp(\frac{-2qn}{T}),\quad \rho_{ij}=0 \text{ in case } i\neq j\neq 2 \text{ or } \neq 3.
\end{aligned} \tag{21}$$

Also here, through the previous calculations of the two-qubit system, we calculate the concurrence $C(\rho)$, the information entropy $H(\sigma_Z)$ and the linear entropy $P_P(t)$.

3. Results and Discussion

Since we calculated the three-particles residual entanglement N_{abc}, the concurrence $C(\rho)$, the information entropy $H(\sigma_Z)$ and the linear entropy $P_P(t)$ as applications to the previous solutions we reached, we now review the definitions and laws of these applications.

4. Three-Particle Residual Entanglement

The three-particle residual entanglement N_{abc} can be defined as the following [39]:

$$N_{abc} = N_{a-bc}(\rho_{abc}) - N_{a-b}(\rho_{ab}) - N_{a-c}(\rho_{ac}), \tag{22}$$

where,

$$\begin{aligned} N_{a-bc}(\rho_{abc}) &= \left\| \rho_{abc}^{T_a} \right\|_1 - 1, \\ N_{a-b}(\rho_{ab}) &= \left\| \rho_{ab}^{T_a} \right\|_1 - 1 = \left\| \rho_{ab}^{T_b} \right\|_1 - 1, \\ N_{a-c}(\rho_{ac}) &= \left\| \rho_{ac}^{T_a} \right\|_1 - 1 = \left\| \rho_{ab}^{T_c} \right\|_1 - 1. \end{aligned} \tag{23}$$

The term $N_{a-bc}(\rho_{abc})$ quantifies the strengths of quantum correlations between the atom "a" and the other two atoms. The term $N_{a-b}(\rho_{ab})$ $(N_{a-c}(\rho_{ac}))$ quantifies the pairwise entanglement between the atom "a" and "b" ("c").

And,

$$\left\| \rho^{T} \right\|_1 = (\sum_i |\mu_i| - \sum_i \mu_i) + 1, \tag{24}$$

where ρ^{T_A} is the fractional transpose of a state ρ according to subsystem A, and $\|.\|_1$ is the trace norm; $\sum\limits_i |\mu_i|$ is the summation of the absolute value for each eigenvalue of ρ^{T_A}.

5. The Information Entropy and the Linear Entropy

The information entropy $H(\sigma_Z)$ of the atomic operator σ_z can be written as follows [40,41]:

$$H(\sigma_Z) = -\sum_{j=1}^{\varepsilon} P_j(\sigma_Z) \ln P_j(\sigma_Z), \tag{25}$$

where the probability distribution $P_j(\sigma_Z)$ for ε probable outcomes of measurements for a random quantum state of an atomic operator σ_Z is

$$P_j(\sigma_Z) = \left\langle \Phi_{Zj} \right| \rho \left| \Phi_{Zj} \right\rangle, \tag{26}$$

where ρ represents the density matrix of the total quantum system and $\left| \Phi_{Zj} \right\rangle$ eigenvector of the atomic operator σ_Z:

$$\sigma_Z \left| \Psi_{Zj} \right\rangle = v_{Zj} \left| \Psi_{Zj} \right\rangle, \; j = 1, 2, ..., \varepsilon, \tag{27}$$

where v_{Zj} is the eigenvalue of the atomic operator σ_β shown in Equation (27).

For a two-qubit, $\epsilon = 4$, but in the case of the three-qubit $\varepsilon = 8$.

The evolution of the linear entropy $P_P(t)$ is given by

$$P_P(t) = 1 - Tr_P(\rho_P^2(t)), \tag{28}$$

where $\rho_P(t) = Tr_Q\, \rho(t)$ denotes the reduced density matrix for bipartite system.

6. Concurrence

In the case of the proposed system, which consists of two quantum bits, the concurrence of the system can be calculated as proposed in [42] to calculate the quantum correlation. Firstly, in the pure state case, we can write the concurrence is as follows:$C(\rho) = \max\{0, \sqrt{\xi_1} - \sqrt{\xi_2} - \sqrt{\xi_3} - \sqrt{\xi_4}\}$, where ξ_k are the eigenvalues of the matrix $\varrho = \rho\tilde{\rho} = \rho(\sigma_y^1 \otimes \sigma_y^2)\rho^{\intercal}(\sigma_y^1 \otimes \sigma_y^2)$, an arrangement in descending order. Furthermore, the transposition of density matrix ρ is denoted by $\rho^{\intercal}$. It is obvious that the value of concurrence is bounded between zero and one.

We are now explaining the results we have been able to deduce through the figures we have made. We discuss the effect of changing parameters on the applications we are studying. In Figure 1, we discuss the case of a system that consists of three quantum bits under the influence of a quantum spin environment. We study the effects of the temperature T on the behavior of the three-particle residual entanglement N_{abc}, the information entropy $H(\sigma_Z)$ and the linear entropy $P_P(t)$ in case of $\epsilon_0 = 2$, $q = q_0 = 1$, $\omega = 0$, $\alpha = 0$ and $\beta = \frac{\pi}{2}$. We study this case at $T = 0.1$, $T = 1$ and $T = 3$. The curves of entanglement N_{abc} at $T = 0.1$ and $T = 1$ have regular and periodic oscillations while at $T = 3$, they lose their regularity. The maximum values of the entanglement decrease with increasing temperature T, while the minimum value remains unchanged. In the case of the information entropy $H(\sigma_Z)$ and the linear entropy $P_P(t)$, we also notice the regularity of the curves and their periodicity, and then they are irregular at $T = 3$. But it is clear here that both the maximum and the minimum value of the information entropy $H(\sigma_Z)$ and the linear entropy $P_P(t)$ increases with increasing temperature T. That is, we note that by increasing the temperature the purity of the atoms and the degree of entanglement between the atoms and some decrease. So it is normal and expected that the increase will occur for the entropy. In Figures 2 and 3, we discuss the case of a two-qubit system, where we get to the case of the two atoms by taking the trace of the density matrix operator ρ_P, ($\rho_{ij}(t) = Tr_k(\rho_P(t))$; i, j and k are equal; a, b, c, $i \neq j \neq k$, $\rho_P = \rho_{abc}$). In Figure 2, we study the influence of the temperature T on the behavior of the concurrence $C(\rho)$, the information entropy $H(\sigma_Z)$ and the linear entropy $P_P(t)$ in case of $\epsilon_0 = 2$, $q = q_0 = 1$, $\omega = 0$, $\alpha = 0$ and $\beta = \frac{\pi}{2}$. We study this case at $T = 0.1$, $T = 1$ and $T = 2$. At $T = 0.1$ and $T = 1$, the curves of the concurrence $C(\rho)$ are almost regular, but at $T = 2$, the regularity of curves is slightly decreased. The maximum values of the concurrence $C(\rho)$ decrease from $C(\rho) = 0.7$ to $C(\rho) = 0.3$ and the minimum value keeps its stability. For the information entropy $H(\sigma_Z)$ and the linear entropy $P_P(t)$, at $T = 0.1$ the curves are regular, and then the regularity of the curves gradually decreases by increasing the temperature T to $T = 1$ and then to $T = 2$. Both the maximum and minimum values increase significantly with increasing temperature T. We also notice an increase in the number of peaks for the linear entropy $P_P(t)$. In Figure 3, we study the influence of the coupling constant ϵ_0 between the spin- qubit system, and an external magnetic field along the Z direction on the behavior of the concurrence $C(\rho)$, the information entropy $H(\sigma_Z)$ and the linear entropy $P_P(t)$ in case of $T = 1$, $q = q_0 = 1$, $\omega = 0$, $\alpha = 0$ and $\beta = \frac{\pi}{2}$. We study this case at $\epsilon_0 = 2$, $\epsilon_0 = 4$ and $\epsilon_0 = 6$. We observe the state of regularity and periodicity of all curves at any value for ϵ_0. By increasing the coupling constant ϵ_0, we notice that the number of oscillations of the concurrence $C(\rho)$ increases significantly; i.e., the interconnections between the atoms increase, and the numbers of oscillations of both the information entropy $H(\sigma_Z)$ and the linear entropy $P_P(t)$ are also significantly reduced. The maximum values of the concurrence $C(\rho)$ increase but the minimum value tends to persist. The increase in the maximum value for the information entropy $H(\sigma_Z)$ and the linear entropy $P_P(t)$ is minor. Additionally, the number of peaks of the linear entropy $P_P(t)$ in this case increases.

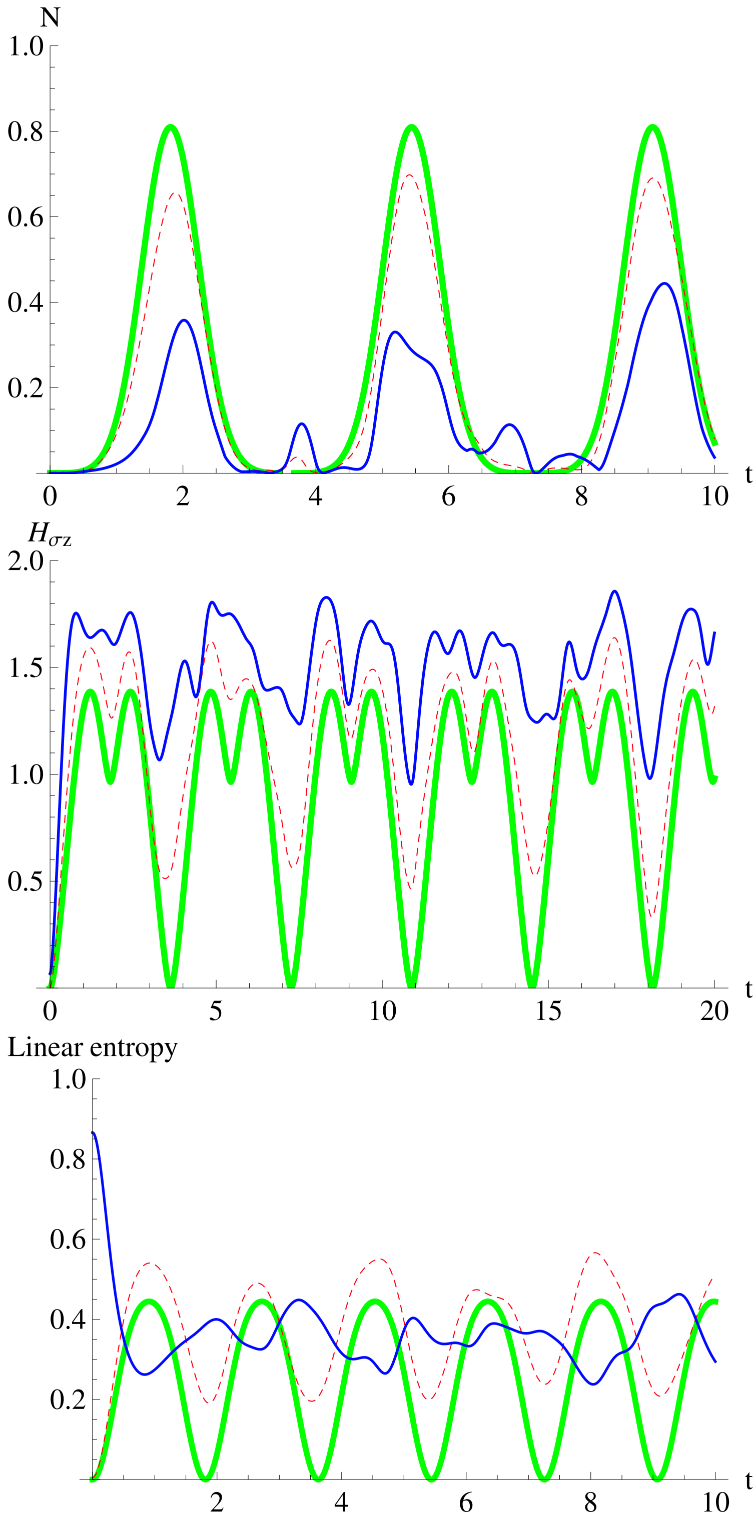

Figure 1. The time evolution of the entanglement N_{abc}, the information entropy $H(\sigma_Z)$ and the linear entropy $P_P(t)$ in the case of three-qubit for parameters $\epsilon_0 = 2$, $\omega = 0$, $q = q_0 = 1$, $\alpha = 0$ and $\beta = \frac{\pi}{2}$. Where solid green, red dots and blue curves correspond, respectively, to $T = 0.1$, 1 and 3.

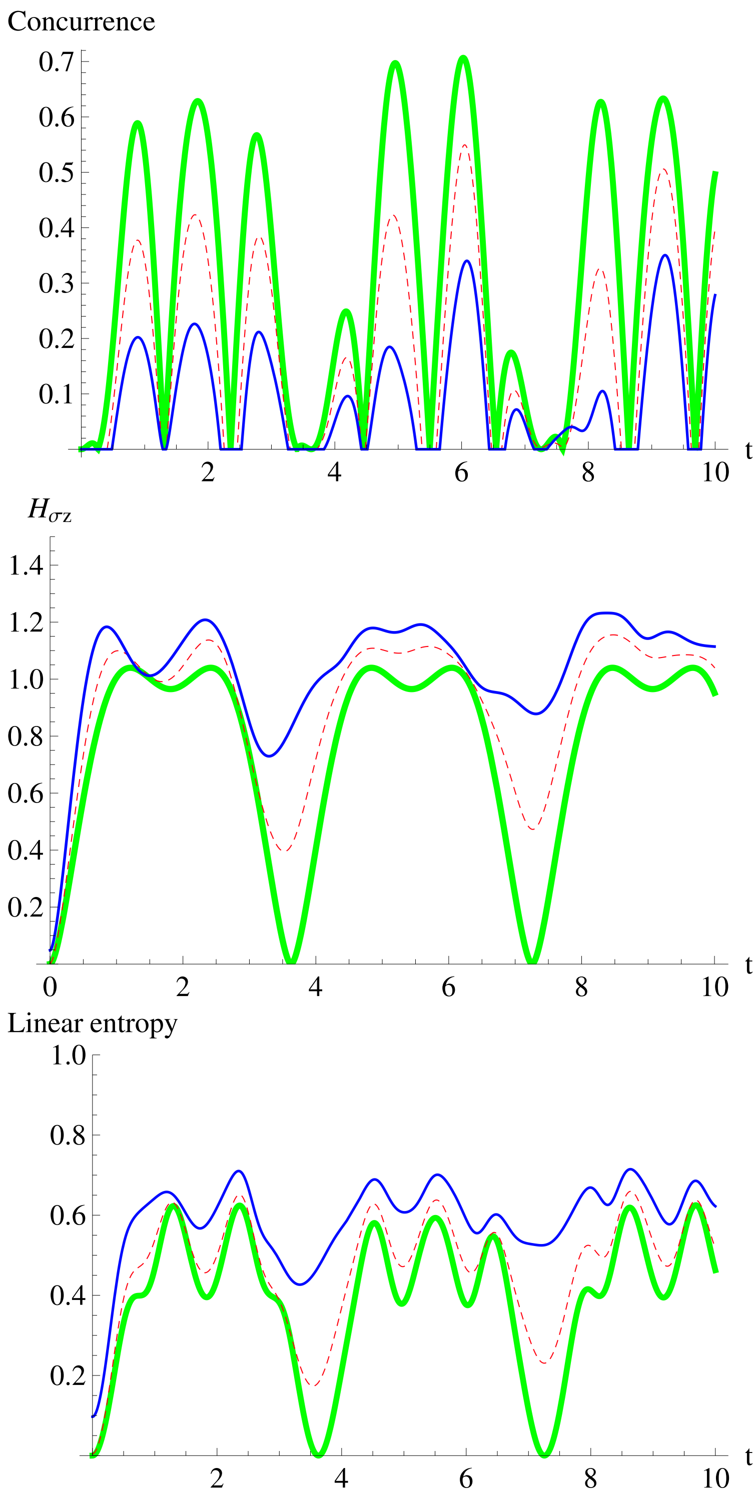

Figure 2. The time evolution of the concurrence $C(\rho)$, the information entropy $H(\sigma_Z)$ and the linear entropy $P_P(t)$ in the case of two-qubit for parameters $\epsilon_0 = 2$, $\omega = 0$, $q = q_0 = 1$, $\alpha = 0$ and $\beta = \frac{\pi}{2}$. Solid green, red dots and blue curves correspond, respectively, to $T = 0.1$, 1 and 2.

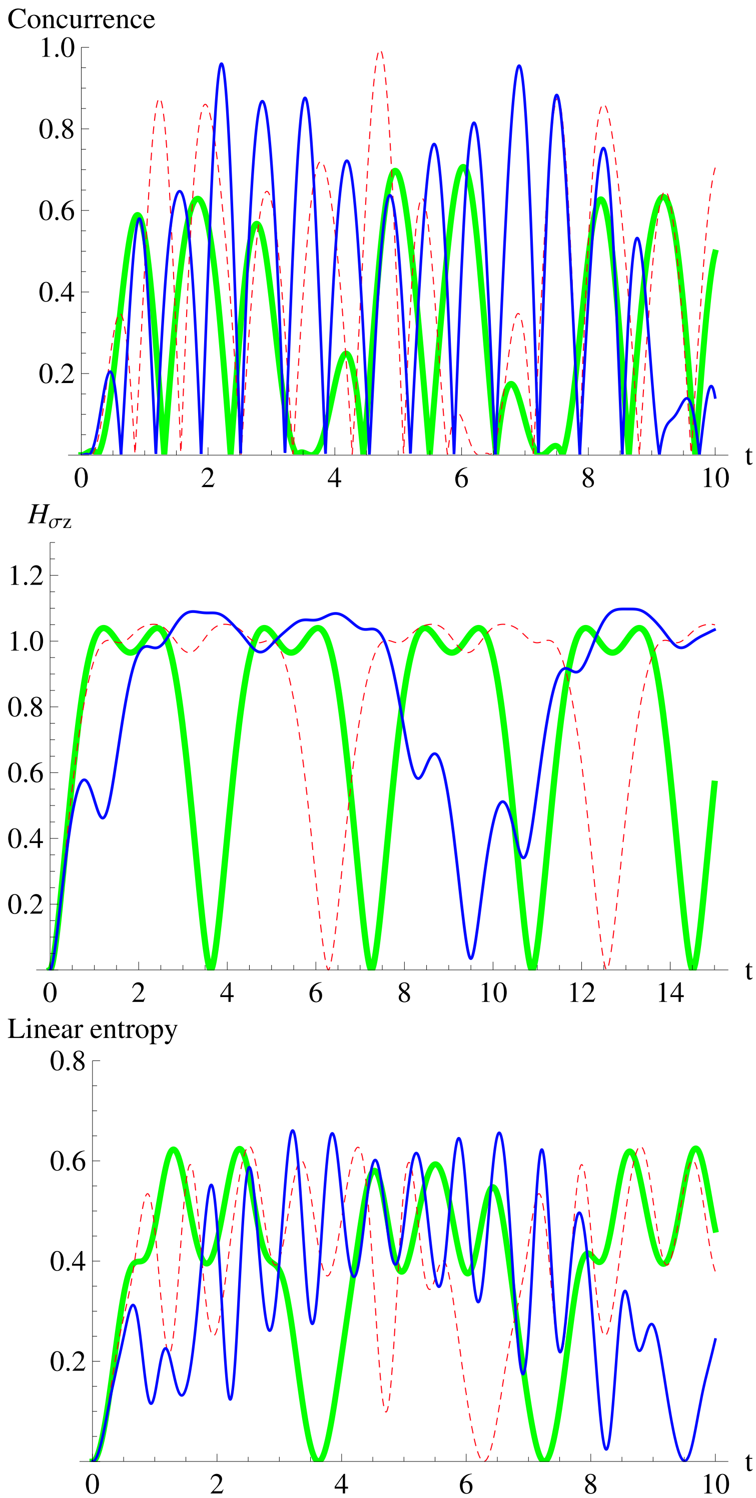

Figure 3. The time evolution of the concurrence $C(\rho)$, the information entropy $H(\sigma_Z)$ and the linear entropy $P_P(t)$ in the case of two-qubit for parameters $T = 2$, $\omega = 0$, $q = q_0 = 1$, $\alpha = 0$ and $\beta = \frac{\pi}{2}$, where solid green, red dots and blue curves correspond, respectively, to $\epsilon_0 = 2$, 4 and 6.

From the above-mentioned discussion, we can observe that oscillations in the results always take indefinite forms. That is because we have chosen an initial entangled state, and this, as we observed, reduces the effect of the surrounding environment, in addition to changing the values of parameters which we control. Thus, the oscillations would continue to take the indefinite appearance, rather than a steady state, in the figures.

7. Conclusions

We analytically solved a quantum system composed of three qubits characterized by a finite temperature within a thermodynamic limit influenced by a quantum spin environment. The case of the two-qubit was obtained by deriving the trace for the density matrix. We studied the effect of the temperature T and the coupling constant ϵ_0 between the spin-qubit system and an external magnetic field on the three-particle residual entanglement N_{abc}, the concurrence $C(\rho)$, the information entropy $H(\sigma_Z)$ and the linear entropy $P_P(t)$. Our work is considered to be an extension of a previous study [32], yet with much larger calculations to reach an exact solution of a non-Markovian case of three-qubit quantum system by using a novel operator technique. We observed that all curves, whether in the case of three-qubit or two-qubit, are regular and periodic at low temperatures T, but at high temperatures T tend to be irregular. We also noticed a significant change in the number of oscillations, increasing or decreasing, for the concurrence $C(\rho)$, the information entropy $H(\sigma_Z)$ and the linear entropy $P_P(t)$ when the coupling constant ϵ_0 changed. Also note the strong relationship between the entanglement, N_{abc} and $C(\rho)$, and the entropy, $H(\sigma_Z)$ and $P_P(t)$. When we saw an increase in the entanglement between the atoms, there was a marked decrease in entropy and vice versa. From there, we proved an inverse relationship between the entanglement and the entropy, that we controlled the degree of entanglement, and subsequently, the degree of entropy by controlling the temperature T and the coupling constant ϵ_0.

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