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Probabilistic Models for the Shear Strength of RC Deep Beams

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Abstract: A new shear strength determination of reinforced concrete (RC) deep beams was proposed by using a statistical approach. The Bayesian–MCMC (Markov Chain Monte Carlo) method was introduced to establish a new shear prediction model and to improve seven existing deterministic models with a database of 645 experimental data. The bias correction terms of deterministic models were described by key explanatory terms identified by a systematic removal process. Considering multi-parameters, the Gibbs sampling was used to solve the high dimensional integration problem and to determine optimum and reliable model parameters with 50,000 iterations for probabilistic models. The model continuity and uncertainty for key parameters were quantified by the partial factor that was investigated by comparing test and model results. The partial factor for the proposed model was 1.25. The proposed model showed improved accuracy and continuity with the mean and coefficient of variation (CoV) of the experimental-to-predicted results ratio as 1.0357 and 0.2312, respectively.

Keywords: deep beams; shear strength; Bayesian–MCMC method; probabilistic prediction; model uncertainty



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1. Introduction

Reinforced concrete (RC) deep beams have been widely used for constructions such as pile caps, transfer girders, shear walls, and floor slabs [1]. In particular, the shear behavior of RC deep beams has emerged as an essential research topic [2]. Many models were developed by simplifying the rules of mechanics or regressing experimental data [3–5]. The most widely used method is the strut-and-tie model, which simplifies the complex stress state in discontinuity regions (D-regions) of concrete members into uniaxial and simple stress paths by incorporating a truss system comprising struts, ties, and nodes [6,7]. Another standard method is the empirical model considering the multiple parameters and experimental data [8]. However, these models provide conservative strength predictions or unpredictable biases due to complex mechanical factors [9,10].

The Bayesian parameter estimation method has been extensively applied to design beams, columns, joints, shear walls and predict probabilistic fatigue strength [11,12]. In contrast, the posterior parameter distribution of Bayesian estimation is mostly high-dimensional and complex, which is difficult to calculate directly [13]. The Markov Chain Monte Carlo (MCMC) method was commonly introduced to samples directly from the posterior distribution, thus solving the problem [14–16]. Reis explored the Bayesian–MCMC methods for evaluating the posterior distributions of flood quantiles and flood risk, which provides complete uncertainty descriptions of parameters, qualities, and performance metrics [17]. Papaioannou applied the Bayesian method, improved by the MCMC method, to update the numerical model of a tunnel in soft soil conditional on settlement measurements [18]. This MCMC method, concerned with the simulation of high dimensional probabilistic distributions, has gained enormous prominence and revolutionized Bayesian statistics [19].

The quantification and explicit inclusion of resistance model uncertainty are essential for reliability investigations of RC structures when the description of structural response is

excessively complex and the operational procedures depend on simplified models [20]. As an indicator of the difference between model predictions and actual structural resistances, the uncertainties in shear strength models for essential variables were investigated and described by the statistical characteristics, including the partial factor, mean, and coefficient of variation (CoV) of the experiment-to-prediction ratios [21,22].

This paper introduced the Bayesian–MCMC method to establish the probabilistic shear strength models with an experimental database and multiple important parameters. The existing deterministic models, including ACI318-14 (2014) [23], GB50010-10 (2010) [24], CSA A23.3-04 (2004) [25], EC2 (2004) [26], CEB-FIP (2010) [27], Foster and Gilbert (1998) [28], Mitchell and Collins (1974) [29], were compared and improved by corresponding bias correction terms. The probabilistic model without prior models was proposed to reduce bias and uncertainty further. Then, the performance of probabilistic models was investigated according to the evaluating indicators containing the mean, CoV, and partial factor.

2. Database

An extensive database (listed in Table 1) of 645 tests was used to develop probabilistic models covering a wide range of structural parameters. Figure 1 provides an overview of the distributions of main influencing parameters, including compressive strength (f_c'), the reinforcement ratio (ρ), stirrup rate (ρ_v), shear span ratio (a/d), shear–depth ratio (l_0/h), and width–depth ratio (b/h). Approximately 457 beams had concrete compressive strengths lower than 40 MPa. Most members had a shear–depth ratio (l_0/h) between 1.5 and 4. More than half of the test specimens had longitudinal reinforcement ratios greater than 1.5%, and 96 beams had less than 1.0% reinforcement ratios. About three quarters of the members had a shear span ratio of 0.6–1.8, and the ratios of 123 beams were 0.2–0.6.

Table 1. Deep-Beam Database.

Literature	Quantity	f_c'/MPa	a/d	$V_{\text{test}}/\text{kN}$
Kong et al., 1970 [30]	33	18.6–26.8	0.35–0.18	78.0–308.0
Tan et al., 1997 [31]	18	86.3–56.2	0.85–1.69	185.0–775.0
Liu et al., 2000 [32]	5	19.6–26.1	0.5–2.50	64.7–180.2
Manuel et al., 1971 [33]	12	30.1–44.8	0.30–1.00	226.9–258.0
Smith et al., 1982 [34]	52	20.4–28.7	0.77–2.01	73.4–178.5
Gong 1982 [35]	39	18.3–30.1	0.36–1.94	67.6–411.6
Mphonde et al., 1984 [36]	10	20.6–83.8	1.50–2.94	87.700–558.1
Fang 1990 [37]	5	33.7–37.0	0.75–0.84	434.0–472.0
Tan et al., 1995 [38]	19	41.1–58.8	0.27–2.70	105.0–675.0
Tan et al., 1997 [39]	15	72.1–64.6	0.28–1.28	150.0–925.0
Foster et al., 1998 [28]	5	120.0–89.0	0.87–0.88	950.0–2000.0
Shin et al., 1999 [40]	13	52.0–73.0	1.50–2.00	90.0–287.1
Tan et al., 1999 [41]	12	30.8–49.1	0.56–1.13	435.0–1636.0
Tan et al., 2008 [42]	4	32.7–37.6	0.42–0.84	331.5–1305.0
Oh et al., 2001 [43]	53	23.7–73.6	0.50–2.00	112.5–745.6
Kani 1967 [44]	5	26.7–31.4	1.00–1.03	155.2–585.4
Rogowsky et al., 1986 [45]	16	26.1–43.2	1.02–2.08	185.0–875.0
Lu et al., 2013 [46]	13	34.6–67.8	0.61–0.83	1156.0–2018.0
Teng et al., 2000 [47]	6	34.0–41.0	1.71	225.0–450.0
Moody et al., 1954 [48]	14	17.2–25.4	1.52	267.6–507.1

Table 1. *Cont.*

Literature	Quantity	f_c'/MPa	a/d	$V_{\text{test}}/\text{kN}$
Laupa 1955 [49]	9	23.5~26.2	1.17~0.56	94.2~260.0
Morrow et al., 1957 [50]	21	11.3~46.8	0.95~1.00	129.0~900.7
Ramakrishnan et al., 1968 [51]	20	10.8~28.4	0.30~1.00	40.0~193.0
Lee 1982 [52]	3	28.0~33.5	1.56~1.70	840.0~967.5
Mathey et al., 1963 [53]	16	21.9~27.0	1.51	179.5~312.9
Leonhardt et al., 1961 [54]	3	32.4	1.00~2.00	80.2~120.3
Subedi 1988 [55]	5	29.6~41.6	0.31~1.53	175.0~797.5
Walraven et al., 1994 [56]	25	13.9~26.4	0.97~1.01	109.0~669.1
Adebar 2000 [57]	6	19.5~21.0	1.43~2.20	330.0~771.0
Yang et al., 2003 [58]	19	31.4~78.5	0.36~1.41	192.1~1029.0
Tanimura et al., 2005 [59]	41	22.5~97.5	0.50~1.50	184.23~739.77
Salamy et al., 2005 [60]	12	29.2~37.8	0.50~1.50	308.0~980.0
Zhang et al., 2007 [61]	12	24.8~32.4	1.10	85.0~775.0
Garay et al., 2008 [62]	2	43.0~44.0	1.19~1.78	1027.5~1373.5
Brena et al., 2009 [63]	7	27.0~34.1	1.00~1.50	211.0~371.0
Kosa 2009 [64]	18	23.0~42.3	1.00	195.0~4198.0
Zhang et al., 2009 [65]	11	38.3~41.2	0.57~1.42	240.1~665.4
Sagaseta et al., 2010 [66]	6	68.4~80.2	1.51	326.0~602.0
Sahoo et al., 2010 [67]	11	36.3~45.2	0.50	303.2~371.2
Senturk et al., 2010 [68]	2	24.4~26.2	1.37	1307.0~1809.0
Mihaylov et al., 2010 [69]	6	29.1~37.8	1.55~2.29	416.0~1162.0
Lin 2011 [70]	4	25.8~30.1	0.86~1.02	260.0~460.0
Hanifi et al., 2012 [71]	8	22.1~35.7	0.50~2.00	65.0~329.0
Aguilar et al., 2002 [72]	4	28.0~32.0	1.14~1.27	1134.0~1357.0
Quintero et al., 2006 [73]	12	22.0~50.3	0.81~1.57	196.0~484.0
Subedi 1988 [74]	12	22.4~29.2	0.43~1.56	78.0~485.0

**Figure 1.** The distributions of main structural parameters.

3. Bayesian–MCMC Method

3.1. Bayesian Method

3.1.1. Probabilistic Model

The Bayesian method established a probabilistic model in the form of

$$C(\mathbf{X}, \boldsymbol{\Theta}) = C_d(\mathbf{X}) + \gamma(\mathbf{X}, \boldsymbol{\Theta}) + \sigma\varepsilon, \quad (1)$$

where $\mathbf{X}$ is the vector of input parameters; $\boldsymbol{\Theta} = (\theta, \sigma)$ denotes the set of unknown model parameters introduced to fit the model to the test results; $C_d(\mathbf{X})$ is the deterministic model; $\gamma(\mathbf{X}, \theta)$ is the correct term for the bias inherent in the deterministic model; ε is the standard random variable; and σ is the unknown model parameter representing the magnitude of the model error.

Since the form of the bias correction term $\gamma(\mathbf{X}, \theta)$ is unknown, it is expressed using a set of “explanatory” functions $h_i(x)$, where $i = 1, \dots, p$, in the form

$$\gamma(\mathbf{X}, \theta) = \sum_{i=1}^p \theta_i h_i(\mathbf{X}). \quad (2)$$

Considering the non-negative nature of the shear strength, the logarithmic variance-stabilizing transformation is selected among other possible transformations to formulate a homoskedastic model, which is written as

$$\ln V(\mathbf{X}, \boldsymbol{\Theta}) = \ln V_d(\mathbf{X}) + \sum_{i=1}^p \theta_i h_i(x) + \sigma\varepsilon. \quad (3)$$

$V_d(\mathbf{X})$ is the deterministic Model for the shear strength, $V(\mathbf{X}, \boldsymbol{\Theta})$ is the probabilistic model for the shear strength.

3.1.2. Bayesian Parameter Estimation

Suppose that $\mathbf{V} = (V_1, V_2, V_3, \dots, V_n)$ is a vector of shear strength whose probability distribution $p(\mathbf{V} | \boldsymbol{\Theta})$ depends on the vector of parameters $\boldsymbol{\Theta} = (\theta, \sigma)$. Suppose also that $\boldsymbol{\Theta}$ itself has a prior distribution $p(\boldsymbol{\Theta})$. Given the observed data $\mathbf{V}$, the conditional distribution of $\boldsymbol{\Theta}$ is

$$p(\boldsymbol{\Theta} | \mathbf{V}) = \frac{p(\mathbf{V} | \boldsymbol{\Theta})p(\boldsymbol{\Theta})}{p(\mathbf{V})}. \quad (4)$$

A probabilistic model was proposed by determining the values of the model parameters $\boldsymbol{\Theta} = (\theta, \sigma)$. The Bayesian approach updated this prior distribution to the posterior distribution $f(\boldsymbol{\Theta})$. The well-known Bayesian updating rule [74] is

$$f(\boldsymbol{\Theta}) = \kappa L(\boldsymbol{\Theta})p(\boldsymbol{\Theta}), \quad (5)$$

where $f(\boldsymbol{\Theta})$ is the posterior distribution representing our updated state of knowledge about $\boldsymbol{\Theta}$; $L(\boldsymbol{\Theta})$ is the likelihood function representing the objective information on $\boldsymbol{\Theta}$ contained in a set of observations; $p(\boldsymbol{\Theta})$ is prior distribution reflecting our state of knowledge about $\boldsymbol{\Theta}$ prior to obtaining the observations; and $\kappa = [\int L(\boldsymbol{\Theta})p(\boldsymbol{\Theta})d(\boldsymbol{\Theta})]^{-1}$ is normalizing factor. The likelihood is a function proportional to the conditional probability of making the observations for a given value of $\boldsymbol{\Theta}$.

3.1.3. The Prior Distribution

When no such information is available, one should use a prior distribution with minimal influence on the posterior distribution so that inferences are unaffected by information external to the observations. For the set of parameters $\boldsymbol{\Theta} = (\theta, \sigma)$, it is generally assumed that θ and σ are approximately independent so that $p(\theta, \sigma) \approx p(\theta)p(\sigma)$. Using Jeffrey’s rule, Box and Tiao have shown that the noninformative prior for the parameters θ is locally uniform such that $p(\theta, \sigma) \cong p(\sigma)$. According to the Bayesian theory, the “noninformative” prior was

employed when no information could be used for the prior distribution of parameters [75]. The noninformative prior of σ then takes the form

$$p(\theta, \sigma) \cong p(\theta)p(\sigma) \propto \frac{1}{\sigma}. \quad (6)$$

3.1.4. Likelihood Functions

The likelihood function $L(\Theta)$ was defined as a function proportional to the conditional probability of the observations for parameters x . One of three kinds of observations can be made from the i th test:

- (1) “failure datum”: the demand V_i was measured at the instant of the failure, i.e., $\sigma\epsilon = \ln V_i - \ln V_d - \gamma(x_i, \theta)$;
- (2) “lower bound datum”: the component did not fail up to the demand level V_i , i.e., $\sigma\epsilon > \ln V_i - \ln V_d - \gamma(x_i, \theta)$;
- (3) “upper bound datum”: the component has failed under a lower demand than measured V_i , i.e., $\sigma\epsilon < \ln V_i - \ln V_d - \gamma(x_i, \theta)$. The likelihood function for the univariate model has the general form

$$L(\Theta) \propto \prod_{\text{failure data}} P[\sigma\epsilon = \ln V_i - \ln V_d(X_i) - \gamma(X_i, \theta)] \times \prod_{\text{lower bound data}} P[\sigma\epsilon > \ln V_i - \ln V_d(X_i) - \gamma(X_i, \theta)] \times \prod_{\text{upper bound data}} P[\sigma\epsilon < \ln V_i - \ln V_d(X_i) - \gamma(X_i, \theta)] \quad (7)$$

Under the assumption of statistically independent tests and the normality of ϵ , the likelihood function was derived [11] as

$$L(\Theta) \propto \prod_{\text{failure data}} \left\{ \frac{1}{\sigma} \left[\frac{\ln V_i - \ln V_d(X_i) - \gamma(X_i, \theta)}{\sigma} \right] \right\} \times \prod_{\text{lower bound data}} \left\{ \Phi \left[- \frac{\ln V_i - \ln V_d(X_i) - \gamma(X_i, \theta)}{\sigma} \right] \right\} \times \prod_{\text{upper bound data}} \left\{ \Phi \left[\frac{\ln V_i - \ln V_d(X_i) - \gamma(X_i, \theta)}{\sigma} \right] \right\} \quad (8)$$

where $\phi(\cdot)$ and $\Phi(\cdot)$, respectively, denote the PDF and the cumulative distribution function (CDF) of the standard normal distribution. Note that the likelihood function can use even the information of lower- and upper-bound data in establishing a probabilistic model.

3.2. MCMC Method

The MCMC method effectively solves high-dimensional problems by random simulation with the R programming language (R). It generated samples from the posterior distribution functions using Markov chains [76]. Finally, various statistical inferences were made to determine (θ, σ) with these samples.

The expectation and variance of arbitrary function $f(x)$ were provided, respectively.

$$E[f(x)] = \frac{1}{n-m} \sum_{t=m+1}^n f(X^{(t)})$$

$$\text{VAR}[f(x)] = \frac{1}{n-m} \sum_{t=m+1}^n [f(X^{(t)})]^2 - \left[\frac{1}{n-m} \sum_{t=m+1}^n f(X^{(t)}) \right]^2 \quad (9)$$

where n is the total number of samples generated by simulation, and m is the number of samples when the Markov chain reaches stationary distribution.

A posteriori distribution function was obtained by synthesizing prior information, general information, and sample information as follows:

$$\pi(\theta, \sigma | Y, X) \propto L(\theta, \sigma | Y, X) \pi(\theta, \sigma) \propto \frac{1}{\sigma^{n+1}} \exp \left\{ -\frac{1}{2\sigma^2} \left[S_n^2 + (\theta - \hat{\theta})^T X^T X (\theta - \hat{\theta}) \right] \right\}. \quad (10)$$

According to the Gibbs sampling [77], the entire conditional distribution of Equation (10) can be expressed as

$$\pi(\theta_i | \theta_{(-i)}, \sigma, Y, X) \propto L(\theta, \sigma | Y, X) \times 1 \propto \frac{1}{\sigma^n} \exp \left\{ -\frac{1}{2\sigma^2} [S_n^2 + (\theta - \hat{\theta})^T X^T X (\theta - \hat{\theta})] \right\}, \quad (11)$$

$$\pi(\sigma | \theta, Y, X) \propto L(\theta, \sigma | Y, X) \pi(\theta, \sigma) \propto \frac{1}{\sigma^{n+1}} \exp \left\{ -\frac{1}{2\sigma^2} [S_n^2 + (\theta - \hat{\theta})^T X^T X (\theta - \hat{\theta})] \right\}, \quad (12)$$

where $\pi(\theta, \sigma | Y, X)$ is the posterior distribution function; $L(\theta, \sigma | Y, X)$ is the likelihood function; $\pi(\theta, \sigma)$ is the prior distribution of parameters.

4. Probabilistic Shear Strength Models

The deterministic models were used as prior models. The Gibbs sampling method sampled parameters (θ, σ) and simulated using the Markov chains. In addition, the posterior estimates of the samples were obtained by the Monte Carlo integral method. Afterward, some critical parameters identified by a systematic removal process were used to simplify probabilistic models.

4.1. Operation of Probabilistic Models

4.1.1. Selection of Explanatory Functions

Identifying critical parameters relevant to the shear behaviors of deep beams was necessary for effective bias correction. Considering the shear transformation mechanism and the existing experimental results, some key parameters were selected [34,78], i.e., (1) the concrete compressive strength, (2) the shear span ratio, (3) the reinforcement ratio, (4) the stirrup rate, (5) the component section size, (6) the yield strength of longitudinal reinforcement, (7) the span–depth ratio. The Bayesian parameter estimation using the form in Equation (3) produced better results when the natural logarithms were applied to the normalized parameters. Therefore, this study chose

$$h_1(x) = \left\{ \ln(f_t), \ln(b), \ln(h), \ln\left(\frac{a}{d}\right), \ln\left(\frac{l_0}{h}\right), \ln(e^{\rho v}), \ln(e^{\rho h}), \ln(\rho) \right\}, \quad (13)$$

$$h_2(x) = \left\{ \ln\left(\frac{f'_c}{f_y}\right), \ln\left(\frac{a}{d}\right), \ln\left(\frac{l_0}{h}\right), \ln\left(\frac{b}{h}\right), \ln(e^{\rho v}), \ln(e^{\rho h}), \ln(\rho) \right\}. \quad (14)$$

4.1.2. Parameter Removal Process

The parameter with the maximum value of the Coefficient of Variation (CoV) was regarded as causing little influence on the shear strength of deep beams and would be removed [11]. The parameter deletion process is as follows:

- (1) Calculate the posterior estimation values of parameters $\theta = [\theta_1, \theta_2, \dots, \theta_p]$ and σ .
- (2) Calculate the coefficient of variation COV for θ_i :

The Coefficient of Variation of each parameter is calculated as

$$\text{COV} = \frac{\sigma_i}{\mu_i}, \quad (15)$$

where σ_i and μ_i are the standard deviations and the mean value of the i th parameter, respectively.

- (3) Remove $h_i(x)$ with the highest coefficient of variation COV. If the COV for the θ_i is the largest, it is considered that the $h_i(x)$ has the most negligible impact and is removed. It is continuing the adjustment with the remaining term $h(x)$.
- (4) Repeat the parameter removal process until a significant increase occurs in the prediction model's standard deviation (SD).

4.1.3. Gibbs Sampling

The parameter θ was sampled from the expressions for the full conditionals by using Gibbs sampling [77]. Suppose the existence of a parameter vector $\theta = [\theta_{(1)}, \dots, \theta_{(8)}]^T$ and a given data set X and Y , as well as a starting point $\theta^{(0)}$ was arbitrarily chosen; then, suppose that the value of θ is $\theta^{(i-1)}$ when the i th iteration begins. The iterative algorithm is shown below:

- (1) $\theta_{(1)}^{(i)}$ was sampled from the full conditional distribution

$$\begin{aligned} \pi\left(\theta_{(1)}^{(i)} \mid Y, X, \theta_{(2)}^{(i-1)}, \dots, \theta_{(8)}^{(i-1)}\right) &= \pi\left(\theta_{(1)}^{(i)} \mid \theta_{(k)}^{(i-1)}, \sigma^*, Y, X_k\right) \\ &\propto \frac{1}{\sigma^{*n}} \exp\left\{-\frac{1}{2\sigma^{*2}}\left[S_n^2 + \sum_{k=2}^8 \left(\left(\theta_{(k)}^{(i-1)} - \hat{\theta}_k\right) X_k^T X_k \left(\theta_{(k)}^{(i-1)} - \hat{\theta}_k\right)\right)\right]\right\}. \end{aligned} \quad (16)$$

- (2) $\theta_{(2)}^{(i)}$ was sampled from the full conditional distribution

$$\begin{aligned} \pi\left(\theta_{(2)}^{(i)} \mid Y, X, \theta_{(1)}^{(i-1)}, \theta_{(2)}^{(i-1)}, \dots, \theta_{(8)}^{(i-1)}\right) &= \pi\left(\theta_{(2)}^{(i)} \mid \theta_{(k)}^{(i-1)}, \sigma^*, Y, X_k\right) \\ &\propto \frac{1}{\sigma^{*n}} \exp\left\{-\frac{1}{2\sigma^{*2}}\left[S_n^2 + \sum_{k=2}^8 \left(\left(\theta_{(1)}^{(i-1)} - \hat{\theta}_k\right) X_1^T X_1 \left(\theta_{(1)}^{(i-1)} - \hat{\theta}_k\right) + \sum_{k=3}^8 \left(\left(\theta_{(k)}^{(i-1)} - \hat{\theta}_k\right) X_k^T X_k \left(\theta_{(k)}^{(i-1)} - \hat{\theta}_k\right)\right)\right]\right\}. \end{aligned} \quad (17)$$

- (3) $\theta_{(8)}^{(i)}$ was sampled from the full conditional distribution

$$\begin{aligned} \pi\left(\theta_{(8)}^{(i)} \mid Y, X, \theta_{(2)}^{(i-1)}, \dots, \theta_{(7)}^{(i-1)}\right) &= \pi\left(\theta_{(8)}^{(i)} \mid \theta_{(k)}^{(i-1)}, \sigma^*, Y, X_k\right) \\ &\propto \frac{1}{\sigma^{*n}} \exp\left\{-\frac{1}{2\sigma^{*2}}\left[S_n^2 + \sum_{k=1}^7 \left(\left(\theta_{(k)}^{(i-1)} - \hat{\theta}_k\right) X_k^T X_k \left(\theta_{(k)}^{(i-1)} - \hat{\theta}_k\right)\right)\right]\right\}. \end{aligned} \quad (18)$$

The iteration sampling was repeated until n samples $\theta^{(1)}, \theta^{(2)}, \dots, \theta^{(n)}$ were created. The calculation process is shown in Figure 2.

4.2. Development of Probabilistic Models

4.2.1. Existing Shear Strength Models

Various shear strength models were proposed by considering different influencing factors. However, due to the complex shear transfer mechanism, each model has different degrees of deviation (listed in Table 2). Explanatory functions h_1 were used to establish probabilistic models to guarantee accuracy and continuity. The existing deterministic models are listed in Table 2, namely ACI 318-14, GB 50010-10, CSA A23.3-04, EC2, CEB-FIP, Foster and Gilbert, and Mitchell and Collins. To consider more parameters, suppose the prior information was the Bayesian hypothesis, and the prior model was “1” for Bayesian posterior parameter estimation with explanatory functions h_2 .

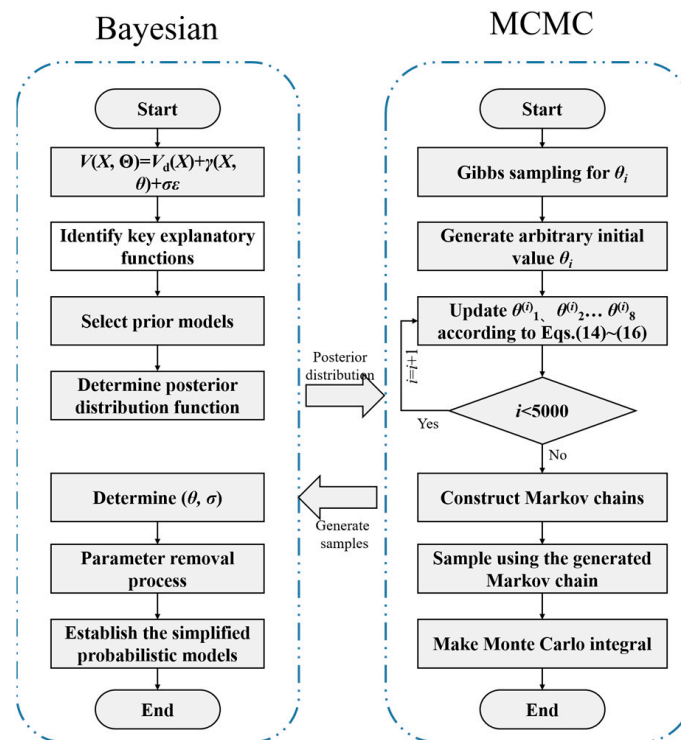


Figure 2. Flow chart of solution procedure for Bayesian–MCMC method.

Table 2. Well-known shear strength models.

Design Provisions	Shear Calculation Model	Mean	SD	CoV
ACI318-14 (2014) [23]	$V_n = 0.85\beta_s f'_c b_w w_s \sin \theta_s$ $\tan \theta_s = \frac{h - \frac{w_t + w'_t}{2}}{a} \geq 0.488$ $w'_t = 0.8w_t$ $w_t = 2(h - h_0)$ $w_s = \frac{2.25w_t \cos \theta_s + [(l_p)_E + (l_p)_p] \sin \theta_s}{2}$	1.1637	0.4148	0.3565
GB 50010-10 (2010) [24]	$V_n = \frac{1.75}{\lambda + 1} f_t b h_0 + \left(\frac{l_0/h - 2}{3} \right) f_{yv} \frac{A_{sv}}{S_h} h_0 + \left(\frac{5 - l_0/h}{6} \right) f_{yh} \frac{A_{sh}}{S_v} h_0$	1.6164	0.6075	0.3758
CSA A23.3-04 (2004) [25]	$V_n = \frac{1}{0.8 + 170\varepsilon_1} f'_c b_w w_s \sin \theta_s$ $\varepsilon_1 = \varepsilon_s + (\varepsilon_s + 0.002) \cot^2 \theta_s$	1.2327	0.6464	0.3690
EC2 (2004) [26]	$V_n = 0.6 \left(1 - \frac{f_{ck}}{250} \right) f'_c b_w w_s \sin \theta_s$	1.2179	0.6310	0.5181
CEB-FIP (2010) [27]	$V_n = 0.55 \left(\frac{30}{f_{ck}} \right)^{1/3} f_{cd} f'_c b_w w_s \sin \theta_s$	1.3641	0.4302	0.3154
Foster and Gilbert (1998) [28]	$\frac{a}{d} < 2, \quad v = 1.25 - \frac{f'_c}{500} - 0.72 \frac{a}{d} + 0.18 \left(\frac{a}{d} \right)^2 \leq 1$ $\frac{a}{d} < 2, \quad v = 0.53 - \frac{f'_c}{500}$	1.0900	0.4594	0.4215
Mitchell and Collins (1974) [29]	$V_n = \frac{3.35}{\sqrt{f'_c}} f'_c b_w w_s \sin \theta_s$	1.2440	0.4015	0.3228

4.2.2. Calculation Results

In the stochastic simulation process, the former 1000 values were removed to eliminate the negative effect of the initial value on chains. Then, the posterior estimate of (θ, σ^2) was obtained by Equation (7). Table 3 illustrates the calculation results of the parameters,

including the mean, standard deviation (SD), naive standard error (SE), time-series SE, and quintiles. Most CoVs were below 0.2, indicating that the sample averages have a relatively concentrated distribution. The naive standard error provided a measure of the potential error in this estimate that decreased as chain length increased. In turn, an appropriate chain length (i.e., 50,000) was determined by comparing it with the target value (i.e., 1/1000). Therefore, this random simulation method can accurately obtain the optimal estimation value and the corresponding reliability of model parameters.

Table 3. Calculation results of probabilistic shear strength model parameters.

Model Parameter	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	θ_7	θ_8	θ_9	σ^2
Mean	−5.770	0.896	1.039	0.835	−0.614	−0.065	0.184	0.056	0.326	0.082
SD	0.221	0.040	0.029	0.033	0.034	0.040	0.029	0.020	0.022	0.005
CoV	0.038	0.045	0.028	0.040	0.054	0.621	0.158	0.358	0.067	0.055
Naive SE ($\times 10^{-4}$)	9.880	1.809	1.299	1.479	1.501	1.797	1.295	9.002	0.969	0.201
Time-series SE ($\times 10^{-4}$)	9.880	1.809	1.305	1.468	1.797	1.797	1.295	9.002	0.969	0.204
2.5% quantile	−6.202	0.817	0.981	0.769	−0.680	−0.143	0.127	0.017	0.283	0.074
97.5% quantile	−5.333	0.975	1.095	0.899	−0.548	0.014	0.240	0.096	0.368	0.092

Figure 3 clearly shows the simulation results of parameters obtained by the iterative process of the Markov chain in Appendix A. The iterative process of two Markov chains with different initial values was shown in trace plots. After 10,000 times of iterations, the two chains were utterly mixed together and stably distributed in a minimal range near the posterior mean. Ergodic mean plots display ergodic mean statistical results of the two Markov chains. With the iterations increasing, the ergodic mean of Markov chains tended to be a horizontal line and converged to the parameter mean. It is noted that Markov chains have reached a stable state and convergence. The posterior frequency histograms were obtained from 50,000 posterior samples after removing the former 1000 values. It can be seen that the majority of the rectangle area was concentrated around the simulated result values, which is consistent with the posterior mean.

4.2.3. Probabilistic Models

The proposed model without a prior model can be obtained by substituting the simulation results of the parameter into Equation (3) and performing the exponential operation. The proposed model took the form

$$V_{\text{MCMC}} = 0.0031 \cdot (f_t)^{0.8957} \cdot (b)^{1.0385} \cdot (h)^{0.8349} \cdot \left(\frac{a}{d}\right)^{-0.6137} \cdot \left(\frac{l_0}{h}\right)^{-0.0647} \cdot e^{0.1837\rho v} \cdot e^{0.0561\rho h} \cdot \rho^{0.3481} \quad (19)$$

The proposed model with many parameters was very complicated. Therefore, some parameters were eliminated to simplify the probabilistic model with the parameter removal process. Table 4 illustrates the stepwise removal process. According to CoVs, θ_6 , θ_7 , θ_8 , and θ_9 were removed sequentially. The removal process stopped after the third removal (SD = 0.2396) because the fourth removal (SD = 0.2914) caused a significant increase in the posterior SD.

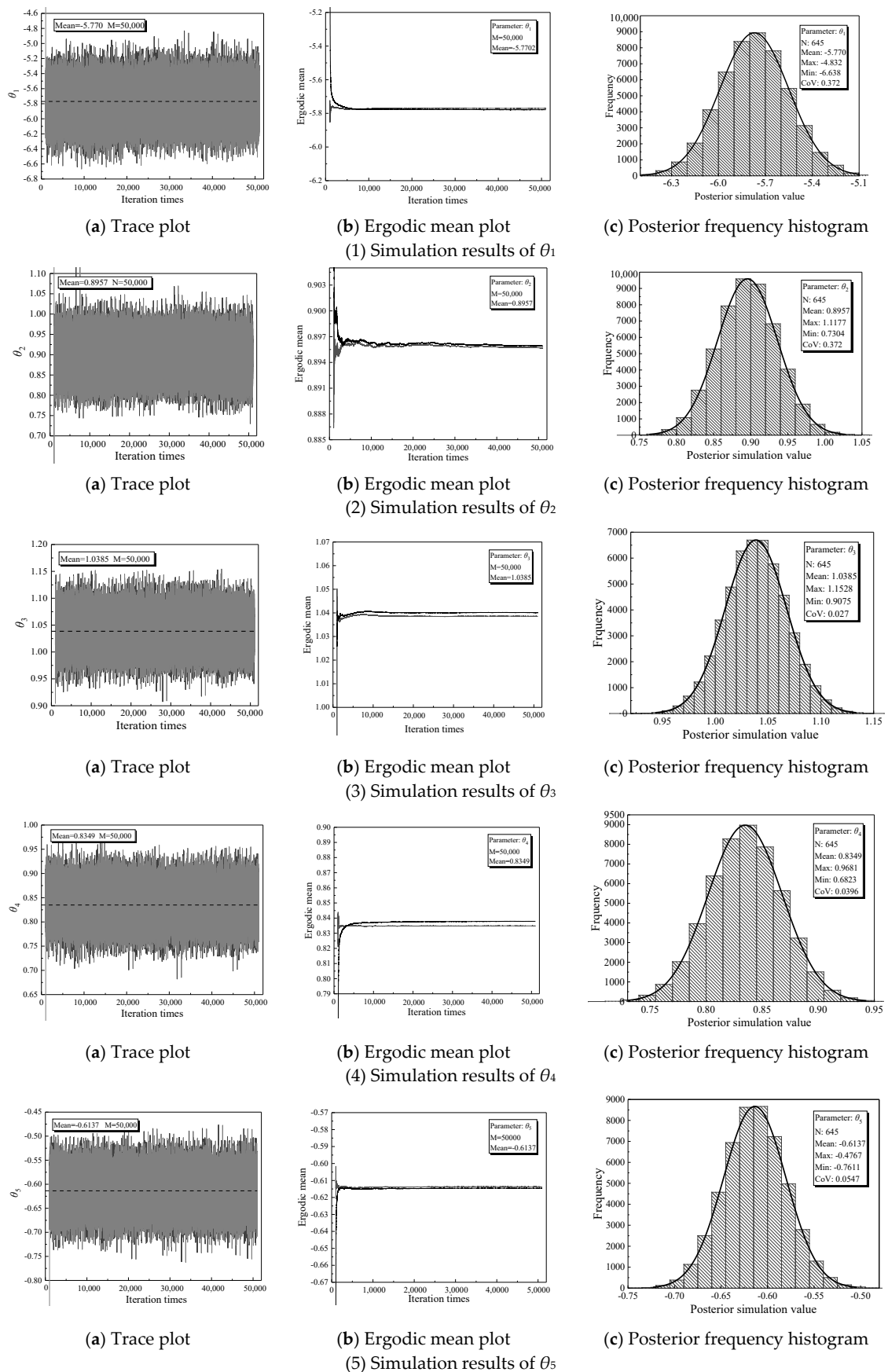


Figure 3. Cont.

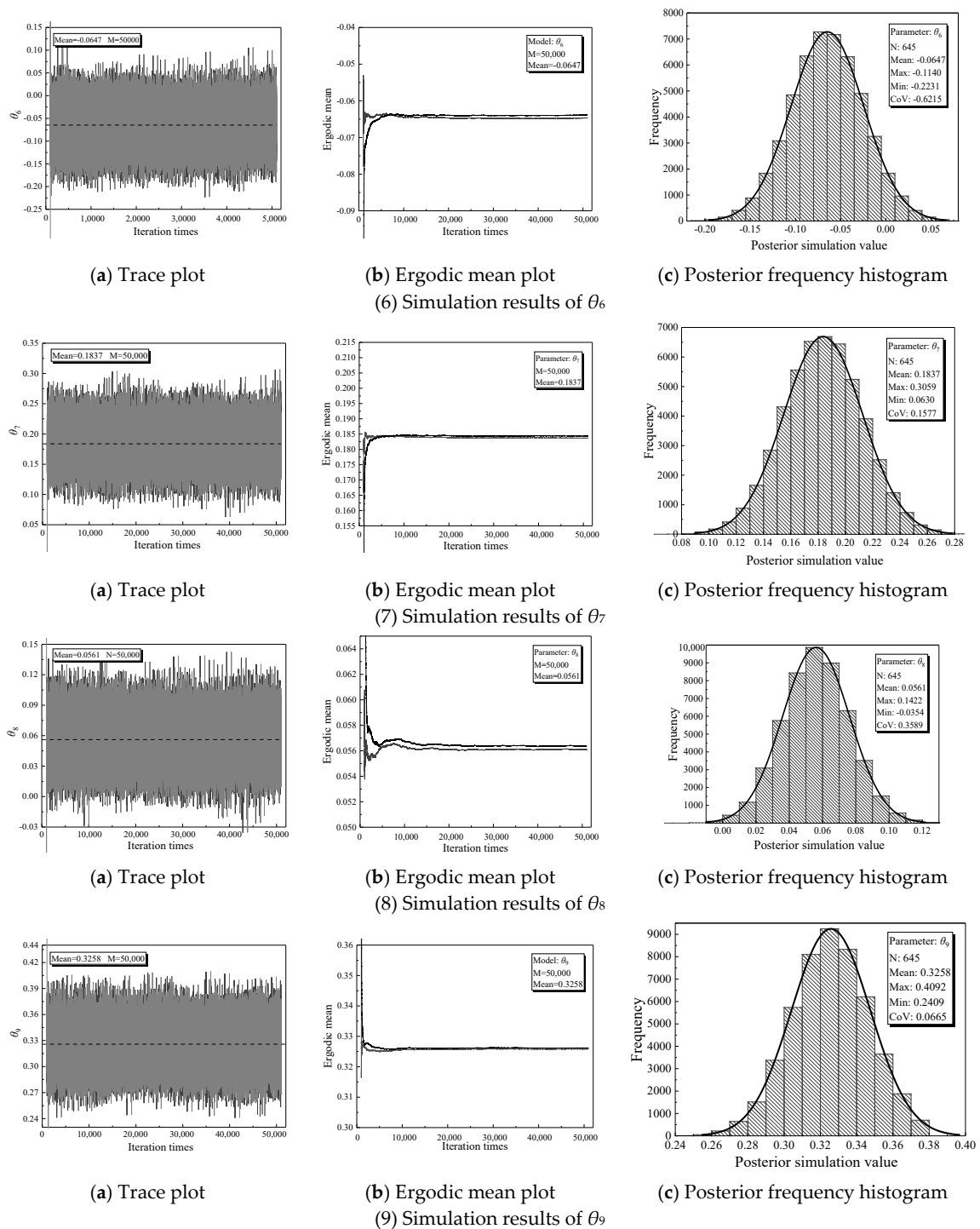


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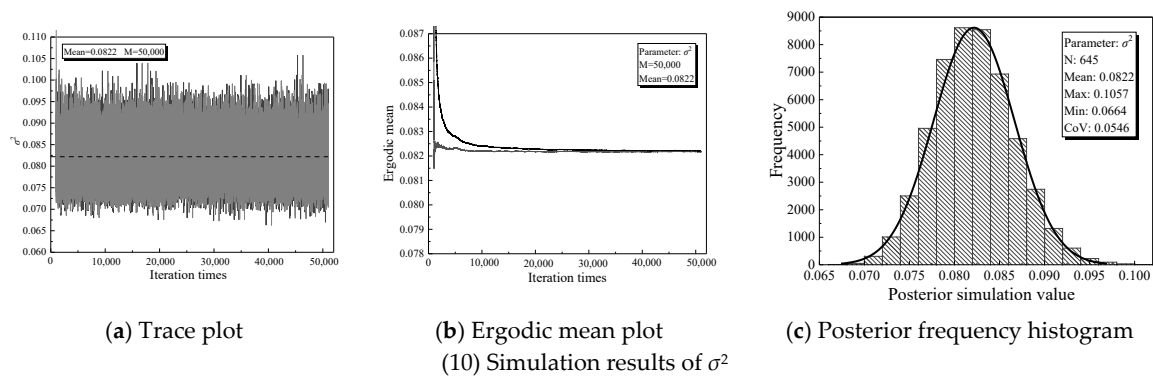


Figure 3. Simulation results of parameters.

Table 4. Step-wise removal process for the proposed model.

Step	Posterior CoV of the Corresponding θ_i									σ
	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	θ_7	θ_8	θ_9	
	(Constant)	($\ln f_i$)	($\ln b$)	($\ln h$)	($\ln a/d$)	($\ln l_0/h$)	(ρ_v)	(ρ_h)	($\ln \rho$)	
Initial	0.0383	0.0492	0.028	0.0396	0.0547	0.6214	0.3677	0.3589	0.0665	0.0546
1st	0.0373	0.0441	0.0280	0.0390	0.0412	×	0.3552	0.3514	0.0674	0.0545
2nd	0.0394	0.0450	0.0292	0.0408	0.0419	×	×	0.3374	0.0647	0.0545
3rd	0.0399	0.0450	0.0290	0.0403	0.0419	×	×	×	0.0646	0.0542
4th	0.2166	0.1187	0.2603	0.1192	0.0789	×	×	×	×	0.0544

Step	Posterior Mean of the Corresponding θ_i									SD
	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	θ_7	θ_8	θ_9	
	(Constant)	($\ln f_i$)	($\ln b$)	($\ln h$)	($\ln a/d$)	($\ln l_0/h$)	(ρ_v)	(ρ_h)	($\ln \rho$)	
Initial	−5.7707	0.8957	1.0385	0.8349	−0.6137	−0.0647	0.1837	0.0561	0.3258	0.2249
1st	−5.8362	0.8761	1.0359	0.8412	−0.6469	×	0.1860	0.0578	0.3205	0.2276
2nd	−5.5820	0.8796	1.0122	0.8256	−0.6523	×	×	0.0618	0.3385	0.2379
3rd	−5.5223	0.8829	0.9899	0.8354	−0.6567	×	×	×	0.3418	0.2396
4th	1.8175	0.5812	0.1994	−0.4868	0.3363	×	×	×	×	0.2914

Note: × represents the removed parameter.

From Table 4, the uncertainty may be reduced for the model using fewer terms of parameters because unnecessary parameters may prevent the model from effectively describing the behavior. On the contrary, removing a key parameter also makes the model perform poorly and thus increases the uncertainty. Table 5 shows the stepwise removal process and explanatory terms that survived the modified models. Some explanatory terms were removed in the process, which implies that the base models have already considered their effect on shear strength or are insignificant. For example, the modified probabilistic model based on ACI 318-14 removed the descriptive terms a/d , ρ_v , and ρ_h . Bias correction terms for simplified probabilistic models are listed in Table 6.

Table 5. Step-wise removal result for other probabilistic models.

Prior Model	Posterior Mean of the Corresponding θ_i							
	Constant	f'_c/f_y (f_{cu}/f_y)	a/d	l_0/h	b/h	ρ_v	ρ_h	ρ
GB50010-2010	1.3460	$\times^1$	−0.2506	$\times^3$	$\times^2$	−0.0894	−0.1960	0.4416
ACI 318-14	0.6310	−0.2403	$\times^2$	−0.4193	−0.1992	$\times^3$	$\times^1$	0.1905
CSA 23.3-04	1.3209	$\times^3$	0.8291	$\times^2$	−0.1432	0.1116	$\times^1$	0.0031
EC2	1.4215	−0.1016	0.6843	−0.1672	$\times^1$	0.1652	$\times^2$	$\times^3$
CEB-FIP	1.2508	$\times^3$	−0.1324	−0.2691	−0.1659	$\times^1$	$\times^2$	0.2033
Foster and Gilbert	0.7606	−0.1244	$\times^3$	−0.5332	−0.2717	$\times^1$	$\times^2$	0.2932
Mitchell and Collins	1.2960	0.1484	$\times^2$	$\times^3$	−0.1780	0.1729	0.0693	$\times^1$

Note: $\times^1$ represents the 1st removed parameter. $\times^2$ represents the 2nd removed parameter. $\times^3$ represents the 3rd removed parameter.

Table 6. Bias correction term.

Model	Bias Correction Term	Mean	SD	COV
The Proposed Model	$0.0040 \cdot (f_t)^{0.8829} \cdot (b)^{0.9899} \cdot (h)^{0.8354} \cdot \left(\frac{a}{d}\right)^{-0.6567} \cdot \rho^{0.3481}$	1.0357	0.2396	0.2312
GB50010-2010	$1.3460 \cdot \left(\frac{a}{d}\right)^{-0.2506} \cdot \left(\frac{b}{h}\right)^{0.3855} \cdot (e^{\rho v})^{-0.0894} \cdot (e^{\rho h})^{-0.1960} \cdot \rho^{0.4416}$	1.0384	0.2908	0.2803
ACI 318-14	$0.6310 \cdot \left(\frac{f'_c}{f_y}\right)^{-0.2403} \cdot \left(\frac{l_0}{h}\right)^{-0.4193} \cdot \left(\frac{b}{h}\right)^{-0.1992} \cdot \rho^{0.1905}$	1.0387	0.2906	0.2798
CSA 23.3-04	$1.3209 \cdot \left(\frac{a}{d}\right)^{-0.6567} \cdot \left(\frac{b}{h}\right)^{-0.1432} \cdot (e^{\rho v})^{-0.1116} \cdot \rho^{-0.0598}$	1.0744	0.3421	0.3184
EC2	$1.4215 \cdot \left(\frac{f'_c}{f_y}\right)^{-0.1016} \cdot \left(\frac{a}{d}\right)^{0.6843} \cdot \left(\frac{l_0}{h}\right)^{-0.1672} \cdot (e^{\rho v})^{0.1652}$	1.0647	0.4129	0.3879
CEB-FIP	$1.2508 \cdot \left(\frac{a}{d}\right)^{-0.1324} \cdot \left(\frac{l_0}{h}\right)^{-0.2691} \cdot \left(\frac{b}{h}\right)^{-0.1659} \cdot \rho^{0.2033}$	1.0322	0.2622	0.2540
Foster and Gilbert	$0.7606 \cdot \left(\frac{f'_c}{f_y}\right)^{-0.1244} \cdot \left(\frac{l_0}{h}\right)^{-0.5332} \cdot \left(\frac{b}{h}\right)^{-0.2717} \cdot \rho^{0.2932}$	1.0557	0.3729	0.3542
Mitchell and Collins	$1.2960 \cdot \left(\frac{f'_c}{f_y}\right)^{-0.1484} \cdot (e^{\rho v})^{-0.1783} \cdot (e^{\rho h})^{0.1729} \cdot \rho^{0.0693}$	1.0396	0.2866	0.2756

5. Discussion

5.1. Comparison with Existing Models

The boxplots of the experiment-to-prediction ratios for different prediction models are provided in Figure 4. The boxplots provided statistical information such as the lower quartile, median, upper quartile, and outliers. Figure 4 shows that the proposed model had less bias (i.e., means and medians closer to 1) and scatter (i.e., shorter boxes) than the other deterministic models. Compared with other models, the outliers of the strength calculation of the proposed model were very few. Although the deterministic models were improved by the Bayesian–MCMC approach in terms of their bias and uncertainties, those modified models still had larger scatters than the proposed model. The mean and CoV of the proposed model were 1.0357 and 0.2312, respectively, which indicates that the strength calculation is more consistent with the experimental data and has minor variability.

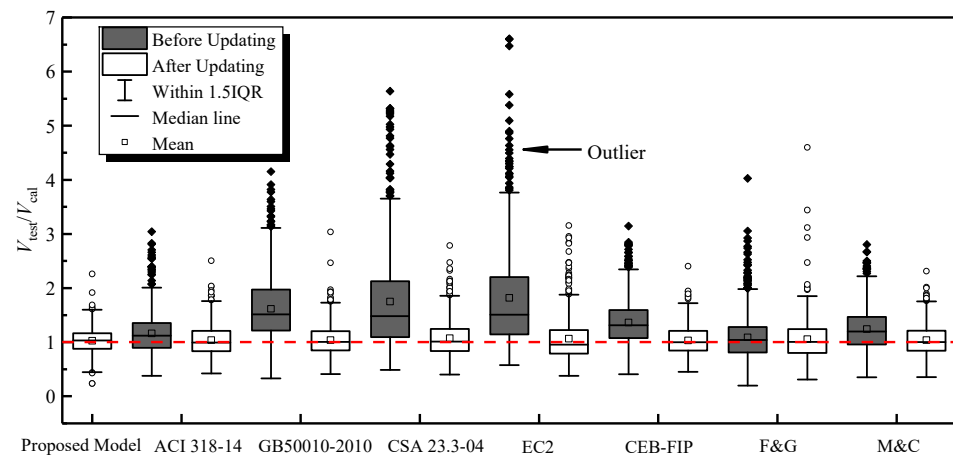


Figure 4. Boxplots of the experiment-prediction ratios.

5.2. Comparison with Experimental Observations

Figure 5 demonstrates the performance of the probabilistic models by comparison with each experimental data point. The test values were sorted according to an increasing order of the mean shear strengths calculated by the probabilistic models. The mean curve was plotted with a shaded area representing mean $\pm$ SD interval. It covered approximately 70% of the probability distribution of the strength. As shown in Figure 5a, the curve of the mean shear strength mostly passed through the center of the experimental points. The mean $\pm$ SD covered most of the experimental data throughout the whole range of strengths in the database. It noted that the proposed model without a deterministic model was reliable and consistent with the experimental data.

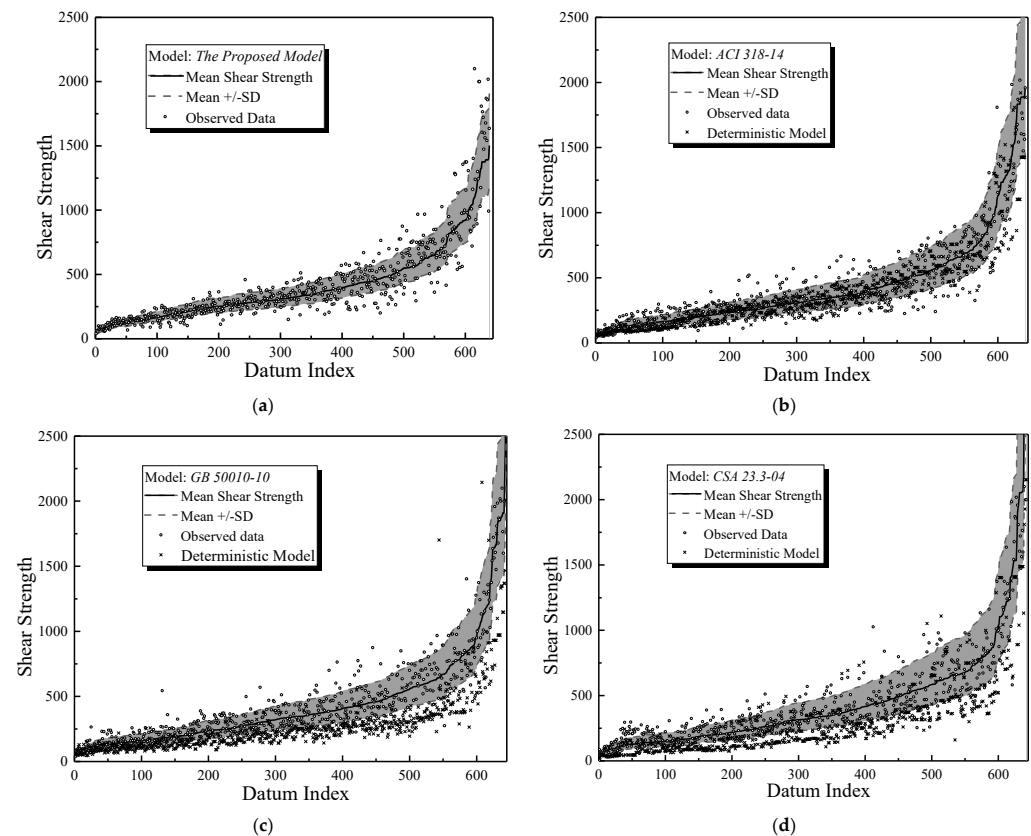


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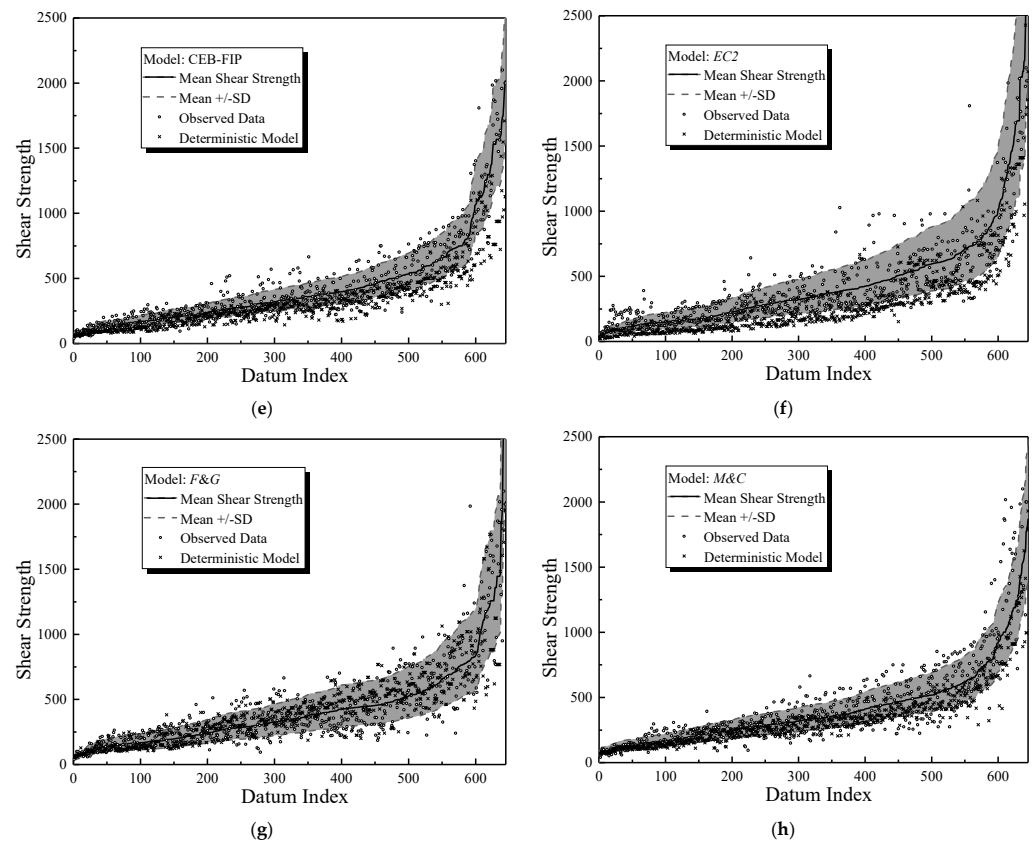


Figure 5. Performance of the probabilistic models at each experimental data. (a) The Proposed Model. (b) ACI 318-14. (c) GB 50010-10. (d) CSA 23.3-04. (e) CEB-FIP. (f) EC2. (g) Foster and Gilbert. (h) Mitchell and Collins.

By contrast, the deterministic models show relatively large scatter or biases. Most of the biases were observed on the conservative side, especially in GB 50010-10, CSA 23.3-04, CEB-FIP, and EC2. After being improved by the Bayesian–MCMC method, most modified probabilistic models showed good performance in bias and scatter. For example, the Mitchell and Collins, CEB-FIP, and ACI 318-14 have higher accuracy and lower discreteness (a narrower shaded area).

5.3. Continuity and Uncertainty

The consistent performance and model uncertainty concerning essential variables were investigated by comparing test and model results. The boxplots of the experiment-to-prediction ratios were made for different ranges of each of the following key explanatory terms: f_c' (Figure 6a), a/d (Figure 6b), b/h (Figure 6c), and ρ (Figure 6d).

The boxplots of the bias of the proposed model, ACI 318-14, and the modified probabilistic model were made for comparison. As shown in Figure 6a, the ACI 318-14 tended to underestimate the shear strengths of normal-strength members and overestimate ultra-strength members. The experiment-prediction ratio decreased as concrete strength increased. Similarly, the ACI 318-14 model tended to underestimate the shear strengths for $0 < \rho \leq 1.0$ members and had great discreteness. Figure 6c depicts the ACI 318-14, which had a very large scatter of prediction for all ranges of b/h , and the bias was not uniform.

The modified probabilistic model based on ACI 318-14 effectively corrected these biases and allowed for more consistent performance over the considered ranges. This may be due to the fact that the bias correction term indirectly considers the effect of f_c' , b/h , and ρ , etc. As for ultra-strength and $0 < b/h < 0.2$ members, the proposed model still had great variability (i.e., CoVs > 0.25), which probably results from uniform distributions of parameters in the database. It is concluded that these boxplots confirmed that the proposed

model performed uniformly well over the entire ranges of the selected key parameters. The Bayesian–MCMC method further improves the consistency of the performance.

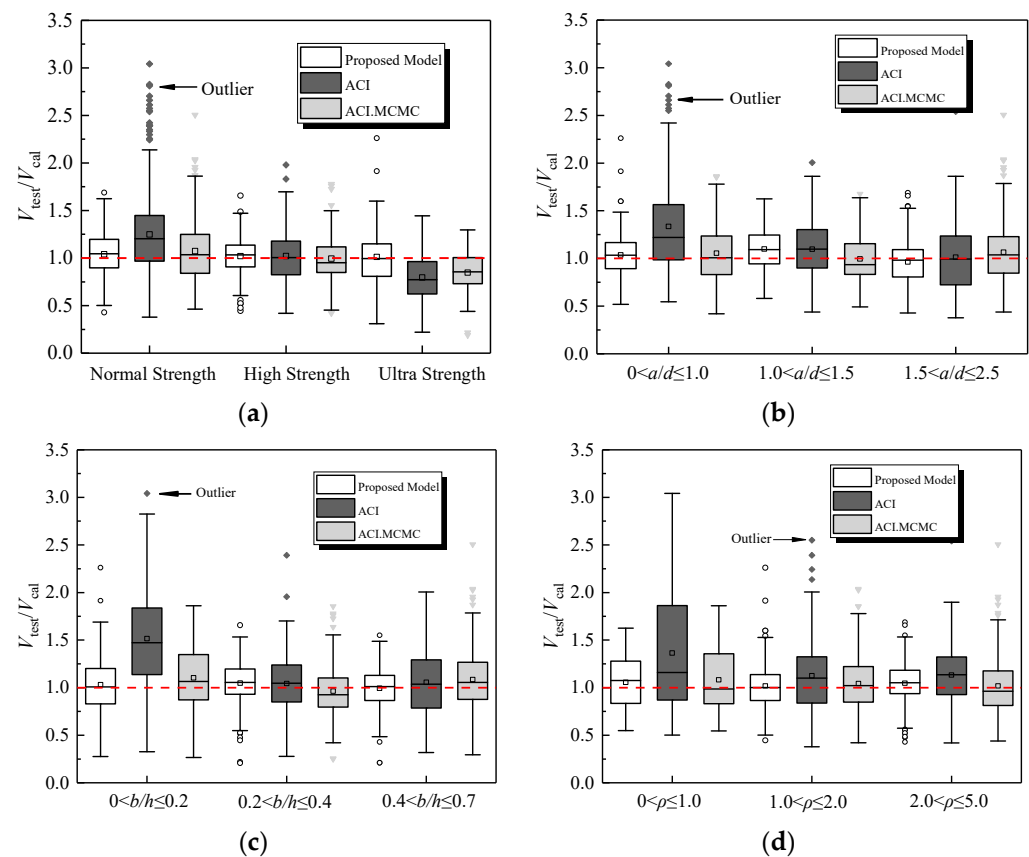


Figure 6. The boxplots of the experiment-prediction ratios. (a) f_c' /MPa. (b) a/d . (c) b/h . (d) $\rho/\%$.

An adequate description of the uncertainties in shear prediction models was recognized as one of the critical issues in reliability investigations of RC structures [79–82]. The partial factor method was used to evaluate model uncertainty. Assuming a lognormal distribution and unity characteristic (nominal) value of model uncertainty, the partial factor γ_{Rd} can be obtained from the following equation:

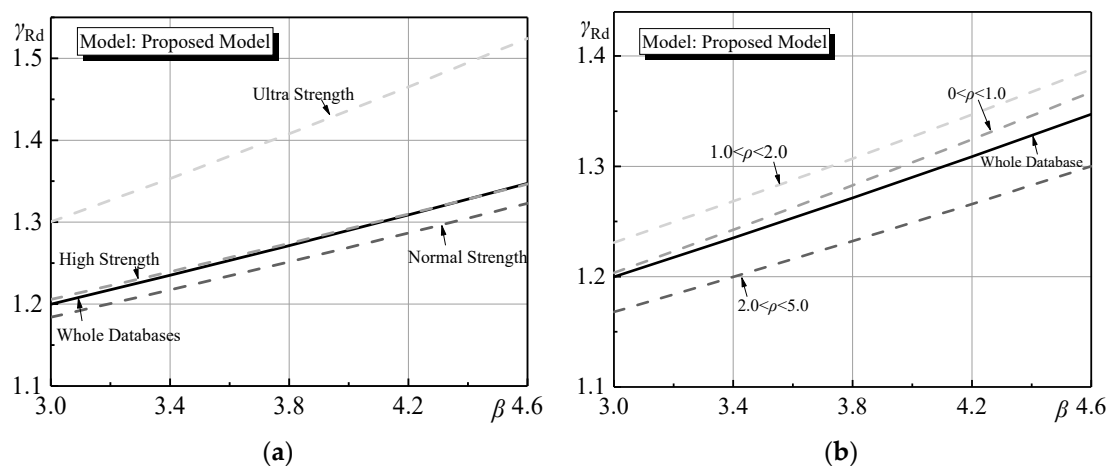
$$\gamma_{Rd} = \frac{1}{\mu \exp(-\alpha_R \beta v)}, \quad (20)$$

where α_R is the FORM (First Order Reliability Method) sensitivity factor; β is the selected target reliability index; μ and v are the mean and coefficient of the experiment-to-prediction ratios, respectively.

Considering the model uncertainty characteristics shown in Table 7, the variation of model uncertainty factor with target reliability β is displayed in Figure 7. $\alpha_R = 0.4 \times 0.8 = 0.32$, that is, “non-dominant resistance variable” according to ISO 2394 (2015) [83] and fib Bulletin 80 (2016) [84]. The range of β represented the target lifetime levels commonly associated with the Ultimate Limit States for new and existing structures.

Table 7. Sample characteristics for different ranges of the key explanatory terms.

Level of Approximation		<i>n</i>	Proposed Model		ACI		ACI.MCMC	
Description of Sample			Mean	CoV	Mean	CoV	Mean	CoV
Whole Databases		645	1.0357	0.2263	1.1637	0.3565	1.0387	0.2798
Concrete compressive strength	Normal strength 10~40 MPa	457	1.0402	0.2169	1.2491	0.3437	1.0733	0.2812
	High strength 40~70 MPa	129	1.0201	0.2156	1.0210	0.2865	0.9941	0.2570
	Ultra strength 70~120 MPa	59	1.0356	0.3101	0.8140	0.2704	0.8686	0.2173
Reinforcement ratio	$0 < \rho \leq 1.0$	96	1.0558	0.2494	1.3640	0.4725	1.0827	0.3050
	$1.0 < \rho \leq 2.0$	273	1.0177	0.2346	1.1250	0.3442	1.0441	0.2694
	$2.0 < \rho \leq 5.0$	276	1.0466	0.2092	1.1325	0.2728	1.0182	0.2795
Component section size	$0 < b/h \leq 0.2$	156	1.0408	0.2752	1.5299	0.3259	1.1142	0.2662
	$0.2 < b/h \leq 0.4$	296	1.0537	0.2082	1.0485	0.2769	0.9683	0.2565
	$0.4 < b/h \leq 0.7$	193	1.0029	0.2106	1.0630	0.3184	1.0935	0.2945
Shear span ratio	$0 < a/d \leq 1.0$	246	1.0367	0.2320	1.3359	0.3608	1.0558	0.2746
	$1.0 < a/d \leq 2.0$	211	1.0991	0.1838	1.0971	0.2688	0.9950	0.2337
	$2.0 < a/d \leq 5.0$	188	0.9633	0.2495	1.0129	0.3477	1.0654	0.3209

**Figure 7.** Variation of model uncertainty factor with target reliability. (a) f_c' / MPa. (b) ρ / %.

As shown in Figure 7, the partial factor γ_{Rd} increased with an increasing target reliability index β . The value of β is 3.8 for ordinary structures with 50 years of design service life. When β was between 3.0 and 4.6, the model uncertainty varied approximately within the following intervals:

- 1.20~1.35 for whole database ($\gamma_{Rd} \approx 1.25$, $\beta = 0.38$)
- 1.18~1.32 for normal-strength members ($\gamma_{Rd} \approx 1.25$, $\beta = 0.38$)
- 1.21~1.35 for high-strength members ($\gamma_{Rd} \approx 1.27$, $\beta = 0.38$)
- 1.30~1.52 for ultra-strength members ($\gamma_{Rd} \approx 1.41$, $\beta = 0.38$)
- 1.20~1.37 for $0 < \rho \leq 1.0$ members ($\gamma_{Rd} \approx 1.28$, $\beta = 0.38$)
- 1.23~1.39 for $1.0 < \rho \leq 2.0$ members ($\gamma_{Rd} \approx 1.31$, $\beta = 0.38$)
- 1.17~1.30 for $2.0 < \rho \leq 5.0$ members ($\gamma_{Rd} \approx 1.21$, $\beta = 0.38$)

6. Conclusions

Probabilistic models of the shear strength of deep beams were developed with an extensive database by using the Bayesian-MCMC method. The following conclusions can be drawn:

- The deterministic methods improved by the Bayesian–MCMC method showed more accurate and robust predictions with experiment-to-prediction ratios between 1.0322 and 1.0744 and CoVs between 0.2540 and 0.3879.
- The proposed model developed without prior models had higher accuracy (the mean was 1.0357) and lower discreteness (the CoV was 0.2312) than other modified probabilistic models. Uncertainties related to the proposed model were described by the following statistical characteristics and partial factors: whole database ($\beta = 0.38$, $\mu = 1.0357$, $V = 0.2263$, and $\gamma_{Rd} \approx 1.25$).
- Gibbs sampling method was introduced to solve high-dimensional and complex integration problems by which the sampling progress was simplified, improving the optimization and reliability of model parameters.
- Explanatory functions such as f_c'/f_y , l_0/h , a/h_0 , b/h , ρ_v , ρ_h , and ρ were identified by different removal processes, which improves the modified probabilistic models.

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Glossary

a = shear span;

a/d = the shear span-to-depth ratio;

A_{sv} = the total area of vertical web reinforcement in the same section;

A_{sh} = the total area of horizontal web reinforcement in the same section;

b = the width of specimens;

b_w = the width of web;

d = the effective depth of beam;

$E[f(x)]$ = the expectation of arbitrary function;

$f(x)$ = the arbitrary function;

f_c' = the specified compressive strength of concrete;

f_{cd} = the design value of cylinder compressive strength of concrete;

f_{ck} = the compressive cylinder strength of concrete;

f_{cu} = the cube compressive strength of concrete;

f_t = the tensile strength of concrete;

f_y = the yield strength of longitudinal reinforcement;

f_{yh} = the yield stress for horizontal web reinforcement in beam;

f_{yv} = the yield stress for vertical web reinforcement in beam;

$f(\Theta)$ = the posterior distribution;

h = the depth of specimens;

h_0 = the effective depth of beam;

$h_i(x)$ = the explanatory functions;
 $L(\theta, \sigma | Y, X)$ = the likelihood function;
 $L(\Theta)$ = the likelihood function;
 l_0/h = the shear–depth ratio;
 $(l_p)_E$ = the width of the bottom loading plate;
 $(l_p)_P$ = the width of the top loading plate;
 S_h = the spacing of horizontal web reinforcement;
 S_n = the variance of arbitrary function;
 S_v = the spacing of vertical web reinforcement;
 M = iteration times
 m = the number of samples when the Markov chain reaches stationary distribution;
 n = the total number of samples generated by simulation;
 p = the number of explanatory functions;
 $p(\Theta)$ = the joint probability density function of a prior distribution;
 $VAR[f(x)]$ = the variance of arbitrary function;
 $V(X, \Theta)$ = a probabilistic model;
 V_d = an existing deterministic model;
 V_i = calculation of probability model of i th test;
 V_n = the shear strength of deep beams with different calculation models;
 w_s = the width of a strut perpendicular to the axis of the strut;
 w_t = the effective height of the bottom nodal zone
 w_t' = the effective height of the top nodal zone;
 X = the vector of input parameters;
 α_R = FORM (First Order Reliability Method) sensitivity factor;
 β = the elected target reliability index;
 β_s = the factor used to account for the effect of cracking and confining reinforcement on the effective compressive strength of the concrete in a strut;
 γ_{Rd} = the partial factor;
 $\gamma(X, \theta)$ = the correction term for the bias;
 ε = the normal random variable with the zero mean and unit variance;
 ε_1 = the main tensile strain;
 ε_s = the longitudinal tensile strain in mid span of beams;
 θ = the vector of uncertain model parameters;
 $\theta_{(-i)}$ = the vector of uncertain model parameters excluding eighth parameter;
 $\theta_{(k)}^{(i)}$ = the i th iterative value of the k th parameter;
 θ_s = the smallest angle between the strut and the adjoining ties;
 $\hat{\theta}$ = the mean value of each uncertain model parameter;
 κ = the normalizing factor;
 λ = the shear span-to-depth ratio;
 μ_i = the mean value of the i th parameter;
 $\pi(\theta, \sigma)$ = the prior distribution of parameters;
 $\pi(\theta, \sigma | Y, X)$ = a posteriori distribution function;
 ρ = longitudinal reinforcement ratio;
 ρ_h = the horizontal web reinforcement ratio;
 ρ_v = the vertical web reinforcement ratio;
 σ = the unknown model parameter representing the magnitude of the model error;
 σ_i = the standard deviation of the i th parameter;
 σ^2 = the variance;
 v = coefficient of the experiment-prediction ratios;
 $\varphi(\cdot)$ = the joint probability density function (PDF) of the standard normal;
 $\Phi(\cdot)$ = the cumulative distribution function (CDF) of the standard normal distribution;
 Θ = a set of model parameters introduced to fit the model to the test results.

Appendix A

The trace plots show the iterative process of two Markov chains with different initial values. Ergodic mean plots display ergodic mean statistical results of the two Markov chains. The posterior frequency histograms show statistical results of 50,000 posterior samples.

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