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Transformation of Trigonometric Functions into Hyperbolic Functions Based on Cable Statics

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Abstract: Trigonometric functions are widely used to express relationships between the sides and angles of triangles, being fundamental in a wide variety of fields in science and engineering. Previous research indicates that the Gudermann function connects hyperbolic and trigonometric functions without requiring complex numbers, while the Mercator projection maps the Earth's spherical surface onto a cylindrical plane. This article presents four key contributions derived from hyperbolic functions, with the main proof applying Newton's first law in the static case of cables. First, a new method relates right triangle formulas to the sides of a right triangle, facilitating vector decomposition along the X and Y axes. Second, a right triangle function with a hyperbolic angle is proposed, relating the three sides of a right triangle and the hyperbolic angle, offering an alternative to the Pythagorean theorem. Third, the law of hyperbolic cosines and the law of hyperbolic tangents is applied to trigonometric problems. Fourth, the hyperbolic Mollweide's formula is used to solve oblique triangles. These results demonstrate the potential of hyperbolic transformations in engineering and mathematical contexts, for both education and research. Future investigations should include experimental and analytical tests to further extend the applications to all branches based on mathematics.

Keywords: law of hyperbolic tangents; law of hyperbolic cosines; trigonometry; triangles; hyperbolic function transformation

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1. Introduction

Pythagoras's theorem and trigonometric functions are widely used to solve right triangles. In contrast, the analysis of oblique triangles relies on alternative methodologies, such as the law of sines and the law of cosines. However, some approaches remain underutilized in education; for example, Mollweide's formulas define the relationships among a triangle's sides and angles [1]. Indeed, the tangent law can be used for the study of oblique triangles and is derived from Mollweide's formulas. In teaching the analysis of oblique triangles, sum and difference trigonometric identities and Euler's formulas are used [1,2]. However, the study of hyperbolic functions has been limited in the context of these triangles, with the focus primarily on their relationship with trigonometric functions.

Trigonometric functions are fundamental for the calculation of hyperbolic functions due to their Eulerian properties, which are related through complex numbers. In fact, the identities that relate these functions are derived using Euler's number [3]. Other, less commonly explored studies have linked trigonometric functions with hyperbolic functions due to their geometric properties [4,5]. Thus, the Gudermann function is derived, which relates these trigonometric and hyperbolic functions without resorting to complex numbers. One application of the Gudermann function is the Mercator map projection,

which involves mapping the Earth's surface by projecting the spherical surface of the Earth onto a cylindrical surface, preserving the true length of the parallels while distorting the equatorial lines in the projection [5,6]. Traditionally, the solution of triangles is approached through trigonometry, overlooking the analysis of triangles with hyperbolic functions in educational contexts [1].

This manuscript is aimed at both mathematics education and research, as it integrates mathematical formulations derived from physical phenomena with practical applications in engineering. By linking traditional trigonometric concepts with innovative methods based on hyperbolic functions, it provides a powerful didactic tool and opens a new field of study that encourages future research, enriching both areas. The research proposes an original methodology that extracts a specific physical phenomenon, cable engineering, to establish a precise relationship between trigonometric and hyperbolic functions [7–10]. This approach not only provides a robust mathematical framework for solving triangles in engineering problems but also facilitates the coherent integration of theory and practice. The results obtained have the potential to enrich mathematics education and drive research exploring the applications of hyperbolic functions derived from physical phenomena in undergraduate mathematics and engineering programs.

The solution of triangles has traditionally been approached through trigonometry, a fundamental discipline in teaching mathematics at the high school and university levels. The handling of triangles relies on the use of trigonometric functions, equations, and identities that facilitate the resolution of angles and the application of classical theorems, such as Pythagoras's theorem, which are foundational for calculus in engineering due to the geometric properties inherent in these figures [9]. However, in the context of engineering, particularly in the statics of cables, physical phenomena emerge where trigonometry is closely related to hyperbolic functions, enabling the approach of complex problems from an innovative perspective [7,10].

This article shows the deduction of the hyperbolic transformation applied to right triangles. The theorem is an alternative method for solving engineering problems and can be easily used in industrial applications and in teaching calculations, for example, to determine distances, in problems of stability of rigid bodies, and in electricity [2,7]. Other alternatives for the use of the proposed transformation are applied to Mollweide's formulas. In the literature, its use is reported for the training of high school and preuniversity students. This is an alternative teaching method for the law of sines and cosines, used in the solution of oblique triangles [1]. The contributions of the transformation of trigonometric functions to hyperbolic functions are as follows:

1. The establishment of right triangle formulas that relate hyperbolic functions to the sides of a right triangle with hyperbolic functions.
2. The right triangle function with a hyperbolic angle that relates the three sides of a right triangle, the hyperbolic angle, and Euler's number, without the need for hyperbolic functions, trigonometric functions, or imaginary numbers.
3. The application of the law of hyperbolic cosines and hyperbolic tangents in the solution of engineering problems.
4. The use of the hyperbolic Mollweide's formulas for the solution of oblique triangles.

The article has been structured as follows: In the first part, the background that has been carried out will be outlined. In the second part, the deduction of the transformation of trigonometric functions into hyperbolic functions will be introduced. The third part will cover the applications, followed by a discussion of the results, and the article will summarize with the conclusions.

2. Background

The Background Section provides a general literature review to demonstrate the novelty of the proposed approach. Various mathematical tools have been developed since the 17th century with great discoveries and scientific contributions. Scientists and mathematicians observed that certain combinations of exponential functions were frequently presented in applications such as the decay of entities such as light, electricity, speed, radioactivity, the fall of a body through a viscous medium [8], the diffusion of a gas in a porous medium, the transmission of heat [3], geodesic triangles [11] and so on. At the beginning of the 18th century, in an attempt to broaden the concept of imaginary and complex, the functions sin φ and cos φ were defined by complex exponential addition and subtraction, and Euler obtained the following expressions for cos φ and sin φ as shown in [2,12]:

$$\cos\varphi = \frac{e^{i\varphi} + e^{-i\varphi}}{2}; \sin\varphi = \frac{e^{i\varphi} - e^{-i\varphi}}{2i}. \tag{1}$$

The success and recognition obtained by Euler for rediscovering the Cotes formula is not entirely undeserved [2]. These combinations of addition and subtraction of exponentials can be deduced from a hyperbole [7,8]; this is how real numbers are related to imaginary and complex numbers. These identities, shown in Equation (2) as (hyperbolic sine of an angle) sinh $i\varphi$ and (hyperbolic cosine of an angle) cosh $i\varphi$ [3,11,12], have been related in an analogous way to the trigonometric functions for a hyperbolic angle x with identities such as (hyperbolic sine) sinh x, (hyperbolic cosine) cosh x, (hyperbolic tangent) tanh x, (hyperbolic cosecant) csch x, (hyperbolic secant) sech x, and (hyperbolic cotangent) coth x [3,11,13–16]. Equation (2) establishes a relationship with Equation (1) using hyperbolic functions. Thus, the similarity between trigonometric and hyperbolic functions is proven [11,15,17–19], as shown below:

$$\cosh i\varphi = \frac{e^{i\varphi} + e^{-i\varphi}}{2}; \quad \sinh i\varphi = \frac{e^{i\varphi} - e^{-i\varphi}}{2}. \tag{2}$$

Hyperbolic functions have their name because the geometry with which they are built is defined on a hyperbole with trigonometric functions, also called circular functions, where their geometry is described with the shape of a circle [14,15]. Hyperbolic functions describe one side of the hyperbola; they are also called aperiodic functions, and the angle of the hyperbola can vary from 0 to ∞ [14].

Hyperbolic functions can be derived because they are related to curve geometries such as hyperbolas, ellipses, and catenaries and are frequently used to describe real-life applications based on engineering concepts. There are several identities for relating hyperbolic functions; one of the most fundamental is shown in [11,13–15,17]:

$$\cosh^2 x - \sinh^2 x = 1. \tag{3}$$

Since hyperbolic functions are derived from the sum of exponentials, their inverse functions can be formulated using logarithmic functions, as shown in Equations (4) and (5) [11,13,17–19]. In these equations, the intervals indicate the domain of the inverse functions, i.e., the set of x values for which the function is defined, not the codomain. Specifically, Equation (4) applies for all $x \in (-\infty, \infty)$, and Equation (5) applies for $x \in [1, \infty)$ [11,13,17–19], as illustrated bellow:

$$\sinh^{-1} x = \ln\left(x + \sqrt{1 + x^2}\right), \quad x \in (-\infty, \infty), \tag{4}$$

and

$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right), \quad x \in [1, \infty). \tag{5}$$

Derivatives and integrals can be calculated by writing the hyperbolic functions in terms of exponential functions. The derivatives and integrals generally yield another hyperbolic function, as presented in Equations (6) and (7) [13,14,16].

$$\frac{d}{dx}\sinh x = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh x. \tag{6}$$

$$\int \sinh x\, dx = \int \frac{e^x - e^{-x}}{2}\, dx = \frac{e^x + e^{-x}}{2} + c = \cosh x + c. \tag{7}$$

Hyperbolic functions have inspired scientists, including engineers, architects, astronomers, and so on, such as the construction of the Shukhov Tower (1920–1922) [20], mirror construction, planetary motion [21], the trajectory of an electron in an electric field, modelling the behaviour of gases [22], in the Loran long-range navigation system [23], and so on. There are many applications and fields of action of hyperbolic functions, so a strict classification is laborious. According to the uses found in the literature, four state-of-the-art classifications are proposed: (1) the Guderman function, (2) the theory of parallels, (3) catenaries, and (4) heat transmission.

2.1. Gudermann Function

Hyperbolic functions have been used in different applications, where the Gudermann function is used as a solution for the inverted pendulum. Christoph Gudermann (1798–1852) was one of the first to relate hyperbolic functions to trigonometric functions without using complex numbers [5,6]. He used hyperbolic series to expand elliptic functions [5]. Another of the applications used is the normal and transverse Mercator projection used in cartography for the creation of maps [3], which is based on the latitude distance from north to south and the arc length. Equations (8)–(10) express the Gudermann function [3,4], as presented in:

$$\cosh u = \sec v, \tag{8}$$

$$\sinh u = \tan v, \tag{9}$$

and

$$\tanh u = \sin v. \tag{10}$$

Equation (11) shows the relationship that has been called the Gudermann function [5], and the Anti-Gudermann function is given by:

$$v = \mathrm{gd}\, u, \quad u = \mathrm{gd}^{-1} v. \tag{11}$$

The derivatives of the Gudermann function and the Anti-Gudermann function are defined in Equations (12) and (13), as an expansion of the series [5]:

$$d\,\mathrm{gd}^{-1} v = \sec v\, dv, \tag{12}$$

and

$$d\,\mathrm{gd}\, u = \mathrm{sech}\, u\, du. \tag{13}$$

Therefore, the Gudermann function establishes a relationship between hyperbolic angles and trigonometric angles without the need for complex numbers. However, the applications of this function are not broad enough to fully explain angles and triangles in trigonometry. On the other hand, the Mercator projection, widely used and accepted in navigation, has not transcended the field of engineering due to its affinity and proximity to the transformation that is intended to be performed. Although this projection is presented in the article, the same approach is not found in the existing literature [3,4].

2.2. Theory of Parallels

In the geometry of Euclid's elements, five postulates are stated from which the other prepositions are deduced. The first four postulates are simple and intuitive; Euclid demonstrated 27 prepositions referring to triangles (their construction, relations between sides and angles of the same triangle or of two different triangles) as well as adjacent angles and angles opposite the vertex. It is stated by the Fifth Postulate that if, in the same plane, a straight line intersects two other lines, referred to as lines a and b, and the sum of the interior angles on the same side of the transversal is less than two right angles (less than 180 degrees), then these two lines, when extended indefinitely, will eventually be intersected on the side where the sum of these angles is less than 180 degrees. To visualize this, a plane with two parallel lines, a and b, that do not intersect is imagined. A third line, known as the transversal, is then considered, which is cut across both parallel lines. According to the Fifth Postulate, if the sum of the interior angles formed on the same side of the transversal is less than π, the two parallel lines will eventually be met at some point when extended. The behaviour of parallel lines in Euclidean geometry is described by this postulate, with a clear condition being established for when lines will intersect, as opposed to remaining parallel indefinitely, as is illustrated in Figure 1 [24].

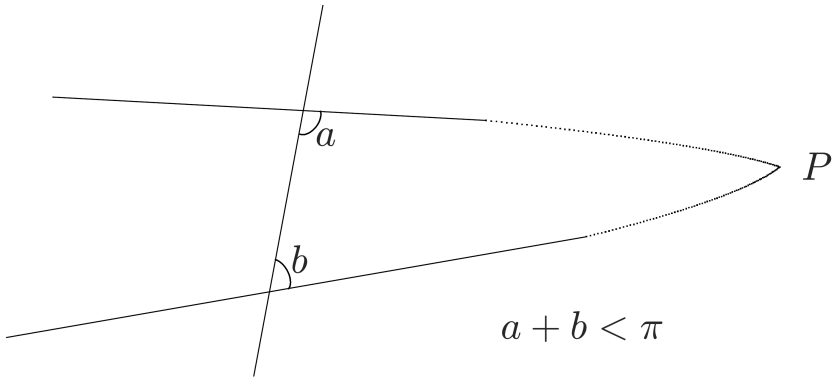


Figure 1. Fifth postulate and birth of non-Euclidean geometries [24].

Euclidean geometry remained unchallenged for several centuries. But in the Renaissance, efforts to understand Euclid's work brought up the "parallel problem" again. The authors Bolyai and Lobachevsky demonstrated that Euclid's fifth postulate is not necessary and independently discovered hyperbolic geometry. The appearance of hyperbolic geometry brought with it the inevitable task of defining which of the two geometries was "true". This meant studying whether material space was Euclidean or hyperbolic. In the end, it was determined that both geometries are consistent and neither is truer than the other [18,19,24].

Generalized hyperbolic distributions have been of great importance in recent years because their parameters have broad particular applications. They have been used to evaluate financial models, using optimal portfolio selection and risk measurement assessment. Non-Euclidean (hyperbolic) geometries allowed for the advancement of new theories such as relativity. Hyperbolic functions have wide application with respect to special relativity, with particular reference to Lorentz transformations and kinematics [25].

2.3. Catenaries

The catenary is obtained in the form of a heavy and flexible hanging wire from a transmission line or a telephone line. The combination of exponentials that describe this trajectory has been defined from hyperbolic functions [10]. Some authors have given credit for hyperbolic functions to mathematician Johan Lambert and other authors who have been

interested in solving the form of the catenary (from the Greek katena, meaning chain) [10]. Galileo mistakenly believed that it might be a parabola; the German mathematician Joachim Jungius (1587–1657) corrected Galileo's claim, although it was not widely accepted by 17th-century mathematicians. Bernoulli proved that the curve satisfies the differential equation $dy/dx = s/k$, where s represents the arc length and k is a constant that depends on the weight per unit length [7,8]. The catenary can also be used for case studies of a fixed-length bar moving on one end of its axis, forming the tractive curve [26].

2.4. Heat Transmission

At the beginning of the 20th century, the parabolic model of heat transfer with infinite velocity and infinite heat fluxes was physically unacceptable. This model is used in engineering because in ordinary applications the results are close to reality. However, with the use of new technologies in which large amounts of heat are applied in small time intervals (such as the use of laser pulses in materials processing), new physical situations have arisen in which there are serious disparities between the results obtained theoretically and experience. This promoted the appearance of a new alternative model called the hyperbolic model that predicts a finite heat transfer rate and finite heat fluxes. This conception led to the formulation of a new heat conduction equation called the modified Fourier Law [27]. It introduced a new term called the relaxation parameter, which expresses the time it takes for heat to be transmitted and produce the heat flow [22,27]. In addition, hyperbolic functions can be applied in mechanical and electrical systems for the analysis of local thermodynamic equilibrium, elasticity with memory, multiphase and phase transition, and in oscillating and non-oscillating systems in linear and nonlinear waves [4,22].

Finally, the transformation of trigonometric functions into hyperbolic functions is deduced using the equation of a catenary in which the Gudermann function and the Mercator projection are obtained. The transformation of trigonometric functions into hyperbolic functions is easy to apply, and its use extends to engineering branches based on mathematics. A non-Eulerian (hyperbolic) solution is presented to problems normally solved with trigonometric (circular) functions. Building on the background literature, the next section details the transformation of hyperbolic functions.

3. Deduction of Hyperbolic Function Transformation

For a cable, it is assumed that a cable differential element is considered, and Newton's first law is applied, as illustrated in Figure 2. The tension is represented by T, and the angle at each end of the cable is measured as θ and $\theta + d\theta$. The difference between the two angles at each end of the cable is $d\theta$. From this, Equations (14) and (15) are derived, assuming the value of μ to be constant. The summation of forces in the X- and Y-directions is expressed as follows:

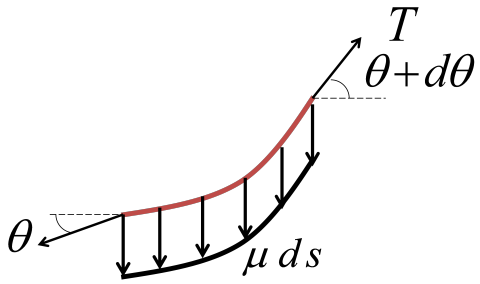


Figure 2. Differential cable element.

$$\sum F_x = 0; \quad (T + dT)\cos(\theta + d\theta) = T\cos\theta, \tag{14}$$

and

$$\sum F_y = 0; \quad (T + dT)\sin(\theta + d\theta) = T\sin\theta + \mu\, ds. \tag{15}$$

In the cable analysis, a force balance is performed on a differential element, where the differential angle $d\theta$ is sufficiently small to justify the small-angle approximation. In this context, the change in tension dT is considered negligible compared to the overall cable tension. Therefore, the standard engineering assumptions are valid: $\sin d\theta \approx d\theta$, $d\theta \times dT \approx 0$, and $\cos d\theta \approx 1$. Consequently, dT and $d\theta$ are small enough to render higher-order terms negligible, thus simplifying the expressions in Equations (16) and (17) [3,7,8,10], as derived below:

$$d(T\cos\theta) = 0, \tag{16}$$

and

$$d(T\sin\theta) = \mu\, ds. \tag{17}$$

From the first condition, it follows that $T\cos\theta$ is equal to a constant whose value is equal to the minimum voltage when $\theta = 0$; therefore, the minimum tension is $T_0 = T\cos\theta$. By substituting this condition into Equation (17), Equation (18) is derived [8]:

$$T_0\, d(\tan\theta) = \mu\, ds. \tag{18}$$

The derivative is the slope of the tangent line to a curve at a point, defined as $\frac{dy}{dx} = \tan\theta$ [7]. Substituting this definition into Equation (18) yields Equation (19):

$$\frac{d^2y}{dx^2} = \frac{\mu}{T_0}\frac{ds}{dx}. \tag{19}$$

The arc length, also called the rectification of a curve, is defined as $(ds)^2 = (dx)^2 + (dy)^2$. Thus, the problem reduces to solving Equation (20). Additionally, both terms of the equation are divided into dx [7]:

$$\frac{d^2y}{dx^2} = \frac{\mu}{T_0}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}. \tag{20}$$

Equation (20) can be easily solved by substituting $p = \frac{dy}{dx}$, which results in Equation (21). Figure 3 illustrates the three main forces: the minimum tension T_0, another point of tension T, and the weight of the cable μs. Furthermore, in relation to the previous Figure 2, the X and Y axes are introduced to represent the minimum tension as a boundary condition:

$$\frac{dp}{\sqrt{1+p^2}} = \frac{\mu}{T_0}\, dx. \tag{21}$$

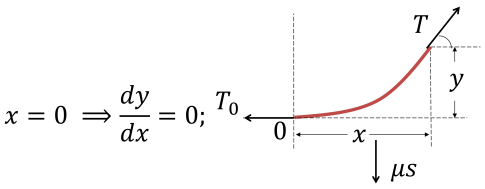


Figure 3. Cable conditions for lowest point.

Integrating and solving leads to Equation (22). The solution can be obtained by checking the derivative of the function in Equation (22), as detailed in Appendix A. Using the boundary condition $p = \frac{dy}{dx} = \tan\theta = 0$ when $x = 0$, it is determined that the constant $C = 0$, according to Figure 3:

$$\ln\left(p+\sqrt{1+p^2}\right)=\frac{\mu x}{T_0}. \tag{22}$$

The clearing of p is performed in Appendix B, obtaining Equation (23). Therefore, $p=\sinh\left(\frac{\mu x}{T_0}\right)$:

$$\tan\theta=\frac{dy}{dx}=\frac{e^{\frac{\mu x}{T_0}}-e^{-\frac{\mu x}{T_0}}}{2}=\sinh\frac{\mu x}{T_0}. \tag{23}$$

For the condition $y=0$ and $x=0$, integrating and solving leads to Equation (24):

$$y=\frac{T_0}{\mu}\left(\cosh\frac{\mu x}{T_0}-1\right). \tag{24}$$

A force balance is performed for half of the cable, as shown on the left side of Figure 4. Each of the forces acting on the cable is indicated, including the minimum tension, the weight of the cable, and the tension at the point where the cable is supported, along with its angle. The principle of force translation along a line of action is applied, forming a triangle, as shown on the right side of Figure 4. The hypotenuse of this triangle represents the tension, the weight of the cable is on the Y-axis, and the minimum tension is on the X-axis.

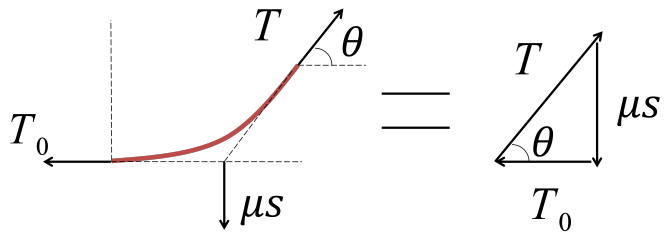


Figure 4. A representation of the three forces by a triangle.

The value of $\frac{dy}{dx}=\tan\theta=\frac{\mu s}{T_0}$, which allows for the determination of the cable length using Equation (25) [7,8]. Specifically,

$$s=\frac{T_0}{\mu}\sinh\frac{\mu x}{T_0}. \tag{25}$$

Using the Pythagorean theorem, Equation (26) is obtained:

$$T^2=\mu s^2+T_0^2. \tag{26}$$

Replacing Equation (25) in Equation (26) yields Equation (27):

$$T^2=T_0^2\left(1+\sinh^2\frac{\mu x}{T_0}\right)=T_0^2\cosh^2\frac{\mu x}{T_0}. \tag{27}$$

Figure 5 shows a proportional relationship between two right triangles. On the left side, a right triangle involving tensions is represented. The hypotenuse is labelled as T, the opposite side is represented by the term μs, and the adjacent side corresponds to T_0. The angle θ represents the angle, and the subscript h indicates that it is a hyperbolic angle associated with this triangle of forces. The triangle can be related by the property of similar angles. On the right side, the general nomenclature for a similar triangle is presented, with a hypotenuse labelled as H, an adjacent side labelled as $C.A.$, and an opposite side labelled as $C.O.$, with the trigonometric angle θ indicated.

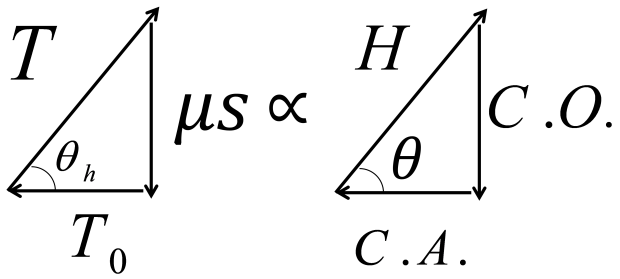


Figure 5. Property of similarity of triangles.

Using Equation (27) and according to the triangle in Figure 5, Equation (28) is included:

$$s = \frac{T_0}{\mu} \sinh \frac{\mu x}{T_0}. \tag{28}$$

Taking $\frac{\mu x}{T_0} = \theta_h$, where $\frac{\mu}{T_0}$ is a constant and x is variable, the angle is represented by θ, with the subscript h indicating that it is a hyperbolic angle, according to the law of similarity of triangles. Moreover, $H = k \times T$ is determined, where k is the similarity constant of triangles, $C.A. = k \times T_0$, and $C.O. = k \times \mu s$, according to Figure 5. Therefore, Equation (29) can be deduced:

$$\frac{1}{\cos \theta} = \cosh \theta_h = \frac{H}{C.A.}. \tag{29}$$

Using Equation (25), $\tan \theta = \frac{\mu_s}{T_0} = \frac{\sinh \mu x}{T_0}$, another relationship is obtained for the sides of the triangle. Using similarity of triangles, Equation (30) is derived:

$$\tan \theta = \sinh \theta_h = \frac{C.O.}{C.A.}. \tag{30}$$

By dividing Equation (30), $\frac{\sin \theta}{\cos \theta} = \sinh \theta_h = \frac{C.O.}{C.A.}$, in Equation (29), $\frac{1}{\cos \theta} = \cosh \theta_h = \frac{H}{C.A.}$, the expression shown in Equation (31) is obtained:

$$\sin \theta = \tanh \theta_h = \frac{C.O.}{H}. \tag{31}$$

The first expressions coincide with the expression known as the Gudermann function [3,4] and in the second part, the triangle theorem is obtained using hyperbolic functions, which are shown in Equations (29)–(31). Therefore, the hyperbolic transformation can be expressed using the right triangle formulas with hyperbolic functions, as shown in Equations (32)–(34):

$$\sinh \theta_h = \frac{C.O.}{C.A.}, \tag{32}$$

$$\cosh \theta_h = \frac{H}{C.A.}, \tag{33}$$

and

$$\tanh \theta_h = \frac{C.O.}{H}. \tag{34}$$

Another expression that relates the sides of the triangle can be obtained by replacing $\theta_h = \frac{\mu x}{T_0}$ and $p = \sinh \theta_h = \tan \theta = \frac{C.O.}{C.A.}$, and, rearranging Equation (22), an expression similar to the Mercator projection is obtained, as shown in Equation (35) and Appendix D. Thus, trigonometric angles are transformed into hyperbolic angles without having to resort to hyperbolic functions. The equation presents the relationship between trigonometric

functions, the hyperbolic angle, and the natural logarithm. In the equation involving the natural logarithm, solving it reveals that Euler's number can be used as a substitute:

$$\theta_h = \ln\left(\frac{C.O.}{C.A.} + \sqrt{1 + \left(\frac{C.O.}{C.A.}\right)^2}\right) = \ln(\tan\theta + \sec\theta). \tag{35}$$

The following expressions can be obtained, called the right triangle function with hyperbolic angle; in Equation (36), all the sides of the triangle are related. Thus, the sides of the right triangle are related to the hyperbolic angle and Euler number, without the need for hyperbolic functions, trigonometric functions, or imaginary numbers:

$$\theta_h = \ln\frac{C.O. + H}{C.A.}. \tag{36}$$

The right triangle function with hyperbolic angle relates the three sides of a right triangle, Euler's number, and the hyperbolic angle, unlike the Pythagorean theorem, which relates the square of the hypotenuse to the sum of the squares of the two legs of the triangle:

$$e^{\theta_h} = \frac{C.O. + H}{C.A.}. \tag{37}$$

The hyperbolic angle can be related to the Pythagorean theorem, and it can be determined by knowing any of the two sides of the triangle and a hyperbolic angle:

$$\theta_h = \ln\frac{\sqrt{H^2 - C.A.^2} + H}{C.A.}, \tag{38}$$

and

$$\theta_h = \ln\frac{C.O. + H}{\sqrt{H^2 - C.O.^2}}. \tag{39}$$

Another relationship that can be obtained is by replacing Equation (38) with Equation (32), where $\sinh\theta_h = \frac{C.O.}{C.A.}$, and Equation (33), where $\cosh\theta_h = \frac{H}{C.A.}$, as shown in Appendix C and Appendix D, where the similarity between the transformation of hyperbolic functions to trigonometric functions is analysed through the multiplication of an imaginary number i by the natural logarithm. In this way, hyperbolic angles are transformed into trigonometric angles without having to resort to trigonometric functions. This equation represents a variant of the Mercator projection:

$$\theta = \sin^{-1}(\tanh\theta_h) = -i\ln(i\tanh\theta_h + \operatorname{sech}\theta_h). \tag{40}$$

By using Euler's number instead of the natural logarithm, the following function is obtained. An example of the application of this function is its use in representing phasors, which are employed in electrical engineering [10]. The expression matches Euler's formula [2]. Therefore, the function can relate the three sides of the right triangle, the trigonometric angle, the imaginary number, and Euler's number:

$$e^{i\theta} = i\tanh\theta_h + \operatorname{sech}\theta_h = \cos\theta_h + i\sin\theta_h = \frac{C.A. + iC.O.}{H}. \tag{41}$$

The sum of angles for trigonometric functions can be expressed from the transformation to hyperbolic functions from the following equations, where the sum can be performed by the analogy to know the domain of the hyperbolic angles, where $\theta_h \in (-\infty, \infty)$, while in trigonometric functions $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, for other intervals $|\theta_A - \theta_B| = \pi$, adjustments are

made in other investigations. Angle values are also tested with $\theta \in (0, \pi)$, and the values are valid for this segment. Equations (43) and (44) are obtained in Appendix E:

$$\theta_A \pm \theta_B = \theta_C, \tag{42}$$

$$\left| \frac{\cosh \theta_{hA} \cosh \theta_{hB}}{1 \mp \sinh \theta_{hA} \sinh \theta_{hB}} \right| = \cosh \pm \theta_{hC}, \tag{43}$$

and

$$\frac{\sinh \theta_{hA} \pm \sinh \theta_{hB}}{\cosh \theta_{hA} \cosh \theta_{hB}} = \tanh \theta_{hC}. \tag{44}$$

The expression for right triangles, in which $\theta_A + \theta_B = \frac{\pi}{2}$ can be reduced to the equations by replacing the terms obtained for Appendix F, is:

$$\sinh \theta_{hA} \sinh \theta_{hB} = 1. \tag{45}$$

Figure 6 presents an oblique triangle with sides a, b, and c. The angles are represented by θ, where the subscript h indicates that they are hyperbolic angles, and the subscripts A, B, and C represent the vertices of the triangle.

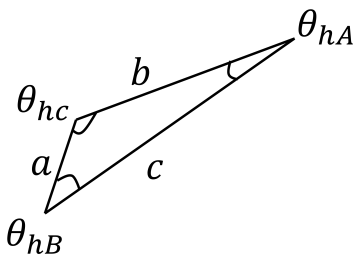


Figure 6. Oblique triangle with hyperbolic functions.

Another expression that can be used is the law of hyperbolic tangents based on the law of sines, where $\theta_{hC} \geq \frac{\pi}{2}$. The angles $\frac{\pi}{2} > \max(\theta_A, \theta_B)$ and can be calculated using the law of hyperbolic cosines, where $c > \max(a, b)$:

$$\frac{a}{\tanh \theta_{hA}} = \frac{b}{\tanh \theta_{hB}} = \frac{c}{\tanh \theta_{hC}}. \tag{46}$$

The law of hyperbolic cosines can be calculated by the expression, just like the previous equation θ_{hC}, which is calculated using Equations (47)–(49):

$$a^2 = b^2 + c^2 - \frac{2bc}{\cosh \theta_{hA}}, \tag{47}$$

$$\theta_{hA} = \cosh^{-1} \frac{2bc}{b^2 + c^2 - a^2}, \tag{48}$$

and

$$\theta_{hB} = \cosh^{-1} \frac{2ab}{a^2 + c^2 - b^2}. \tag{49}$$

For the angle θ_{hC}, the property $\cosh \theta_{hC} = \cosh -\theta_{hC}$ is used:

$$\theta_{hC} = \cosh^{-1} - \frac{2ab}{a^2 + b^2 - c^2}. \tag{50}$$

To ensure a positive value, the expression can be generalized as:

$$\theta_{hC} = \cosh^{-1} \left| \frac{2ab}{a^2 + b^2 - c^2} \right|. \tag{51}$$

Another expression is the dot product, the deduction of which is presented in Appendix G:

$$\vec{a} \cdot \vec{b} = \frac{2ab}{\cosh|\theta_{hC}|}. \tag{52}$$

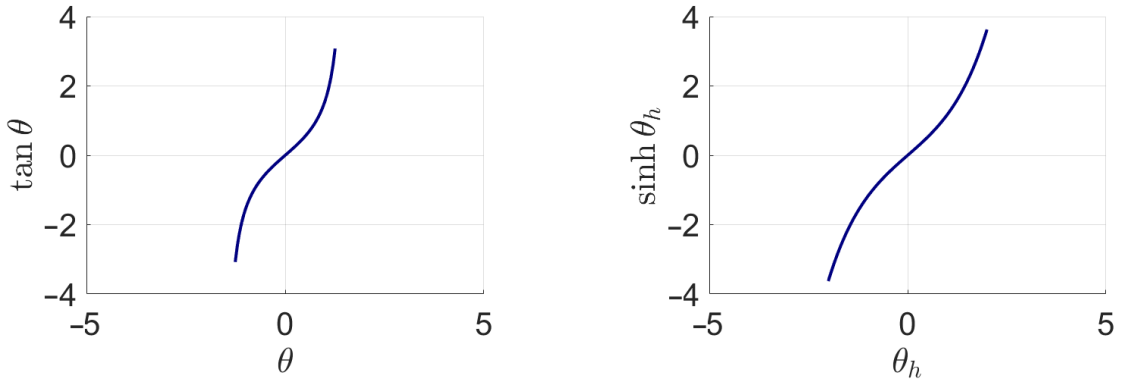
The cross product expression according to Appendix H is:

$$\vec{a} \times \vec{b} = ab \tanh \theta_{hC}. \tag{53}$$

The hyperbolic Mollweide's formulas are solved in Appendix I, where the two trigonometric expressions are simplified into a single expression:

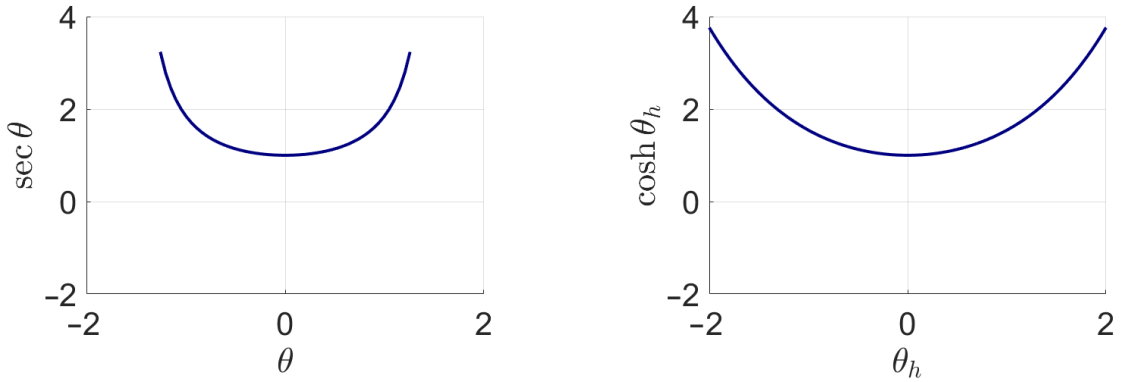
$$\frac{\cosh(\theta_{hA} \pm \theta_{hB}) \pm 1}{\cosh\theta_{hA} \cosh\theta_{hB}(1 \pm \operatorname{sech}\theta_{hC})} = \left(\frac{a \pm b}{c}\right)^2. \tag{54}$$

According to the deduction, the analogy can be made to know the domain of the hyperbolic angles where $\theta_h \in (-\infty, \infty)$, while in trigonometric functions $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. The graphs in Figure 7 show the comparison between the functions $\tan\theta$ and $\sinh\theta_h$, the graphs in Figure 8 present the comparison between the functions $\sec\theta$ and $\cosh\theta_h$, and the graphs in Figure 9 show the comparison between the functions $\sin\theta$ and $\tanh\theta_h$. It is concluded that similar trends are presented in Equations (32)–(34) for the comparison between trigonometric functions and hyperbolic functions, with the differences in curvature due to the fact that the angle domains are different.



(**a**) (**b**)

Figure 7. Function comparison, $\tan\theta$ vs. $\sinh\theta_h$: (**a**) line graph of $\tan\theta$; (**b**) line graph of $\sinh\theta_h$.



(**a**) (**b**)

Figure 8. Function comparison, $\sec\theta$ vs. $\cosh\theta_h$: (**a**) line graph of $\sec\theta$; (**b**) line graph of $\cosh\theta_h$.

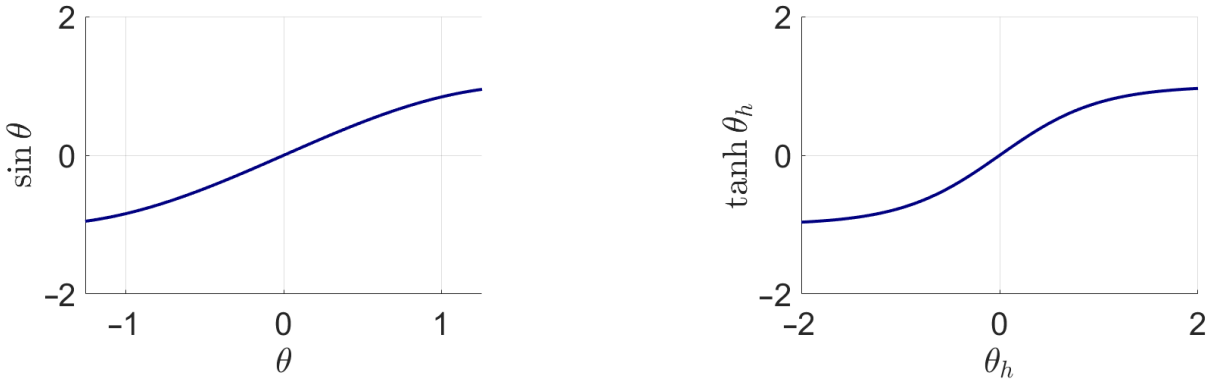


Figure 9. Function comparison, $\sin\theta$ vs. $\tanh\theta_h$: (**a**) line graph of $\sin\theta$; (**b**) line graph of $\tanh\theta_h$.

Once the mathematical analysis of hyperbolic functions is complete, the next section examines their applications in engineering and mathematics.

4. Application of Hyperbolic Function Transformation

The transformation of hyperbolic functions is easy to apply, and its solution can be extended to basic engineering subjects such as calculus, statics, electricity, and trigonometry. The results are validated using trigonometric functions, the Pythagorean theorem, and Mollweide's formula to solve real-world problems, as these functions have been widely used in solving such problems. Due to their extensive application and prior validation, no experimental studies are conducted. Below are some engineering examples, which can be solved by transforming hyperbolic functions, which will be solved in parallel using other solution methods as trigonometric functions and Mollweide's formula [1].

4.1. Static

A force of 300 kN is applied to the top of a truss, as shown in Figure 10. For the symmetrical structure, the value of the force is to be determined for F_{AB} and F_{AD}. In addition, the force of element BD is equal to zero [7,8].

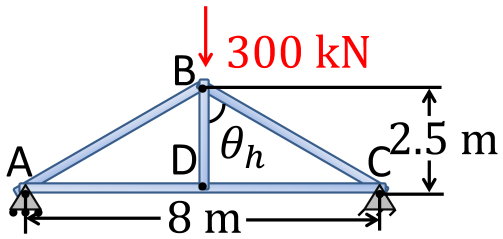


Figure 10. Structure with point force.

4.1.1. Use of Hyperbolic Functions

The value of the hyperbolic angle is calculated as shown in Equation (55), while the forces F_{AB} and F_{AD} are determined by Equations (56) and (57), respectively. The truss, subjected to a force of 300 kN, consists of members defined by the vertices A, B, C, and D, with subscripts indicating the direction of the force. The dimensions of the truss are 8 m and 2.5 m. The objective is to determine the forces in the elements, where the letter (C) represents compression and the letter (T) denotes tension in the corresponding element:

$$\theta_h = \sinh^{-1}\frac{4\text{ m}}{2.5\text{ m}} = 1.249, \tag{55}$$

$$F_{AB} = 300\ \text{kN}\ \cosh 1.249 = 566\ \text{kN(C)}, \tag{56}$$

and

$$F_{AD} = 300\ \text{kN}\ \sinh 1.249 = 480\ \text{kN(T)}. \tag{57}$$

4.1.2. Validation Using Trigonometric Functions

The angle is determined in degrees, as shown in Equation (58). This angle determines the value of the force F_{AB}, as given in Equation (59), and the value of the force F_{AD}, as shown in Equation (60):

$$\theta = \tan^{-1} \frac{4\ \text{m}}{2.5\ \text{m}} = 57.99^\circ, \tag{58}$$

$$F_{AB} = \frac{300\ \text{kN}}{\cos 57.99^\circ} = 566\ \text{kN(C)}, \tag{59}$$

and

$$F_{AD} = 566\ \text{kN}\ \sin 57.99^\circ = 480\ \text{kN(T)}. \tag{60}$$

4.2. Calculation

A fire brigade has an 18 m ladder that is used for emergencies. Given that the ladder is positioned 3 m away from the building, determine the height it can reach, as illustrated in Figure 11.

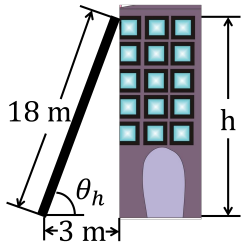


Figure 11. Ladder leaning against building.

4.2.1. Solution Using Hyperbolic Function Transformation

The height of the building can be determined using the hyperbolic function theorem. The hyperbolic angle is calculated in Equation (61), and the height is calculated in Equation (62):

$$\theta_h = \cosh^{-1} \frac{18\ \text{m}}{3\ \text{m}} = 2.478, \tag{61}$$

and

$$h = 18\ \text{m}\ \tanh 2.478 = 17.75\ \text{m}. \tag{62}$$

The right triangle function with the hyperbolic angle is replaced by the obtained values, and the equality is verified:

$$e^{2.478} = \frac{17.75\ \text{m} + 18\ \text{m}}{3\ \text{m}}. \tag{63}$$

$$11.916 = 11.916$$

4.2.2. Solution Using Pythagorean Theorem

The height h is determined as shown in Equation (64):

$$h = \sqrt{18^2 - 3^2}\ \text{m} = 17.75\ \text{m}. \tag{64}$$

4.3. Electricity

Determine the voltage of the system knowing that $I = (2 + j3)$ A and $Z = (30 - j70)\ \Omega$, where $V = IZ$, and the hyperbolic angle is derived as:

$$\theta_{hI} = \sinh^{-1} \frac{C.O._I}{C.A._I} = \sinh^{-1} \frac{3\text{ A}}{2\text{ A}} = 1.1947. \tag{65}$$

The value of I is determined as:

$$|I| = \frac{C.O._I}{\tanh \theta_{hI}} = \frac{3\text{ A}}{\tanh 1.1947} = 3.605\text{ A}. \tag{66}$$

The hyperbolic angle of the impedance is determined as:

$$\theta_{hZ} = \sinh^{-1} \frac{C.O._Z}{C.A._Z} = \sinh^{-1} -\frac{70\ \Omega}{30\ \Omega} = -1.5835. \tag{67}$$

The magnitude of the impedance is calculated as:

$$|Z| = \frac{C.O._Z}{\tanh \theta_{hZ}} = -\frac{70\ \Omega}{\tanh -1.5835} = 76.1577\ \Omega. \tag{68}$$

For the sum of angles, the following expression is used:

$$\theta_{hV} = \tanh^{-1} \frac{\sinh \theta_{hI} + \sinh \theta_{hZ}}{\cosh \theta_{hI} \cosh \theta_{hZ}} = -0.184. \tag{69}$$

The following equation expresses the voltage value as the product of the current and the impedance:

$$V = IZ = |I||Z|(\operatorname{sech} \theta_{hV} + i \tanh \theta_{hV}). \tag{70}$$

By substituting the values, the result is obtained. The values have been calculated with the decimals of a fx-9750GII calculator from the manufacturer Casio (Tokyo, Japan) to obtain the exact value:

$$V = IZ = |3.605\text{ A}||76.1577\ \Omega|(\operatorname{sech} -0.184 + i \tanh -0.184) = (270 - i50)\text{ V}. \tag{71}$$

To verify the solution, the following calculation can be used:

$$V = (2 + i3)\text{ A} \times (30 - i70)\ \Omega = (270 - i50)\text{ V}. \tag{72}$$

4.4. Trigonometry

Four types of solutions for oblique triangles are explored below. In the first, the solution using the law of sines and cosines is presented; in the second, the law of hyperbolic cosines and tangents is presented; in the third, an alternative method to the law of sines and cosines, called Mollweide's formulas, is presented; and in the fourth, the method called the hyperbolic Mollweide formula is presented. The second and fourth solutions are methods proposed in this research.

4.4.1. Law of Sines and Cosines

Below are the steps to solve a triangle with sides $a = 9$, $b = 5$, and $c = 12$, using the law of sines and the law of cosines. The angles are represented by θ, with A, B, and C denoting the vertices, which are reflected in the subscript of each angle, forming the following notation: θ_A, θ_B, and θ_C. The following conventions are adopted to represent the angles: $\theta_A = \angle CAB$, $\theta_B = \angle ABC$, and $\theta_C = \angle BCA$. The triangle is shown in Figure 12. Thus, the angles satisfy the equation $\theta_A + \theta_B + \theta_C = \pi$.

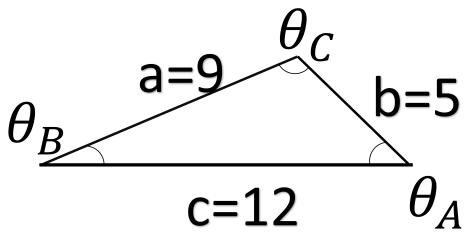


Figure 12. Oblique triangle, showing its sides and angles.

Law of Cosines

The law of cosines is useful when the three sides of the triangle are known. It can be used to find an angle θ_C when the three sides are known. The formula is:

$$\cos\theta_C = \frac{a^2 + b^2 - c^2}{2ab}. \tag{73}$$

By substituting the values of the sides of the triangle, the following is obtained:

$$\cos\theta_C = \frac{9^2 + 5^2 - 12^2}{2 \times 9 \times 5} = -\frac{19}{45}. \tag{74}$$

Now, to find θ_C, the inverse cosine is taken:

$$\theta_C = \cos^{-1} -\frac{19}{45} \approx 114.975^\circ. \tag{75}$$

Law of Sines

Now that θ_C has been found, the law of sines can be used to find the angle θ_A of the triangle. The law of sines states that:

$$\frac{a}{\sin\theta_A} = \frac{b}{\sin\theta_B} = \frac{c}{\sin\theta_C}. \tag{76}$$

This formula is used to determine θ_A, where $\sin\theta_A$ is then isolated:

$$\sin\theta_A = \frac{a\sin\theta_C}{c}. \tag{77}$$

The known values are substituted:

$$\theta_A = \sin^{-1}\frac{9\sin 114.947^\circ}{12} = 42.833^\circ. \tag{78}$$

The values are substituted to find θ_B:

$$\theta_B = \sin^{-1}\frac{5\sin 114.947^\circ}{12} = 22.192^\circ. \tag{79}$$

The angles of the triangle are substituted into the law of sines:

$$\frac{9}{\sin 42.833^\circ} = \frac{5}{\sin 22.192^\circ} = \frac{12}{\sin 114.975^\circ}. \tag{80}$$

The values are substituted into the law of sines:

$$13.238 = 13.238 = 13.238.$$

4.4.2. Law of Hyperbolic Cosines and Tangents

The relationship between the sides of the triangle and the hyperbolic angles can be verified for a triangle with the following sides: $a = 9$, $b = 5$, and $c = 12$. In this section, the solution is obtained using the law of hyperbolic cosines and tangents. The angles are represented by θ, where the subscript h indicates that they are hyperbolic angles. The subscripts A, B, and C are added, corresponding to the vertices of the triangle. Therefore, the three hyperbolic angles are denoted as θ_{hA}, θ_{hB}, and θ_{hC}. The convention for the hyperbolic angles is as follows: $\theta_{hA} = \angle CAB$, $\theta_{hB} = \angle ABC$, and $\theta_{hC} = \angle BCA$. The triangle is shown in Figure 13.

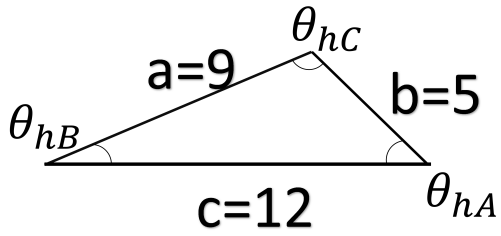


Figure 13. A triangle with the representation of hyperbolic angles.

The calculation of the hyperbolic angle A is performed:

$$\theta_{hA} = \cosh^{-1} \frac{2bc}{b^2 + c^2 - a^2}. \tag{81}$$

The values are replaced, and the result is obtained:

$$\theta_{hA} = \cosh^{-1} \frac{2 \times 5 \times 12}{5^2 + 12^2 - 9^2} = 0.8289. \tag{82}$$

The hyperbolic angle B is calculated:

$$\theta_{hB} = \cosh^{-1} \frac{2ac}{a^2 + c^2 - b^2}. \tag{83}$$

The values are substituted, yielding the result:

$$\theta_{hB} = \cosh^{-1} \frac{2 \times 9 \times 12}{9^2 + 12^2 - 5^2} = 0.3974. \tag{84}$$

The hyperbolic angle of vertex C is determined:

$$\theta_{hC} = \cosh^{-1} \left| \frac{2ab}{a^2 + b^2 - c^2} \right|. \tag{85}$$

The values are inserted, and the result is obtained:

$$\theta_{hC} = \cosh^{-1} \left| \frac{2 \times 9 \times 5}{9^2 + 5^2 - 12^2} \right| = 1.508. \tag{86}$$

Proportionalities are proven with the law of hyperbolic tangents. By calculating the sides with the value of the hyperbolic tangents, it can be verified that the proportionalities are maintained; therefore, the law of hyperbolic tangents can be used to determine the relationships between the sides of a triangle with its hyperbolic angles:

$$\frac{a}{\tanh \theta_{hA}} = \frac{b}{\tanh \theta_{hB}} = \frac{c}{\tanh \theta_{hC}}. \tag{87}$$

By substituting the values of the angles, the result is obtained:

$$\frac{9}{\tanh 0.8289} = \frac{5}{\tanh 0.3974} = \frac{12}{\tanh 1.508}. \tag{88}$$

The values allow the equality to be verified:

$$13.238 = 13.238 = 13.238.$$

4.4.3. Mollweide's Formulas

Mollweide's formulas are stated and used to establish a relationship between the internal angles and the sides of an oblique triangle. These formulas are fundamental in trigonometry, since they allow the calculation of the sides and angles of a non-right triangle, given certain conditions. In particular, Mollweide's formulas are used when two angles and an opposite side, or two sides and an angle not included, are known. In this way, the resolution of oblique triangles, which cannot be solved using only the traditional formulas for right triangles, is facilitated:

$$\frac{\sin \frac{\theta_A - \theta_B}{2}}{\cos \frac{\theta_C}{2}} = \frac{a-b}{c}, \tag{89}$$

and

$$\frac{\cos \frac{\theta_A - \theta_B}{2}}{\sin \frac{\theta_C}{2}} = \frac{a+b}{c}. \tag{90}$$

The angles obtained in Section 4.4.1 from the law of sines and cosines are replaced, and the values of the sides are taken from Figure 12. These values are then substituted into Mollweide's formulas to obtain the corresponding results:

$$\frac{\sin \frac{42.833^\circ - 22.192^\circ}{2}}{\cos \frac{114.975^\circ}{2}} = \frac{9-5}{12},$$

and

$$\frac{\cos \frac{42.833^\circ - 22.192^\circ}{2}}{\sin \frac{114.975^\circ}{2}} = \frac{9+5}{12}.$$

The values are substituted into Mollweide's formulas. These values are stored in a fx-9750GII calculator to minimize truncation errors, and the following results are obtained for the two Mollweide's formulas, respectively:

$$\frac{1}{3} = \frac{1}{3}, \quad \text{and} \quad \frac{7}{6} = \frac{7}{6}.$$

4.4.4. Hyperbolic Mollweide's Formula

Mollweide's formula is considered an essential foundation in trigonometry for oblique triangles. The transformation of hyperbolic functions is key to establishing the relationship between the three sides of the triangle and their hyperbolic angles. Due to its origin, this formula can be referred to as the "hyperbolic Mollweide's formula", and its advantage lies in the fact that two equations are reduced to a single equation, unlike its trigonometric version. The complete formulation is presented in Appendix I:

$$\frac{\cosh(\theta_{hA} \pm \theta_{hB}) \pm 1}{\cosh \theta_{hA} \cosh \theta_{hB} (1 \pm \operatorname{sech} \theta_{hC})} = \left(\frac{a \pm b}{c} \right)^2. \tag{91}$$

The hyperbolic angles obtained in Section 4.4.2 using the law of hyperbolic cosines and the law of hyperbolic tangents are replaced, and the values of the sides are taken from Figure 13. Subsequently, these values are substituted into the hyperbolic Mollweide's formula, both for the positive and negative cases, yielding the following results:

$$\frac{\cosh(0.8289 \pm 0.3974) \pm 1}{\cosh 0.8289 \cosh 0.3974(1 \pm \operatorname{sech} 1.508)} = \left(\frac{9 \pm 5}{12}\right)^2. \tag{92}$$

The hyperbolic Mollweide's formula yields the following results. The equality is analysed for both the positive and negative cases, and to minimize truncation errors, these values are stored in an fx-9750GII calculator. First, the negative solution is presented, followed by the positive solution:

$$\frac{1}{9} = \frac{1}{9}, \quad \text{and} \quad \frac{49}{36} = \frac{49}{36}. \tag{93}$$

Now that the analytical applications are complete, the next section discusses the results.

5. Discussion of Results

In this section, the results obtained from hyperbolic angle functions are discussed, along with key observations regarding the study and its correlation with functions reported in the literature. This article deems it necessary to clarify that hyperbolic angles are being used in the context of hyperbolic functions, as trigonometric functions are being developed in parallel, in order to avoid confusion between the two types of angles. However, for simpler use, in other research, the term "angle" may be employed when working with hyperbolic functions.

The trigonometric functions are related by Gudermann to the hyperbolic functions. The angle associated with the trigonometric function is referred to as the Gudermann angle, denoted as gd x, in honor of Christoph Gudermann. This angle represents the same value as the trigonometric functions, but in this article, the term is simplified to "hyperbolic angle" in order to facilitate understanding. An innovative approach to the demonstration of the Gudermann function is presented in this work, based on engineering concepts. The statics of cables, used in transmission lines, is employed as the foundation for the solution to the Gudermann function, which enables a better understanding of mathematics through the analogy with physical phenomena [5].

Previous research has shown that in hyperbolic triangles, the sum of their interior angles differs from π, and the relationship between these angles is addressed through Euclidean trigonometry. In other words, hyperbolic trigonometry is solved using Euclidean geometry. Unlike these studies, the focus of this work is on the resolution of Euclidean triangles, where the sum of the vertex angles is equal to π. The proposed solution employs non-Euclidean functions, specifically hyperbolic functions, which constitutes an innovative finding and establishes a bridge between the resolution of triangles in Euclidean trigonometry, where Euclidean triangles are solved using hyperbolic functions [28].

The application of the Mercator projection involves representing the spherical surface of the Earth on a plane, which produces a distortion of the areas as they move away from the equator. This research presents an innovative approach to deriving the Mercator projection from cable statics, emphasizing the advantage that trigonometric angles are converted into hyperbolic angles without the need for hyperbolic functions. Another benefit of this variant of the Mercator projection is that hyperbolic angles are transformed into trigonometric angles without the need for trigonometric functions. Indeed, for both cases, Euler's number is replaced by the natural logarithm introduced in this study.

In this research, several applications are carried out in which numerical results are obtained by comparing the use of hyperbolic functions with trigonometric functions, storing all decimal values of the angles. Exact values for the sides are obtained. That is, the only numerical errors that could arise when using hyperbolic functions are attributable to rounding errors in the computing systems used. A significant contribution of this research is the use of hyperbolic functions to solve trigonometric problems related to triangles, with applications that can be extended to teaching in undergraduate and graduate programs. In general terms, it is found that both methods are valid, and it cannot be stated that one method is superior to the other.

This research presents the following contributions: (1) The right triangle formulas with hyperbolic functions establish a relationship between the sides of a right triangle and the hyperbolic angles, based on the principles of cable statics; (2) a right triangle function with a hyperbolic angle is introduced, which relates the three sides of the triangle, the hyperbolic angle, and Euler's number; (3) deriving the law of hyperbolic cosines and hyperbolic tangents using hyperbolic functions; (4) the derivation of Mollweide's formula with hyperbolic functions, which can also be referred to as the second law of hyperbolic cosines, as it relates the sides of a non-right triangle to their respective hyperbolic angles, linking the hyperbolic cosines of the triangle to each of its sides.

This article is intended for both mathematics education and research in the field, since it integrates a mathematical formulation derived from physical phenomena with practical applications in engineering. By linking traditional trigonometry concepts with innovative methods based on hyperbolic functions, the findings show applications such as the following: (1) the transformation of hyperbolic functions that is used in right angles, (2) the law of hyperbolic tangents and cosines that is used to solve trigonometry problems. Calculus and electricity problems are also solved; therefore, the presented formulation is validated in different fields of engineering and mathematics, allowing for its multidisciplinary use and being a didactic tool that can be adopted in future research.

6. Conclusions

The transformation of hyperbolic functions is generated from Newton's first law by performing the balance of forces for a cable, where the relationship between trigonometric functions and hyperbolic functions is demonstrated, arriving at the Gudermann function and the Mercator projection. In addition, the right triangle formulas involving hyperbolic functions establish a relationship between the sides of the right triangle and the hyperbolic angles. This effectively links the sides of the triangle to their corresponding hyperbolic properties. Furthermore, a new right triangle function is developed, integrating a hyperbolic angle and directly relating the three sides of the triangle, the hyperbolic angle, and Euler's number. Notably, this formulation achieves the relationship without relying on hyperbolic functions, trigonometric functions, or imaginary numbers, providing a simplified yet robust mathematical tool for various applications. The transformation of hyperbolic functions offers an alternative approach to classical trigonometry, facilitating problem solving in geometric and applied mathematical contexts. These relationships result in the law of hyperbolic cosines, the law of hyperbolic tangents, and the hyperbolic Mollweide's formula. Numerical solutions based on the interrelationships among hyperbolic functions are employed to address engineering problems in calculus, statics, electrical engineering, and trigonometry. The outcomes are consistent with those obtained via traditional methods using trigonometric functions, complex numbers, and the law of sines and cosines. Any minor discrepancies observed are primarily attributed to rounding errors inherent in the computational systems used. In the future, the transformation of hyperbolic functions can

be further explored, and its applications can be expanded to areas that form the foundation of engineering and mathematics.

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Conflicts of Interest: The author declares no conflicts of interest.

Nomenclature

Symbol	Description
$C.A.$	Adjacent leg
$C.O.$	Opposite leg
F	Force (N)
H	Hypotenuse
S	Cable length (m)
T	Cable tension (N)
T_0	Minimum cable tension (N)
θ	Trigonometric angle
θ_h	Hyperbolic angle
μ	Weight per unit of length (N/m)
I	Imaginary number

Appendix A

To verify that the solution is correct, the derivative is taken:

$$\ln\left(p+\sqrt{1+p^2}\right). \tag{A1}$$

The chain rule in Equation (A2) is used to derive:

$$\frac{d}{dp}\ln\left(p+\sqrt{1+p^2}\right)=\frac{1}{p+\sqrt{1+p^2}}\frac{d}{dp}\left(p+\sqrt{1+p^2}\right). \tag{A2}$$

The derivative of $p+\sqrt{1+p^2}$ is given by:

$$1+\frac{p}{\sqrt{1+p^2}}. \tag{A3}$$

By substituting this expression into Equation (A2), the following result is obtained:

$$\frac{1}{p+\sqrt{1+p^2}}\times\left(1+\frac{p}{\sqrt{1+p^2}}\right).$$

Simplification leads to:

$$\frac{1+\frac{p}{\sqrt{1+p^2}}}{p+\sqrt{1+p^2}}=\frac{\sqrt{1+p^2}+p}{\sqrt{1+p^2}\times\left(p+\sqrt{1+p^2}\right)}.$$

The numerator and denominator cancel out, leaving:

$$\frac{1}{\sqrt{1+p^2}}.$$

Therefore:

$$\int \frac{dp}{\sqrt{1+p^2}} = \ln\left(p+\sqrt{1+p^2}\right). \tag{A4}$$

Appendix B

The solution is expressed in terms of p:

$$\ln\left(p+\sqrt{1+p^2}\right) = \frac{\mu x}{T_0}. \tag{A5}$$

The exponential function with base e is applied to both sides of the equation:

$$p+\sqrt{1+p^2} = e^{\frac{\mu x}{T_0}}. \tag{A6}$$

By moving p to the other side of the equation through subtraction, the following expression is obtained:

$$\sqrt{1+p^2} = e^{\frac{\mu x}{T_0}} - p.$$

Raising both sides of the equation to the power of 2:

$$1+p^2 = \left(e^{\frac{\mu x}{T_0}} - p\right)^2.$$

Expanding the squared binomial in the equation:

$$1+p^2 = e^{\frac{2\mu x}{T_0}} - 2pe^{\frac{\mu x}{T_0}} + p^2.$$

Solving for p after eliminating the p^2 terms:

$$p = \frac{e^{\frac{2\mu x}{T_0}} - 1}{2e^{\frac{\mu x}{T_0}}}. \tag{A7}$$

The numerator and denominator are multiplied by $e^{-\frac{\mu x}{T_0}}$:

$$p = \frac{\left(e^{\frac{2\mu x}{T_0}} - 1\right) \times e^{-\frac{\mu x}{T_0}}}{2e^{\frac{\mu x}{T_0}} \times e^{-\frac{\mu x}{T_0}}}.$$

The Euler function in the denominator cancels:

$$p = \frac{e^{\frac{\mu x}{T_0}} - e^{-\frac{\mu x}{T_0}}}{2}. \tag{A8}$$

The following expression represents the hyperbolic sine (sinh):

$$p = \sinh\frac{\mu x}{T_0}. \tag{A9}$$

Appendix C

Based on the following relationship:

$$\sin\theta = \tanh\theta_h. \tag{A10}$$

using the change of variable $w = \tanh\theta_h$:

$$\sin\theta = w. \tag{A11}$$

The Euler constant expression with complex numbers is used:

$$\frac{e^{\theta i} - e^{-\theta i}}{2i} = w.$$

The Euler constant expression with complex numbers is used:

$$\frac{e^{2\theta i} - 1}{2i \times e^{\theta i}} = w. \tag{A12}$$

The expression is multiplied, obtaining, by solving for 1:

$$e^{2\theta i} - 2i \times e^{\theta i} w = 1. \tag{A13}$$

w^2 is added to both sides of the equation:

$$e^{2\theta i} - 2i \times e^{\theta i} w - w^2 = 1 - w^2.$$

The sum of squares is completed:

$$\left(e^{\theta i} - i \times w\right)^2 = 1 - w^2. \tag{A14}$$

The equation is solved for $e^{\theta i}$ for positive values, yielding:

$$e^{\theta i} = i \times w + \sqrt{1 - w^2}, \tag{A15}$$

and θ is solved for, using the relation $-i = 1/i$:

$$\theta = -i \ln\left(i \times w + \sqrt{1 - w^2}\right). \tag{A16}$$

By replacing $w = \tanh\theta_h$, it is obtained that:

$$\theta = -i \ \ln\left(i \tanh\ \theta_h + \sqrt{1 - \tanh^2\theta_h}\right). \tag{A17}$$

By replacing the identity $1 - \tanh^2\theta_h = \mathrm{sech}^2\theta_h$:

$$\theta = -i \ \ln(i \ \tanh\theta_h + \mathrm{sech}\,\theta_h). \tag{A18}$$

Finally, the identity can be expressed in terms of $\sinh\theta_h$ and $\cosh\theta_h$:

$$\theta = -i \ \ln\frac{i \ \sinh\theta_h \pm 1}{\cosh\theta_h}. \tag{A19}$$

Appendix D

The equation $\ln(p + \sqrt{1+p^2}) = \frac{\mu x}{T_0}$ from Appendix B is taken. The change of variable $p = \tan\theta$ and $\frac{\mu x}{T_0} = \theta_h$ is made, yielding:

$$\theta_h = \ln\left(\tan\theta + \sqrt{1+\tan^2\theta}\right). \tag{A20}$$

The trigonometric identity $1 + \tan^2\theta = \sec^2\theta$ is used, yielding:

$$\theta_h = \ln(\tan\theta + \sec\theta). \tag{A21}$$

From Appendix C, the following equation is obtained:

$$\theta i = \ln(i\ \tanh\theta_h + \operatorname{sech}\theta_h). \tag{A22}$$

The functions from Equations (29)–(31) are used, and the following result is obtained:

$$\theta i = \ln\frac{C.A + C.O_i}{H}. \tag{A23}$$

By rearranging the equation, the following result is obtained:

$$e^{\theta i} = \operatorname{sech}\theta_h + i\ \tanh\ \theta_h. \tag{A24}$$

The functions from Equations (29)–(31) are used, and the following result is obtained:

$$e^{\theta i} = \cos\theta + i\sin\theta. \tag{A25}$$

By isolating the hyperbolic angle in Equation (A21), the following result is derived:

$$e^{\theta_h} = \sec\theta + \tan\theta. \tag{A26}$$

The functions from Equations (29)–(31) are used, and the following result is obtained:

$$e^{\theta_h} = \cosh\theta_h + \sinh\theta_h. \tag{A27}$$

Appendix E

An analogy can be made to determine the domain of the hyperbolic angles, where $\theta_h \in (-\infty, \infty)$, whereas for trigonometric functions, $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$:

$$\theta_A + \theta_B = \theta_C. \tag{A28}$$

The sum of angles can be expressed as:

$$-i\ln\left(\frac{1 + i\ \sinh\theta_{h,A}}{\cosh\theta_{hA}}\right) - i\ \ln\left(\frac{1 + i\ \sinh\theta_{h,B}}{\cosh\theta_{hB}}\right) = \theta_C. \tag{A29}$$

For the sum of two terms with the property of logarithms, the following result is obtained:

$$\ln\left(\frac{1 + i\ \sinh\theta_{h,A}}{\cosh\theta_{hA}} \times \frac{1 + i\ \sinh\theta_{h,B}}{\cosh\theta_{hB}}\right) = \theta_C i.$$

To eliminate the logarithm, both sides of the equation are raised to the exponent:

$$\frac{1 + i\ \sinh\theta_{hA}}{\cosh\theta_{hA}} \times \frac{1 + i\ \sinh\theta_{hB}}{\cosh\theta_{hB}} = e^{\theta_C i}. \tag{A30}$$

By substituting the identity $e^{\theta i} = \cos\theta + i\,\sin\theta$ and multiplying both sides of the equation, the following result is obtained:

$$\frac{(1 + i\,\sinh\theta_{hA})(1 + i\,\sinh\theta_{hB})}{\cosh\theta_{hA}\cosh\theta_{hB}} = \cos\theta_C + i\,\sin\theta_C. \tag{A31}$$

By expanding the terms in the numerator on the left-hand side, the following result is obtained:

$$(1 + i\,\sinh\theta_{hA})(1 + i\,\sinh\theta_{hB}) = 1 - \sinh\theta_{hA}\sinh\theta_{hB} + i\,\sinh\theta_{hA} + i\,\sinh\theta_{hB}. \tag{A32}$$

The equation is now given by:

$$\frac{1 - \sinh\theta_{hA}\sinh\theta_{hB} + i\sinh\theta_{hA} + i\sinh\theta_{hB}}{\cosh\theta_{hA}\cosh\theta_{hB}} = \cos\theta_C + i\sin\theta_C. \tag{A33}$$

By separating the real part of the equation, the following result is obtained:

$$\frac{1 - \sinh\theta_{hA}\sinh\theta_{hB}}{\cosh\theta_{hA}\cosh\theta_{hB}} = \cos\theta_C. \tag{A34}$$

By separating the imaginary part of the equation, the following result is derived:

$$\frac{\sinh\theta_{hA} + \sinh\theta_{hB}}{\cosh\theta_{hA}\cosh\theta_{hB}} = \sin\theta_C. \tag{A35}$$

By extending the property for the sum and difference of angles, taking $\theta_A \pm \theta_B = \theta_C$, and using the properties $\sin\pm\theta_B = \pm\sin\theta_B$ and $\tanh\pm\theta_{hB} = \pm\tanh\theta_{hB}$, the following equation is obtained:

$$\frac{1 \mp \sinh\theta_{hA}\sinh\theta_{hB}}{\cosh\theta_{hA}\cosh\theta_{hB}} = \cos\theta_C. \tag{A36}$$

Using the relationship from Equation (29), the following value is obtained:

$$\frac{1 \mp \sinh\theta_{hA}\sinh\theta_{hB}}{\cosh\theta_{hA}\cosh\theta_{hB}} = \operatorname{sech}\theta_{hC}. \tag{A37}$$

By applying the property $\cosh\pm\theta_B = \cosh\theta_B$, the following result is derived:

$$\left|\frac{\cosh\theta_{hA}\cosh\theta_{hB}}{1 \mp \sinh\theta_{hA}\sinh\theta_{hB}}\right| = \cosh\theta_{hC}. \tag{A38}$$

Using the property $\sin\pm\theta_B = \pm\sin\theta_B$, by extending the equation $\theta_A \pm \theta_B = \theta_C$:

$$\frac{\sinh\theta_{hA} \pm \sinh\theta_{hB}}{\cosh\theta_{hA}\cosh\theta_{hB}} = \sin\theta_C. \tag{A39}$$

The following value is derived using the relationship found in Equation (31):

$$\frac{\sinh\theta_{hA} \pm \sinh\theta_{hB}}{\cosh\theta_{hA}\cosh\theta_{hB}} = \tanh\theta_{hC}. \tag{A40}$$

Appendix F

By taking $\theta_C = \frac{\pi}{2}$ from Appendix E, the following equations are obtained:

$$\theta_A + \theta_B = \frac{\pi}{2}. \tag{A41}$$

Replacing Equation (A34), the following is obtained:

$$\frac{1-\sinh\theta_{hA}\sinh\theta_{hB}}{\cosh\theta_{hA}\cosh\theta_{hB}}=0. \tag{A42}$$

The relationship of hyperbolic angles is determined:

$$\sinh\theta_{hA}\sinh\theta_{hB}=1. \tag{A43}$$

Another relationship between hyperbolic angles can be determined from Equation (A35):

$$\sinh\theta_{hA}+\sinh\theta_{hB}=\cosh\theta_{hA}\cosh\theta_{hB}. \tag{A44}$$

Appendix G

For the dot product, the hyperbolic cosine law can be used:

$$c^2=a^2+b^2-\frac{2ab}{\cosh\theta_{hC}}. \tag{A45}$$

Values can be determined from the dot product:

$$c\cdot c=a\cdot a+b\cdot b-2a\cdot b. \tag{A46}$$

This is determined using the associative property:

$$c\cdot c=(a-b)\cdot(a-b), \tag{A47}$$

where $\vec{c}=\vec{a}-\vec{b}$, and the following directions are assigned to the vectors. The given vectors have been defined to illustrate the subtraction of vector $\vec{a}$ minus vector $\vec{b}$, as shown in Figure A1.

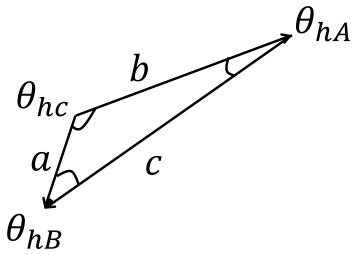

Figure A1. Subtract vector a minus b.

Therefore, it is deduced that:

$$\vec{a}\cdot\vec{b}=\frac{ab}{\cosh|\theta_{hC}|}. \tag{A48}$$

Appendix H

For the dot product, the hyperbolic cosine law can be used:

$$c^2=a^2+b^2-\frac{2ab}{\cosh\theta_{hC}}. \tag{A49}$$

For the cross product, the hyperbolic tangent law is used:

$$\frac{a}{\tanh\theta_{hA}}=\frac{b}{\tanh\theta_{hB}}=\frac{c}{\tanh\theta_{hC}}. \tag{A50}$$

The terms of the equation are inverted:

$$\frac{\tanh\theta_{hA}}{a} = \frac{\tanh\theta_{hB}}{b} = \frac{\tanh\theta_{hC}}{c}.$$

The three terms are multiplied by abc:

$$bc\ \tanh\theta_{hA} = ac\ \tanh\theta_{hB} = ab\ \tanh\theta_{hC}. \tag{A51}$$

The value of equality can be represented by the cross product.

$$|\vec{b}\times\vec{c}| = |\vec{c}\times\vec{a}| = |\vec{a}\times\vec{b}|. \tag{A52}$$

where $\vec{a}+\vec{b}+\vec{c}=0, \quad \vec{c}=-(\vec{a}+\vec{b})$, and the following directions are taken for the vectors. The given vectors have been defined to illustrate that the sum of $\vec{a}$, $\vec{b}$, and $\vec{c}$ is equal to zero, as they return to the same starting point, as shown in Figure A2.

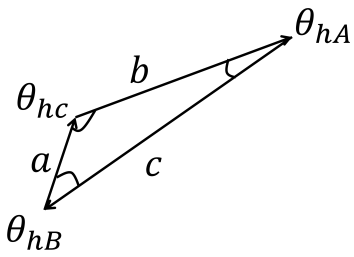

Figure A2. Vector sum equal to zero.

Replacing the sum of vectors equal to zero, Equation (A53) is obtained:

$$|\vec{b}\times-(\vec{a}+\vec{b})| = |-(\vec{a}+\vec{b})\times\vec{a}| = |\vec{a}\times\vec{b}|. \tag{A53}$$

Replacing the sum of vectors equal to zero, Equation (A54) is determined:

$$|\vec{b}\times-\vec{a}| = |-\vec{b}\times\vec{a}| = |\vec{a}\times\vec{b}|. \tag{A54}$$

Equality is to be verified:

$$|\vec{a}\times\vec{b}| = |\vec{a}\times\vec{b}| = |\vec{a}\times\vec{b}|. \tag{A55}$$

Therefore, the following equations are presented:

$$|\vec{b}\times\vec{c}| = bc\ \tanh\theta_{hA}, \quad |\vec{c}\times\vec{a}| = ca\ \tanh\theta_{hB}, \text{ and } \quad |\vec{a}\times\vec{b}| = ab\ \tanh\theta_{hC}. \tag{A56}$$

Appendix I

By using Mollweide's formulas, Equations (A57) and (A58) are obtained [1], where $\theta_A+\theta_B+\theta_C=\pi$.

$$\frac{\cos\frac{\theta_A-\theta_B}{2}}{\sin\frac{\theta_C}{2}} = \frac{a+b}{c}, \tag{A57}$$

and

$$\frac{\sin\frac{\theta_A-\theta_B}{2}}{\cos\frac{\theta_C}{2}} = \frac{a-b}{c}, \tag{A58}$$

where the trigonometric identity for the internal angles can be expressed as:

$$\cos\frac{\theta_A-\theta_B}{2}=\pm\sqrt{\frac{1+\cos\theta_A\cos\theta_B+\sin\theta_A\ \sin\theta_B}{2}}. \tag{A59}$$

The trigonometric identity for the internal angles is presented as:

$$\sin\frac{\theta_A-\theta_B}{2}=\pm\sqrt{\frac{1-\cos\theta_A\cos\theta_B-\sin\theta_A\sin\theta_B}{2}}. \tag{A60}$$

Considering that the transformation to hyperbolic functions is valid for values where $\theta\geq\frac{\pi}{2}$, the relation $\theta_C\geq\frac{\pi}{2}$ must hold. Therefore, the following relation is used for angles greater than $\frac{\pi}{2}$, which is valid when performing the calculation with hyperbolic functions. The subscript 2 in $\theta_{2C}\leq\frac{\pi}{2}$ indicates that this angle is valid for this condition, where $\theta_C+\theta_{2C}=\pi$:

$$\cos\frac{\pi-\theta_{2C}}{2}=\pm\sqrt{\frac{1+\cos(\pi-\theta_{2C})}{2}}. \tag{A61}$$

By using the trigonometric identity for the difference of angles, the following is obtained:

$$\cos(\pi-\theta_{2C})=-\cos\theta_{2C}=-\operatorname{sech}\theta_{hC}. \tag{A62}$$

Substituting into the equation, the following is obtained:

$$\cos\frac{\pi-\theta_{2C}}{2}=\pm\sqrt{\frac{1-\cos\theta_{2C}}{2}}=\pm\sqrt{\frac{1-\operatorname{sech}\theta_{hC}}{2}}. \tag{A63}$$

The function $\sin\frac{\pi-\theta_2C}{2}$ can be represented by the following identity:

$$\sin\frac{\pi-\theta_{2C}}{2}=\pm\sqrt{\frac{1+\cos\theta_{2C}}{2}}=\pm\sqrt{\frac{1+\operatorname{sech}\theta_{hC}}{2}}. \tag{A64}$$

In Equations (A65) and (A66), the replacements of the Gudermann function are made:

$$\cos\frac{\theta_A-\theta_B}{2}=\pm\sqrt{\frac{1+\operatorname{sech}\theta_{hA}\operatorname{sech}\theta_{hB}+\tanh\theta_{hA}\ \tanh\theta_{hB}}{2}}. \tag{A65}$$

and

$$\sin\frac{\theta_A-\theta_B}{2}=\pm\sqrt{\frac{1-\operatorname{sech}\theta_{hA}\operatorname{sech}\theta_{hB}-\tanh\theta_{hA}\ \tanh\theta_{hB}}{2}}. \tag{A66}$$

Equations (A65) and (A66) of the Mollweide's formulas can be simplified into a single expression:

$$\frac{\pm\sqrt{\frac{1\pm\operatorname{sech}\theta_{hA}\operatorname{sech}\theta_{hB}\pm\tanh\theta_{hA}\ \tanh\theta_{hB}}{2}}}{\pm\sqrt{\frac{1\pm\operatorname{sech}\theta_{hC}}{2}}}=\frac{a\pm b}{c}. \tag{A67}$$

The numerator and denominator are divided by 2; thus, they cancel out, and by applying the identities for $\operatorname{sech}\theta_{hA}$, $\operatorname{sech}\theta_{hB}$, $\tanh\theta_{hA}$, and $\tanh\theta_{hB}$, the following expression is obtained:

$$\pm\sqrt{\frac{\cosh\theta_{hA}\cosh\theta_{hB}\pm1\pm\sinh\theta_{hA}\sinh\theta_{hB}}{\cosh\theta_{hA}\cosh\theta_{hB}(1\pm\operatorname{sech}\theta_{hC})}}=\frac{a\pm b}{c}. \tag{A68}$$

The identity for the sum and difference of hyperbolic angles of the hyperbolic cosine function is stated below:

$$\cosh(\theta_{hA} \pm \theta_{hB}) = \cosh\theta_{hA}\cosh\theta_{hB} \pm \sinh\theta_{hA}\sinh\theta_{hB}. \tag{A69}$$

Substituting into the identity and eliminating the square root, the hyperbolic Mollweide's formula expression is obtained:

$$\frac{\cosh(\theta_{hA} \pm \theta_{hB}) \pm 1}{\cosh\theta_{hA}\cosh\theta_{hB}(1 \pm \operatorname{sech}\theta_{hC})} = \left(\frac{a \pm b}{c}\right)^2. \tag{A70}$$

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