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Performance Analysis and Parameter Optimization of the Optimal Fixed-Point Quantum Search

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Abstract: The optimal fixed-point quantum search (OFPQS) algorithm [Phys. Rev. Lett. 113, 210501 (2014)] achieves both the fixed-point property and quadratic speedup over classical algorithms, which gives a sufficient condition on the number of iterations to ensure the success probability is no less than a given lower bound (denoted by $1 - \delta^2$). However, this condition is approximate and not exact. In this paper, we derive the sufficient and necessary condition on the number of feasible iterations, based on which the exact least number of iterations can be obtained. For example, when $\delta = 0.8$, iterations can be saved by almost 25%. Moreover, to find a target item certainly, setting directly $1 - \delta^2 = 100\%$, the quadratic advantage of the OFPQS algorithm will be lost, then, applying the OFPQS algorithm with $1 - \delta^2 < 100\%$ requires multiple executions, which leads to a natural problem of choosing the optimal parameter δ . For this, we analyze the extreme and minimum properties of the success probability and further analytically derive the optimal δ which minimizes the query complexity. Our study can be a guideline for both the theoretical and application research on the fixed-point quantum search algorithms.

Keywords: fixed-point quantum search; quadratic speedup; success probability

1. Introduction

Grover search [1,2] provides a quadratic speedup over classical search algorithms, and has been proven optimal [3–6]. However, there still exists the soufflé problem [7], i.e., the success probability will decline if the algorithm iterates too many times. Therefore, the Grover algorithm can only apply to the case where the optimal number of iterations [8] can be determined.

Based on the quantum amplitude amplification [9–12] and phase-matching methods [13–16], a fixed-point quantum search algorithm [17] has been proposed, where the final state of the algorithm converges to the target states and the success probability increases as the number of iterations grows. This algorithm applies to the case where the lower bound (denoted by λ_0) of the fraction of target items (denoted by λ) is known. However, the advantage of quadratic speedup is lost [18,19].

Fortunately, Yoder et al. developed the optimal fixed-point quantum search (OFPQS) algorithm [20], which achieves both the fixed-point property and optimal query complexity, where the success probability for any unknown $\lambda \geq \lambda_0$ can be always no less than a given value (denoted by $1 - \delta^2$) between 0 and 1, as long as the given condition on the number of iterations is satisfied. However, this condition is approximate and not tight, i.e., the required number of iterations might be further reduced.

In addition, the lower bound of success probability $1 - \delta^2$ of the OFPQS algorithm is lower than 100%, because setting $\delta = 0$, the algorithm will change back to the original fixed-point

algorithm [17] and thus lose the quadratic speedup [20]. However, in practical search problems, it is often desired to find a target item eventually, rather than just succeed with probability above a lower bound. For this, a natural strategy is to make multiple trials of the OFPQS algorithm with $\delta > 0$ until success. This brings up a new problem, that is, how to choose the optimal parameters to find a target item as soon as possible.

In this paper, we expect to give the minimum feasible number of iterations, analyze the extreme and minimum properties of the success probability, and further derive the optimal parameter of the OFPQS algorithm to find a target item with the minimum query complexity.

The paper is organized as follows: In Section 2, we briefly introduce the OFPQS algorithm. In Section 3, we derive the optimal number of iterations and analyze the properties of the success probability of OFPQS algorithm. In Section 4, we give the selection method of optimal parameter δ . Section 5 comprises of discussions about the effects of the optimal number of iterations and the optimal δ , as well as the upper bound of expected queries of applying the OFPQS algorithm to find a target item certainly. Finally, a brief conclusion is given in Section 6.

2. OFPQS Algorithm Revisited

Based on the multi-phase matching method [21,22], the OFPQS algorithm [20] overcomes the soufflé problem in the original Grover algorithm [1] as well as the loss of quadratic speedup in the original fixed-point quantum search algorithm [17].

The initial state of the OFPQS algorithm is prepared to be $|\psi\rangle = A|0\rangle$, where A can be an arbitrary unitary operator. Denote the equal superposition of all target and nontarget states as $|\alpha\rangle$ and $|\beta\rangle$, respectively, i.e.,

$$|\alpha\rangle = \frac{1}{\sqrt{M}} \sum_{x \in f^{-1}(1)} |x\rangle, \quad (1)$$

$$|\beta\rangle = \frac{1}{\sqrt{N-M}} \sum_{x \in f^{-1}(0)} |x\rangle, \quad (2)$$

where M is the number of target items in the database of size N , $f(x)$ is a Boolean function identifying the target state(s), i.e., if $|x\rangle$ is a target state, then $f(x) = 1$ —otherwise, $f(x) = 0$, with $A = H^{\otimes n}$ and H being the Hadamard transform, $|\psi\rangle$ can be written as

$$|\psi\rangle = \sqrt{\lambda} |\alpha\rangle + \sqrt{1-\lambda} |\beta\rangle, \quad (3)$$

where $\lambda = M/N$ represents the fraction of target items.

The sequence of operations performed on $|\psi\rangle$ is given by (see also Equation (2.6) of [21])

$$S_L = G(\phi_L, \varphi_L) G(\phi_{L-1}, \varphi_{L-1}) \cdots G(\phi_1, \varphi_1), \quad (4)$$

where $L-1 = 2l$ represents the query complexity of sequence S_L , as each generalized Grover operation $G(\phi_j, \varphi_j)$ requires two Oracle queries [20], and

$$G(\phi_j, \varphi_j) = -AS_0^{\phi_j} A^\dagger S_f^{\varphi_j}, \quad (5)$$

where $S_0^{\phi_j}$ and $S_f^{\varphi_j}$ are the selective phase shifts ($i = \sqrt{-1}$) (see also Equations (1) and (2) of [13]),

$$S_0^{\phi_j} = I - (1 - e^{i\phi_j}) |0\rangle \langle 0|, \quad (6)$$

$$S_f^{\varphi_j} = I - (1 - e^{i\varphi_j}) \sum_{x \in f^{-1}(1)} |x\rangle \langle x|. \quad (7)$$

Under the multiphase matching condition (see also Equation (11) of [20])

$$\phi_j = \varphi_{l-j+1} = -2 \cot^{-1} \left(\tan(2\pi j/L) \sqrt{1-\gamma^2} \right), \quad 1 \leq j \leq l, \quad (8)$$

the final state can be obtained as

$$S_L |\psi\rangle = \sqrt{P_L} |\alpha\rangle + \sqrt{1-P_L} |\beta\rangle, \quad (9)$$

where $\gamma = T_{1/L}^{-1}(1/\delta)$, P_L is the success probability of the algorithm (see also Equation (1) of [20]),

$$P_L = 1 - \delta^2 T_L^2 \left[T_{1/L}(1/\delta) \sqrt{1-\lambda} \right], \quad (10)$$

and $T_L(x)$ is the L th Chebyshev polynomial of the first kind [23],

$$T_L(x) = \begin{cases} \cos(L \arccos(x)), & \text{if } |x| \leq 1, \\ \cosh(L \operatorname{arccosh}(x)), & \text{if } x \geq 1, \\ (-1)^L \cosh(L \operatorname{arccosh}(-x)), & \text{if } x \leq -1. \end{cases} \quad (11)$$

To ensure the success probability $P_L(\lambda)$ no less than a given lower bound $1 - \delta^2$ for any $\delta \in (0, 1]$, an approximated condition of L is given as

$$L \geq \frac{\ln(2/\delta)}{\sqrt{\lambda}}, \quad (12)$$

which is sufficient but not necessary, as shown below.

3. Performance Analysis of the OFPQS Algorithm

After deep theoretical analysis, we can obtain three new properties on the number of iterations and success probability of the OFPQS algorithm, which are as follows.

Property 1. Given a lower bound λ_0 of the fraction λ of target items, the least exact number of iterations l (denoted by l_{opt}) of the OFPQS algorithm that enables the success probability $P_L(\lambda) \geq 1 - \delta^2$ for any $\lambda \geq \lambda_0$ with $L = 2l + 1$, can be given in the form

$$l_{opt} = \left\lceil \frac{\operatorname{arccosh}(1/\delta)}{2 \operatorname{arccosh}(1/\sqrt{1-\lambda_0})} - \frac{1}{2} \right\rceil. \quad (13)$$

Proof. Based on Equation (10), for arbitrary $\lambda \in (0, 1)$ and $\delta \in (0, 1]$, a necessary and sufficient condition to make the success probability $P_L(\lambda) \geq 1 - \delta^2$ can be given as follows,

$$|T_{1/L}(1/\delta)| \sqrt{1-\lambda} \leq 1, \quad (14)$$

due to the fact that

$$|T_L(x)| \leq 1 \Leftrightarrow |x| \leq 1. \quad (15)$$

Note that if $|x| > 1$, then, $\operatorname{arccosh}(|x|) > 0$ and $|T_L(x)| = \cosh(L \operatorname{arccosh}(|x|)) > 1$; if $|x| \leq 1$, then, $|T_L(x)| = |\cos(L \arccos(x))| \leq 1$. Due to $T_{1/L}(1/\delta) \geq 1$ and $\cosh(x)$ monotonically increases for $x \geq 0$, Equation (14) can be further reduced to

$$L \geq \frac{\operatorname{arccosh}(1/\delta)}{\operatorname{arccosh}(1/\sqrt{1-\lambda})}. \quad (16)$$

Then, according to $L = 2l + 1$, we have

$$l \geq \left\lceil \frac{\operatorname{arcosh}(1/\delta)}{2\operatorname{arcosh}(1/\sqrt{1-\lambda})} - \frac{1}{2} \right\rceil. \quad (17)$$

Therefore, to ensure that $P_L(\lambda) \geq 1 - \delta^2$ for any $\lambda \geq \lambda_0$, the least number of iterations, expressed by Equation (13), can be finally obtained. $\square$

Property 2. The extreme properties of the success probability $P_L(\lambda)$ as a function of $\lambda \in (0, 1)$ can be given as follows (see Proof in Appendix A):

For $\delta \in (0, 1]$, when $L > 1$, $P_L(\lambda)$ has $(L - 1)/2 \equiv l$ local maximum points

$$\lambda_{\max,j} = 1 - \gamma^2 \cos^2 \left(\frac{2j-1}{2L} \pi \right), j = 1, 2, \dots, l, \quad (18)$$

and l local minimum points

$$\lambda_{\min,j} = 1 - \gamma^2 \cos^2 \left(\frac{j}{L} \pi \right), j = 1, 2, \dots, l. \quad (19)$$

When $L = 1$, there are no local extreme points.

Property 3. The success probability $P_L(\lambda)$ with $L = 2l_{\text{opt}} + 1 \equiv L_{\text{opt}}$ has a minimum value (denoted by $P_{\min}$) on the range $\lambda \in [\lambda_0, 1)$ for $\lambda_0 \leq 4/5$, which can be written as (for a proof, see Appendix B)

$$\begin{aligned} P_{\min} &\equiv \min \{ P_{L_{\text{opt}}}(\lambda) : \lambda \geq \lambda_0 \} \\ &= \begin{cases} 1 - \delta^2, & \text{if } \delta < \sqrt{1 - \lambda_0}, \\ \lambda_0, & \text{if } \delta \geq \sqrt{1 - \lambda_0}. \end{cases} \end{aligned} \quad (20)$$

Note that the case $\lambda_0 > 4/5$ can already be well disposed of by classical search.

4. Optimization of Parameters of the OFPQS Algorithm

In practical applications of the OFPQS algorithm, if set $\delta = 0$, then, from Equation (12) or Equation (13) it can be found that the quadratic speedup over classical algorithms will be lost; while, if set $\delta > 0$, the output of the OFPQS algorithm is not necessarily a target item. Inspired by [4], which achieves about 12% reduction of the expected queries through stopping the Grover algorithm short of the optimal number of iterations and restarting again in case of failure, to find a target item as soon as possible, a natural strategy is to set the lower bound of success probability $1 - \delta^2 < 100\%$ and repeat the OFPQS algorithm until it succeeds. Under this strategy, in order to find the optimal parameter δ , we shall estimate the expected number of Oracle queries of the OFPQS algorithm before a target item is found.

First, a single execution of the OFPQS algorithm requires at least $2l_{\text{opt}}$ queries, since l_{opt} iterations are required in the sequence S_L of Equation (4). Second, judging whether the algorithm is successful from the measurement result also takes one query. Then, define $L_{\text{opt}} = 2l_{\text{opt}} + 1$, we can get the expected number of Oracle queries as

$$L_E = \sum_{j=1}^{\infty} p_j L_{\text{opt}}, \quad (21)$$

where

$$p_j = \left[1 - P_{L_{\text{opt}}}(\lambda) \right]^{j-1} \quad (22)$$

is the probability of occurrence of the j -th execution of the OFPQS algorithm. From Equations (20)–(22), we can further obtain that

$$\begin{aligned} L_E &\leq \sum_{j=1}^{\infty} (1 - P_{min})^{j-1} L_{opt} \\ &= (2l_{opt} + 1) / P_{min} \equiv L_E^{Up}, \end{aligned} \quad (23)$$

where L_E^{Up} represents the upper bound of L_E .

We can define the optimal δ as the one that makes the upper bound of expected number of queries L_E^{Up} as few as possible. Detailed analysis shows that such optimal δ (denoted by δ_{opt}) exists and can be analytically written in the following form (Proof see Appendix C),

$$\delta_{opt} = \begin{cases} \delta_k, & \text{if } \lambda_0 \in [\lambda_{0,k}, \lambda_{0,k-1}], k \geq 1, \\ \forall \delta \in [\delta_0, 1), & \text{if } \lambda_0 \in [\lambda_{0,0}, 4/5], \end{cases} \quad (24)$$

where

$$\delta_k = T_{2k+1}^{-1} \left(1 / \sqrt{1 - \lambda_0} \right), \quad (25)$$

$T_L(x)$ is defined by Equation (11),

$$\lambda_{0,k-1} = 1 - \cosh^{-2}(y_{k-1}), k \geq 1, \quad (26)$$

y_{k-1} is the unique solution of equation

$$\sinh^2(4ky) + \sinh^2(2y) - 4k \sinh(4ky) \sinh(2y) = 0, \frac{x}{4k+2} < y < \frac{x}{4k-2}, \quad (27)$$

and x satisfies

$$\sinh(x) - 2x = 0, x > 0. \quad (28)$$

Note that, when $\lambda_0 = \lambda_{0,k}$ ($k \geq 0$) or $\lambda_{0,0} \leq \lambda_0 \leq 4/5$, the corresponding δ_{opt} has multiple values, as shown in Figure 1.

Figure 1. (Color online.) The optimal parameter δ_{opt} as a function of the lower bound λ_0 of the fraction of target items. The red solid curves, red dotted vertical lines and the yellow (gray) area represent δ_k , $\lambda_{0,k}$ ($k \geq 0$), and the value of δ_{opt} for $\lambda_{0,0} \leq \lambda_0 \leq 4/5$, respectively.

Note that, with the above optimal parameters, we can obtain the corresponding upper bound of the expected number of queries, i.e.,

$$L_{E,opt}^{Up}(\lambda_0) \equiv L_E^{Up}(\lambda_0, l_{opt}, \delta_{opt}). \quad (29)$$

5. Discussion

In this section, the effects of optimal parameters l_{opt} of Equation (13) and δ_{opt} of Equation (24), and the complexity of $L_{E,opt}^{Up}(\lambda_0)$ of Equation (29) are discussed as follows.

5.1. Effects of l_{opt} and δ_{opt}

Different from our optimal number of iterations l_{opt} of Equation (13), to achieve a success probability no less than $1 - \delta^2$ for any $\lambda \geq \lambda_0$ in the OFPQS algorithm [20], another number of iterations can be obtained from Equation (12), denoted by

$$l_{min}^{Yoder's} \equiv \left\lceil \frac{\ln(2/\delta)}{2\sqrt{\lambda_0}} - \frac{1}{2} \right\rceil. \quad (30)$$

As a comparison, Figure 2 shows some examples of the success probability $P_L(\lambda)$ for the different number of iterations l with $L = 2l + 1$. We can see that both $l = l_{min}^{Yoder's}$ and $l = l_{opt}$ can achieve the goal $P_L(\lambda) \geq 1 - \delta^2$ for $\lambda \geq \lambda_0$; while $l = l_{opt} - 1$ can't. Therefore, l_{opt} is just the least number of iterations required. When $\lambda_0 \ll 1$, $\text{arcosh}(1/\sqrt{1-\lambda_0}) \approx \sqrt{\lambda_0}$, then, from Equations (13) and (30) it follows that

$$l_{opt}/l_{min}^{Yoder's} \approx \frac{\text{arcosh}(1/\delta)}{\ln(2/\delta)}, \quad (31)$$

which is shown in the inset of Figure 2.

Figure 2. (Color online.) The success probability P_L versus the fraction of target items λ for different number of iterations l with $L = 2l + 1$, $\lambda \geq \lambda_0 = 0.125$ (dotted vertical line) and $P_L \geq 1 - \delta^2 = 0.9$ (dotted horizontal line). The red dashed-dotted, blue solid, and green dashed curves correspond to $l = l_{min}^{Yoder's} = 3$, $l = l_{opt} = 2$ and $l = l_{opt} - 1$, respectively. Inset: We plot $l_{opt}/l_{min}^{Yoder's}$ against δ with $\lambda_0 \ll 1$.

We can see that when $\delta \rightarrow 1$, $l_{opt}/l_{min}^{Yoder's} \rightarrow 0$. For example, when $\delta = 0.8$, $l_{opt}/l_{min}^{Yoder's} \approx 75.65\%$, almost a quarter of iterations can be saved.

For any given lower bound $\lambda_0 \in (0, 4/5]$ of the fraction of target items, by theoretical analysis we have obtained the analytical expression of the optimal δ , i.e., δ_{opt} defined by Equation (24). In order

to compare the results of $\delta = \delta_{opt}$ and $\delta \neq \delta_{opt}$ with $l = l_{opt}(\delta)$, we define $L_{E,rel}^{Up,nor}(\lambda_0, \delta)$ to be the normalized relative value of $L_E^{Up}(\lambda_0, \delta)$ with respect to $L_E^{Up}(\lambda_0, \delta_{opt})$, i.e.,

$$L_{E,rel}^{Up,nor}(\lambda_0, \delta) = \frac{L_E^{Up}(\lambda_0, \delta)}{\max \{L_E^{Up}(\lambda_0, \delta) : 0 < \delta \leq 1\}}, \quad (32)$$

where

$$L_{E,rel}^{Up}(\lambda_0, \delta) = \ln \ln \left(L_E^{Up}(\lambda_0, \delta) \right) - \ln \ln \left(L_E^{Up}(\lambda_0, \delta_{opt}) \right). \quad (33)$$

The dependence of $L_{E,rel}^{Up,nor}(\lambda_0, \delta)$ on λ_0 and δ is shown in Figure 3, where the darker the color, the smaller the value. For ease of comparison, δ_{opt} is marked by the white dashed curves. We can see that the color in the area corresponding to δ_{opt} is darkest, which indicates that the optimal parameter δ_{opt} indeed enables the upper bound of expected queries to reach the minimum.

Figure 3. (Color online.) The normalized relative value $L_{E,rel}^{Up,nor}(\lambda_0, \delta)$ as a function of δ and λ_0 . The white dashed curves correspond to the optimal parameter δ_{opt} of Equation (24).

5.2. Complexity of $L_{E,opt}^{Up}(\lambda_0)$

Based on Equations (13), (20), (23), (24), and (29), we can obtain the expression of $L_{E,opt}^{Up}(\lambda_0)$ as below:

$$L_{E,opt}^{Up}(\lambda_0) = \begin{cases} \frac{2k+1}{1-\delta_k^2}, & \text{if } \lambda_0 \in [\lambda_{0,k}, \lambda_{0,k-1}), k \geq 1, \\ \frac{1}{1-\delta_0^2}, & \text{if } \lambda_0 \in [\lambda_{0,0}, 4/5], \end{cases} \quad (34)$$

where δ_k and $\lambda_{0,k-1}$ are defined by Equations (25) and (26), respectively. To analyze the complexity of $L_{E,opt}^{Up}(\lambda_0)$, first, for any known $\lambda_0 \ll 1$, we can determine an integer

$$k = \left\lceil \frac{x}{4 \operatorname{arcosh}(1/\sqrt{1-\lambda_0})} - \frac{1}{2} \right\rceil, \quad (35)$$

such that $\lambda_0 \in [\lambda'_{0,k}, \lambda'_{0,k-1})$, where $\lambda'_{0,k}$ is as defined by Equation (A26) and $x > 0$ is the solution of Equation (28). From Equations (26) and (27) it follows that

$$\lambda'_{0,k} \leq \lambda_{0,k-1} < \lambda'_{0,k-1}, \quad (36)$$

then, we have $\lambda_0 \in [\lambda_{0,k}, \lambda_{0,k-2})$. When $\lambda_0 \ll 1$, $k \rightarrow \infty$, $k \approx k-1$, thus,

$$L_{E,opt}^{Up}(\lambda_0) \approx \frac{2k+1}{1-\delta_k^2}, \quad (37)$$

where

$$\delta_k \approx \cosh^{-1}\left(\frac{x}{2}\right) \quad (38)$$

due to $k \approx \frac{x}{4\text{arccosh}(1/\sqrt{1-\lambda_0})} - \frac{1}{2}$, and

$$k \approx \frac{x}{4\sqrt{\lambda_0}} - \frac{1}{2} \quad (39)$$

due to $\text{arccosh}(1/\sqrt{1-\lambda_0}) \approx \sqrt{\lambda_0}$. Combining Equations (37), (38), and (39), we can obtain

$$\begin{aligned} L_{E,opt}^{Up}(\lambda_0) &\approx \frac{x}{2\left[1 - \cosh^{-2}\left(\frac{x}{2}\right)\right]} \frac{1}{\sqrt{\lambda_0}} \\ &\equiv L_{E,opt}^{Up,\approx}(\lambda_0). \end{aligned} \quad (40)$$

Therefore, we conclude that $L_{E,opt}^{Up}(\lambda_0) = O(1/\sqrt{\lambda_0})$. The corresponding graphs of $L_{E,opt}^{Up}(\lambda_0)$ and $L_{E,opt}^{Up,\approx}(\lambda_0)$ as functions of λ_0 are shown in Figure 4, which shows a good agreement. Note that, for an unknown λ and a given λ_0 , there also exists an upper bound of the expected queries of the classical search, namely, $1/\lambda_0$. This means that the OFPQS algorithm with the optimal parameters achieves a quadratic speedup over classical algorithms.

Figure 4. (Color online.) The optimal upper bound of expected queries $L_{E,opt}^{Up}(\lambda_0)$ and its approximation $L_{E,opt}^{Up,\approx}(\lambda_0)$ as functions of the lower bound $\lambda_0 \in [10^{-4}, 10^{-1}]$ of the fraction of target items. The blue dashed and red solid curves represent $L_{E,opt}^{Up}(\lambda_0)$ and $L_{E,opt}^{Up,\approx}(\lambda_0)$, respectively.

6. Conclusions

In summary, we have analyzed the performance and optimized the parameters of the OFPQS algorithm. We derived the least number of iterations (denoted by l_{opt}) of the OFPQS algorithm to ensure the success probability for a given lower bound of the fraction of target items no less than $1 - \delta^2$. Moreover, all extreme points as well as the minimum value of the success probability of the OFPQS algorithm were analyzed. In addition, we calculated the upper bound of expected queries of repeatedly executing the OFPQS algorithm to find a target item, and further analytically derived the optimal parameter that minimizes this upper bound. Compared with the minimum number of iterations given by [20], our optimal number of iterations l_{opt} has a significant reduction, e.g., when $\delta = 0.8$, almost a quarter of iterations can be saved. Our study can provide a guideline for the research and application of the OFPQS algorithm.

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Abbreviations

The following abbreviation is used in this manuscript:

OFPQS Optimal fixed-point quantum search

Appendix A. Proof of the Extreme Properties of $P_L(\lambda)$ on $(0, 1)$

Based on Equation (10), we can get the derivative of $P_L(\lambda)$ with respect to λ as

$$\frac{\partial P_L}{\partial \lambda} = -\frac{\delta^2}{2\gamma\sqrt{1-\lambda}} \frac{\sin[2L \arccos(x)]}{\sin[\arccos(x)]}, \quad (\text{A1})$$

where $\gamma = T_{1/L}^{-1}(1/\delta)$ and

$$x \equiv T_{1/L}(1/\delta) \sqrt{1-\lambda}. \quad (\text{A2})$$

Then, for any $\lambda \in (0, 1)$ and $\delta \in (0, 1]$,

$$\frac{\partial P_L}{\partial \lambda} = 0 \Leftrightarrow \sin[2L \arccos(x)] = 0 \text{ and } \arccos(x) \neq 0. \quad (\text{A3})$$

Note that $\frac{\partial P_L}{\partial x} = -2L\delta^2 \neq 0$ when $\arccos(x) = 0$. Furthermore, we can get

$$x = \cos\left(\frac{k\pi}{2L}\right) \in (0, 1). \quad (\text{A4})$$

Note that, $x \neq 1$ due to $\arccos(x) \neq 0$, and $x > 0$ for $\lambda \in (0, 1)$ and $\delta \in (0, 1]$.

For the case $L > 1$, from Equations (10), (A2), and (A4), it follows that when $k = 2j - 1$, $P_L(\lambda) = 1$, where $j = 1, 2, \dots, l \equiv (L - 1)/2$, thus, the local maximum points can be obtained as

$$\lambda_{max,j} = 1 - \gamma^2 \cos^2\left(\frac{2j-1}{2L}\pi\right), \quad j = 1, 2, \dots, l. \quad (\text{A5})$$

Additionally, because $P_L(\lambda = 0) = 0$ and $P_L(\lambda = 1) = 1$, P_L has the same number of local maximum points and minimum points. Thus, when $k = 2j$, $j = 1, 2, \dots, l$, the corresponding local minimum points can be found, denoted by

$$\lambda_{min,j} = 1 - \gamma^2 \cos^2 \left(\frac{j}{L} \pi \right), j = 1, 2, \dots, l. \quad (A6)$$

For the case $L = 1$, there is no available k for Equation (A4), namely, P_L has no extreme points.

Appendix B. Proof of the Minimum of $P_{L,opt}$ on $[\lambda_0, 1)$ of Equation (20)

Based on Equation (13), for any $l \geq l_{opt}$, if the local minimum point $\lambda_{min,l}$ (defined by Equation (19)) exists and $\lambda_{min,l} \in [\lambda_0, 1)$, then the minimum of $P_L(\lambda)$ with $L = 2l + 1$ can be obtained as $1 - \delta^2$, due to $P_L(\lambda) \geq 1 - \delta^2$ for any $\lambda \geq \lambda_0$, and $P_L(\lambda_{min,j}) = 1 - \delta^2$ for any $1 \leq j \leq l$. Note that, $\lambda_{min,l} = \max \{ \lambda_{min,j} : 1 \leq j \leq l \}$. While if $\lambda_{min,l}$ doesn't exist or $\lambda_{min,l} \notin [\lambda_0, 1)$, then $P_L(\lambda)$ monotonically increases on the range $\lambda \in [\lambda_0, 1)$, thus, the minimum success probability is $P_L(\lambda_0)$. Based on these, we can give the proof of P_{min} (defined by Equation (20)) for $l = l_{opt}$ and $\lambda_0 \leq \frac{4}{5}$ as follows.

In the case of $\delta \geq \sqrt{1 - \lambda_0}$, from Equation (13) it follows that $l_{opt} = 0$, thus, for $l = l_{opt}$, $\lambda_{min,l}$ does not exist and

$$P_{min} = P_L(\lambda_0) = \lambda_0. \quad (A7)$$

In the case of $\delta < \sqrt{1 - \lambda_0}$, from Equation (13) it follows that $l_{opt} > 0$. Then, for a given λ_0 and $l = l_{opt}$, $\lambda_{min,l}$ exists and can be written as a step function with respect to δ , i.e., for $\delta \in [\delta_k, \delta_{k+1})$,

$$\lambda_{min,l} = 1 - T_{\frac{1}{2k+1}}^{-2} \left(\frac{1}{\delta} \right) \cos^2 \left(\frac{k\pi}{2k+1} \right), k \geq 1, \quad (A8)$$

where $T_L(x)$ and δ_k are defined by Equations (11) and (25), respectively. Note that, $\delta_0 = \sqrt{1 - \lambda_0}$ and

$$\bigcup_{k \geq 1} [\delta_k, \delta_{k+1}) = (0, \delta_0). \quad (A9)$$

From Equation (A8), we can see that $\lambda_{min,l}$ monotonically decreases on the range $\delta \in [\delta_k, \delta_{k+1})$, then,

$$\begin{aligned} & \min \{ \lambda_{min,l} : \delta_k \leq \delta < \delta_{k+1} \} \\ &= \lim_{\delta \rightarrow \delta_{k+1}^-} \lambda_{min,l} \\ &= 1 - T_{\frac{1}{2k+1}}^{-2} \left(1/\sqrt{1 - \lambda_0} \right) \cos^2 \left(\frac{k}{2k+1} \pi \right). \end{aligned} \quad (A10)$$

Moreover, $\lim_{\delta \rightarrow \delta_{k+1}^-} \lambda_{min,l}$ increases as k grows. Therefore,

$$\begin{aligned} & \min \{ \lambda_{min,l} : 0 < \delta < \delta_{cri} \} \\ &= \lim_{\delta \rightarrow \delta_0^-} \lambda_{min,l} \\ &= 1 - T_{1/3}^{-2} \left(1/\sqrt{1 - \lambda_0} \right) \cos^2(\pi/3). \end{aligned} \quad (A11)$$

Note that

$$\lambda_0 \leq \lim_{\delta \rightarrow \delta_0^-} \lambda_{min,l} \Leftrightarrow 0 < \lambda_0 \leq 4/5. \quad (A12)$$

Then, when $\lambda_0 \leq \frac{4}{5}$, $\lambda_{min,l} \in [\lambda_0, 1)$, and thus we can obtain that $P_{min} = 1 - \delta^2$.

Appendix C. Proof of the Optimal Parameter δ_{opt} of Equation (24)

Based on Equations (20) and (23), for a given λ_0 , the upper bound of the expected number of queries L_E^{Up} as a function of δ can be obtained as follows:

$$L_E^{Up}(\delta) = \begin{cases} (2l_{opt} + 1) / (1 - \delta^2) \equiv L_{E,left}^{Up}(\delta), & \text{if } \delta < \sqrt{1 - \lambda_0}, \\ (2l_{opt} + 1) / \lambda_0 \equiv L_{E,right}^{Up}(\delta), & \text{if } \delta \geq \sqrt{1 - \lambda_0}, \end{cases} \quad (\text{A13})$$

where l_{opt} is defined by Equation (13). We can see that $L_{E,left}^{Up}(\delta)$ is a step function with δ , i.e.,

$$L_{E,left}^{Up}(\delta) = \frac{2m + 1}{1 - \delta^2}, \text{ if } \delta_m \leq \delta < \delta_{m-1}, m \geq 1, \quad (\text{A14})$$

due to $l_{opt} = m$ for $\delta \in [\delta_m, \delta_{m-1})$, where $\delta_m = T_{2m+1}^{-1}(1/\delta_{cri})$, consistent with Equation (25). From Equation (A14) it follows that $L_{E,left}^{Up}(\delta)$ monotonically increases on range $[\delta_m, \delta_{m-1})$, then

$$\begin{aligned} & \min \{ L_{E,left}^{Up}(\delta) : \delta_m \leq \delta < \delta_{m-1} \} \\ &= L_{E,left}^{Up}(\delta_m) = \frac{2m + 1}{1 - \delta_m^2}. \end{aligned} \quad (\text{A15})$$

In addition, $L_{E,right}^{Up}(\delta) = 1/\lambda_0 = L_{E,left}^{Up}(\delta_0)$, therefore,

$$\min \{ L_E^{Up}(\delta) : 0 < \delta < 1 \} = \min \{ L_{E,left}^{Up}(\delta_m) : m \geq 0 \}. \quad (\text{A16})$$

Note that if $L_{E,left}^{Up}(\delta_k)$ ($k > 0$) is minimum, then δ_k is optimal; while if $L_{E,left}^{Up}(\delta_0)$ is minimum, then arbitrary $\delta \in [\delta_0, 1)$ is optimal.

To determine $\min \{ L_{E,left}^{Up}(\delta_m) : m \geq 0 \}$, we define

$$L_{E,left}^{Up,\approx}(\delta) \equiv \frac{\text{arcosh}(1/\delta)}{(1 - \delta^2) \text{arcosh}(1/\sqrt{1 - \lambda_0})}. \quad (\text{A17})$$

From Equations (13), (A13), and (A17), it follows that

$$L_{E,left}^{Up}(\delta) \geq L_{E,left}^{Up,\approx}(\delta), \quad (\text{A18})$$

$$L_{E,left}^{Up}(\delta_m) = L_{E,left}^{Up,\approx}(\delta_m). \quad (\text{A19})$$

Moreover, we can get the derivative of $L_{E,left}^{Up,\approx}(\delta)$ with respect to δ , i.e.,

$$\frac{\partial L_{E,left}^{Up,\approx}}{\partial \delta} = \frac{-(1 - \delta^2)^{-2} g(\delta)}{\text{arcosh}(1/\sqrt{1 - \lambda_0})}, \quad (\text{A20})$$

where

$$g(\delta) \equiv \frac{\sqrt{1 - \delta^2}}{\delta} - 2\delta \ln \left(\frac{1 + \sqrt{1 - \delta^2}}{\delta} \right). \quad (\text{A21})$$

Solving $g(\delta) = 0$ gives rise to a local minimum point, denoted by $\delta_{min}^{left,\approx}$, which satisfies

$$\delta_{min}^{left,\approx} = \cosh^{-1} \left(\frac{x}{2} \right), \quad (\text{A22})$$

where $x > 0$ is the unique solution of Equation (28). Then, we can obtain ($k \geq 1$)

$$\min \left\{ L_{E,left}^{Up}(\delta_m) : m \geq 0 \right\} = \begin{cases} L_{E,left}^{Up}(\delta_0), & \text{if } \delta_{min}^{left,\approx} \geq \delta_0, \\ \min \left\{ L_{E,left}^{Up}(\delta_k), L_{E,left}^{Up}(\delta_{k-1}) \right\}, & \text{if } \delta_k \leq \delta_{min}^{left,\approx} < \delta_{k-1}. \end{cases} \quad (A23)$$

Note that for $0 < \lambda_0 \leq 4/5$,

$$\delta_{min}^{left,\approx} \geq \delta_0 \Leftrightarrow \lambda'_{0,0} \leq \lambda_0 \leq 4/5, \quad (A24)$$

$$\delta_k \leq \delta_{min}^{left,\approx} < \delta_{k-1} \Leftrightarrow \lambda'_{0,k} \leq \lambda_0 < \lambda'_{0,k-1}, \quad (A25)$$

where

$$\lambda'_{0,k} = 1 - T_{2k+1}^{-2} \left(1/\delta_{min}^{left,\approx} \right), \quad k \geq 0. \quad (A26)$$

Define

$$L_{E,left}^{Up,\Delta}(\lambda_0) = L_{E,left}^{Up}(\delta_k) - L_{E,left}^{Up}(\delta_{k-1}), \quad k \geq 1. \quad (A27)$$

Then, through further analysis about $L_{E,left}^{Up,\Delta}$, we can find the following two results: (1) On the range $\lambda_0 \in [\lambda'_{0,k}, \lambda'_{0,k-1})$ ($k \geq 1$), there is a solution for λ_0 of equation $L_{E,left}^{Up,\Delta} = 0$, denoted by $\lambda_{0,k-1}$. (2) $L_{E,left}^{Up,\Delta} < 0$ for $\lambda'_{0,k} \leq \lambda_0 < \lambda_{0,k-1}$, and $L_{E,left}^{Up,\Delta} > 0$ for $\lambda_{0,k-1} < \lambda_0 < \lambda'_{0,k-1}$. Corresponding reasons are given as follows:

(1) From Equations (A15) and (A27) it follows that

$$\begin{aligned} L_{E,left}^{Up,\Delta} &= \frac{2k+1}{1-\delta_k^2} - \frac{2k-1}{1-\delta_{k-1}^2} \\ &= \frac{T_{2k-1}^{-2}(\delta_0) T_{2k+1}^{-2}(\delta_0)}{2(1-\delta_k^2)(1-\delta_{k-1}^2)} h_k(y), \end{aligned} \quad (A28)$$

where

$$y = \text{arcosh} \left(1/\sqrt{1-\lambda_0} \right), \quad (A29)$$

and

$$h_k(y) = \sinh^2(4ky) + \sinh^2(2y) - 4k \sinh(4ky) \sinh(2y). \quad (A30)$$

Note that y increases as λ_0 grows, and according to Equations (A22), (A26), and (A29), $y = \frac{x}{4k+2} \equiv y'_k$ for $\lambda_0 = \lambda'_{0,k}$, and $y = \frac{x}{4k-2} \equiv y'_{k-1}$ for $\lambda_0 = \lambda'_{0,k-1}$. When k is sufficiently large, i.e., $k \rightarrow +\infty$, $y'_k, y'_{k-1} \rightarrow 0$, then simple algebra shows that,

$$h_k(y'_k) \approx \frac{2x^2}{2k+1} [2 - \cosh(x)] < 0, \quad (A31)$$

$$h_k(y'_{k-1}) \approx \frac{2x^2}{2k-1} [\cosh(x) - 2] > 0, \quad (A32)$$

which can also be numerically proven for finite k , for example $1 \leq k \leq 10^5$. Therefore, based on the intermediate value theorem (See p. 271 of [24]), we confirm that there exists a solution of $h_k(y) = 0$ between y'_k and y'_{k-1} , denoted by y_{k-1} . Correspondingly, from Equation (A29), the solution of $L_{E,left}^{Up,\Delta} = 0$ for λ_0 , denoted by $\lambda_{0,k-1}$, can be finally obtained as defined in Equation (26).

(2) Based on Equation (29), we can see that $L_{E, left}^{Up, \Delta}$ and $h_k(y)$ have the same sign. Moreover, we can obtain the derivative of $h_k(y)$ with respect to y , as below,

$$\begin{aligned} \frac{\partial h_k(y)}{\partial y} &= 8k \sinh(4ky) \cosh(4ky) + 4 \sinh(2y) \cosh(2y) \\ &\quad - 4k [4k \sinh(2y) \cosh(4ky) + 2 \sinh(4ky) \cosh(2y)] \\ &\equiv h'_k(y). \end{aligned} \quad (A33)$$

Due to $\sinh(x), \cosh(x)$ monotonically increases for $x > 0$ and $y'_k \leq y < y'_{k-1}$ when k is sufficiently large, i.e., $k \rightarrow +\infty$, we have $y'_k, y'_{k-1} \rightarrow 0$, and

$$\begin{aligned} h'_k(y) &> 8k \sinh(4ky'_k) \cosh(y'_k) + 4 \sinh(2y'_k) \cosh(2y'_k) \\ &\quad - 4k [4k \sinh(2y'_{k-1}) \cosh(4ky'_{k-1}) + 2 \sinh(4ky'_{k-1}) \cosh(2y'_{k-1})] \\ &\approx 2k [\sinh(2x) - 4 \sinh(x)] - 2x [4 \cosh(x) + 5 \sinh^2(x) + 2] \\ &\geq 0, \text{ when } k \geq \left\lceil \frac{x [4 \cosh(x) + 5 \sinh^2(x) + 2]}{\sinh(2x) - 4 \sinh(x)} \right\rceil = 12. \end{aligned} \quad (A34)$$

Therefore, $h_k(y)$ monotonically increases on the range $y \in [y'_k, y'_{k-1})$, yielding that $h_k(y) < 0$ for $y'_k \leq y < y_{k-1}$ and $h_k(y) > 0$ for $y_{k-1} \leq y < y'_{k-1}$. Then, the corresponding results about $L_{E, left}^{Up, \Delta}$ follow immediately, which can also be numerically proven to hold when k is small, for example, $k = 1, 2, \dots, 12$.

Finally, combining Equations (A16) and (A23) with the above results (1) and (2), we can obtain that

$$\min \left\{ L_E^{Up}(\delta) : 0 < \delta < 1 \right\} = \begin{cases} L_E^{Up, left}(\delta_k), & \text{if } \lambda_0 \in [\lambda_{0,k}, \lambda_{0,k-1}], k \geq 1, \\ L_E^{Up, left}(\delta_0), & \text{if } \lambda_0 \in [\lambda_{0,0}, 4/5], \end{cases} \quad (A35)$$

and thus δ_{opt} of Equation (24) is just the optimal δ .

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