

Article

A New Criterion for Judging Tunnel Failure by the Strength Reduction Method

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Abstract

Recently, the application of the strength reduction method (SRM) to tunnel stability research has become a trend. The failure criteria for the SRM in tunnels often draw from those used in slopes. The applicability of the newly proposed energy mutation criterion from slopes to the tunnels is explored. Additionally, a new criterion—the variational criterion—is suggested for tunnels. The effectiveness of the aforementioned work is validated based on unlined tunnels, lined tunnels, and constructed tunnels. The results indicate that the energy or variational value can serve as a new indicator for tunnel stability. For the variational criterion, the applicability is not coincidental with respect to mesh densities, strength reduction factor intervals, geometric parameters, and mechanical parameters. In specific tunnels, the variational criterion exhibits several relatively distinct characteristics of simplicity, comprehensiveness, clarity, reliability, and uniformity. This criterion provides a theoretical basis and mechanical explanation for tunnel failure.

Keywords: tunnel stability; strength reduction method; failure criterion; variational criterion

1. Introduction

Tunnel stability represents one of the critical issues in geotechnical engineering. The factor of safety serves as a key quantitative indicator for assessing stability. For determining the factor of safety of the geotechnical engineering, the strength reduction method (SRM) has been proposed [1]. Particularly in slopes, it has been demonstrated that results of the SRM are very close to those of the limit equilibrium method and the limit analysis method [2]. In tunnels, it currently lacks a safety factor to evaluate the overall stability of a tunnel [3,4]. The introduction of the SRM into tunnels has become a trend [5–7]. Some scholars employed the SRM as the primary approach for calculating the safety factor of tunnels [8–14]. Some researchers integrated SRM with limit analysis methods to derive safety factors for tunnel faces under various conditions, such as groundwater, rock bolt support, and complex structural planes [15–20]. Additionally, some researchers utilized the computational results of SRM as references in their studies or adopted experimental approaches to analyze the tunnel stability through the SRM [21,22].

The application of SRM lies in the selection of an appropriate failure criterion. Currently, the choice of failure criteria for SRM in tunnels often references practices from slopes. In slopes, three commonly used failure criteria include Criterion I: the displacement at the characteristic point mutation [1,23], Criterion II: the plastic zone penetration [24,25],



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and Criterion III: the numerical program non-convergence [26,27]. Furthermore, Criterion IV of energy mutation has been proposed in recent years [28,29]. Specifically, Criterion IV₁ represents the gravitational potential energy loss mutation. Criterion IV₂ denotes the dissipated energy increment mutation. Criterion IV₃ signifies the kinetic energy increment mutation. Research indicates that Criterion IV demonstrates satisfactory applicability in slopes [30–32]. Based on these criteria, scholars have adopted Criterion I [33–35], Criterion II [36], and Criterion III [8,37] when applying the SRM to tunnels.

The core failure of the criterion lies in determining stability indices as well as identifying instability methods. Traditionally, the stability indices for tunnels or underground engineering are primarily divided into three categories. The instability is deemed to occur when these indices exceed a certain critical value. The first category comprises the stress, including indicators such as strength-to-stress ratio [9,38], stress concentration factor [39,40], brittle shear ratio [41,42], depth of yield zone [43,44], and depth of failure [45]. The second category consists of the strain, which cover indicators like Stacey's extension strain [46], Sakurai's critical strain [47], Aydan's squeezing assessment strain [48], geometrically accumulated deformation [49], strain-strength [50], critical tensile strain [51,52], mean stress–volumetric strain [53], plastic damage index [54,55], and displacement or relative displacement [56]. The third category pertains to indicators considering the combined effect of stress and strain, including elastic energy [57], strain energy [58,59], dissipated energy [60–63], radiated energy [64], and elastic energy–dissipated energy ratio [65]. However, for different types of tunnels, conventional failure criteria require determining different stability indices and critical values. Thus, it is difficult to establish a unified framework for the conventional criteria.

Indeed, Criteria I, II, III, and IV of the SRM in tunnels fall within the category of conventional criteria, and there is also no unified standard for them. For the stability indices, the stability indicator of Criterion I—the displacement of characteristic points—falls within the category of strain. The stability indicator of Criterion II—the plastic zone—belongs to the stress category. The stability indicator of Criterion III—the convergence of the computational program—pertains to either stress or strain. The stability indicator of Criterion IV—the energy—is essentially the integral of the partial stress–strain product. The method of identifying instability is determining the mutation of the displacement curves in Criterion I. It is determining the penetration of some plastic zones in Criterion II. It is determining the occurrence of the computational program in Criterion III. It is determining the mutation of the energy curves in Criterion IV.

Some scholars have treated the tunnel as an integrated system and analyze its stability based on the principle of minimum potential energy [66–68]. The principle of minimum potential energy allows for a comprehensive consideration of the energy transformation processes within a structure. Its essential implication lies in the total potential energy of the structure attaining a minimum value. It implies that the sign of the second-order variational value of the total potential energy can serve as a unified criterion for identifying the stability or instability of various tunnels. Considering that, there is currently limited literature exploring the applicability of Criterion IV in tunnels when the SRM is used. The application of the variational method in tunnels has primarily focused on theoretical derivations [69,70]; limited studies have ventured into the exploration of numerical indicators. Moreover, there is currently a lack of more comprehensive stability indicators and unified identifying instability methods for tunnels. As noted by Heidarzadeh et al. [71], it can be explored by integrating the stress and strain using the FISH language coding in the finite difference procedure—FLAC^{3D} software version 5.0. This study utilizes the FLAC^{3D} software version 3.0 to investigate the performance of Criterion IV and propose the new failure criterion for tunnels based on the SRM.

This paper focuses on two main objectives: (1) to investigate the applicability of the energy mutation criterion in tunnels, and (2) to propose a new failure criterion—the variational criterion. The effectiveness of these works is subsequently verified through case studies involving unlined tunnels, lined tunnels, engineering tunnels, and complex loading tunnels. The structure of this paper is organized as follows. Section 2 introduces the SRM and various failure criteria. Section 3 validates the applicability and generality of both the energy mutation criterion and the variational criterion using unlined, lined, constructed, and complex loading tunnel examples. Section 4 provides a discussion of the findings. Finally, Section 5 presents the conclusions derived from this study. Based on this study, it provides a new stability indicator, the variational value, and a new identifying instability method, determining the sign of the variational value for tunnels. Through the variational value changing, it offers a unified mechanical explanation for the failure of specific tunnels.

2. Materials and Methods

2.1. Strength Reduction Method

The numerical model of a tunnel is established in the FLAC^{3D} software. The strength parameters of the effective cohesion and the internal friction angles are concerned. The initial strength parameters of the material are continuously reduced. The new strength parameters are substituted into the model to calculate until the model fails. The strength reduction factor (SRF) corresponding to the model failure is taken as the factor of safety of the model. As shown in Equation (1), the initial strength parameters (denoted as c and φ) are divided by the SRF (denoted as k) to obtain the new strength parameters (denoted as c' and φ'), respectively.

$$\begin{cases} c' = \frac{c}{k} \\ \varphi' = \arctan\left(\frac{\tan\varphi}{k}\right) \end{cases} \quad (1)$$

2.2. Conventional Failure Criteria

The criterion to determine the model failure is the failure criterion. As is stated above, there are mainly four types of criteria. They are Criterion I, Criterion II, Criterion III, and Criterion IV. (i) Criterion I, the displacement at the characteristic point catastrophe. In this criterion, the relationship curve between the displacement of monitoring points and the SRF is plotted. The SRF at the catastrophe point of the curve is defined as the safety factor. (ii) Criterion II, the plastic zone penetration. In this criterion, the SRF at the moment when there is a certain degree of plastic zone penetration is defined as the safety factor. (iii) Criterion III, the numerical program non-convergence. It is realized by the FOS command in the software FLAC^{3D} in this paper. (iv) Criterion IV, the energy catastrophe. In this criterion, the relationship curves between various energy losses or increments and the SRFs are plotted, respectively. The SRF corresponding to the catastrophe point on the curve is defined as the safety factor of the model. Specifically, IV₁ represents the gravitational potential energy loss catastrophe, IV₂ represents the dissipated energy increment catastrophe, and IV₃ represents the kinetic energy increment catastrophe. Relative calculation is shown in Equation (2).

$$\begin{cases} \Delta(E_g) + \Delta(E_e) + \Delta(E_k) + \Delta(E_d) = 0 \\ E_g = \int_0^V \rho g z d\Omega \\ E_e = \frac{1}{2} \int_0^V \sigma_{ij} \varepsilon_{ij} d\Omega \\ E_k = \frac{1}{2} \int_0^V \rho v^2 d\Omega \\ E_d = E_g - E_e - E_k \end{cases} \quad (2)$$

where E_g is the gravitational potential energy of the model; E_e is the elastic energy of the model; E_k is the kinetic energy of the model; E_d is the dissipated energy of the model;

Δ represents the changes in various energy components of the model under a specific SRF compared to the initial state; ρ is the density of the element; g is the gravitational acceleration; z is the settlement of the center of the element; σ_{ij} is the stress tensor of the element; ε_{ij} is the strain tensor of the element; v is the velocity of the center of the element; Ω represents the volume of the element; and V represents the total volume of the model.

2.3. The Proposed Variational Criterion

Assuming that there exists a numerical model in the FLAC^{3D} software, the loading conditions of the model are as follows: There is only the gravitational loading. The boundary conditions of the model are either prescribed stress boundaries or non-working boundaries. If the numerical model is regarded as a whole system, the stability of the model can be judged by the second-order variational value of the total potential energy (denoted as E_t) of the model. According to the definition, E_t is the algebraic sum of the internal force work (denoted as U) and the external force work (denoted as W). When small virtual displacements compatible with the boundary constraints are applied on the model, E_t remains unchanged. In mathematical expressions, the first-order variational value of E_t is always equal to zero. That is, E_t always takes the local maximum or minimum. As is known, the principle of minimum potential energy is universal. When the model is in a stable state, the local minimum of E_t is taken. It means that the second-order variational value of E_t is greater than zero. When the model is in an unstable state, the local maximum of E_t is taken. It means that the second-order variational value of E_t is less than zero. In conclusion, when the model is stable, the second-order variational value of E_t is greater than zero. When the model is unstable, the second-order variational value of E_t is less than zero. The relevant derivation formulas are given by Equations (3) and (4).

$$E_t = U - W \quad (3)$$

$$\delta E_t = \delta U - \delta W = \int [U(\varepsilon_{ij}^0 + \delta \varepsilon_{ij}) - U(\varepsilon_{ij}^0)] d\Omega - \left\{ \int [\bar{f}_i(u_i^0 + \delta u_i) - \bar{f}_i(u_i^0)] d\Omega + \int [\bar{T}_i(u_i^0 + \delta u_i) - \bar{T}_i(u_i^0)] dS \right\} \quad (4)$$

where δ represents the first-order variation; U , $\bar{f}_i$, and $\bar{T}_i$ are the strain energy, body force, and surface force, respectively; u_i^0 and ε_{ij}^0 are the initial displacement tensor and strain tensor, respectively; δu_i and $\delta \varepsilon_{ij}$ are the first-order variational values of displacement tensor and strain tensor, respectively; and S and Ω are the area of the model boundary and the volume of the model body, respectively.

After expanding U , $\bar{f}_i$, and $\bar{T}_i$ according to Taylor's series and ignoring the second-order minute quantities, the first-order variational value expression of E_t can be transformed into Equation (5).

$$\delta E_t = \int \sigma_{ij}^0 \delta \varepsilon_{ij} d\Omega - \left[\int \bar{f}_i(\delta u_i) d\Omega + \int \bar{T}_i(\delta u_i) dS \right] \quad (5)$$

The second-order variational value of E_t can be calculated according to Equation (6).

$$\delta^2 E_t = \int \delta \sigma_{ij} \delta \varepsilon_{ij} d\Omega - \int \delta [\bar{f}_i(\delta u_i)] d\Omega - \int \delta [\bar{T}_i(\delta u_i)] dS \quad (6)$$

For a certain model, the body force $\bar{f}_i$ is constant. So, the second-order variational value of $\bar{f}$ is zero. That is, there are $\delta \bar{f}_i = 0$ and $\delta [\bar{f}_i(\delta u_i)] = (\delta \bar{f}_i)(\delta u_i) = 0$. Because the boundaries are with either the prescribed stress or the non-working effect, there are $\delta \bar{T}_i = 0$

or $\bar{T}_i(\delta u_i) = 0$. That is, there are $\delta[\bar{T}_i(\delta u_i)] = (\delta \bar{T}_i)(\delta u_i) = 0$ or $\delta[\bar{T}_i(\delta u_i)] = \delta(0) = 0$. So, Equation (6) can be transformed into Equation (7).

$$\delta^2 E_t = \int \delta \sigma_{ij} \delta \varepsilon_{ij} d\Omega \quad (7)$$

Whether the model is stable or not depends on the sign symbol of the variational value. To unify the order of magnitude of various variational values of different models, the mean variational value (denoted as v_v) is defined. The “variational value” mentioned in the following text is the mean variational value. As is shown in Equation (8), this value is obtained by dividing the total variational value by the volume of the model body. It does not change the sign symbol of the total variational value.

$$v_v = \frac{\int \delta \sigma_{ij} \delta \varepsilon_{ij} d\Omega}{\int d\Omega} \quad (8)$$

The variational value under each SRF can be calculated by the following method. At a certain SRF, the stress and strain values of each element are recorded and saved after the calculation is completed. Then, one more time step is conducted, and the stress and strain values of each element are recorded and saved again. Subtracting the previously recorded values from the later-recorded values, the differences can be taken as the first-order variational values of the stress and strain of each element. According to Equations (3)–(8), the variational value can be calculated. By repeating the above steps, the variational values under different SRFs can be obtained. Finally, the state of the model can be judged. If the variational value is positive, the model is stable. If the variational value is negative, the model is unstable. To ensure the safety, the SRF before failure occurring is determined as the FOS of the model. This is the proposed variational criterion in this paper. The special application process of the variational criterion is shown in Figure 1.

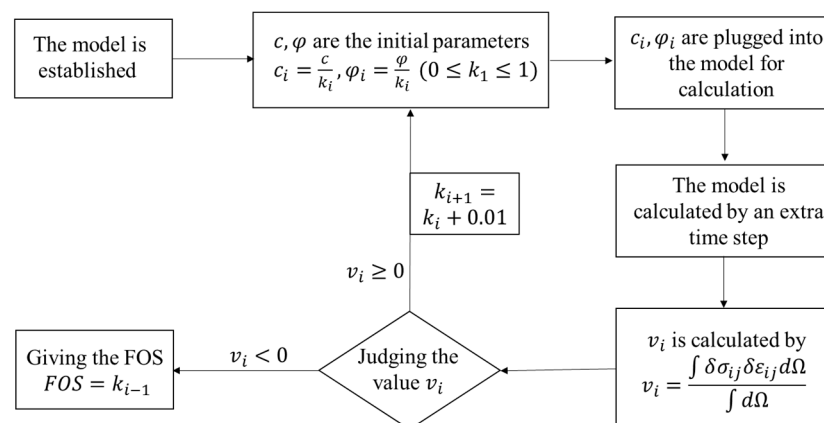


Figure 1. Flow chart of the application of the proposed variational criterion.

3. Results

In this section, an unlined tunnel (Example 1), a lined tunnel (Example 2), and a tunnel under construction (Example 3) are employed to compare results of various failure criteria. By this way, the applicability and generality of the energy mutation criterion (Criterion IV) and the proposed variational criterion are verified.

3.1. Example 1: An Unlined Tunnel

According to the paper [72], a numerical model of unlined tunnel ($S = 3$, $H_w = 1.5$, $H = 30$, $RSR = \frac{1}{2}$, $d_1 = 0.4$, $d_2 = 4$) is established. The numerical model is established according to an actual unlined loess tunnel. The geometric dimensions of the tunnel and

the locations of monitoring points are shown in Figure 2. The mechanical parameters of the soil are presented in Table 1. The soil material is modeled as the ideal elastoplastic material. The material follows the Mohr–Coulomb yield criterion and non-associated flow rule. The distance from the tunnel to the left side, right side, and bottom of the model is set at five times the span, respectively. The model is constrained in horizontal displacement on the left and right sides, fixed at the bottom, and free at the top. The longitudinal length of the tunnel is set to 1 m.

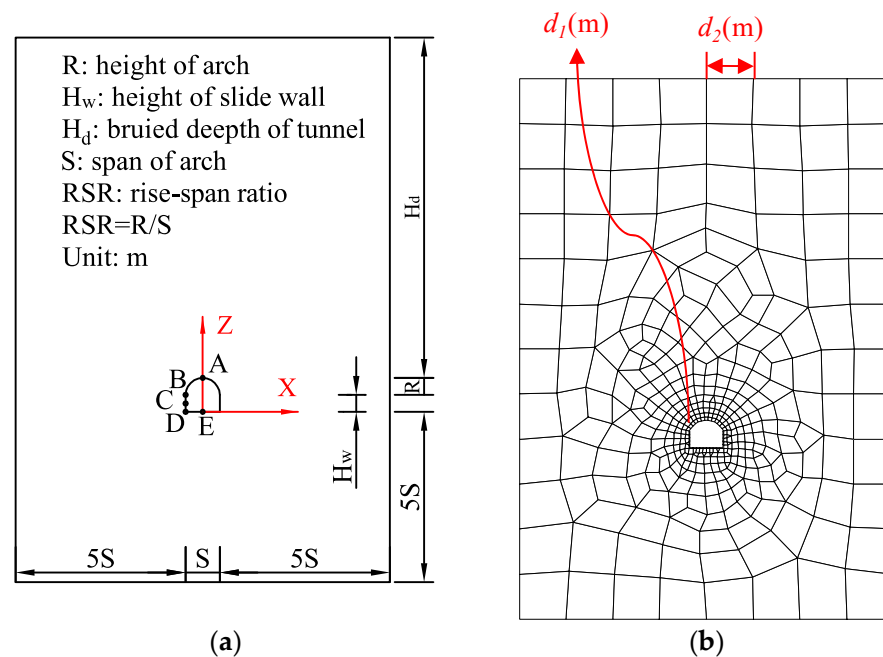


Figure 2. Sketch and numerical model of the unlined tunnel. (a) Sketch of the unlined tunnel, the arrows and the red lines represent the coordinate axes; (b) numerical model of the unlined tunnel, the arrows and red lines indicate the mesh sizes in different regions of the model.

Table 1. Mechanical parameters of the soil for the unlined tunnel.

Young's Modulus E/MPa	Poisson's Ratio ν	Unit Weight $\gamma/(\text{kN}/\text{m}^3)$	Cohesion c/MPa	Internal Friction Angle $\varphi/(^{\circ})$	Tensile Strength t/MPa
40	0.35	17	0.05	25	0.02

3.1.1. Comparisons of Different Criteria for the Unlined Tunnel

Based on the above example, the SRM and various criteria are applied to determine the safety factor of the tunnel.

(1) Criterion I

As shown in Figure 3, the mutation characteristics of displacement curves vary significantly across different monitoring points and displacement directions. In terms of the X-displacement, the arch shoulder and arch waist show the most pronounced mutations, while the arch springing exhibits slightly weaker mutations, and the arch crown and arch base show no mutations. Regarding the Z-displacement, the arch crown demonstrates the most significant mutations, followed by the arch shoulder and arch base, while the arch waist and arch foot exhibit the weakest mutations. Therefore, selecting appropriate monitoring points and displacement directions for this criterion involves a considerable workload. Moreover, the manual identification of mutations is prone to subjective errors.

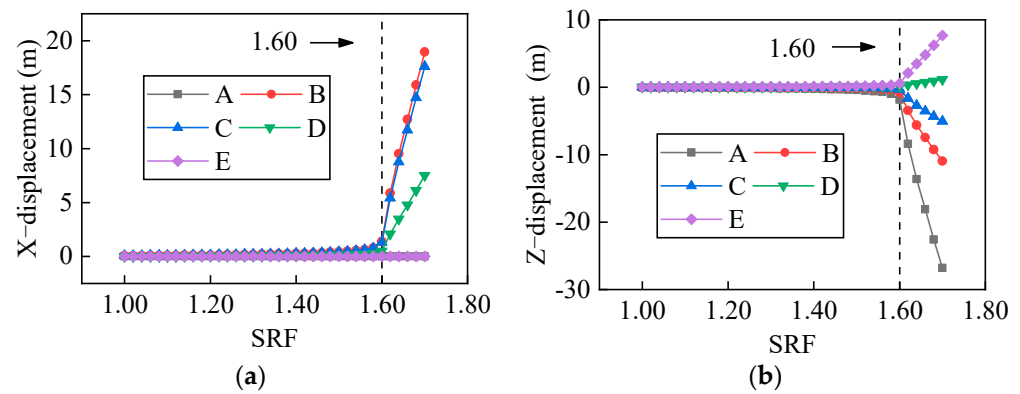


Figure 3. Curves between the displacements of the monitoring points and the SRFs for the unlined tunnel. (a) The X-displacement; (b) the Z-displacement.

(2) Criterion II

As shown in Figure 4, when the SRF increases, the plastic zone around the tunnel gradually expands. When the SRF reaches 1.59–1.61, the plastic zones on both sides of the tunnel become fully connected, creating conditions for an overall collapse. Particularly at the SRF of 1.61, a significantly large and continuous plastic zone emerges at the crown of the tunnel, indicating the onset of an overall collapse.

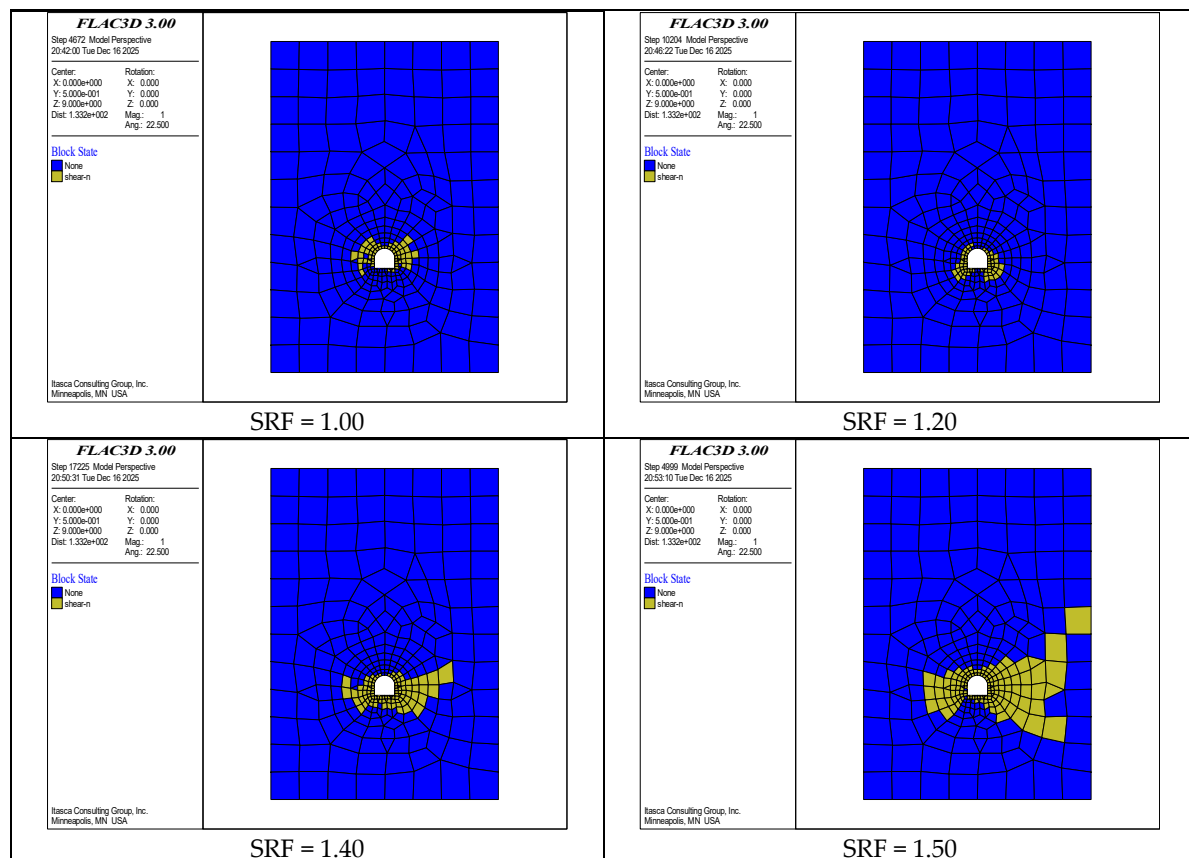


Figure 4. Cont.

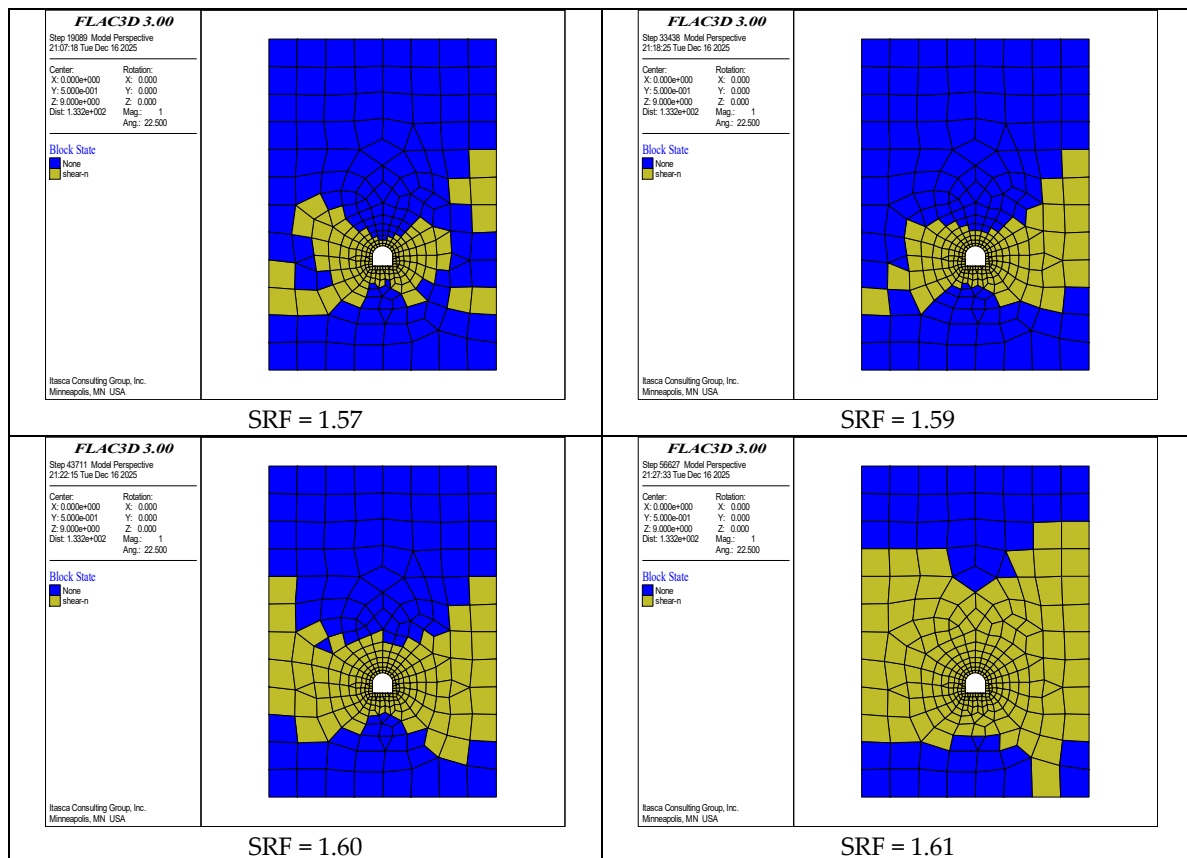


Figure 4. Distribution of plastic elements (yellow elements) in the numerical model under different SRFs for the unlined tunnel.

(3) Criterion III

As shown in Figure 5, the FOS command directly yields a safety factor of 1.61 for the model. At this point, the shear strain increment of elements is relatively high across the entire model region. The nodal velocities in the overlying soil above the tunnel are significant, while those in the surrounding rock around the tunnel are even greater and more concentrated. It indicates the onset of overall instability.

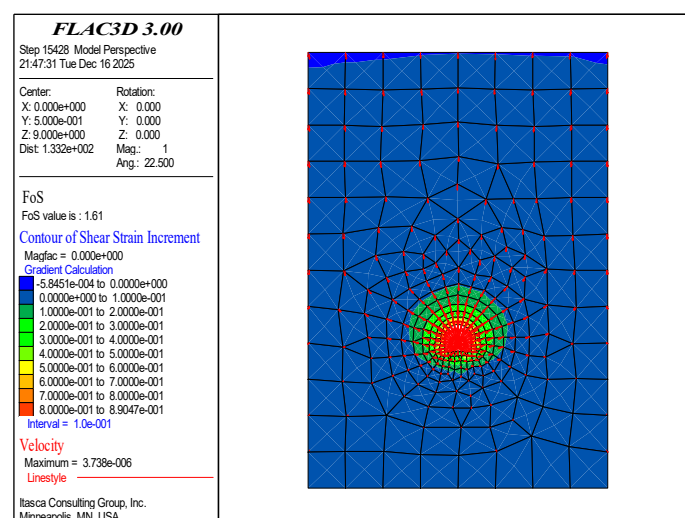


Figure 5. Results of the FOS command including the shear strain increment of elements and the velocity of nodes for the unlined tunnel.

(4) Criterion IV

As shown in Figure 6, three types of energy curves exhibit similar mutation characteristics. Around the SRF of 1.60, all the curves undergo relatively obvious mutations. This is largely consistent with the patterns revealed by other criteria. It indicates that the energy criterion is well applicable for unlined tunnels.

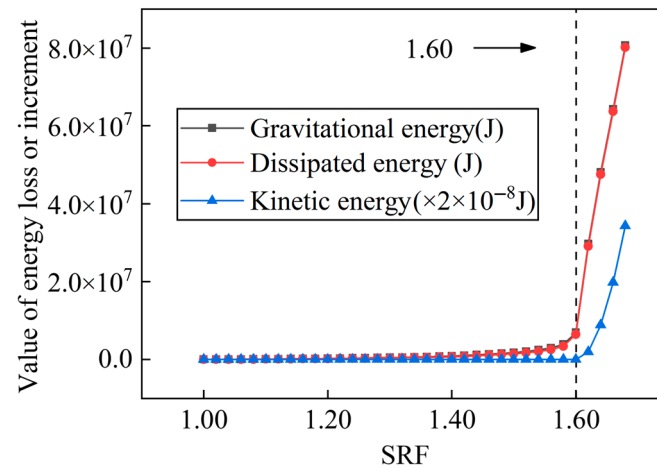


Figure 6. Curves between the various energy values and the SRFs for the unlined tunnel.

(5) Variational criterion

As shown in Table 2, when the SRF ranges from 1.00 to 1.60, the variational values of the model remain positive. It indicates that instability has not occurred. When the strength reduction factor reaches 1.61, the variational value of the model turns negative for the first time. It signals the onset of instability.

Table 2. Variational values under different SRFs for the unlined tunnel.

SRF	Variational Value	SRF	Variational Value	SRF	Variational Value
1.00	1.99×10^{-8}	1.21	1.24×10^{-8}	1.42	1.16×10^{-8}
1.01	1.59×10^{-8}	1.22	1.70×10^{-8}	1.43	1.06×10^{-8}
1.02	1.72×10^{-8}	1.23	1.36×10^{-8}	1.44	1.21×10^{-8}
1.03	2.81×10^{-8}	1.24	2.29×10^{-8}	1.45	1.23×10^{-8}
1.04	2.06×10^{-8}	1.25	1.96×10^{-8}	1.46	1.88×10^{-8}
1.05	2.31×10^{-8}	1.26	1.69×10^{-8}	1.47	1.20×10^{-8}
1.06	1.60×10^{-8}	1.27	1.52×10^{-8}	1.48	1.61×10^{-8}
1.07	1.52×10^{-8}	1.28	1.51×10^{-8}	1.49	1.84×10^{-8}
1.08	1.59×10^{-8}	1.29	1.51×10^{-8}	1.50	2.28×10^{-8}
1.09	1.82×10^{-8}	1.30	1.37×10^{-8}	1.51	1.50×10^{-8}
1.10	2.15×10^{-8}	1.31	1.68×10^{-8}	1.52	1.57×10^{-8}
1.11	1.87×10^{-8}	1.32	1.25×10^{-8}	1.53	2.96×10^{-8}
1.12	2.06×10^{-8}	1.33	1.93×10^{-8}	1.54	1.94×10^{-8}
1.13	2.44×10^{-8}	1.34	1.11×10^{-8}	1.55	2.05×10^{-8}
1.14	1.79×10^{-8}	1.35	1.39×10^{-8}	1.56	2.28×10^{-8}
1.15	1.95×10^{-8}	1.36	1.22×10^{-8}	1.57	1.17×10^{-8}
1.16	1.87×10^{-8}	1.37	1.55×10^{-8}	1.58	2.06×10^{-8}
1.17	1.92×10^{-8}	1.38	1.22×10^{-8}	1.59	9.01×10^{-9}
1.18	1.98×10^{-8}	1.39	1.13×10^{-8}	1.60	1.11×10^{-8}
1.19	1.62×10^{-8}	1.40	1.39×10^{-8}	1.61	7.35×10^{-7}
1.20	1.39×10^{-8}	1.41	1.94×10^{-8}	1.62	-1.91×10^{-7}

(6) Comparisons

Results of different criteria are summarized in Table 3. As shown in Table 3, results of the three types of energy catastrophe criteria are close to those of the three commonly used failure criteria. The maximum absolute and relative errors between them are 0.01 and 0.6%, respectively. It indicates that for unlined tunnels, there exists a favorable correspondence between the three commonly used failure criteria and the three energy catastrophe criteria. It is consistent with observations in slopes. The result of the variational criterion is close to those of the six typical failure criteria. There are the maximum absolute and relative errors of 0.02 and 1.3%, respectively. It demonstrates that the variational criterion exhibits good consistency with the six typical failure criteria in unlined tunnels.

Table 3. Factors of safety under different criteria for the unlined tunnel.

Criteria	Factors of Safety	Absolute Error		Relative Error	
Criterion I (1)	1.60	0.01	(17-11)	0.6%	(17-11/1)
Criterion II (2)	1.59	0.02	(17-21)	1.3%	(17-21/2)
Criterion III (3)	1.61	0.00	(17-31)	0.0%	(17-31/3)
Criterion IV ₁ (4)	1.60	0.01	(17-41)	0.6%	(17-41/4)
Criterion IV ₂ (5)	1.60	0.01	(17-51)	0.6%	(17-51/5)
Criterion IV ₃ (6)	1.60	0.01	(17-61)	0.6%	(17-61/6)
Variational criterion (7)	1.61	—	—	—	—

3.1.2. Analysis of Influencing Factors

As demonstrated earlier through the typical example ($S = 3$, $H_w = 1.5$, $H = 30$, $RSR = \frac{1}{2}$, $d_1 = 0.4$, $d_2 = 4$), the results of the FOS command (Criterion III) are close to those of other criteria (Criteria I, II, IV₁, IV₂, and IV₃). However, applying Criteria I, II, IV₁, IV₂, and IV₃ involves a substantial amount of work. To save time, the subsequent analysis will use results of the FOS command in place of those of the six conventional criteria (Criteria I, II, III, IV₁, IV₂, and IV₃). The mesh densities, strength reduction factor intervals, geometric parameters, and mechanical parameters are changed. The generality of the variational criterion is verified by comparing results of the FOS command with those of the variational criterion under different conditions.

(1) Mesh densities

As shown in Table 4, under different mesh densities, results of the variational criterion consistently align closely with those of the FOS command. There is an average relative error of 1.2% between them, indicating good agreement. This demonstrates that the applicability of the variational criterion in unlined tunnels is not coincidental due to mesh densities.

(2) Strength reduction factor intervals

As shown in Table 5, results with a strength reduction factor interval of 0.001 consistently remain close to those with an interval of 0.01. There is an average relative error of 0.9% between them, indicating good agreement. This demonstrates that the applicability of the variational criterion in unlined tunnels is not coincidental due to strength reduction factor intervals.

(3) Geometric parameters

As shown in Table 6, under different geometric parameters, results of the variational criterion consistently align closely with those of the FOS command. There is an average relative error of 1.3% between them, indicating good agreement. This demonstrates that the applicability of the variational criterion in unlined tunnels is not limited to specific geometric parameters.

Table 4. Factors of safety under different mesh densities for unlined tunnels.

RSR	Result of the FOS Command, Result of the Variational Criterion, (d_1, d_2)								
0–0	1.68,	1.71,	1.73,	1.69,	1.72,	1.73,	1.71,	1.73,	1.75,
	1.69,	1.71,	1.72,	1.70,	1.73,	1.74,	1.87,	1.73,	1.78,
	(0.3,3)	(0.3,4)	(0.3,5)	(0.4,3)	(0.4,4)	(0.4,5)	(0.5,3)	(0.5,4)	(0.5,5)
1–6	1.67,	1.69,	1.72,	1.68,	1.70,	1.73,	1.70,	1.72,	1.76,
	1.66,	1.71,	1.71,	1.70,	1.73,	1.72,	1.74,	1.73,	1.79,
	(0.3,3)	(0.3,4)	(0.3,5)	(0.4,3)	(0.4,4)	(0.4,5)	(0.5,3)	(0.5,4)	(0.5,5)
1–3	1.60,	1.61,	1.64,	1.61,	1.63,	1.65,	1.62,	1.64,	1.67,
	1.65,	1.62,	1.65,	1.63,	1.64,	1.67,	1.64,	1.67,	1.70,
	(0.2,2)	(0.2,3)	(0.2,4)	(0.3,2)	(0.3,3)	(0.3,4)	(0.4,2)	(0.4,3)	(0.4,4)
1–2	1.61,	1.62,	1.63,	1.62,	1.64,	1.65,	1.64,	1.66,	1.68,
	1.61,	1.63,	1.63,	1.66,	1.64,	1.68,	1.66,	1.67,	1.71,
	(0.4,4)	(0.4,5)	(0.4,6)	(0.5,4)	(0.5,5)	(0.5,6)	(0.6,4)	(0.6,5)	(0.6,6)
2–3	1.52,	1.53,	1.55,	1.53,	1.56,	1.57,	1.53,	1.56,	1.58,
	1.54,	1.56,	1.57,	1.55,	1.57,	1.59,	1.55,	1.56,	1.60,
	(0.2,3)	(0.2,4)	(0.2,5)	(0.3,3)	(0.3,4)	(0.3,5)	(0.4,3)	(0.4,4)	(0.4,5)

Table 5. Factors of safety under different strength reduction factor intervals for unlined tunnels.

RSR	Result with the Strength Reduction Factor Interval of 0.01, Result with the Strength Reduction Factor Interval of 0.001, (d_1, d_2)								
0–0	1.69,	1.71,	1.72,	1.70,	1.73,	1.74,	1.87,	1.73,	1.78,
	1.688,	1.701,	1.730,	1.704,	1.721,	1.737,	1.714,	1.744,	1.753,
	(0.3,3)	(0.3,4)	(0.3,5)	(0.4,3)	(0.4,4)	(0.4,5)	(0.5,3)	(0.5,4)	(0.5,5)
1–6	1.66,	1.71,	1.71,	1.70,	1.73,	1.72,	1.74,	1.73,	1.79,
	1.666,	1.701,	1.719,	1.686,	1.702,	1.736,	1.705,	1.726,	1.763,
	(0.3,3)	(0.3,4)	(0.3,5)	(0.4,3)	(0.4,4)	(0.4,5)	(0.5,3)	(0.5,4)	(0.5,5)
1–3	1.65,	1.62,	1.65,	1.63,	1.64,	1.67,	1.64,	1.67,	1.70,
	1.618,	1.627,	1.634,	1.617,	1.628,	1.658,	1.612,	1.650,	1.684,
	(0.2,2)	(0.2,3)	(0.2,4)	(0.3,2)	(0.3,3)	(0.3,4)	(0.4,2)	(0.4,3)	(0.4,4)
1–2	1.61,	1.63,	1.63,	1.66,	1.64,	1.68,	1.66,	1.67,	1.71,
	1.611,	1.630,	1.636,	1.637,	1.647,	1.662,	1.648,	1.660,	1.680,
	(0.4,4)	(0.4,5)	(0.4,6)	(0.5,4)	(0.5,5)	(0.5,6)	(0.6,4)	(0.6,5)	(0.6,6)
2–3	1.54,	1.56,	1.57,	1.55,	1.57,	1.59,	1.55,	1.56,	1.60,
	1.542,	1.538,	1.555,	1.552,	1.572,	1.576,	1.554,	1.576,	1.595,
	(0.2,3)	(0.2,4)	(0.2,5)	(0.3,3)	(0.3,4)	(0.3,5)	(0.4,3)	(0.4,4)	(0.4,5)

(4) Mechanical parameters

As shown in Table 7, under different mechanical parameters, results of the variational criterion consistently align closely with those of the FOS command. There is an average relative error of 0.9% between them, demonstrating good agreement. This indicates that the applicability of the variational criterion in unlined tunnels is not coincidental due to specific mechanical parameters.

Table 6. Factors of safety under different geometric parameters for unlined tunnels.

S	RSR	Result of the FOS Command, Result of the Variational Criterion								
		30			H_d 35			40		
		H_w			H_w			H_w		
		1.5	2.5	3.5	1.5	2.5	3.5	1.5	2.5	3.5
3	1/2	1.61,	1.52,	1.44,	1.53,	1.42,	1.35,	1.48,	1.39,	1.28,
		1.61	1.54	1.47	1.56	1.44	1.37	1.50	1.41	1.33
	1/3	1.65,	1.55,	1.49,	1.59,	1.46,	1.39,	1.52,	1.43,	1.33,
		1.67	1.57	1.53	1.60	1.49	1.40	1.57	1.43	1.34
	1/6	1.72,	1.59,	1.53,	1.62,	1.49,	1.41,	1.56,	1.46,	1.33,
		1.71	1.59	1.54	1.64	1.49	1.43	1.59	1.46	1.37
4	1/2	1.53,	1.45,	1.40,	1.49,	1.33,	1.34,	1.45,	1.37,	1.31,
		1.54	1.46	1.43	1.53	1.38	1.39	1.47	1.38	1.32
	1/3	1.57,	1.50,	1.44,	1.54,	1.38,	1.39,	1.49,	1.42,	1.35,
		1.70	1.50	1.46	1.54	1.39	1.39	1.50	1.44	1.36
	1/6	1.61,	1.53,	1.47,	1.57,	1.40,	1.42,	1.53,	1.45,	1.37,
		1.61	1.55	1.51	1.57	1.40	1.42	1.53	1.46	1.38
5	1/2	1.43,	1.35,	1.31,	1.39,	1.44,	1.28,	1.38,	1.32,	1.27,
		1.45	1.36	1.33	1.40	1.45	1.29	1.37	1.33	1.31
	1/3	1.48,	1.40,	1.35,	1.44,	1.49,	1.32,	1.42,	1.36,	1.31,
		1.48	1.40	1.36	1.44	1.50	1.32	1.42	1.41	1.34
	1/6	1.51,	1.42,	1.38,	1.47,	1.51,	1.35,	1.45,	1.39,	1.33,
		1.52	1.43	1.38	1.48	1.58	1.35	1.52	1.40	1.35

Table 7. Factors of safety under different mechanical parameters for unlined tunnels.

S	RSR	Result of the FOS Command, Result of the Variational Criterion								
		0.04			c (MPa) 0.05			0.06		
		φ (°)			φ (°)			φ (°)		
		20	25	30	20	25	30	20	25	30
3	1/2	1.27,	1.47,	1.68,	1.40,	1.61,	1.83,	1.52,	1.74,	1.96,
		1.28	1.49	1.70	1.41	1.61	1.85	1.53	1.76	1.97
	1/3	1.30,	1.51,	1.71,	1.44,	1.65,	1.87,	1.56,	1.79,	2.02,
		1.32	1.53	1.72	1.45	1.67	1.90	1.57	1.80	2.05
	1/6	1.36,	1.57,	1.79,	1.49,	1.72,	1.95,	1.63,	1.86,	2.10,
		1.38	1.57	1.78	1.52	1.71	1.99	1.62	1.86	2.09

3.2. Example 2: A Lined Tunnel

According to the paper [73], a numerical model of lined tunnel ($H = 40, D = 25.6, d = 0.2, s_1 = 2, s_2 = 10$) is established. The geometric dimensions of the tunnel and the locations of monitoring points are shown in Figure 7. The mechanical parameters of the surrounding rock and the lining are presented in Table 8. Both the surrounding rock and the lining are modeled as ideal elastoplastic materials. The materials follow the Mohr–Coulomb yield criterion and non-associated flow rule. Herein, the surrounding rock is classified as Class III, while the lining is constructed using the C25 concrete. The distance from the tunnel to the left side, right side, and bottom of the model is set at five times the span, respectively. The model is constrained in horizontal displacement on the left and

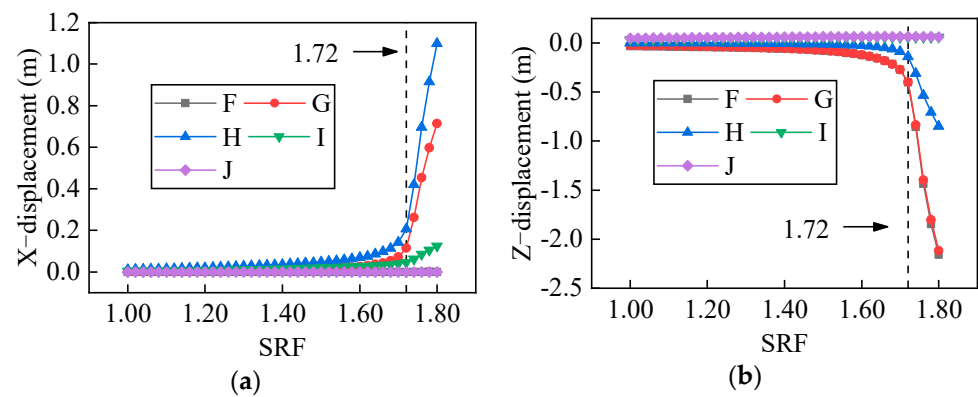


Figure 8. Curves between the displacements of the monitoring points and the SRFs for the lined tunnel. (a) The X-displacement; (b) the Z-displacement.

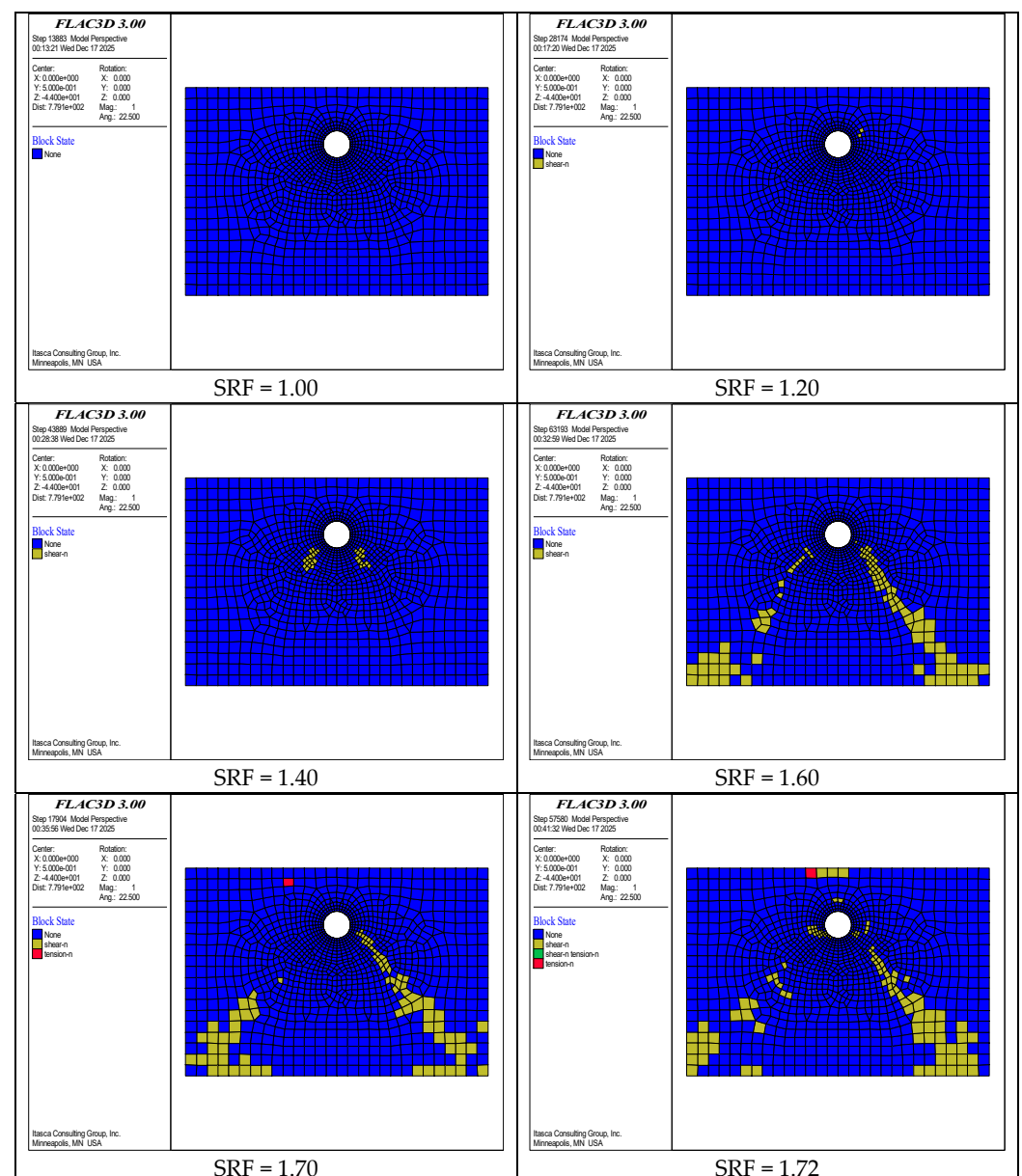


Figure 9. Cont.

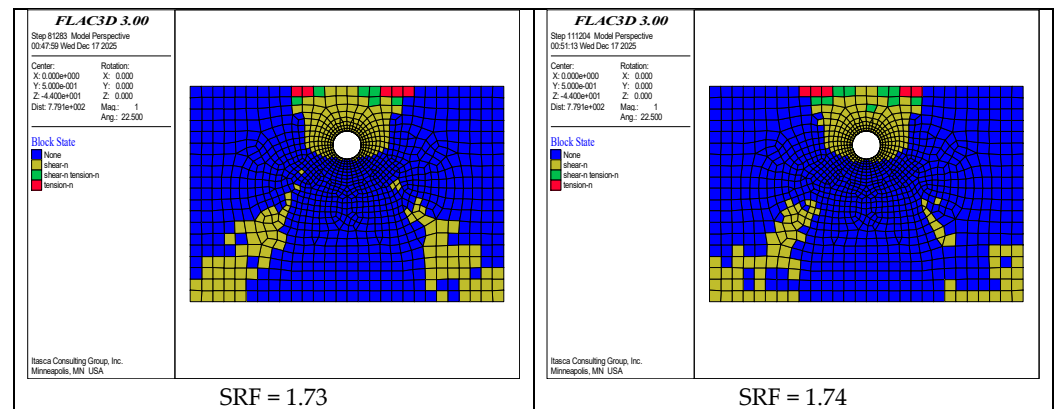


Figure 9. Distribution of plastic elements (red, green, and yellow elements) in the numerical model under different SRFs for the lined tunnel.

(3) Criterion III

As shown in Figure 10, the FOS command indicates a safety factor of 1.75 for the tunnel. In the region from the tunnel crown to the ground surface, both the shear strain increment and velocity are relatively high. Among these, the velocity around the tunnel is particularly significant. It indicates the occurrence of overall collapse. This is generally consistent with the patterns revealed by other criteria.

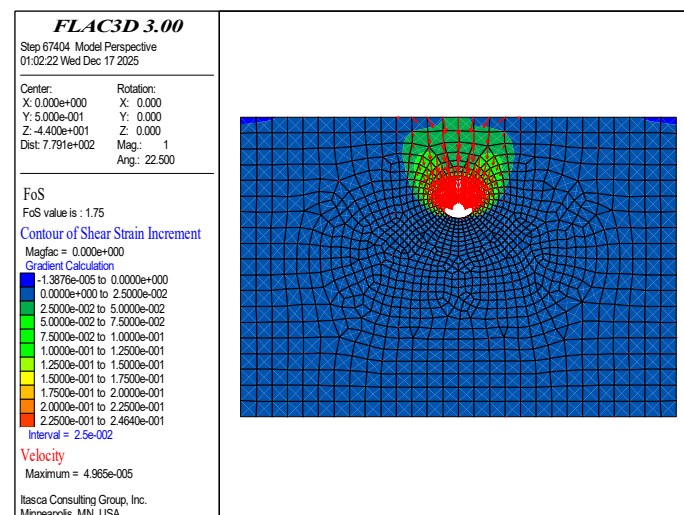


Figure 10. Results of the FOS command including the shear strain increment of elements and the velocity of nodes for the lined tunnel.

(4) Criterion IV

As shown in Figure 11, the gravitational and dissipated energy curves appear smooth near the mutation point, making it difficult to accurately identify the mutation point. In contrast, the kinetic energy curve exhibits a distinct mutation characteristic, with a clear and identifiable mutation point. This demonstrates a limitation of the energy catastrophe criterion: it requires manual identification of the mutation point, which can be challenging and less accurate for smooth curves.

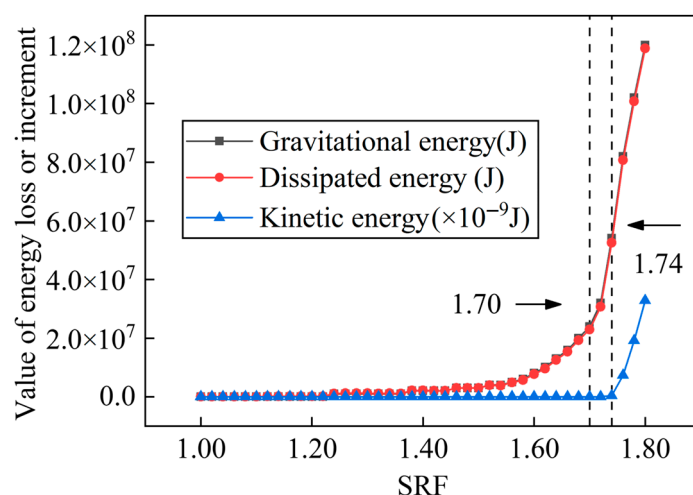


Figure 11. Curves between the various energy values and the SRFs for the lined tunnel.

(5) Variational criterion

As shown in Table 9, when the SRF ranges from 1.00 to 1.75, the variational values of the model remain positive. It indicates that instability has not occurred. When the SRF reaches 1.76, the variational value of the model turns negative for the first time. It signals the onset of instability.

Table 9. Variational values under different SRFs for the lined tunnel.

SRF	Variational Value	SRF	Variational Value	SRF	Variational Value
1.00	1.84×10^{-8}	1.26	2.01×10^{-8}	1.52	1.05×10^{-8}
1.01	1.72×10^{-8}	1.27	1.86×10^{-8}	1.53	1.05×10^{-8}
1.02	1.84×10^{-8}	1.28	1.73×10^{-8}	1.54	1.02×10^{-8}
1.03	2.02×10^{-8}	1.29	1.53×10^{-8}	1.55	1.10×10^{-8}
1.04	2.01×10^{-8}	1.30	1.51×10^{-8}	1.56	1.01×10^{-8}
1.05	2.07×10^{-8}	1.31	1.48×10^{-8}	1.57	9.45×10^{-9}
1.06	2.01×10^{-8}	1.32	1.46×10^{-8}	1.58	7.09×10^{-9}
1.07	1.85×10^{-8}	1.33	1.48×10^{-8}	1.59	7.13×10^{-9}
1.08	1.93×10^{-8}	1.34	1.57×10^{-8}	1.60	5.96×10^{-9}
1.09	1.83×10^{-8}	1.35	1.62×10^{-8}	1.61	8.30×10^{-9}
1.10	1.83×10^{-8}	1.36	1.54×10^{-8}	1.62	6.92×10^{-9}
1.11	1.57×10^{-8}	1.37	1.34×10^{-8}	1.63	6.98×10^{-9}
1.12	1.72×10^{-8}	1.38	1.39×10^{-8}	1.64	7.87×10^{-9}
1.13	1.64×10^{-8}	1.39	1.32×10^{-8}	1.65	7.85×10^{-9}
1.14	1.73×10^{-8}	1.40	1.37×10^{-8}	1.66	8.40×10^{-9}
1.15	1.78×10^{-8}	1.41	1.30×10^{-8}	1.67	9.77×10^{-9}
1.16	1.65×10^{-8}	1.42	1.44×10^{-8}	1.68	9.49×10^{-9}
1.17	1.58×10^{-8}	1.43	1.31×10^{-8}	1.69	1.18×10^{-8}
1.18	1.77×10^{-8}	1.44	1.20×10^{-8}	1.70	1.57×10^{-8}
1.19	1.95×10^{-8}	1.45	1.19×10^{-8}	1.71	1.18×10^{-8}
1.20	1.97×10^{-8}	1.46	1.21×10^{-8}	1.72	1.18×10^{-8}
1.21	1.84×10^{-8}	1.47	1.21×10^{-8}	1.73	2.41×10^{-8}
1.22	2.07×10^{-8}	1.48	1.12×10^{-8}	1.74	6.72×10^{-8}
1.23	2.11×10^{-8}	1.49	1.10×10^{-8}	1.75	1.18×10^{-6}
1.24	1.93×10^{-8}	1.50	1.13×10^{-8}	1.76	-3.18×10^{-7}
1.25	2.16×10^{-8}	1.51	1.11×10^{-8}		

(6) Comparisons

Results of different criteria are summarized in Table 10. As shown in Table 10, results of the three types of energy catastrophe criteria are close to those of the three commonly used failure criteria. The maximum absolute and relative errors between them are 0.05 and 2.9%, respectively. It indicates that for lined tunnels, there exists a favorable correspondence between the three commonly used failure criteria and the three energy catastrophe criteria. It is consistent with observations in slopes. Results of the variational criterion are close to those of the six typical failure criteria. There are the maximum absolute and relative errors of 0.05 and 2.9%, respectively. This demonstrates that the variational criterion exhibits good consistency with the six typical failure criteria in lined tunnels.

Table 10. Factors of safety under different criteria for the lined tunnel.

Criteria	Factor of Safety	Absolute Error		Relative Error	
Criterion I (1)	1.72	0.03	(17-11)	1.7%	(17-11/1)
Criterion II (2)	1.72	0.03	(17-21)	1.7%	(17-21/2)
Criterion III (3)	1.75	0.00	(17-31)	0.0%	(17-31/3)
Criterion IV ₁ (4)	1.70	0.05	(17-41)	2.9%	(17-41/4)
Criterion IV ₂ (5)	1.70	0.05	(17-51)	2.9%	(17-51/5)
Criterion IV ₃ (6)	1.74	0.01	(17-61)	0.6%	(17-61/6)
Variational criterion (7)	1.75	—	—	—	—

3.2.2. Analysis of Influencing Factors

As demonstrated earlier through the typical example ($H = 40, D = 25.6, d = 0.2, s_1 = 2, s_2 = 10$), results of the FOS command (Criterion III) are close to those of other criteria (Criteria I, II, IV₁, IV₂, and IV₃). However, applying Criteria I, II, IV₁, IV₂, and IV₃ involves a substantial amount of work. To save time, the subsequent analysis will use the results of the FOS command in place of those of the six conventional criteria (Criteria I, II, III, IV₁, IV₂, and IV₃). The mesh densities, strength reduction factor intervals, geometric parameters, and mechanical parameters are changed. The generality of the variational criterion is verified by comparing results of the FOS command with those of the variational criterion under different conditions.

(1) Mesh densities

As shown in Table 11, under different mesh densities, results of the variational criterion consistently align closely with those of the FOS command. There is an average relative error of 0.9% between them, indicating good agreement. This demonstrates that the applicability of the variational criterion in lined tunnels is not coincidental due to mesh densities.

Table 11. FOSs under different mesh densities for lined tunnels.

<i>d</i>	Result of the FOS Command, Result of the Variational Criterion, (<i>s</i> ₁ , <i>s</i> ₂)								
0.2	1.75,	1.75,	1.75,	1.74,	1.76,	1.77,	1.76,	1.78,	1.79,
	1.73,	1.74,	1.76,	1.73,	1.79,	1.76,	1.76,	1.74,	1.79,
	(2,8)	(2,10)	(2,12)	(2.5,8)	(2.5,10)	(2.5,12)	(3,8)	(3,10)	(3,12)
0.3	1.81,	1.82,	1.82,	1.82,	1.82,	1.83,	1.83,	1.84,	1.85,
	1.80,	1.83,	1.81,	1.79,	1.83,	1.81,	1.81,	1.83,	1.83,
	(2,8)	(2,10)	(2,12)	(2.5,8)	(2.5,10)	(2.5,12)	(3,8)	(3,10)	(3,12)
0.4	1.86,	1.87,	1.87,	1.87,	1.89,	1.89,	1.88,	1.89,	1.91,
	1.84,	1.85,	1.84,	1.85,	1.86,	1.87,	1.86,	1.90,	1.90,
	(2,8)	(2,10)	(2,12)	(2.5,8)	(2.5,10)	(2.5,12)	(3,8)	(3,10)	(3,12)

(2) Strength reduction factor intervals

As shown in Table 12, results with a strength reduction factor interval of 0.001 consistently remain close to those with an interval of 0.01. There is an average relative error of 0.7% between them, indicating good agreement. This demonstrates that the applicability of the variational criterion in lined tunnels is not coincidental due to strength reduction factor intervals.

Table 12. Factors of safety under different strength reduction factor intervals for lined tunnels.

<i>d</i>	Result with the Strength Reduction Factor Interval of 0.01, Result with the Strength Reduction Factor Interval of 0.001, (<i>s</i>₁,<i>s</i>₂)								
0.2	1.73,	1.74,	1.76,	1.73,	1.79,	1.76,	1.76,	1.74,	1.79,
	1.725,	1.734,	1.731,	1.735,	1.750,	1.745,	1.742,	1.749,	1.773,
	(2,8)	(2,10)	(2,12)	(2.5,8)	(2.5,10)	(2.5,12)	(3,8)	(3,10)	(3,12)
0.3	1.80,	1.83,	1.81,	1.79,	1.83,	1.81,	1.81,	1.83,	1.83,
	1.786,	1.793,	1.789,	1.792,	1.806,	1.809,	1.814,	1.816,	1.829,
	(2,8)	(2,10)	(2,12)	(2.5,8)	(2.5,10)	(2.5,12)	(3,8)	(3,10)	(3,12)
0.4	1.84,	1.85,	1.84,	1.85,	1.86,	1.87,	1.86,	1.90,	1.90,
	1.841,	1.852,	1.848,	1.851,	1.863,	1.867,	1.869,	1.877,	1.886,
	(2,8)	(2,10)	(2,12)	(2.5,8)	(2.5,10)	(2.5,12)	(3,8)	(3,10)	(3,12)

(3) Geometric parameters

As shown in Table 13, under different geometric parameters, results of the variational criterion consistently align closely with those of the FOS command. There is an average relative error of 0.7% between them, indicating good agreement. This demonstrates that the applicability of the variational criterion in lined tunnels is not coincidental due to specific geometric parameters.

Table 13. Factors of safety under different geometric parameters for lined tunnels.

<i>d</i>	Result of the FOS Command, Result of the Variational Criterion								
	40			50			60		
	<i>D</i>			<i>D</i>			<i>D</i>		
	15	20	25.6	15	20	25.6	15	20	25.6
0.2	2.30,	1.99,	1.75,	2.22,	1.91,	1.69,	2.15,	1.86,	1.65,
	2.27	2.01	1.75	2.19	1.92	1.69	2.15	1.85	1.65
0.3	2.40,	2.08,	1.82,	2.31,	1.98,	1.75,	2.22,	1.93,	1.70,
	2.39	2.10	1.81	2.28	1.97	1.77	2.23	1.90	1.69
0.4	2.51,	2.15,	1.87,	2.40,	2.05,	1.80,	2.31,	1.98,	1.75,
	2.49	2.12	1.85	2.39	2.04	1.78	2.30	1.96	1.74

(4) Mechanical parameters

As shown in Table 14, under different mechanical parameters, results of the variational criterion consistently align closely with those of the FOS command. There is an average relative error of less than 1.0% between them, indicating good agreement. This demonstrates that the applicability of the variational criterion in lined tunnels is not coincidental due to specific mechanical parameters.

Table 14. Factors of safety under different mechanical parameters for lined tunnels.

c_1 (MPa)	φ_1 (°)	Result of the FOS Command, Result of the Variational Criterion							
		$d = 0.2$				$d = 0.2$			
		c_2 (MPa)				c_2 (MPa)			
		0.2		0.3		0.2		0.3	
		φ_2 (°)		φ_2 (°)		φ_2 (°)		φ_2 (°)	
		20	30	20	30	20	30	20	30
1	30	1.12,	1.37,	1.41,	1.68,	1.19,	1.44,	1.46,	1.73,
		1.12	1.38	1.40	1.66	1.17	1.45	1.44	1.75
	40	1.13,	1.38,	1.42,	1.68,	1.21,	1.46,	1.48,	1.75,
		1.12	1.42	1.42	1.66	1.20	1.45	1.46	1.74
2	30	1.18,	1.44,	1.46,	1.73,	1.30,	1.56,	1.56,	1.84,
		1.20	1.43	1.45	1.73	1.28	1.55	1.59	1.81
	40	1.21,	1.46,	1.48,	1.75,	1.34,	1.61,	1.59,	1.87,
		1.19	1.45	1.47	1.75	1.36	1.59	1.60	1.85

In summary, for unlined and lined tunnels, a favorable correspondence exists between the three commonly used failure criteria and the three energy catastrophe criteria. It is consistent with observations in slopes. Furthermore, the variational criterion also demonstrates applicability in this context. This applicability is not coincidental due to mesh densities, strength reduction factor intervals, geometric parameters, and mechanical parameters.

3.3. Example 3: A Constructed Tunnel

According to the paper [74,75], a numerical model of constructed tunnel using the annular excavation with the reserved core soil method is established. The geometric dimensions of the tunnel and the numerical model are shown in Figure 12. Both the surrounding rock and the lining are modeled as ideal elastoplastic materials. The materials follow the Mohr–Coulomb yield criterion and non-associated flow rule. The mechanical parameters of materials are presented in Table 15. In this context, the strength parameters of the surrounding rock are determined with reference to the research of Tang et al. [74]. The lining adopts C25 concrete, with its strength parameters determined based on the research of Xiao et al. [75]. The construction steps of the annular excavation with reserved core soil method are listed in Table 16. The distance from the tunnel to the left side, right side, and bottom of the model is set at five times the span, respectively. The model is constrained in horizontal displacement on the left and right sides, fixed at the bottom, and free at the top. The longitudinal length of the tunnel is set to 1 m.

Table 15. Mechanical parameters of the material for the constructed tunnel.

Materials	Thickness (m)	Young's Modulus E /kPa	Poisson's Ratio μ	Unit Weight γ /(kN/m ³)	Cohesion c /kPa	Internal Friction Angle φ /(°)
Surrounding rock	—	5×10^5	0.40	19.0	343.0	14.2
Lining	0.28	1×10^8	0.15	25.0	1810	43.1

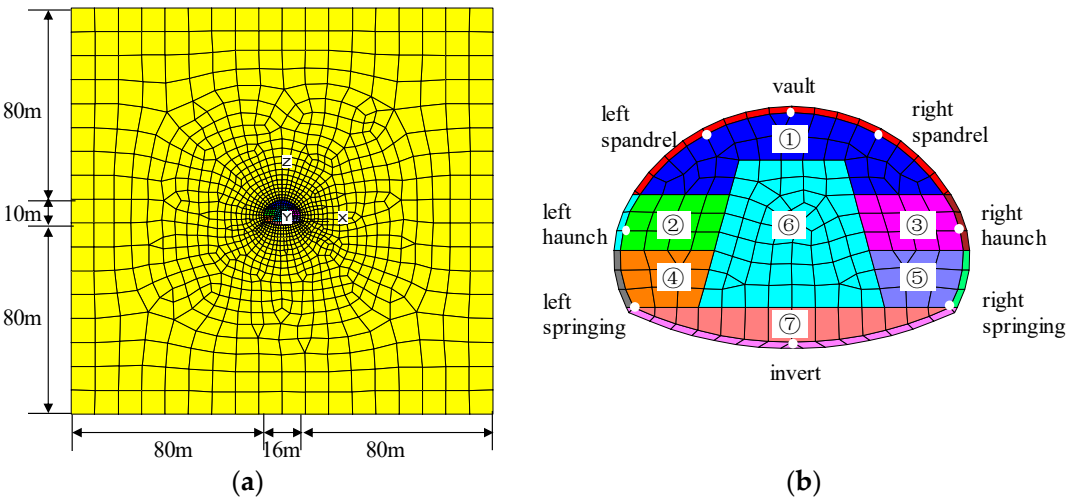


Figure 12. Overall and detailed drawings of the numerical model of the constructed tunnel. (a) Overall drawings of the model; (b) detailed drawings of the model, the tunnel interior is divided into seven excavation sections and these are represented by seven distinct colors and serial numbers.

Table 16. Divisions of construction stages for the constructed tunnel.

Construction Stage	Work Description
1	Remove section ① and apply the primary support around it.
2	Remove section ② and apply the primary support around it.
3	Remove section ③ and apply the primary support around it.
4	Remove section ④ and apply the primary support around it.
5	Remove section ⑤ and apply the primary support around it.
6	Remove section ⑥.
7	Remove section ⑦ and apply the primary support around it.

3.3.1. Comparisons of Different Criteria Under the Typical Condition

Taking construction stage 3 as a typical condition, the SRM and various criteria are applied to determine the safety factor of the tunnel.

(1) Criterion I

As shown in Figure 13, the characteristics of mutations in the curves vary across different monitoring points and displacement directions. Regarding the X-displacement, the most pronounced mutation occurs at the left haunch, followed by a slightly weaker mutation at the left spandrel, while no mutation is observed at the vault. For the Z-displacement, the most significant mutations are noted at the vault and left spandrel, with a relatively weaker mutation at the left haunch. Under this condition, a free face is not formed at the left springing and the invert. Therefore, the characteristics of their displacement curves are not analyzed. It is evident that the effective application of this criterion requires the selection of appropriate monitoring points and displacement directions.

(2) Criterion II

As shown in Figure 14, when the SRF ranges from 1.00 to 1.80, almost no failure elements appear around the tunnel. As the SRF increases to 1.97–1.99, a significant number of failure elements begin to develop at the left and right arch waists of the tunnel. These elements extend upward and downward along approximately a 45-degree angle relative to the horizontal. Concurrently, tensile failure elements start to emerge near the ground surface above the tunnel. When the SRF reaches 2.00, both tensile and shear failure elements above the tunnel increase substantially. It means that there is a condition for the overall collapse.

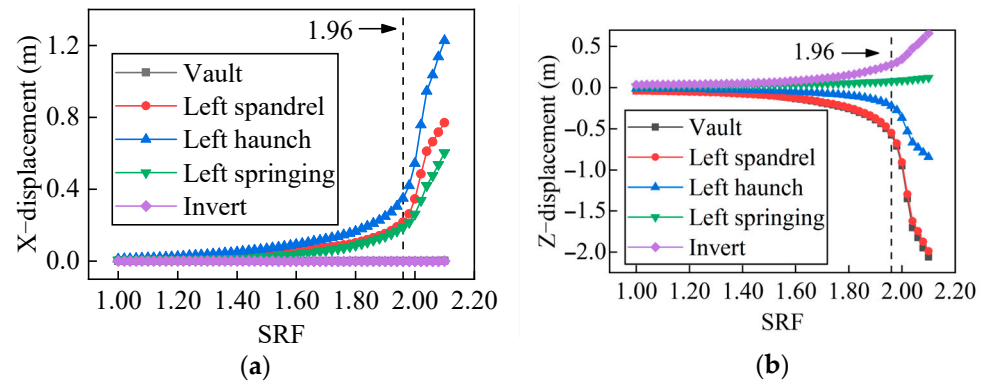


Figure 13. Curves between the displacements of the monitoring points and the SRFs for the constructed tunnel. (a) The X-displacement; (b) the Z-displacement.

(3) Criterion III

As shown in Figure 15, the FOS command yields a safety factor of 2.03 for the tunnel. When the model becomes unstable, significant node velocities are observed both surrounding and overlying regions of the tunnel. It aligns with the mechanism of overall collapse and is largely consistent with the patterns revealed by other criteria.

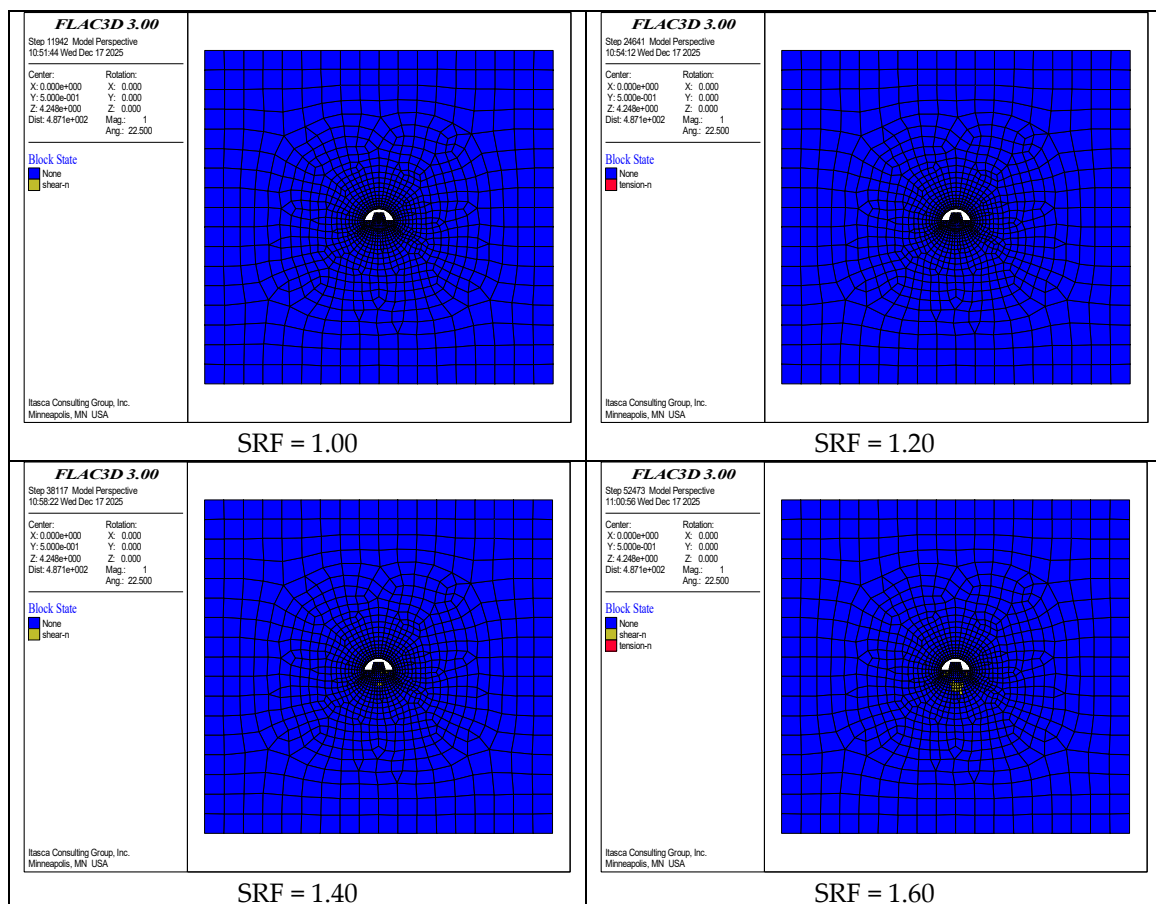


Figure 14. Cont.

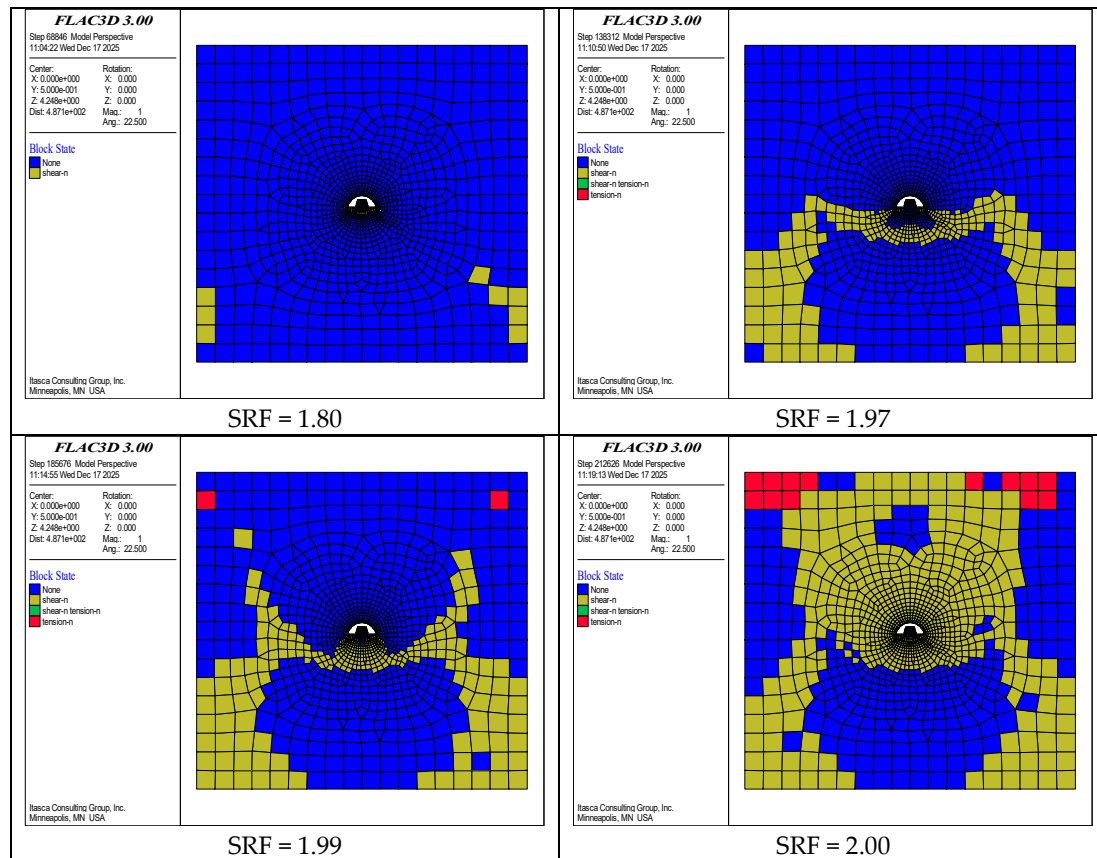


Figure 14. Distribution of plastic elements (red, green, and yellow elements) in the numerical model under different SRFs for the constructed tunnel.

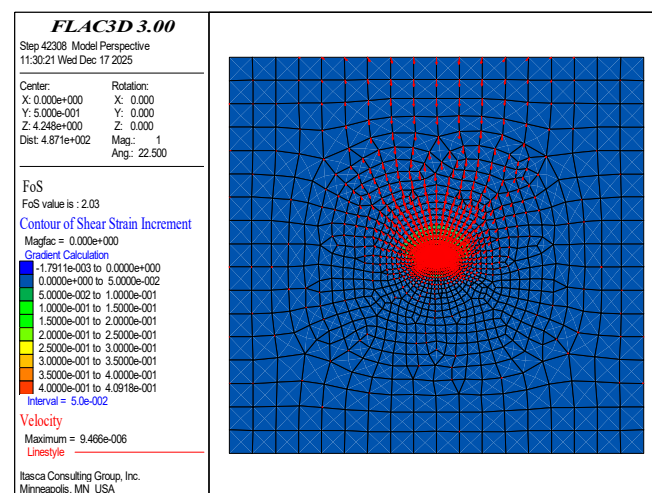


Figure 15. Results of the FOS command including the shear strain increment of elements and the velocity of nodes for the constructed tunnel.

(4) Criterion IV

As illustrated in Figure 16, the mutation points in the gravitational and dissipated energy curves are not distinct, making them difficult to identify. In contrast, the mutation point in the kinetic energy curve is relatively pronounced and easier to recognize. This highlights a limitation of the energy catastrophe criterion: the identification of mutation points relies on manual interpretation, which may introduce human error, particularly when the curves are relatively smooth.

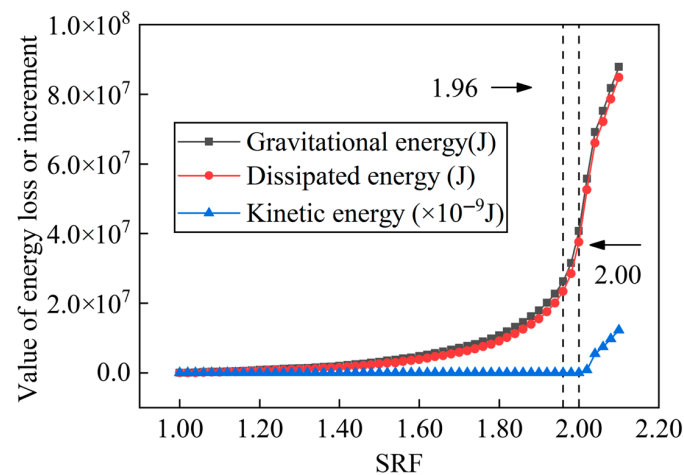


Figure 16. Curves between the various energy values and the SRFs for the constructed tunnel.

(5) Variational criterion

As shown in Table 17, when the SRF ranges from 1.00 to 1.99, the variational values of the model remain positive. It indicates that the instability has not occurred. When the SRF reaches 2.00, the variational value of the model turns negative for the first time. It signals the onset of instability.

Table 17. Variational values under different SRFs for the constructed tunnel.

SRF	Variational Value	SRF	Variational Value	SRF	Variational Value
1.00	1.65×10^{-8}	1.34	3.19×10^{-8}	1.68	1.69×10^{-8}
1.01	1.53×10^{-8}	1.35	3.16×10^{-8}	1.69	1.99×10^{-8}
1.02	1.67×10^{-8}	1.36	3.52×10^{-8}	1.70	2.19×10^{-8}
1.03	1.87×10^{-8}	1.37	3.51×10^{-8}	1.71	2.07×10^{-8}
1.04	1.65×10^{-8}	1.38	3.31×10^{-8}	1.72	1.67×10^{-8}
1.05	2.04×10^{-8}	1.39	3.83×10^{-8}	1.73	1.68×10^{-8}
1.06	1.85×10^{-8}	1.40	3.69×10^{-8}	1.74	2.01×10^{-8}
1.07	1.87×10^{-8}	1.41	3.23×10^{-8}	1.75	2.13×10^{-8}
1.08	1.86×10^{-8}	1.42	3.61×10^{-8}	1.76	2.38×10^{-8}
1.09	1.76×10^{-8}	1.43	3.23×10^{-8}	1.77	1.86×10^{-8}
1.10	1.73×10^{-8}	1.44	2.90×10^{-8}	1.78	2.02×10^{-8}
1.11	1.73×10^{-8}	1.45	2.62×10^{-8}	1.79	1.78×10^{-8}
1.12	1.63×10^{-8}	1.46	2.79×10^{-8}	1.80	2.03×10^{-8}
1.13	1.67×10^{-8}	1.47	3.06×10^{-8}	1.81	1.70×10^{-8}
1.14	1.69×10^{-8}	1.48	2.78×10^{-8}	1.82	1.86×10^{-8}
1.15	1.76×10^{-8}	1.49	2.60×10^{-8}	1.83	2.91×10^{-8}
1.16	1.81×10^{-8}	1.50	2.72×10^{-8}	1.84	2.25×10^{-8}
1.17	1.88×10^{-8}	1.51	2.83×10^{-8}	1.85	2.32×10^{-8}
1.18	1.88×10^{-8}	1.52	2.96×10^{-8}	1.86	2.04×10^{-8}
1.19	2.05×10^{-8}	1.53	2.60×10^{-8}	1.87	1.84×10^{-8}
1.20	1.83×10^{-8}	1.54	2.74×10^{-8}	1.88	1.95×10^{-8}
1.21	1.94×10^{-8}	1.55	2.76×10^{-8}	1.89	2.43×10^{-8}
1.22	1.90×10^{-8}	1.56	2.48×10^{-8}	1.90	2.94×10^{-8}
1.23	2.04×10^{-8}	1.57	2.39×10^{-8}	1.91	3.08×10^{-8}
1.24	2.09×10^{-8}	1.58	2.42×10^{-8}	1.92	3.21×10^{-8}
1.25	2.09×10^{-8}	1.59	2.31×10^{-8}	1.93	2.55×10^{-8}
1.26	2.27×10^{-8}	1.60	2.50×10^{-8}	1.94	2.04×10^{-8}
1.27	2.32×10^{-8}	1.61	2.46×10^{-8}	1.95	1.53×10^{-8}
1.28	2.81×10^{-8}	1.62	2.49×10^{-8}	1.96	1.35×10^{-8}
1.29	2.38×10^{-8}	1.63	2.58×10^{-8}	1.97	1.22×10^{-8}
1.30	2.76×10^{-8}	1.64	2.33×10^{-8}	1.98	9.25×10^{-9}
1.31	2.65×10^{-8}	1.65	2.45×10^{-8}	1.99	9.01×10^{-9}
1.32	2.78×10^{-8}	1.66	2.21×10^{-8}	2.00	-2.54×10^{-8}
1.33	2.89×10^{-8}	1.67	1.99×10^{-8}		

(6) Comparisons

Results of different criteria are summarized in Table 18. As shown in Table 18, results of the three types of energy catastrophe criteria are close to those of the three commonly used failure criteria. There are maximum absolute and relative errors of 0.04 and 2.0%, respectively. This indicates that in engineering tunnels, a favorable correspondence exists between the three commonly used failure criteria and the three energy catastrophe criteria. It is consistent with observations in slopes. Furthermore, results of the variational criterion are close to those of the six typical failure criteria. There are the maximum absolute and relative errors of 0.04 and 2.0%, respectively. It demonstrates that the variational criterion exhibits good consistency with the six typical failure criteria in tunnels under construction.

Table 18. Factors of safety under different criteria for the constructed tunnel.

Criteria	Factor of Safety	Absolute Error	Relative Error
Criterion I (1)	1.96	0.03 (17-11)	1.5% (17-11/1)
Criterion II (2)	1.99	0.00 (17-21)	0.0% (17-21/2)
Criterion III (3)	2.03	0.04 (17-31)	2.0% (17-31/3)
Criterion IV ₁ (4)	1.96	0.03 (17-41)	1.5% (17-41/4)
Criterion IV ₂ (5)	1.96	0.03 (17-51)	1.5% (17-51/5)
Criterion IV ₃ (6)	2.00	0.01 (17-61)	0.5% (17-61/6)
Variational criterion (7)	1.99	—	—

3.3.2. Safety Factors Under All Conditions

As is shown in Figure 17, the change curves of safety factors under different construction stages are plotted. (1) As construction progresses, the safety factor of the tunnel decreases. When the construction reaches its later stages, the safety factor remains essentially unchanged. This is likely because, during the initial excavation phase, the free surface area of the tunnel is small, and it results in a higher safety factor. During the intermediate excavation phase, the free surface area increases, and it leads to a lower safety factor. In the later excavation stages, a greater amount of primary support is applied. It effectively resists tunnel deformation and stabilizes the safety factor. (2) As the construction stages advance, there are maximum absolute and relative errors of 0.05 and 2.5% between results of the variational criterion and those of the FOS command. It indicates consistent agreement between the two. (3) Results of the variational criterion are generally lower than those of the FOS command. It suggests that the variational criterion is more sensitive to stability identification. Consequently, results of the variational criterion are more conservative and thus safer for engineering applications.

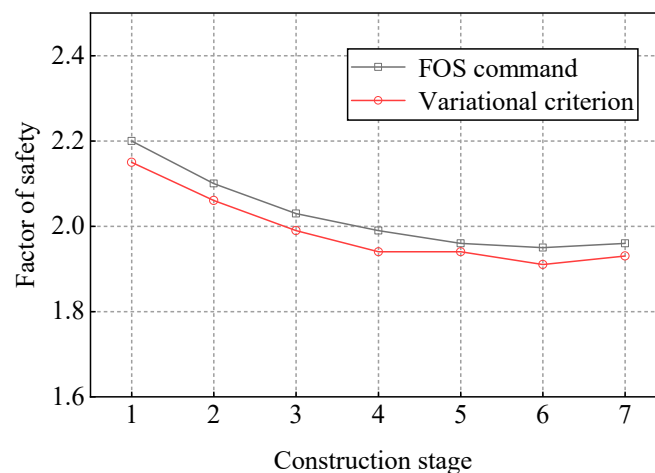


Figure 17. Factors of safety of the tunnel under different construction stages.

In summary, the variational criterion is applicable for stability evaluation of engineering tunnels. The study reveals that, compared to indicators such as displacement and plastic zone, engineering tunnels exhibit greater sensitivity to changes in the variational value. This is further evidenced by the generally lower results of the variational criterion. Additionally, the variational value is not influenced by the complex geometric shapes or stress conditions of tunnels. It demonstrates broad application prospects.

3.4. Example 4: A Tunnel with Complex Loading Conditions

To further verify the applicability of the variational criterion, a tunnel example under complex loading conditions is selected. Tschuchnigg et al. [8] employed the SRM and limit analysis method to provide specific values for the undrained stability safety factor of a tunnel face under specific conditions. In their study, the SRM was implemented using total stress analysis based on the Plaxis 2D software-2011. The specific conditions refer to a tunnel burial depth equal to the tunnel diameter ($H_T/D_T = 1$), a tunnel face support pressure of 75 kPa ($\sigma_T = 75$ kPa), and multiple combinations of effective cohesion and internal friction angle (different c' and ϕ'). Based on the above classical example, the verification is carried out. Noting that, in this example, the supporting pressure can be regarded as a prescribed stress boundary. Under the undrained condition, the saturated soil is treated as a special type of soil with non-zero cohesion and zero internal friction angle. Therefore, the derived variational value calculation formula and the proposed variational criterion remain applicable.

3.4.1. Establishment of the Numerical Model

The geometric dimensions of the tunnel are presented in Figure 18a. In addition to the mechanical parameters presented in Figure 18a, the remaining parameters are listed in Tables 19 and 20. As shown in Figure 18b, a corresponding numerical model is established in the FLAC^{3D} software based on the shown geometric dimensions. To achieve reliable computational accuracy and efficiency, a finer mesh with an element size of 0.2 m is adopted around the tunnel, while a coarser mesh with an element size of 2 m is used near the model boundaries. The application process of the SRM based on total stress analysis is illustrated in Figure 19.

Table 19. Undrained shear strength parameters ($\psi' = 0$).

$\rho D_T/s_{u0}$		0.00	0.40	1.20	2.00	4.00	6.00	8.00	12.00
$H_T/D_T = 1$	$s_{u0} : \text{kPa}$	60.80	43.43	27.64	20.27	12.16	8.69	6.76	4.68
	$\rho : \text{kPa/m}$	0.00	2.17	4.15	5.07	6.08	6.51	6.76	7.02

Table 20. Effective shear strength parameters ($\psi' = 0$).

$\rho D_T/s_{u0}$		0.00	0.40	1.20	2.00	4.00	6.00	8.00	12.00
$H_T/D_T = 1$	$c' : \text{kPa}$	60.80	43.75	28.54	21.37	13.27	9.67	7.62	5.36
	$\phi' : \text{degrees}$	0.00	6.99	14.43	18.48	23.58	26.06	27.55	29.25

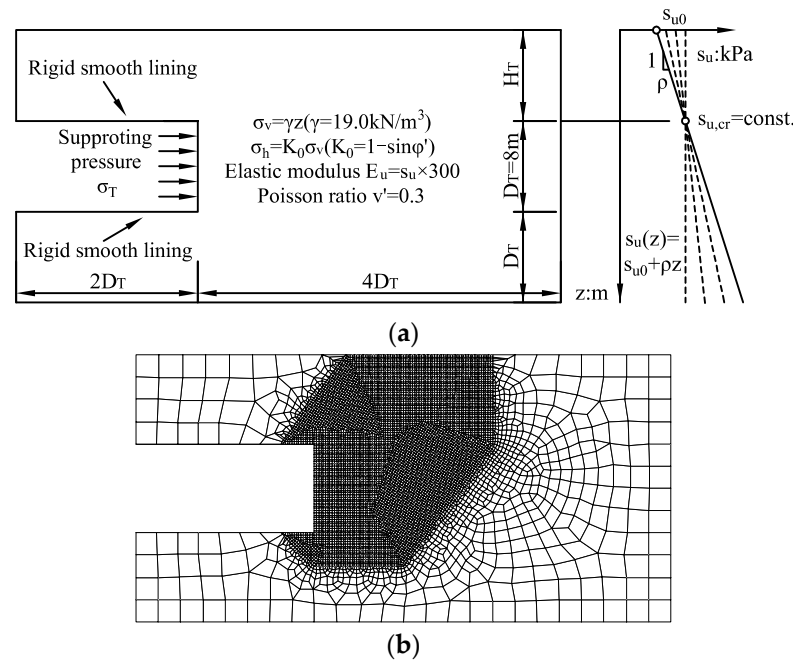


Figure 18. Sketch and numerical model of the complex loading tunnel. (a) Sketch of the tunnel, dashed lines represent the different values of ρ ; (b) numerical model of the tunnel.

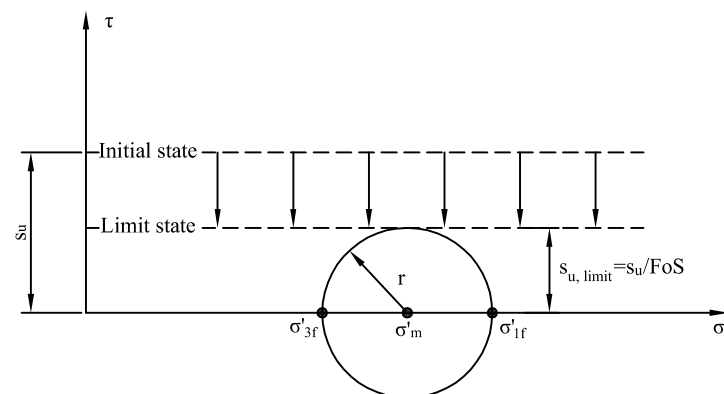


Figure 19. Mohr's circle at failure (total stress analysis).

3.4.2. Verification of the Variational Criterion

Based on the above computational conditions, a FISH language code is developed to calculate the safety factor of the tunnel face using the variational criterion. The results are compared with those from the reference paper [8] to validate the applicability of the variational criterion. The relative results are summarized in Table 21. As shown in Table 21, results of the variational criterion in the FLAC^{3D} are close to those of the convergence criterion in the PLAXIS and the limit analysis method. The average relative errors are 1.6% and 2.0%, respectively. It indicates that the variational criterion remains highly applicable and accurate for complex loading tunnels.

Table 21. Results under different methods and relative errors among them.

$\rho D_T/s_{u0}$	Limit Analysis Method	SRM		Relative Error	
	Average of the Upper and Lower Bounds [8] (1)	PLAXIS-Convergence Criterion [8] (2)	FLAC ^{3D} -Variational Criterion (This Paper) (3)	3-1 /1	3-2 /2
0.00	1.60	1.60	1.61	0.6%	0.6%
0.40	1.68	1.68	1.71	1.8%	1.8%
1.20	1.74	1.73	1.77	1.7%	2.3%
2.00	1.76	1.75	1.79	1.7%	2.3%
4.00	1.77	1.76	1.80	1.7%	2.3%
6.00	1.78	1.77	1.80	1.1%	1.7%
8.00	1.78	1.77	1.82	2.2%	2.8%
12.00	1.78	1.77	1.81	1.7%	2.3%
Average	—	—	—	1.6%	2.0%

4. Discussion

Based on the above analysis, it can be verified that Criterion IV is applicable. The energy value can be a new indicator for tunnel stability when the SRM is used. In tunnels, there are some drawbacks of Criteria I, II, and III which are similar to those in slopes [76–78]. Specifically, the validity of Criterion I is influenced by the location of monitoring points and the direction of monitoring displacements. Improper selection of these conditions will lead to incorrect results (Figures 3, 8 and 13). Criterion II requires an understanding of the failure mechanism of the tunnel. It carries certain uncertainties, as different types of tunnels have different failure mechanisms, leading to varied standards for defining plastic zone connectivity (Figures 4, 9 and 14). For Criterion III, there is currently no clear and unified standard for the computational program non-convergence. For instance, You et al. [37] used the finite difference program FLAC, while Tschuchnigg et al. [8] employed the finite element program PLAXIS. The convergence mechanisms of the two programs differ. In summary, there is a lack of unified stability indicators and identifying instability methods for the three criteria above. Criterion IV, to a certain extent, unifies the stability indicator—specifically, the various energies of the model. It also partially standardizes the identifying instability method by determining the mutation of the energy curves (Figures 6, 11 and 16). Furthermore, it is found that for engineering tunnels, Criterion IV and the proposed variational criterion demonstrate higher sensitivity in identifying instability compared with other criteria (Table 18 and Figure 17). However, Criterion IV also demonstrates some disadvantages. Some of the shortcomings have been pointed out by scholars [78]: (1) it requires integral calculations over all elements of the numerical model, resulting in high computational load and slow processing speed; (2) the identification of mutation of the energy curves relies on manual judgment, introducing subjective errors. Others are revealed in this paper: (i) the various energy values are calculated and analyzed separately (Equation (2); Figures 6, 11 and 16), and the complete stress–strain information of the model cannot be considered simultaneously; (ii) it is difficult to identify the mutation point when the curve is smooth (Figures 11 and 16).

The proposed variational criterion inherits the advantages and addresses the disadvantages of Criterion IV. For inheriting the advantages, the variational criterion more effectively standardizes the stability indicator—specifically, the variational value. This criterion also better unifies the identifying instability method by examining the positive or negative of the variational value. For addressing the disadvantages, the variational value is calculated over

a virtual time step. It requires minimal computational effort and delivers fast processing speed. By the variational criterion, it quantifies the qualitative identification of “mutation of the energy curve,” thereby reducing human errors. Furthermore, the variational value calculation formula incorporates all kinds of energies (Equations (3)–(8)). So, the variational value accounts for the complete stress–strain information of the model at the same time. Finally, in the variational criterion, there is no need to consider whether the energy curve is smooth or steep. Certainly, it is observed that results of the variational criterion are slightly influenced by the convergence criteria. This influence leads to approximately $\pm 2\%$ relative errors in the results. In this study, the convergence criteria generally adopt a limit time step number of 10,000 and a maximum unbalanced force ratio of 1.0×10^{-5} . How to establish more reasonable convergence criteria to obtain more accurate results based on the variational criterion remains a further investigation.

Overall, displacement is the integral of strain, and Criterion I reflects the effect of strain in some elements of the model. Generally, plastic yielding is governed by the stress function, and Criterion II reflects the effect of stress in some elements of the model. The convergence criteria in general computational procedures are set based on stress or strain, and Criterion III reflects the effect of stress or strain in all elements of the model. Energy is the integral of the product of stress and strain, and Criterion IV (Criterion IV₁, or Criterion IV₂, or Criterion IV₃) reflects the effect of partial stress–strain products in all elements of the model. None of these criteria can simultaneously account for the full stress–strain product effects across the entire model region. Consequently, it lacks a comprehensive indicator, and there is no unified standard for determining instability in tunnels. The variational criterion addresses these shortcomings.

In specific tunnels, the proposed variational criterion exhibits several relatively distinct characteristics. (1) Simplicity: It eliminates the setting for monitoring points, the analysis of plastic zone development, the determining of computational program non-convergence, and the calculating of energy values. (2) Comprehensiveness: The variational value accounts for the stress–strain conditions of all elements in the model simultaneously, overcoming the limitations of traditional criteria. (3) Clarity: It enables quantitative measurement of “instability”, reducing the subjectivity and ambiguity of other criteria. (4) Reliability: It is derived based on the principle of minimum potential energy, providing a solid mechanical basis. (5) Uniformity: It is independent of various conditions, applying the same computational method, procedure, and process to different geometric and mechanical conditions of tunnels. In summary, the variational criterion establishes a one-to-one correspondence between the tunnel stability and the sign of the variational value. This criterion provides a theoretical basis and mechanical explanation for tunnel failure. Therefore, the variational value can also be a new indicator for tunnel stability.

5. Conclusions

In this paper, the energy catastrophe (Criterion IV) of the strength reduction method (SRM) is introduced to tunnels. A new failure criterion of the SRM—the variational criterion—is proposed. By comparing with the monitoring point displacement mutation (Criterion I), the plastic zone penetration (Criterion II), and the numerical program non-convergence (Criterion III) through different tunnels, the applicability and generality of the Criterion IV and the variational criterion are validated. The main conclusions are as follows:

(1) Compared to indicators such as displacement and plastic zone, engineering tunnels exhibit greater sensitivity to changes in the energy or variational value. The energy or variational value can serve as a new indicator for tunnel stability. Results of the varia-

tional criterion are generally lower, making them more conservative and thus safer for engineering applications.

(2) The variational criterion is applicable to stability identification in unlined tunnels, lined tunnels, engineering tunnels, and complex loading tunnels. This applicability is not coincidental due to mesh densities, strength reduction factor intervals, geometric parameters, and mechanical parameters.

(3) In specific tunnels, the variational criterion exhibits the following relatively distinct characteristics. (a) Simplicity: It relies on only one indicator. (b) Comprehensiveness: It synthesizes the evolutionary trends of all elements. (c) Clarity: It enables quantitative measurement of “instability”. (d) Reliability: It is derived based on the principle of minimum potential energy. (e) Uniformity: It is independent of various geometric and mechanical conditions of tunnels.

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