

Article

Impulse Controllability for Singular Hybrid Coupled Systems

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Abstract: This study examines the concept of impulse controllability within singular hybrid coupled systems through the utilisation of decentralised proportional plus derivative (P-D) output feedback. By employing the Differential Mean Value Theorem, the nonlinear model can be converted into a linear parameter-varying large-scale system. Our analysis leads to the establishment of algebraic conditions that are both necessary and sufficient for the existence of a decentralised P-D output feedback controller that can guarantee impulse controllability in these complex systems. Moreover, we address the issue of admissibility within these systems by employing matrix trace inequalities. We present a novel sufficient condition for impulse controllability, which offers a new perspective on addressing this challenging problem. To validate our findings, we present numerical examples that demonstrate the effectiveness of the proposed methodologies in practice.

Keywords: singular hybrid coupled systems; nonlinear singular large-scale systems; impulse controllability; admissibility; decentralised proportional plus derivative output feedback controller



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1. Introduction

Over the past few decades, there has been a notable increase in interest within academic circles regarding the exploration of intricate and dynamic networks. The concept is attracting attention from a variety of scientific fields, including physics, biology, and the social sciences. For further information on this topic, please refer to references [1–3]. In addition to examples such as the internet and power sources, there are numerous other instances that support this concept. These include food chains, brain and communication networks, and the emerging networks of collaborative authorship and references. Scientists are easily recognisable and distinguished [4]. It is essential that the entire network be accurately modelled. When considering the entire network, it is important to view it as a complex system in which each node operates dynamically and to give greater consideration to the issue of decentralisation in intricate scenarios. For instance, a time-delay method was proposed for managing large networked control systems in one study [5], which involved linking numerous local communication networks with sensors, controllers, and actuators. Additionally, reference [6] delved into the challenges of impulse synchronisation in singular hybrid coupled systems with time-varying nonlinear disturbances. Reference [7] also investigated the robust exponential synchronisation of a specific type of neutral complex network. Moreover, the utilisation of the Lyapunov–Krasovskii function in the descriptor-model transformation framework plays a crucial role in assessing the robust stability and H control of a specific category of uncertain T-S fuzzy systems. This is particularly important when dealing with time delays that occur simultaneously in both state and control inputs [8].

When modeling complex real-world networks, it is important to take into account algebraic constraints. One example of this is the need to assign different levels of user privilege based on the limited communication resources available. Therefore, it is crucial to include certain limitations when distributing resources. Nevertheless, it is a widely

recognised fact that the presence of algebraic constraints can lead to sudden spikes in system reactions, potentially leading to control saturation or even complete system breakdown in real-world scenarios. Hence, it is essential to tackle the issue of impulsive behavior in intricate networks with algebraic constraints by employing feedback controllers.

Reference [9] discusses the issue of removing impulsive modes using state feedback. Demonstrating impulsive controllability has been identified as both a necessary and sufficient requirement for achieving this. The idea of impulsive controllability was initially presented in reference [10] and has since been a key focus in the field of elimination. Further research has been conducted in references [11,12], confirming the findings that proportional-derivative techniques have been successfully implemented. In order to eliminate impulsive behavior, it is necessary to implement state feedback. The system can be controlled impulsively. In reference [13], a new method was introduced using the null space approach. standards for characterising the ability to control impulsive behavior in linear systems. In the study [14], the focus was on addressing impulse elimination issues by proposing a new condition that assumes the presence of feedback and its regulations centered on decentralised output. These conditions are crucial in ensuring that the closed-loop system remains free of impulses and maintains maximum dynamic order [15]. Additionally, reference [16] introduced the idea of a structured P-D feedback controller and put forward a necessary and sufficient condition to uphold the regularity and impulse-free nature of the closed-loop system. Furthermore, references [17–19] delved into the correlation between impulsive responses and impulse controllability and observability in descriptor systems. Specifically, they analysed these aspects in detail for large-scale interconnected descriptor systems with two subsystems operating under derivative feedback. Reference [20] explored how a delay can have a stabilising effect on complex neural networks made up of uniform systems using pinning impulsive control. Reference [21] examined the development of non-fragile decentralised H_∞ controllers for interconnected systems of integer order based on the existing literature.

This paper delves into the issue of controlling impulses in singular systems. A decentralised P-D output feedback controller is employed in hybrid coupled systems. In order to effectively handle the system's nonlinearity, a specific technique is employed. According to the Differential Mean Value Theorem (DMVT), the term that is removed necessitates the implementation of a universal Lipschitz constraint and circumvents the use of high-gain control mechanisms. The process of transformation causes a change in the nonlinear singular large-scale system, transforming it into a linear one. This is a singular large-scale system with varying parameters, designed for the analysis of nonlinear systems. The analysis of behaviour is conducted using the tenets of linear system theory.

Firstly, to preserve the impulse controllability in singular hybrid coupled, the primary outcome is the existence of a decentralised output feedback controller, which depends on the necessary and sufficient algebraic conditions we derived. In addition, we provide a novel sufficient condition for the admissibility of an impulse-controllable singular hybrid coupled system. Finally, we present illustrative examples to demonstrate the effectiveness of this method.

This paper proceeds as follows: in Section 2, we introduce the notations, definitions, and lemmas required for the subsequent sections. Section 3 presents a necessary and sufficient algebraic condition for impulsive controllability using a decentralised derivative output feedback controller. Section 4 provides illustrative examples demonstrating the feasibility of the proposed theorems. Finally, Section 5 offers a summary of the paper.

2. Notations and Preliminaries

This section serves to re-examine some of the fundamental symbols and findings presented previously. The set $\mathbf{R}^{n \times m}(C^{n \times m})$ is defined as the collection of $n \times m$ real (complex) matrices. Let us consider the transpose of M , denoted by M^T , as the matrix with elements $[m_{ij}]_{n \times m}$, where M is an nm matrix belonging to R_{nm} . Similarly, let us denote the identity matrix as I_m . Let $\max_K(M)$ be used to indicate the general rank of

matrix M , which represents the highest rank that can be achieved by matrix M with a specific variable set K . Let $\mathcal{N}$ be defined as a collection of the natural numbers from 1 to N , inclusive. Let φ be a non-empty subset of N with elements arranged in ascending order, i.e., $\{1, 2, \dots, j-1, j+1, \dots, N\}$ and φ . In this context, B_φ and C_φ are defined as follows:

$$C_\varphi = [C_{i_1} \ C_{i_2} \ \cdots \ C_{i_s}]^T,$$

and

$$B_\varphi = [B_{i_1} \ B_{i_2} \ \cdots \ B_{i_s}].$$

Furthermore, $P(\mathcal{N})$ denotes the power set of $\mathcal{N}$, encompasses the complete collection of all subsets of the set $\mathcal{N}$, and

$$\mathcal{N} - \varphi = \{x : x \in \mathcal{N}, x \notin \varphi\}.$$

Let

$$S_c = \{K : K = \text{block diag}[K_1, K_2, \dots, K_N], K_i \in \mathbf{R}^{l \times (Nn)}, i \in \mathcal{N}\}.$$

Examine a category of singular hybrid coupled systems of the following form :

$$\begin{cases} E\dot{x}_i(t) = Ax_i(t) + f_i(x_i(t)) + c \sum_{j=1}^N b_{ij}\Gamma x_j(t) + Bu_i, \\ y_i = \bar{C}_i x(i = 1, 2, \dots, N) \end{cases}, \quad (1)$$

where the matrix E is found to be non-invertible, $0 < \text{rank}(E) = r < n$. $A \in \mathbf{R}^{n \times n}$, $B \in \mathbf{R}^{n \times l}$, and $\bar{C} \in \mathbf{R}^{l \times Nn}$ are the constant matrices. Let N represent the quantity of interconnected nodes. $x_i = (x_{i1}, x_{i2}, \dots, x_{in})^T \in \mathbf{R}^n$ are state variables of node i , and u_i is the control input. $f(x_i(t)) \in \mathbf{R}^n$ is a nonlinear function that varies with time and takes vector values. The constant $c > 0$ indicates the level of connection between two entities, and $\Gamma = \text{diag}(\gamma_1, \gamma_2, \dots, \gamma_n) \in \mathbf{R}^{n \times n}$ is a 0 – 1 diagonal matrix with specific $\gamma_i = 1$ or 0 for others. $(b_{ij})_{N \times N}$ denotes the coupling configuration of the entire network. If there is a connection between node i and node j , then $b_{ij} = b_{ji} = 1 (i \neq j)$; otherwise, $b_{ij} = b_{ji} = 0 (i \neq j)$.

$$b_{ii} = - \sum_{j=1, j \neq i}^N, b_{ij} = - \sum_{j=1, j \neq i}^N b_{ji}, i = 1, 2, \dots, N.$$

The singular hybrid coupled systems (1) can be described as a nonlinear singular large-scale systems model,

$$\begin{cases} \bar{E}\dot{x}(t) = \bar{A}x(t) + f(x(t)) + \sum_{j=1}^N \bar{B}_i u_i, \\ y_i = \bar{C}_i x(i = 1, 2, \dots, N) \end{cases}, \quad (2)$$

where $\bar{B}_i \in \mathbf{R}^{Nn \times l}$, $f(x(t))$ is a function that is not linear, $f(0, 0, \dots, 0) = 0$,

$$\bar{E} = \begin{bmatrix} E & 0 & \cdots & 0 \\ 0 & E & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & E \end{bmatrix}, \bar{A} = \begin{bmatrix} A + cb_{11}\Gamma & cb_{12}\Gamma & \cdots & cb_{1N}\Gamma \\ cb_{21}\Gamma & A + cb_{22}\Gamma & \cdots & cb_{2N}\Gamma \\ \vdots & \vdots & \ddots & \vdots \\ cb_{N1}\Gamma & cb_{N2}\Gamma & \cdots & A + cb_{NN}\Gamma \end{bmatrix}, \text{ and}$$

$$\bar{B}_i = \begin{bmatrix} 0 \\ \vdots \\ B \\ \vdots \\ 0 \end{bmatrix} \leftarrow i, f(x(t)) = \begin{bmatrix} f_1(x_1(t)) \\ f_2(x_2(t)) \\ \vdots \\ f_N(x_N(t)) \end{bmatrix}, x = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_N(t) \end{bmatrix}, u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix}.$$

System (2) is a nonlinear singular large-scale systems model with n dimensions.

Lemma 1 ([22]). Suppose $f : \mathbf{R}^n \rightarrow \mathbf{R}^q$, with $a, b \in \mathbf{R}^n$. It is assumed that f is differentiable on $\text{Co}(a, b)$. Then, there are constant vectors $c_1, c_2, \dots, c_q \in \text{Co}(a, b)$, $c_i \neq a, c_i \neq b$ for $i = 1, 2, \dots, q$, $\text{Co}(a, b) = \{\lambda a + (1 - \lambda)b, \lambda \in [0, 1]\}$ is a convex hull. such that

$$f(a) - f(b) = \left(\sum_{i,j=1}^{q,n} e_q(i) e_n^T(j) \frac{\partial f_i}{\partial x_j}(c_i) \right) (a - b). \quad (3)$$

We set

$$A_{ii} = A + cb_{ii}\Gamma \quad i = 1, \dots, N,$$

and

$$A_{ij} = cb_{ij}\Gamma \quad i \neq j, i, j = 1, \dots, N.$$

By Lemma 1 and noting $f(0) = 0$, the nonlinear systems (2) can be transformed into linear singular large-scale systems, which can be represented in the following form:

$$\begin{cases} \bar{E}\dot{x}_i(t) = (\bar{A} + \Sigma)x(t) + \sum_{j=1}^N \bar{B}_i u_j \\ y_i = \bar{C}_i x(i = 1, 2, \dots, N) \end{cases} \quad (4)$$

where

$$\bar{A} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1N} \\ A_{21} & A_{22} & \cdots & A_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ A_{N1} & A_{N2} & \cdots & A_{NN} \end{bmatrix}, \quad (5)$$

$$\begin{aligned} \Sigma &= \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \cdots & \Sigma_{1n} \\ \Sigma_{21} & \Sigma_{22} & \cdots & \Sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma_{n1} & \Sigma_{n2} & \cdots & \Sigma_{nn} \end{bmatrix} \\ &= \sum_{i=1}^N \sum_{k=1}^{n_i} \sum_{j=1}^n e_{\eta} \left(\sum_{l=1}^{i-1} n_l + k \right) e_{\eta}^T(j) \frac{\partial f_{ik}}{\partial x_j}(z_{ik}(t)), z_{ik}(t) \in \text{Co}(0, x(t)), \end{aligned} \quad (6)$$

and $f_{ik}: \mathbf{R}^n \rightarrow \mathbf{R}$ is the k -th component of f_i .

Then,

$$h_{ik}^j(t) = \frac{\partial f_{ik}}{\partial x_j}(z_{ik}(t)),$$

$$h(t) = (h_{11}^1, \dots, h_{11}^{n_1}, \dots, h_{Nn_N}^1, \dots, h_{Nn_N}^{n_N}),$$

and

$$\bar{A}(h(t)) = \bar{A} + \Sigma.$$

In this context, $e_n(i)$ and $e_n(j)$ represent the standard basis vectors of the vector space R^n and can be written as follows:

$$E_n = \left\{ e_n(i) | e_n(i) = (0, \dots, 0, \underbrace{1}_i, 0, \dots, 0)^T, i = 1, \dots, n \right\},$$

where the variable $h_{ik}^j(t)$ represents the derivative of the nonlinear function where z_{ik} denotes a time-varying parameter, and n is equal to $n = Nn$.

The system (4) has the potential to be rephrased as

$$\begin{cases} \bar{E}\dot{x}_i(t) = \bar{A}(h(t))x(t) + \sum_{j=1}^N \bar{B}_j u_j \\ y_i = \bar{C}_i x(i = 1, 2, \dots, N). \end{cases} \quad (7)$$

It is generally believed that the functions $h_{ik}^j(t)$ are constrained within certain limits.

$$\max |h_{ik}^j(t)| < +\infty.$$

It is important to recognise that this assumption is not limiting in nature. In fact, it holds true for a significant category of nonlinear descriptor systems. Based on this hypothesis, the parameter $h(t)$ progresses within a constrained domain, denoted as H_{n^2, n^2} , which is characterised by $2^{n^2 \times n^2}$ vertices defined as follows:

$$\mathcal{V}_{H_{n^2, n^2}} = \left\{ (\alpha_{11}^1, \dots, \alpha_{11}^n, \dots, \alpha_{NnN}^1, \dots, \alpha_{NnN}^n) \mid \alpha_{ik}^j \in \{\underline{h}_{ik}^j, \bar{h}_{ik}^j\} \right\}, \quad (8)$$

where

$$\bar{h}_{ik}^j = \max_t (h_{ik}^j(t)), \underline{h}_{ik}^j = \min_t (h_{ik}^j(t)).$$

$$\eta = Nn.$$

If system (4) is impulse controllable, a state feedback can be implemented to ensure the resulting closed-loop system is free of impulses [23]. In cases where a singular system lacks impulse controllability, a decentralised P-D output feedback can be initially introduced to eliminate impulses.

$$u_i = -L_i y_i + K_i \dot{y}_i + v_i, \quad i = 1, 2, \dots, N. \quad (9)$$

Applying the feedback (9) to the system (7), we obtain the resultant closed-loop system

$$\left(\bar{E} + \sum_{i=1}^N \bar{B}_i K_i \bar{C}_i \right) \dot{x}(t) = \left(\bar{A}(h(t)) + \sum_{i=1}^N \bar{B}_i L_i \bar{C}_i \right) x(t) + \sum_{i=1}^N \bar{B}_i v_i. \quad (10)$$

Consider the following singular systems

$$\begin{cases} E\dot{x}(t) = Ax(t) + Bu \\ y = Cx \end{cases}, \quad (11)$$

where $x \in \mathbf{R}^n$ is the state vector, $u \in \mathbf{R}^r$ is the control input, $y \in \mathbf{R}^l$ is the controlled output, and $E, A \in \mathbf{R}^{n \times n}$, $B \in \mathbf{R}^{n \times r}$ with limited conditions that the constant matrices and $\text{rank}(E) = n_0 \leq n$.

Definition 1 ([24]). A system (11) is considered regular if there is a constant scalar $s \in \mathbb{C}$ that satisfies

$$\det(sE - A) \neq 0. \quad (12)$$

Lemma 2 ([24]). If the system (11) is regular, we have the following results:

(1) The system (11) is free from impulses only when the following relation is satisfied:

$$\text{rank} \begin{pmatrix} 0 & E \\ E & A \end{pmatrix} = n + \text{rank}(E). \quad (13)$$

(2) The system (11) is impulse controllable only when the following relation is satisfied:

$$\text{rank} \begin{pmatrix} 0 & E & 0 \\ E & A & B \end{pmatrix} = n + \text{rank}(E). \quad (14)$$

(3) The system (11) is impulse observable only when the following relation is satisfied:

$$\text{rank} \begin{pmatrix} 0 & E \\ E & A \\ 0 & C \end{pmatrix} = n + \text{rank}(E). \quad (15)$$

Lemma 3 ([15]). Let $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{m \times h}$, and $C \in \mathbb{R}^{l \times n}$ $K \in \mathbb{R}^{h \times l}$,

$$\max_K (A + BKC) = \min \left\{ \text{rank} \begin{pmatrix} A & B \end{pmatrix}, \text{rank} \begin{pmatrix} A \\ C \end{pmatrix} \right\}. \quad (16)$$

$$S_R = \left\{ K : \text{rank}(A + BKC) = \max_{K \in \mathbb{R}^{h \times l}} (A + BKC) \right\}.$$

Lemma 4 ([25]). Let $A \in \mathbb{R}^{m \times n}$, $P_i \in \mathbb{R}^{m \times h_i}$, and $Q_i \in \mathbb{R}^{l_i \times n}$ be fixed real matrices, and $K_i \in \mathbb{R}^{h_i \times l_i}$ ($i = 1, 2, \dots, N$). In that way,

$$\begin{aligned} & \max_{K_1, \dots, K_N} (A + \sum_{i=1}^N P_i K_i Q_i) \\ &= \min \left\{ \max_{K_1, \dots, K_{N-1}} \left(A + \sum_{i=1}^{N-1} P_i K_i Q_i \quad P_N \right), \max_{K_1, \dots, K_{N-1}} \left(A + \sum_{i=1}^{N-1} P_i K_i Q_i \right) \right\}. \quad (17) \end{aligned}$$

Lemma 5 ([26]). If $M \in \mathbb{R}^{m \times n}$, $q = \text{rank}(M)$, and there exists a reversible matrix P, Q , then

$$PMQ = \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix},$$

and

$$E\{1\} = \left\{ Q \begin{bmatrix} I_q & X \\ Y & Z \end{bmatrix} P \right\}.$$

Lemma 6 ([27]). If $A \in \mathbb{R}^{n \times n}$, and $\text{tr}[H(A)] < -\sqrt{(n-1)\text{tr}[H(A)]^2}$, then the eigenvalues of A are all distributed in the left half of the complex plane.

3. Results

In this part, we establish an algebraic condition that is both necessary and sufficient for the presence of a decentralised P-D out feedback controller in singular hybrid coupled systems to achieve impulse controllability.

By Lemma 2, it is easy to obtain the following Theorem 1.

Theorem 1. Consider system (7).

The system (7) is regular and impulse-free, if and only if

$$\text{rank} \left(\begin{bmatrix} 0 & \bar{E} \\ \bar{E} & \tilde{A}(h(t)) \end{bmatrix} \right) = Nn + \text{rank}(\bar{E}). \quad (18)$$

The primary goal of this paper is to establish an algebraic condition that is essential and complete for determining whether a decentralised P-D output feedback control controller (9) exists, which guarantees that the singular system (7) is impulse controllable. This paper offers an algebraic criterion that is both necessary and sufficient for the presence of a decentralised proportional-derivative (P-D) output feedback controller (9) that guarantees the closed-loop system satisfies certain requirements.

$$\max_{\substack{K_N \\ L_N}} \left(\begin{bmatrix} 0 & \bar{E} + \sum_{i \in \mathcal{N}} \bar{B}_i L_i \bar{C}_i & 0 \\ \bar{E} + \sum_{i \in \mathcal{N}} \bar{B}_i L_i \bar{C}_i & \tilde{A}(h(t)) + \sum_{i \in \mathcal{N}} \bar{B}_i K_i \bar{C}_i & \bar{B}_j \end{bmatrix} \right) = Nn + \max_{L_N} (\bar{E} + \sum_{i \in \mathcal{N}} \bar{B}_i L_i \bar{C}_i). \quad (19)$$

Theorem 2. If a feedback controller (9) exists for the system (7) such that the closed-loop system (10) is impulse controllable, then the closed-loop system (10) will satisfy condition (19), if and only if

$$\text{rank} \left(\begin{bmatrix} 0 & \bar{E} & 0 & 0 & \bar{B}_{\varphi_1} & 0 \\ \bar{E} & \tilde{A}(h(t)) & \bar{B}_j & \bar{B}_{\varphi} & 0 & \bar{B}_{\varphi_2} \\ 0 & \bar{C}_{\mathcal{N}-\varphi} & 0 & 0 & 0 & 0 \\ 0 & \bar{C}_{\varphi-\varphi_1} & 0 & 0 & 0 & 0 \\ \bar{C}_{(\mathcal{N}-\varphi)-\varphi_2} & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right) \geq Nn + r, \quad (20)$$

for all $\varphi \in P(\mathcal{N})$, $\varphi_1 \in P(\varphi)$, $\varphi_2 \in P(\mathcal{N} - \varphi)$, with

$$r \triangleq \min_{\varphi \in P(\mathcal{N})} \text{rank} \left(\begin{bmatrix} \bar{E} & \bar{B}_{\varphi} \\ \bar{C}_{\mathcal{N}-\varphi} & 0 \end{bmatrix} \right),$$

Proof. We note that

$$\begin{aligned} & \begin{bmatrix} 0 & \bar{E} + \sum_{i \in \mathcal{N}} \bar{B}_i L_i \bar{C}_i & 0 \\ \bar{E} + \sum_{i \in \mathcal{N}} \bar{B}_i L_i \bar{C}_i & \tilde{A}(h(t)) + \sum_{i \in \mathcal{N}} \bar{B}_i K_i \bar{C}_i & \bar{B}_j \end{bmatrix} \\ &= \begin{bmatrix} 0 & \bar{E} + \sum_{i \in \mathcal{N}} \bar{B}_i L_i \bar{C}_i & 0 \\ \bar{E} + \sum_{i \in \mathcal{N}} \bar{B}_i L_i \bar{C}_i & \tilde{A}(h(t)) & \bar{B}_j \end{bmatrix} + \sum_{i \in \varphi} \begin{bmatrix} 0 \\ \bar{B}_i \end{bmatrix} L_i \begin{bmatrix} 0 & \bar{C}_i & 0 \end{bmatrix}. \end{aligned} \quad (21)$$

By Lemma 4 and (21), we have two inequalities that are true:

$$\begin{aligned} & \max_{\substack{K_1, \dots, K_{j-1}, K_{j+1}, \dots, K_{N-1} \\ L_N}} \left(\begin{bmatrix} 0 & \bar{E} + \sum_{i \in \mathcal{N}} \bar{B}_i L_i \bar{C}_i & 0 & 0 \\ \bar{E} + \sum_{i \in \mathcal{N}} \bar{B}_i L_i \bar{C}_i & \tilde{A}(h(t)) + \sum_{i=1}^{N-1} \bar{B}_i L_i \bar{C}_i & \bar{B}_j & \bar{B}_N \end{bmatrix} \right) \\ & \geq Nn + \text{rank}(\bar{E} + \sum_{i \in \mathcal{N}} \bar{B}_i L_i \bar{C}_i), \end{aligned} \quad (22)$$

and

$$\max_{\substack{K_1, \dots, K_{j-1}, K_{j+1}, \dots, K_{N-1} \\ L_N}} \left(\begin{bmatrix} 0 & \bar{E} + \sum_{i \in \mathcal{N}} \bar{B}_i L_i \bar{C}_i & 0 \\ \bar{E} + \sum_{i \in \mathcal{N}} \bar{B}_i L_i \bar{C}_i & \tilde{A}(h(t)) + \sum_{i=1}^{N-1} \bar{B}_i L_i \bar{C}_i & \bar{B}_j \\ 0 & \bar{C}_N & 0 \end{bmatrix} \right) \quad (23)$$

$$\geq Nn + \text{rank}(\bar{E} + \sum_{i \in \mathcal{N}} \bar{B}_i L_i \bar{C}_i).$$

Using Lemma 4, (22) and (23), it is easy to conclude that

$$\begin{aligned} & \max_{\substack{K_N \\ L_N}} \left(\begin{bmatrix} 0 & \bar{E} + \sum_{i \in \mathcal{N}} \bar{B}_i L_i \bar{C}_i & 0 \\ \bar{E} + \sum_{i \in \mathcal{N}} \bar{B}_i L_i \bar{C}_i & \tilde{A}(h(t)) + \sum_{i \in \mathcal{N}} \bar{B}_i K_i \bar{C}_i & \bar{B}_j \end{bmatrix} \right) \\ &= \text{rank} \left(\begin{bmatrix} 0 & \bar{E} + \sum_{i \in \mathcal{N}} \bar{B}_i L_i \bar{C}_i & 0 & 0 \\ \bar{E} + \sum_{i \in \mathcal{N}} \bar{B}_i L_i \bar{C}_i & \tilde{A}(h(t)) & \bar{B}_j & \bar{B}_\varphi \\ 0 & \bar{C}_{\mathcal{N}-\varphi} & 0 & 0 \end{bmatrix} \right). \end{aligned} \quad (24)$$

We note that since $\varphi \cap \mathcal{N} - \varphi \neq \emptyset$, we have

$$\begin{aligned} & \begin{bmatrix} 0 & \bar{E} + \sum_{i \in \varphi} \bar{B}_i L_i \bar{C}_i & 0 & 0 \\ \bar{E} + \sum_{i \in \mathcal{N}-\varphi} \bar{B}_i L_i \bar{C}_i & \tilde{A}(h(t)) & \bar{B}_j & \bar{B}_\varphi \\ 0 & \bar{C}_{\mathcal{N}-\varphi} & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} I & 0 & \tilde{B}L \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} 0 & \bar{E} + \sum_{i \in \mathcal{N}} \bar{B}_i L_i \bar{C}_i & 0 & 0 \\ \bar{E} + \sum_{i \in \mathcal{N}} \bar{B}_i L_i \bar{C}_i & \tilde{A}(h(t)) & \bar{B}_j & \bar{B}_\varphi \\ 0 & \bar{C}_{\mathcal{N}-\varphi} & 0 & 0 \end{bmatrix} \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ \tilde{L}C & 0 & 0 & I \end{bmatrix}. \end{aligned} \quad (25)$$

By Lemma 4, it is easy to conclude that

$$\begin{aligned} & \max_{\substack{K_N \\ L_N}} \left(\begin{bmatrix} 0 & \bar{E} + \sum_{i \in \mathcal{N}} \bar{B}_i L_i \bar{C}_i & 0 \\ \bar{E} + \sum_{i \in \mathcal{N}} \bar{B}_i L_i \bar{C}_i & \tilde{A}(h(t)) + \sum_{i \in \mathcal{N}} \bar{B}_i K_i \bar{C}_i & \bar{B}_j \end{bmatrix} \right) \\ &= \text{rank} \left(\begin{bmatrix} 0 & \bar{E} & 0 & 0 & \bar{B}_{\varphi_1} & 0 \\ \bar{E} & \tilde{A}(h(t)) & \bar{B}_j & \bar{B}_\varphi & 0 & \bar{B}_{\varphi_2} \\ 0 & \bar{C}_{\mathcal{N}-\varphi} & 0 & 0 & 0 & 0 \\ 0 & \bar{C}_{\varphi-\varphi_1} & 0 & 0 & 0 & 0 \\ \bar{C}_{(\mathcal{N}-\varphi)-\varphi_2} & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right), \end{aligned} \quad (26)$$

for all $\varphi \in P(\mathcal{N})$, $\varphi_1 \in P(\varphi)$, $\varphi_2 \in P(\mathcal{N} - \varphi)$.

By Lemmas 3 and 4, we are aware that the following statement is true for the vast majority of $\tilde{L} \in S_C$:

$$\text{rank}(\bar{E} + \sum_{i \in \mathcal{N}} \bar{B}_i \tilde{L}_i \bar{C}_i) = \max_{L_N} (\bar{E} + \sum_{i \in \mathcal{N}} \bar{B}_i L_i \bar{C}_i) = r. \quad (27)$$

The same theory proves that

$$\max_{L_N} (\bar{E} + \sum_{i \in \mathcal{N}} \bar{B}_i L_i \bar{C}_i) = \min_{\varphi \in P(\mathcal{N})} \text{rank} \left(\begin{bmatrix} \bar{E} & \bar{B}_\varphi \\ \bar{C}_{\mathcal{N}-\varphi} & 0 \end{bmatrix} \right) = r. \quad (28)$$

The system (10) is impulse controllable if and only if (19) holds. Based on our analysis, we can infer that in order for statement (20) to hold true, it is essential that all $\varphi \in P(\mathcal{N}), \varphi_1 \in P(\varphi), \varphi_2 \in P(\mathcal{N} - \varphi)$

$$\text{rank} \left(\begin{bmatrix} 0 & \bar{E} & 0 & 0 & \bar{B}_{\varphi_1} & 0 \\ \bar{E} & \tilde{A}(h(t)) & \bar{B}_j & \bar{B}_{\varphi} & 0 & \bar{B}_{\varphi_2} \\ 0 & \bar{C}_{\mathcal{N}-\varphi} & 0 & 0 & 0 & 0 \\ 0 & \bar{C}_{\varphi-\varphi_1} & 0 & 0 & 0 & 0 \\ \bar{C}_{(\mathcal{N}-\varphi)-\varphi_2} & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right) \geq Nn + r. \quad (29)$$

□

Remark 1. The proof of Theorem 2 can obviously be concluded by repeated use of Lemma 4 and the impulse controllability conditions.

Corollary 1. Consider the system (4). According to Lemma 2(3), the closed-loop system (10) is impulse observable if and only if there is a feedback controller (9) that satisfies this condition for all $\varphi \in P(\mathcal{N}), \varphi_1 \in P(\varphi), \varphi_2 \in P(\mathcal{N} - \varphi)$.

$$\text{rank} \left(\begin{bmatrix} 0 & \bar{E} & 0 & \bar{B}_{\varphi_1} & 0 \\ \bar{E} & \tilde{A}(h(t)) & \bar{B}_{\varphi} & 0 & \bar{B}_{\varphi_2} \\ 0 & \bar{C}_{\mathcal{N}-\varphi} & 0 & 0 & 0 \\ 0 & \bar{C}_{\varphi-\varphi_1} & 0 & 0 & 0 \\ \bar{C}_{(\mathcal{N}-\varphi)-\varphi_2} & 0 & 0 & 0 & 0 \\ 0 & \bar{C}_j & 0 & 0 & 0 \end{bmatrix} \right) \geq Nn + r \quad (30)$$

Remark 2. In [4], the authors considered the stability problem of singular complex networks. However, for singular systems, the impulsive behavior always has a great influence on the system. Impulsive behavior may cause the system to have poor stability and can even destroy the system. Therefore, it is very important to study pulse controllability, pulse observability, and the elimination of a pulse.

Theorem 3. The system (1) is admissible, if

$$\bar{E}^- = Q \begin{bmatrix} I_{\tilde{q}} & 0 \\ 0 & 0 \end{bmatrix} P \in \bar{E}\{1\}, \alpha \in \mathcal{V}_{H_{n^2, n^2}}, \quad (31)$$

and

$$\text{tr} H(\bar{E}^- \tilde{A}(\alpha)) < -\sqrt{(\tilde{q} - 1) \text{tr}[H(\bar{E} - \tilde{A}(\alpha))]^2}, \quad (32)$$

where

$$P \bar{E} Q = \begin{bmatrix} I_{\tilde{q}} & 0 \\ 0 & 0 \end{bmatrix}, \tilde{q} = \sum_{i=1}^N r_{e_i},$$

$$\tilde{A}(\alpha) = \tilde{A}(h(t)) + \Omega, \Omega = \sum_{i=1}^N \sum_{k=1}^{n_i} \sum_{j=1}^n e_n(i) \left(\sum_{l=1}^{i-1} n_l + k \right) e_n^T(j).$$

Proof. According to the convexity principle, as detailed by [28] the inequality presented in (33) suggests that

$$\text{tr} H(\bar{E}^- \tilde{A}(h(t))) < -\sqrt{(\tilde{q} - 1) \text{tr}[H(\bar{E} - \tilde{A}(h(t)))]^2} \quad (33)$$

for all $h(t) \in \mathcal{V}_{H_{n^2, n^2}}$, based on a novel restricted system equivalence between the matrix,

$$P\bar{E}Q = \begin{bmatrix} I_{\tilde{q}} & 0 \\ 0 & 0 \end{bmatrix}, P\tilde{A}(h(t))Q = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n-\tilde{q}} \end{bmatrix},$$

where $A_1 \in \mathbf{R}^{q \times q}$, and

$$\det(s\bar{E} - \tilde{A}(h(t))) = 0 \Leftrightarrow \det(P(s\bar{E} - \tilde{A}(h(t)))Q) = 0 \Leftrightarrow \det(sI_{\tilde{q}} - A_1) = 0. \quad (34)$$

Therefore,

$$\sigma(\bar{E}, \tilde{A}(h(t))) = \sigma(A_1).$$

On the other hand, by Lemma 5, when

$$\bar{E}^- = Q \begin{bmatrix} I_{\tilde{q}} & 0 \\ 0 & 0 \end{bmatrix} P,$$

then

$$\bar{E}^- \tilde{A}(h(t)) = Q \begin{bmatrix} I_{\tilde{q}} & 0 \\ 0 & 0 \end{bmatrix} P P^{-1} \begin{bmatrix} A_1 & 0 \\ 0 & I_{n-\tilde{q}} \end{bmatrix} Q^{-1} = Q \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} Q^{-1},$$

we have

$$\text{tr}(\bar{E}^- \tilde{A}(h(t))) = \text{tr} A_1, \text{tr}[H(\bar{E}^- \tilde{A}(h(t)))] = \text{tr}[H(A_1)]. \quad (35)$$

By (35) and (33), we can obtain

$$\text{tr}[H(A_1)] < -\sqrt{(\tilde{q} - 1)\text{tr}[H(A_1)]^2}, \quad (36)$$

by Lemma 6, we have

$$\sigma(\bar{E}, \tilde{A}(h(t))) = \sigma(A_1) \subset C^-,$$

the system (1) is admissible. $\square$

4. Illustrative Examples

This section explores some illustrative examples to demonstrate the efficacy of the proposed method.

Example 1. Consider the singular hybrid coupled system (1) with the following parameters:

$$\begin{aligned} x_i(t) &= (x_i^{(1)}, x_i^{(2)}, x_i^{(3)})^T, f_i(x_i(t)) = (\sin(x_i^{(1)}(t)), \sin(x_i^{(2)}(t)), \sin(x_i^{(3)}(t)))^T \\ b_{11} &= -1, b_{12} = 1, b_{13} = 0, b_{21} = 1, b_{22} = -2, b_{23} = 1, b_{31} = 0, b_{32} = 1, b_{33} = -1, \\ c &= 1 \ (i, j = 1.2.3), \end{aligned}$$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{bmatrix}, \text{ and } \Gamma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The singular hybrid coupled systems (1) are described as nonlinear singular large-scale systems models (2) with the following parameters:

$$\bar{E} = \begin{bmatrix} E & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & E \end{bmatrix}, \bar{A} = \begin{bmatrix} A + cb_{11}\Gamma & cb_{12}\Gamma & cb_{13}\Gamma \\ cb_{21}\Gamma & A + cb_{22}\Gamma & cb_{23}\Gamma \\ cb_{31}\Gamma & cb_{32}\Gamma & A + cb_{33}\Gamma \end{bmatrix}, f(x(t)) = \begin{pmatrix} f(x_1(t), t) \\ f(x_2(t), t) \\ f(x_3(t), t) \end{pmatrix}$$

$$B_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, B_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$C_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

By Lemma 1, we have

$$f_i(x_i(t), t) = \begin{bmatrix} \cos \alpha(t) & 0 & 0 \\ 0 & \cos \beta(t) & 0 \\ 0 & 0 & \cos \gamma(t) \end{bmatrix} x_i, \quad \begin{matrix} \alpha(t) \in [0, x_i^1], \\ \beta(t) \in [0, x_i^2], \\ \gamma(t) \in [0, x_i^3]. \end{matrix} \quad (37)$$

By Theorem 1 considering the nonlinear effect, we have

$$\text{rank} \left(\begin{bmatrix} 0 & \bar{E} \\ \bar{E} & \tilde{A}(h(t)) \end{bmatrix} \right) = 11, Nn + \text{rank}(\bar{E}) = 12,$$

$$\text{rank} \left(\begin{bmatrix} 0 & \bar{E} \\ \bar{E} & \tilde{A}(h(t)) \end{bmatrix} \right) \neq Nn + \text{rank}(\bar{E}),$$

there is the presence of an impulse within the system (2).

By Theorem 2 and (37), we have

$$\begin{aligned} & \max_{\substack{K_1, \dots, K_3 \\ L_1, \dots, L_3}} \left(\begin{bmatrix} 0 & \bar{E} + \sum_{i=1}^3 \bar{B}_i L_i C_i & 0 \\ \bar{E} + \sum_{i=1}^3 \bar{B}_i L_i C_i & \tilde{A}(h(t)) + \sum_{i=1}^3 \bar{B}_i K_i C_i & \bar{B}_j \end{bmatrix} \right) \\ &= \text{rank} \left(\begin{bmatrix} 0 & \bar{E} & 0 & 0 & \bar{B}_{\varphi_1} & 0 \\ \bar{E} & \tilde{A}(h(t)) & \bar{B}_j & \bar{B}_{\varphi} & 0 & \bar{B}_{\varphi_2} \\ 0 & \bar{C}_{\mathcal{N}-\varphi} & 0 & 0 & 0 & 0 \\ 0 & \bar{C}_{\varphi-\varphi_1} & 0 & 0 & 0 & 0 \\ \bar{C}_{(\mathcal{N}-\varphi)-\varphi_2} & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right) \\ &= 7 + \\ & \text{rank} \left(\begin{bmatrix} 2 + \cos \beta(t) & 0 & 1 & 0 & 0 & 0 \\ 0 & 5 + \cos \gamma(t) & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 + \cos \beta(t) & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 + \cos \gamma(t) & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 5 + \cos \gamma(t) \end{bmatrix} \right) \\ &= 12, \end{aligned}$$

$$Nn + r = 9 + 3 = 12,$$

$$\max_{\substack{K_1, \dots, K_3 \\ L_1, \dots, L_3}} \left(\begin{bmatrix} 0 & \bar{E} + \sum_{i=1}^3 \bar{B}_i L_i C_i & 0 \\ \bar{E} + \sum_{i=1}^3 \bar{B}_i L_i C_i & \tilde{A}(h(t)) + \sum_{i=1}^3 \bar{B}_i K_i C_i & \bar{B}_j \end{bmatrix} \right) = Nn + r, \quad (38)$$

Equation (38) satisfies the necessary and sufficient algebraic condition of impulse controllability. Furthermore, if condition (38) is met, it is possible to establish a decentralised derivative output feedback controller that ensures the closed-loop system exhibits impulse controllability.

$$u_i = -L_i y_i + K_i \dot{y}_i + v_i, \quad (i = 1, 2, 3.),$$

when

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

K_i ($i = 1, 2, 3$) is the arbitrary parameters matrix.

We have

$$\begin{aligned} & \text{rank} \left(\begin{bmatrix} 0 & \bar{E} + \sum_{i=1}^3 \bar{B}_i L_i \bar{C}_i & 0 \\ \bar{E} + \sum_{i=1}^3 \bar{B}_i L_i \bar{C}_i & \tilde{A}(h(t)) + \sum_{i=1}^3 \bar{B}_i K_i \bar{C}_i & \bar{B}_j \end{bmatrix} \right) \\ &= 7 + \text{rank} \left(\begin{bmatrix} 2 + \cos \beta(t) & 0 & 0 & 1 & 0 & 0 \\ 0 & 5 + \cos \gamma(t) & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 + \cos \beta(t) & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 + \cos \gamma(t) & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 5 + \cos \gamma(t) \end{bmatrix} \right) \\ &= 12, \end{aligned}$$

$$\begin{aligned} & Nn + r \\ &= 9 + \text{rank} \left(\begin{bmatrix} E + B_1 K_1 C_1 & 0 & 0 \\ 0 & E + B_2 K_2 C_2 & 0 \\ 0 & 0 & E \end{bmatrix} \right) \\ &= 9 + \text{rank} \left(\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right) \\ &= 12, \end{aligned}$$

$$\max_{\substack{K_1, \dots, K_3 \\ L_1, \dots, L_3}} \left(\begin{bmatrix} 0 & \bar{E} + \sum_{i=1}^3 \bar{B}_i L_i \bar{C}_i & 0 \\ \bar{E} + \sum_{i=1}^3 \bar{B}_i L_i \bar{C}_i & \tilde{A}(h(t)) + \sum_{i=1}^3 \bar{B}_i K_i \bar{C}_i & \bar{B}_j \end{bmatrix} \right) = Nn + r.$$

based on the derivative feedback controller used, it can be inferred that the closed-loop system is impulse controllable.

Example 2. Using the parameters as in [4], consider the singular hybrid coupled systems (1) with the following parameters:

$$\begin{aligned} & x_i(t) = (x_i^{(1)}, x_i^{(2)})^T, f_i(x_i(t)) = \left(\frac{1}{15} \tanh(x_i^{(1)}(t)), \frac{1}{15} \tanh(x_i^{(2)}(t)) \right)^T \\ & b_{11} = -5, b_{12} = 1, b_{13} = 1, b_{14} = 1, b_{15} = 1, b_{16} = 1, b_{21} = 1, b_{22} = -4, b_{23} = 1, b_{24} = 1, \\ & b_{25} = 1, b_{26} = 0, b_{31} = 1, b_{32} = 1, b_{33} = -4, b_{34} = 1, b_{35} = 0, b_{36} = 1, b_{41} = 1, b_{42} = 1, \\ & b_{43} = 1, b_{44} = -4, b_{45} = 1, b_{46} = 0, b_{51} = 1, b_{52} = 1, b_{53} = 0, b_{54} = 1, b_{55} = -4, b_{56} = 1, \\ & b_{61} = 1, b_{62} = 0, b_{63} = 1, b_{64} = 0, b_{65} = 1, b_{66} = -3, \\ & c = 1 \ (i, j = 1, 2, 3, 4, 5, 6), \end{aligned}$$

$$E = \begin{bmatrix} 8 & 0 \\ 0 & 0 \end{bmatrix}, A = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}, \Gamma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

the singular hybrid coupled systems (1) are described as nonlinear singular large-scale systems models (2) with the following parameters:

$$\bar{E} = \begin{bmatrix} E & 0 & 0 & 0 & 0 & 0 \\ 0 & E & 0 & 0 & 0 & 0 \\ 0 & 0 & E & 0 & 0 & 0 \\ 0 & 0 & 0 & E & 0 & 0 \\ 0 & 0 & 0 & 0 & E & 0 \\ 0 & 0 & 0 & 0 & 0 & E \end{bmatrix}, f(x(t)) = \begin{pmatrix} f(x_1(t), t) \\ f(x_2(t), t) \\ f(x_3(t), t) \\ f(x_4(t), t) \\ f(x_5(t), t) \\ f(x_6(t), t) \end{pmatrix},$$

$$\bar{A} = \begin{bmatrix} A + cb_{11}\Gamma & cb_{12}\Gamma & cb_{13}\Gamma & cb_{14}\Gamma & cb_{15}\Gamma & cb_{16}\Gamma \\ cb_{21}\Gamma & A + cb_{22}\Gamma & cb_{23}\Gamma & cb_{24}\Gamma & cb_{25}\Gamma & cb_{26}\Gamma \\ cb_{31}\Gamma & cb_{32}\Gamma & A + cb_{33}\Gamma & cb_{34}\Gamma & cb_{35}\Gamma & cb_{36}\Gamma \\ cb_{41}\Gamma & cb_{42}\Gamma & cb_{43}\Gamma & A + cb_{44}\Gamma & cb_{45}\Gamma & cb_{46}\Gamma \\ cb_{51}\Gamma & cb_{52}\Gamma & cb_{53}\Gamma & cb_{54}\Gamma & A + cb_{55}\Gamma & cb_{56}\Gamma \\ cb_{61}\Gamma & cb_{62}\Gamma & cb_{63}\Gamma & cb_{64}\Gamma & cb_{65}\Gamma & A + cb_{66}\Gamma \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$C_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$C_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$C_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$C_6 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, B_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, B_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, B_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, B_6 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

By Lemma 1, we have

$$f_i(x_i(t), t) = \begin{bmatrix} \frac{1}{15}(1 - \tanh^2(\alpha)) & 0 \\ 0 & \frac{1}{15}(1 - \tanh^2(\beta)) \end{bmatrix} x_i, \alpha(t) \in [0, x_i^1], \beta(t) \in [0, x_i^2], \quad (39)$$

by Theorem 1, considering the nonlinear effect, we have

$$\text{rank} \left(\begin{bmatrix} 0 & \bar{E} \\ \bar{E} & \bar{A}(h(t)) \end{bmatrix} \right) = 17, Nn + \text{rank}(\bar{E}) = 18,$$

$$\text{rank} \left(\begin{bmatrix} 0 & \bar{E} \\ \bar{E} & \bar{A}(h(t)) \end{bmatrix} \right) \neq Nn + \text{rank}(\bar{E}),$$

there is the presence of an impulse within the system (2).

By Theorem 2 and (39), we have

$$\begin{aligned}
 & \max_{\substack{K_1, \dots, K_6 \\ L_1, \dots, L_6}} \left(\begin{bmatrix} 0 & \bar{E} + \sum_{i=1}^6 \bar{B}_i L_i C_i & 0 \\ \bar{E} + \sum_{i=1}^6 \bar{B}_i L_i C_i & \tilde{A}(h(t)) + \sum_{i=1}^6 \bar{B}_i K_i C_i & \bar{B}_j \\ \bar{E} + \sum_{i=1}^6 \bar{B}_i L_i C_i & \tilde{A}(h(t)) + \sum_{i=1}^6 \bar{B}_i K_i C_i & \bar{B}_j \end{bmatrix} \right) \\
 & = \text{rank} \left(\begin{bmatrix} 0 & \bar{E} & 0 & 0 & \bar{B}_{\varphi_1} & 0 \\ \bar{E} & \tilde{A}(h(t)) & \bar{B}_j & \bar{B}_{\varphi} & 0 & \bar{B}_{\varphi_2} \\ 0 & \tilde{C}_{\mathcal{N}-\varphi} & 0 & 0 & 0 & 0 \\ 0 & \tilde{C}_{\varphi-\varphi_1} & 0 & 0 & 0 & 0 \\ \tilde{C}_{(\mathcal{N}-\varphi)-\varphi_2} & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right) \\
 & = 13 + \text{rank} \left(\begin{bmatrix} \Delta & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 + \Delta & 1 & 0 \\ 1 & 1 & 1 & 1 + \Delta & 0 \\ 1 & 1 & 0 & 1 & 1 + \Delta \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} \right) \\
 & = 18,
 \end{aligned}$$

$$\Delta = \frac{1}{15}(1 - \tanh^2(\beta)),$$

$$Nn + r = 18,$$

$$\max_{\substack{K_1, \dots, K_6 \\ L_1, \dots, L_6}} \left(\begin{bmatrix} 0 & \bar{E} + \sum_{i=1}^6 \bar{B}_i L_i C_i & 0 \\ \bar{E} + \sum_{i=1}^6 \bar{B}_i L_i C_i & \tilde{A}(h(t)) + \sum_{i=1}^6 \bar{B}_i K_i C_i & \bar{B}_j \end{bmatrix} \right) = Nn + r. \quad (40)$$

Equation (40) satisfies the necessary and sufficient algebraic condition of impulse controllability. Furthermore, if (40) is satisfied, there exists the decentralised derivative output feedback controller such that the closed-loop system is impulse controllable.

$$u_i = -L_i y_i + K_i \dot{y}_i + v_i, \quad (i = 1, 2, 3, 4, 5, 6),$$

when

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

K_i ($i = 1, 2, 3, 4, 5, 6$) is the arbitrary parameters matrix.

We have

$$\begin{aligned}
 & \text{rank} \left(\begin{bmatrix} 0 & \bar{E} + \sum_{i=1}^6 \bar{B}_i L_i \bar{C}_i & 0 \\ \bar{E} + \sum_{i=1}^6 \bar{B}_i L_i \bar{C}_i & \tilde{A}(h(t)) + \sum_{i=1}^6 \bar{B}_i K_i \bar{C}_i & \bar{B}_j \end{bmatrix} \right) \\
 &= 13 + \text{rank} \left(\begin{bmatrix} \Delta & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 + \Delta & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 + \Delta & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 + \Delta & 1 \\ 1 & 0 & 1 & 0 & 1 & 2 + \Delta \end{bmatrix} \right) \\
 &= 18,
 \end{aligned}$$

$$\begin{aligned}
 & Nn + r \\
 &= 12 + \text{rank} \left(\begin{bmatrix} E + B_1 L_1 C_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & E & 0 & 0 & 0 & 0 \\ 0 & 0 & E + B_3 L_3 C_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & E + B_4 L_4 C_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & E + B_5 L_5 C_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & E + B_3 L_6 C_6 \end{bmatrix} \right) \\
 &= 12 + \text{rank} \left(\begin{bmatrix} 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \right) \\
 &= 18,
 \end{aligned}$$

$$\max_{\substack{K_1, \dots, K_6 \\ L_1, \dots, L_6}} \left(\begin{bmatrix} 0 & \bar{E} + \sum_{i=1}^6 \bar{B}_i L_i \bar{C}_i & 0 \\ \bar{E} + \sum_{i=1}^6 \bar{B}_i L_i \bar{C}_i & \tilde{A}(h(t)) + \sum_{i=1}^6 \bar{B}_i K_i \bar{C}_i & \bar{B}_j \end{bmatrix} \right) = Nn + r,$$

based on the derivative feedback controller used, it can be inferred that the closed-loop system is impulse controllable.

5. Conclusions

In the beginning of this paper, we widely studied the problem of impulse controllability in singular hybrid coupled systems using a P-D output feedback controller. Additionally, we investigated the admissibility of these systems based on the matrix trace inequality. In the assumption of the existence of a decentralised output feedback controller that achieves impulse controllability in singular hybrid coupled systems, we evidently derived its necessary and sufficient conditions. Furthermore, we also established a new sufficient condition for the admissibility of impulse-controllable singular hybrid coupled systems. Finally, we provided illustrative examples to demonstrate the effectiveness of the proposed

approach. At present, there exists a paucity of research concerning impulse control within singular hybrid coupled systems. The notion of impulse controllability, as introduced in this paper, seeks to address a substantial gap in the existing literature. The mitigation of impulses via derivative feedback constitutes one of the most significant challenges faced by singular hybrid coupled systems. Moreover, investigating strategies to mitigate impulses through derivative feedback in nonlinear singular hybrid coupled systems will serve as a primary focus of forthcoming research initiatives.

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