

Article

The Process of Reconstructing the Ancient Magnetic Field Direction: A New Approach to Paleomagnetic Data for a Better Estimate of Accuracy

Andrey Khokhlov ^{1,*}  and Georgy Gvozdk ²¹ IEPT Russian Academy of Sciences, 84/32, Profsoyuznaya, 117997 Moscow, Russia² Geophysical Center RAS, 3, Molodezhnaya Str., 119296 Moscow, Russia

* Correspondence: fbmotion@yandex.ru

Featured Application: For a given paleomagnetic data sample from a lava flow, we identify outliers in the sample and construct a confidence area that localizes the shared direction of the ancient field.

Abstract: Traditionally, the accuracy of paleomagnetic data obtained from samples of igneous rocks relies on the widely known method a95. We propose here a novel statistical method to estimate the ancient field direction using information from Zijderveld diagrams. We show a way to detect outliers in a sample of directions by constructing a confidence domain (convex, but complex in shape) on the direction sphere. Such a region statistically localizes the common direction of the ancient field over a given set of specimens from the lava flow. Often, even for a small sample, this confidence domain is much smaller than the confidence domain that the a95 method gives for the average direction over the sample. Improving the accuracy is obviously important for describing the evolution of the magnetic field.



Citation: Khokhlov, A.; Gvozdk, G. The Process of Reconstructing the Ancient Magnetic Field Direction: A New Approach to Paleomagnetic Data for a Better Estimate of Accuracy. *Appl. Sci.* **2023**, *13*, 4717. <https://doi.org/10.3390/app13084717>

Academic Editors: Paolo Mauriello and Domenico Patella

Received: 7 February 2023

Revised: 28 March 2023

Accepted: 5 April 2023

Published: 9 April 2023



Copyright: © 2023 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

Keywords: probability distributions; statistical modeling; paleomagnetic data; Zijderveld diagrams

1. Introduction

As is well-known, the residual magnetism in a rock sample can only be recovered with some uncertainty, commonly referred to as “error”. This is attributable to inhomogeneities that result during its formation as well as specific aspects that are not fully controllable during the laboratory measurement process. Information about the direction of the residual magnetization vector is important, and can be determined more reliably than its length (i.e., the magnetization intensity). The technique for determining the remanence vector has been well-described (see for example [1,2]) and is based on a series of successive partial demagnetizations and accompanying intermediate measurements. The direction of the ancient field is established by taking both the entire series and possible subseries into account.

The available databases of paleomagnetic directions have been compiled over decades from scientific publications that do not report intermediate details, but instead report only the accuracy of the final direction as calculated using the accepted standardized approaches. The natural way to present the directional precision is to set the cone within which the true paleomagnetic direction is expected to lie with high confidence (0.95 percent or so).

Here, it is necessary to note the difference in the work with sedimentary rock samples and samples of erupted rocks: in the first case, the exact mutual synchronization of specimen ages is hardly possible. Therefore, the specificity of sedimentary rocks in terms of meaning gives only the averaging of the ancient magnetic field over a certain period of deposition, much longer than the period of fixation of the ancient magnetic field in the lava flow. Partly for this reason, for sedimentary rocks, the characteristic of the principal component

analysis MAD (Maximal Angular Deviation) [3] was given in the publications instead of the confidence cone of the recovered direction; at the intuitive level, the MAD indicates the “noisiness” properties of the studied intermediate demagnetization stages. Direct recalculation of the MAD into a confidence cone is not possible [4] without knowing all the details of demagnetization (usually inaccessible from early publications). On the other hand, within a sufficiently simple and intuitive probabilistic description of the demagnetization process, the necessary recalculation is quite possible, see [5].

Turning now to methods for estimating the accuracy of directions determined from erupted rocks, we have a completely different picture. It is possible to select several specimens from the same lava flow so that the implied vector of ancient magnetization in them will share the same direction. The apparent inconsistencies in the responses in different specimens should be attributed to uncontrolled errors of measurements in the demagnetization process. Because of this consideration, it is worth seeking a method to compensate for these errors by means of averaging all the values. Thus, the corresponding method is based on some assumptions about the statistical nature of directions. In a nutshell, it is assumed that a sample of all possible estimates of ancient magnetic fields that can be extracted from samples belonging to a fixed lava flow at a given location (=the site) corresponds to an axisymmetric continuous distribution of points on the sphere [6]. Specifically, the scatter corresponds to the Fisher probability density $f_K(\theta)$ with some concentration parameter K :

$$f_K(\theta) = \frac{K e^{K \cos \theta}}{2\pi(e^K - e^{-K})} \quad (1)$$

The practical a95 method mimics the classical statistical method of estimating the parameters of a one-dimensional Gaussian law from a sample of values. It consists of an approximate calculation of the average direction of n unit vectors, the 0.95-confidence cone axis and a solid angle at its apex. This sheds light on the question of where the true direction of the supposed Fisher axis of distribution and the ancient field lie: both belong to this confidence cone. Various software packages can process data related to demagnetization paleomagnetic measurements and estimate values of MAD, Fisherian quantile angle α_{95} and the corresponding cone axis. That includes for example PMGSC freeware, developed at the Geological Survey of Canada by R.Enkin. For additional software, see, e.g., [7,8].

Note that the practical implementation of the a95 method is affected by outliers, and therefore this method implicitly requires a prior (albeit, heuristic) separation of possible outliers in the data sample.

In this new study, we will propose a way to identify outliers in the sample and at the same time construct a confidence area, which is no longer a circular cone. This new confidence area localizes the shared direction of the ancient field for a given set of specimens from a lava flow. In a fairly general situation, even for a small sample, the localization of this common direction turns out to be much more accurate than that predicted by the a95 method. This improvement in accuracy will be important for studying the evolution of the magnetic field over time and therefore deserves a detailed presentation. Briefly, the idea is to use the statistical model (see [5]) to estimate the confidence cone for each specimen, while using a confidence level p above the traditional 0.95. Then, the intersection of, say, two such cones defines the region in which the true direction lies with probability p^2 and so on. For example, at $p = 0.99$ and five specimens, we find a region on the sphere of directions with the corresponding confidence level not less than 0.95. The size of this region is, however, often smaller than the size of the circle predicted by the a95 method, see Figure 1. That is because the a95 method does not use the accuracy of directions of the specimen from the sample at all, which makes it difficult to pre-screen outliers among specimens. Indeed, if the accuracies in individual directions are inhomogeneous, then the basic assumption of the a95 method that directions share the same Fisher’s law (i.e., with a fixed concentration parameter) is, generally speaking, incorrect [9].

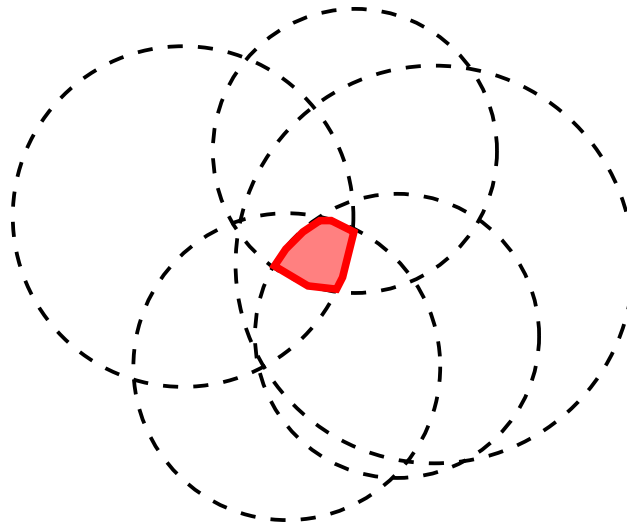


Figure 1. Scheme: $p = 0.99$ -confidence regions of five specimens and the corresponding intersection where the true direction supposed to lie.

2. Basic Ideas: Random Walk Process Model of the Demagnetization Steps

Here, we recall the statistical model that was first presented in Section 2 of [5]. Working with magmatic samples, it is assumed that, at the microscopic level, all magnetic carriers were initially magnetized under the influence of a common magnetic field. Successive measurements during demagnetization change the number of such magnetized parts. Accordingly, a 3D graph (Zijderveld plot) of successive measurements yields a sequence of 3D vectors, each representing an estimate of the initial field direction. Each individual step is certainly inaccurate, so for the statistical stability of the answer, the paleomagnetist, on the one hand, tends to assign a large number of demagnetizations and, on the other hand, tries to make all steps of comparable length. This is partly a consequence of some subjective decisions of the operator. To build a mathematical model of the demagnetization process, let us set that each such intermediate vector (increment vector) is a superposition of two vectors: a vector along a fixed direction (the ancient field) and a random perturbation related to the uncontrolled properties of the sample. The following definitions are quoted from [5]:

We assume each step $\mathbf{S}_i$ in the 3D Zijderveld plots as a systematic step of length δ along the direction of the paleodirection to be recovered, which we arbitrarily assume to be the $(1, 0, 0)$ direction, plus a random unbiased isotropic error of $(\alpha_1, \alpha_2, \alpha_3)_i$, representing the uncertainty with which the corresponding magnetic carriers will have recorded this direction. In other words,

$$\mathbf{S}_i = \mathbf{D} + \mathcal{A}_i = (\delta, 0, 0) + (\alpha_1, \alpha_2, \alpha_3)_i, \quad (2)$$

where each α_j is Gaussian with expectancy $E(\alpha_j) = 0$ and for all $k, j = 1, 2, 3$ components α_k and α_j are uncorrelated with standard deviation $\sigma(\alpha_j) = \sigma$. Note, however, that the relative size of the step δ with respect to σ , $d = \delta/\sigma$, is then the only parameter controlling the directional error at each step in the model. Because we are only interested in recovering directions and their errors, set $\sigma(\alpha_j) = 1$ (and therefore $\delta = d$) and use d and n as the only two parameters controlling the uncertainty directional error.

2.1. The Principal Component Analysis in Application to Zijderveld Plot

Here, we recall the construction in a Cartesian coordinate system according to the description in [2] (see the basic idea in [3]). We start from Zijderveld plot data and then apply versions (named *PC* and *aPC*) of principal component analysis to it.

Let the Zijderfeld plot consist of n data points, each data point k is defined by its Cartesian coordinates (x_{1k}, x_{2k}, x_{3k}) . Compute the "center of mass" of these points, of coordinates $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$, where $\bar{x}_j = \frac{1}{n} \sum_{k=1}^n x_{jk}$, and the new coordinates $(x'_{1k}, x'_{2k}, x'_{3k}) = (x_{1k} - \bar{x}_1, x_{2k} - \bar{x}_2, x_{3k} - \bar{x}_3)$ with respect to this center of mass. The following matrix is known as the *orientation tensor* T [2]

$$T = \begin{pmatrix} \sum x'_{1k}x'_{1k} & \sum x'_{1k}x'_{2k} & \sum x'_{1k}x'_{3k} \\ \sum x'_{2k}x'_{1k} & \sum x'_{2k}x'_{2k} & \sum x'_{2k}x'_{3k} \\ \sum x'_{3k}x'_{1k} & \sum x'_{3k}x'_{2k} & \sum x'_{3k}x'_{3k} \end{pmatrix}, \quad (3)$$

where all sums are from $k = 1$ to n . Additionally, a slightly different version of tensor T reads

$$T = \begin{pmatrix} \sum x_{1k}x_{1k} & \sum x_{1k}x_{2k} & \sum x_{1k}x_{3k} \\ \sum x_{2k}x_{1k} & \sum x_{2k}x_{2k} & \sum x_{2k}x_{3k} \\ \sum x_{3k}x_{1k} & \sum x_{3k}x_{2k} & \sum x_{3k}x_{3k} \end{pmatrix}, \quad (4)$$

we will refer to it as the "anchored" version *aPC* [8,10]. This differs from initial version *PC* due to a special requirement: the best fit to the data points strictly go through the origin of the Zijderfeld plot.

By construction, the matrix T is positive and self-adjoint, and according to the principal components method, the obtained demagnetization direction acts as a normalized eigenvector of this matrix. Because of the Gaussian nature of the three-dimensional random walk model, we have a three-dimensional Gaussian distribution for arbitrary (x_{1k}, x_{2k}, x_{3k}) , and the distribution of eigenvalues is a much more subtle issue, see [5].

2.2. Early Results: 0.95-Confidence Cones

The statistical model described above allowed us in [5] to estimate directional distributions using the Monte Carlo method, and in particular derive the 0.95-quantile angle of deviation from the true direction, Table 1. This angle obviously corresponds to the 0.95 confidence cone describing the statistically estimated accuracy of the demagnetization process for the given parameters d and n . For a small n , this angle is large, but as shown in Figure 1, combining several samples together can greatly reduce the size of the uncertainty region around their common true direction. We aim to use the Monte Carlo method to calculate the necessary quantiles to use in combination. Note that 0.95-quantiles for five specimens can guarantee only an unacceptable level of confidence of 0.77. We conclude that other quantiles are needed for the resulting direction restored in the demagnetization process.

Table 1. Empirical 0.95-quantile angles $Q_{0.95}$ for the principal component (PC), anchored principal component (aPC) expressed in degrees. Results refer to $3 \cdot 10^5$ simulations with $d = 5$ and $d = 10$. Note that empirical results are accurate only to within a few unit changes in the last digit.

n	$Q_{0.95}(PC[5])$	$Q_{0.95}(aPC[5])$	$Q_{0.95}(PC[10])$	$Q_{0.95}(aPC[10])$
3	20.12°	16.89°	9.93°	8.39°
4	16.49°	14.78°	8.21°	7.33°
5	14.43°	13.24°	7.16°	6.63°
6	12.98°	12.15°	6.44°	6.06°
7	11.88°	11.28°	5.92°	5.64°
8	11.08°	10.59°	5.52°	5.30°
9	10.40°	10.03°	5.19°	5.01°
10	9.84°	9.55°	4.90°	4.75°
11	9.39°	9.11°	4.67°	4.53°
12	8.97°	8.72°	4.48°	4.35°
13	8.57°	8.37°	4.29°	4.18°
14	8.26°	8.10°	4.13°	4.05°
15	7.98°	7.82°	3.98°	3.91°
16	7.73°	7.59°	3.86°	3.80°
100	3.08°	3.07°	1.54°	1.54°

3. Methods: Estimation of the New Set of Confidence Cones

The simulation of the n -stage demagnetization process data by Monte Carlo method was implemented as follows:

1. Generation of independent Gaussian increments of the demagnetization process.
2. Reconstruction of the measured magnetization trajectory (random walk with drift). The simulation of the n -stage demagnetization process data points are sequential steps in random walk in $\mathbb{R}^3$. Each step is taken from three-dimensional (symmetric) Gaussian distribution $\mathcal{N}_3(\vec{a}, \mathbf{E})$, with $\vec{a} = (d, 0, 0)$ as the mathematical expectation and unit matrix $\mathbf{E}$). The sum of such steps is again a Gaussian vector with expectation $n \cdot d$, and its central projection onto the unit sphere gives the angular Gaussian distribution or Bingham distribution, since Bingham was the first to explicitly derive the probability density function as an infinite series [11] (see also the implicit formula in [12]).
3. Derivation of two versions of T -matrices: usual PC version Equation (3), and the “anchored” version aPC Equation (4).
4. Computation of the eigenvectors corresponding to the largest eigenvalue of self-adjoint matrices T .

By virtue of the random nature of the distribution of the increments, the coordinates of the eigenvectors in Euclidean coordinates are also characterized by their distribution laws. The directions of these eigenvectors will give the distribution on the sphere of directions, and the goal is to determine the confidence cones with the accuracy required for the practically researched data samples. In contrast to the conclusions of [5], quantile values are now needed for at least the 0.975 and 0.99 confidence levels. The density of the axisymmetric continuous distribution of points on the sphere can be compared with the density of the projections to the axis direction. In particular, the uniform distribution of points on the sphere corresponds to the uniform distribution of points on its diameter. The axisymmetric density on the sphere is thus characterized as a function of the angle of deviation from the axis or as a function of the cosine of this angle, in particular this feature is used in the formula for the Fisher distribution. For the directions on the sphere corresponding to the simulations of the demagnetization process, the explicit density formula obviously depends on parameters d and n , but for small angles of deviation from the axis (or for cosines close to 1) we can speak about asymptotics of density, and it turns out that the density asymptotically coincides with the density of the Fisher distribution density function, see Figure 2. An explicit expression of the asymptotics through the parameters d and n and expressions of the corresponding quantiles will be published later elsewhere, and here we will only outline its derivation.

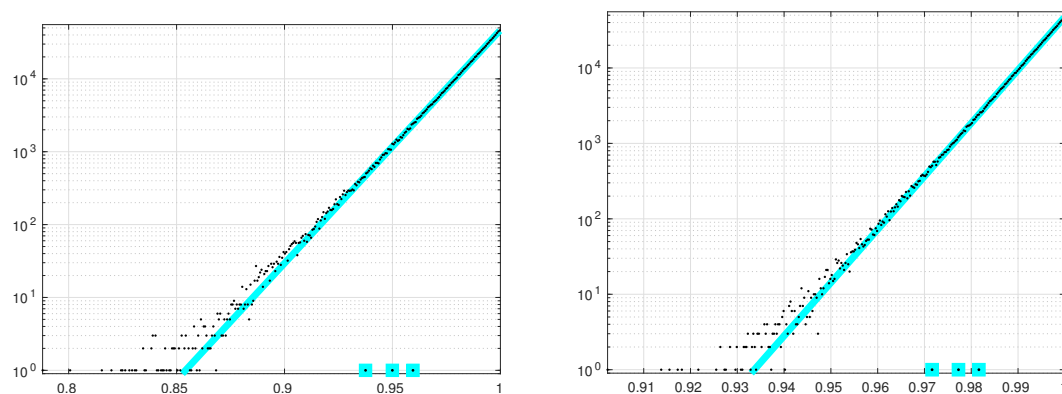


Figure 2. Histogram 10^6 of the cosines of the residual magnetization directions by the PC method for $d = 5$ at $n = 4$ (left) $n = 8$ (right), number of simulations 10^6 . Also shown are the asymptotics (coinciding with the Fisher density $\text{const} \cdot \exp(K \cos \theta)$) and the values of the cosines of the quantile angles $Q_{0.99}(PC[5])$, $Q_{0.975}(PC[5])$ and $Q_{0.95}(PC[5])$.

All the elements of the T -matrices are random variables; in Section 5.1 of [5], we found corresponding explicit analytical expressions (via parameters n and d) for their distribution laws. This allowed us to explicitly estimate the properties of the eigenvalue ratios, in particular, to estimate the values of MAD and aMAD, and to check that the largest eigenvalue is significantly larger than the others. In the present case, we are interested in the coordinates of the eigenvector $\vec{v}$ of unit length $\|\vec{v}\| = 1$ corresponding to the largest eigenvalue of the selfadjoint matrix T . It is clear from the construction that this eigenvector is well approximated by the vector $\vec{v}_0 = (1, 0, 0)$. According to the well-known Arnoldi iteration of selfadjoint matrices, the sequence of vectors of the form

$$\vec{v}_m = \frac{T^m \vec{v}_0}{\|T^m \vec{v}_0\|}$$

converges to $\vec{v}$ rather quickly. A detailed study shows that the resulting directional distribution law $\vec{v}$ is well-approximated when $n \geq 3$ by the Fisher distribution formula (see Equation (1) above). Note that the probability density function for angular Gaussian distributions are also well-approximated by the Fisher distribution formula—see their visual comparison in Figure 2 in [5].

The values of distribution quantiles can be approximated analytically using the asymptotic density formula, but can also be directly estimated from numerical simulations. The second approach to estimate 0.975 and 0.99 quantiles with any precision requires a lot of simulations (a million or more). We therefore calculated the quantiles by both methods and controlled for differences in the responses so as to ensure accuracy to tenths of a degree.

4. Numerical Results for Practical Use

Quantiles and their dependence on the parameters n d are shown in Table 2. The practical value is determined by the availability of information about the demagnetization process (see Section 5 below).

Table 2. Empirical quantile angles Q_p for the principal component (PC) and anchored principal component (aPC) for d equal to 5 and 10 and various n .

n	$Q_{0.95}(PC[10])$	$Q_{0.975}(PC[10])$	$Q_{0.99}(PC[10])$	$Q_{0.95}(aPC[10])$	$Q_{0.975}(aPC[10])$	$Q_{0.99}(aPC[10])$
3	9.9°	11.0°	12.3°	8.4°	9.3°	10.4°
4	8.1°	9.1°	10.1°	7.3°	8.1°	9.1°
5	7.1°	7.9°	8.9°	6.6°	7.3°	8.2°
6	6.4°	7.1°	8.0°	6.0°	6.7°	7.5°
7	5.9°	6.6°	7.3°	5.6°	6.3°	7.0°
8	5.5°	6.1°	6.8°	5.3°	5.9°	6.6°
9	5.2°	5.8°	6.4°	5.0°	5.6°	6.2°
10	4.9°	5.4°	6.1°	4.8°	5.3°	5.9°
11	4.6°	5.2°	5.8°	4.5°	5.0°	5.6°
12	4.4°	5.0°	5.5°	4.3°	4.8°	5.4°
13	4.3°	4.8°	5.3°	4.2°	4.6°	5.2°
14	4.1°	4.6°	5.1°	4.0°	4.5°	5.0°
15	4.0°	4.4°	4.9°	3.9°	4.3°	4.8°
16	3.8°	4.3°	4.8°	3.8°	4.2°	4.7°
n	$Q_{0.95}(PC[5])$	$Q_{0.975}(PC[5])$	$Q_{0.99}(PC[5])$	$Q_{0.95}(aPC[5])$	$Q_{0.975}(aPC[5])$	$Q_{0.99}(aPC[5])$
3	20.1°	22.1°	24.7°	16.9°	18.6°	20.8°
4	16.4°	18.2°	20.4°	14.7°	16.3°	18.2°
5	14.3°	15.9°	17.8°	13.2°	14.7°	16.4°
6	12.9°	14.3°	16.0°	12.1°	13.5°	15.0°
7	11.8°	13.1°	14.7°	11.2°	12.5°	14.0°
8	11.0°	12.3°	13.7°	10.5°	11.7°	13.1°
9	10.4°	11.5°	12.9°	10.0°	11.1°	12.4°
10	9.8°	10.9°	12.2°	9.5°	10.5°	11.8°
11	9.3°	10.4°	11.6°	9.1°	10.1°	11.2°
12	8.9°	9.9°	11.1°	8.7°	9.7°	10.8°
13	8.6°	9.5°	10.6°	8.4°	9.3°	10.4°
14	8.2°	9.2°	10.2°	8.1°	9.0°	10.0°
15	7.9°	8.8°	9.9°	7.8°	8.7°	9.7°
16	7.7°	8.6°	9.6°	7.5°	8.4°	9.4°

For completeness, the results for the value $d = 3$ are shown in Table 3 and Figure 3. The corresponding data are, however, really noisy and therefore unlikely to be of any practical value.

Table 3. Empirical quantile angles Q_p for the principal component (PC) and anchored principal component (aPC) for $d = 3$ and various n .

n	$Q_{0.95}(PC[3])$	$Q_{0.975}(PC[3])$	$Q_{0.99}(PC[3])$	$Q_{0.95}(aPC[3])$	$Q_{0.975}(aPC[3])$	$Q_{0.99}(aPC[3])$
3	30.4°	33.2°	37.2°	25.5°	28.0°	31.3°
4	24.9°	27.4°	30.6°	22.1°	24.4°	27.3°
5	21.6°	23.8°	26.7°	20.0°	22.0°	24.6°
6	19.4°	21.5°	24.1°	18.3°	20.2°	22.6°
7	17.9°	19.8°	22.1°	16.9°	18.8°	21.0°
8	16.6°	18.4°	20.6°	16.0°	17.6°	19.7°
9	15.6°	17.3°	19.3°	15.1°	16.6°	18.6°
10	14.8°	16.4°	18.3°	14.3°	15.8°	17.7°
11	14.0°	15.6°	17.4°	13.7°	15.1°	16.9°
12	13.5°	14.9°	16.6°	13.0°	14.5°	16.2°
13	12.9°	14.3°	16.0°	12.6°	14.0°	15.6°
14	12.4°	13.8°	15.4°	12.1°	13.5°	15.0°
15	12.0°	13.3°	14.8°	11.7°	13.0°	14.6°
16	11.6°	12.8°	14.4°	11.4°	12.6°	14.1°

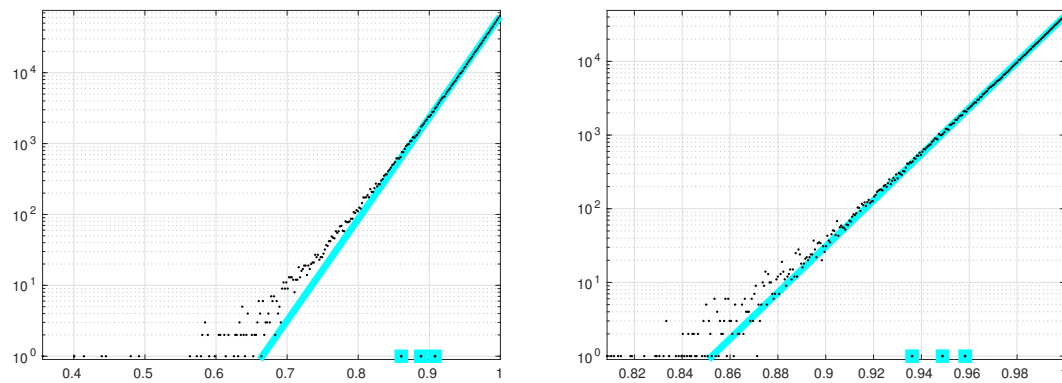


Figure 3. Histogram 10^6 of the cosines of the residual magnetization directions by the PC method for $d = 3$ at $n = 4$ (left) $n = 8$ (right), number of simulations 10^6 . Also shown are the asymptotics (coinciding with the Fisher density $\text{const} \cdot \exp(K \cos \theta)$) and the values of the cosines of the quantile angles $Q_{0.99}(PC[3])$, $Q_{0.975}(PC[3])$ and $Q_{0.95}(PC[3])$.

5. Discussion

Within the framework of modern publication requirements, it is expected that only the information about the value n will be available in the accompanying materials of modern articles. Indeed, the value of the parameter d by definition is statistical by nature, but in the real situation it is hardly possible to perform statistical estimates on the basis of a relatively small amount of data. Past publications of the field directions in igneous rocks indicated only the number of specimens in the sample, but information on the Zijderveld plot for each specimen has not traditionally been provided.

Thus, Table 2 is intended for researchers working with raw data. The traditional software is specified for computing the MAD value for a given specimen, and we may consider an indirect way to derive (approximately) the value of d from MAD (see also Equation (5) below). Let us first recall the main result of [5] (see Table 8 there):

There exist (as a function of number of steps n in a Zijderveld plot) scaling factors C_{MAD} and C_{aMAD} to convert MAD and aMAD angles into α_{95} -confidence angle estimates.

Unfortunately, this scaling transformation is also statistical in nature—both the MAD and the confidence angle of the cone are the result of statistical modeling, and a linear relationship between them exists, but only as a relationship between the rms values of the two random quantities. Hence, knowing the parameter n , the type of method used (PC or

aPC) and value of MAD (or aMAD), it is possible to find the d value using Table 4 and then find an approximately close angle $Q_{0.95}(PC[d])$ (or $Q_{0.95}(aPC[d])$) in Table 2. Thus, one must choose between $d = 5$ or $d = 10$ in order to subsequently detect the corresponding $Q_{0.99}$ or $Q_{0.975}$.

Table 4. Recommended C_{MAD} and C_{aMAD} scaling factors to convert MAD and aMAD angles into α_{95} estimates.

n	C_{MAD}	C_{aMAD}	n	C_{MAD}	C_{aMAD}
3	7.69	6.00	10	2.54	4.14
4	3.90	5.00	11	2.51	4.12
5	3.18	4.63	12	2.48	4.11
6	2.88	4.43	13	2.46	4.08
7	2.71	4.31	14	2.44	4.08
8	2.63	4.24	15	2.43	4.06
9	2.57	4.18	16	2.43	4.05

For cases of very noisy data (that usually correspond to a large MAD and therefore at small values n in the demagnetization process are often rejected), we add here approximations to the leading order in n (see [5]):

$$\text{rms(MAD)} \sim \frac{72.5}{d\sqrt{n}} \quad \text{rms(aMAD)} \sim \frac{44.4}{d\sqrt{n}} \quad (5)$$

where rms angles are expressed in degrees.

We have formulated above how to localize the direction of the ancient magnetic field within the intersection of several confidential cones with a common apex and axes that (generally speaking) do not coincide. According to the method, this localization is by nature probabilistic. Comparing the traditional a95 method with the new process described here, we see in Figure 1 that a possible benefit of the new method may be the size of the confidence region. To statistically test this property, we would need a sufficiently large data collection of lava flows, and process it in two ways:

1. By the traditional a95 method;
2. By a new method proposed here that accounts for the accuracy of individual samples.

For comparison, it is necessary to agree on exactly how the size of the confidence region will be estimated from its shape on the sphere of directions.

Processing a new large collection of lava flows would be a very significant effort. A statistical comparison of the two methods on original samples of erupted rocks is beyond the scope of this article but will be the content of a future publication. We present here only the basic ideas of a comparative statistical analysis.

Area is an obvious parameter for characterizing the size of a region on the surface of a directional sphere. A computational procedure will be required. By the set of confidence cones corresponding to the individual specimens in the sample, one can determine the area corresponding to their intersection on the sphere, this will clarify the compatibility of the specimens in the sample. The numerical implementation of this procedure utilizes a quasi-uniform grid on the surface of the sphere.

The appropriate estimation of the confidence domain for the standard a95 approach is straightforward, even analytically. Note, however, that in the standard approach the question of compatibility with the Fisher distribution is usually solved visually, although a formal test exists. Additionally, the traditional method ignores the accuracies of individual directions and, consequently, in practice, it is plausible that a95 might be applied to incompatible specimens. This must be taken into account when comparing the statistical efficiency of the two methods.

When comparing the new approach with the traditional one, we must also keep in mind the semantic difference in the meaning of the indicators. While the new method indicates the area where the common direction for all specimens lies, the traditional method α_{95} indicates only the area where the Fisher distribution axis falls. Of course, with a small number of specimens, this yields different areas, and therefore information about their intersection is also meaningful when comparing approaches.

6. Conclusions

As mentioned in the discussion above, a detailed statistical comparative study essentially takes us beyond the scope of this short publication and will therefore be published separately. On the other hand, accounting for the accuracy of individual samples in the traditional α_{95} procedure substantially changes the meaning of the answer. Indeed, the traditional method assumes the absolute accuracy of individual directions. Therefore, generally speaking, the new approach serves only as a source of additional knowledge about the configuration of the ancient field, while it does not influence the results of the traditional approach in any way.

The proposed suggestion to utilize the available data from existing publications is far from perfect. For example, because, for lava flows, the detailed information on n and MAD values is usually not available. It is to be hoped that our study will serve as an additional argument for publishing the n and MAD parameters for each sample, thereby opening up a broad opportunity for meaningful improvements in the accuracy of paleomagnetic databases.

Author Contributions: A.K. created the original idea, and developed and implemented the global approach along with all the numerical algorithms. G.G. implemented independent computational methods to control the results. Both authors were involved in the discussion of the results and planning the further implementation of the project, both provided critical feedback about the manuscript. All authors have read and agreed to the published version of the manuscript.

Funding: The first author was financed by the grant № 22-17-00114 of the Russian Science Foundation, <https://rscf.ru/project/22-17-00114/>, accessed on 4 April 2023.

Conflicts of Interest: The authors declare no conflict of interest. The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript; or in the decision to publish the results.

References

1. Hill, M.J.; Gratton, M.N.; Shaw J. A comparison of thermal and microwave palaeomagnetic techniques using lava containing laboratory induced remanence. *Geoph. J. Int.* **2002**, *151*, 157–163. [CrossRef]
2. Tauxe, L.; Banerjee, S.K.; Butler, R.F.; van der Voo, R. *Essentials of Paleomagnetism*, 3rd Web ed.; University of California Press: Berkeley, CA, USA, 2014. Available online: <http://earthref.org/MAGIC/books/Tauxe/Essentials/> (accessed on 4 April 2023).
3. Kirschvink, J.L. The least-squares line and plane and the analysis of paleomagnetic data. *Geophys. J. R. Astr. Soc.* **1980**, *62*, 699–718. [CrossRef]
4. Kent, J.T.; Briden, J.C.; Mardia, K.V. Linear and planar structure in ordered multivariate data as applied to progressive demagnetization of palaeomagnetic remanence. *Geophys. J. R. Astr. Soc.* **1983**, *75*, 593–621. [CrossRef]
5. Khokhlov, A.; Hulot, G. Principal component analysis of palaeomagnetic directions: Converting a Maximum Angular Deviation (MAD) into an α_{95} angle. *Geophys. J. Int.* **2016**, *204*, 274–291. [CrossRef]
6. Fisher, R. Dispersion on a sphere. *Proc. R. Soc. Lond. A* **1953**, *217*, 295–305. [CrossRef]
7. Cogné, J.-P. PaleoMac: A Macintosh (TM) application for treating paleomagnetic data and making plate reconstructions. *Geochem. Geophys. Geosyst.* **2003**, *4*, 1007. [CrossRef]
8. Mazaud, A. User-friendly software for vector analysis of the magnetization of long sediment cores. *Geochem. Geophys. Geosyst.* **2005**, *6*, Q12006. [CrossRef]
9. Chou, R. Small Sample Theory of the Langevin Distribution. *Austral. J. Statist.* **1986**, *28*, 335–344. [CrossRef]
10. Butler, R.F. *Paleomagnetism: Magnetic Domains to Geological Terranes*; Blackwell Scientific Publications: Boston, MA, USA, 1992.

11. Bingham, C. A series expansion for angular gaussian distribution. In *Watson C. Statistics on Spheres*; Wiley-Interscience: Hoboken, NJ, USA, 1983; pp. 226–231.
12. Pukkila, T.M.; Rao, C.R. Pattern recognition based on scale invariant discriminant functions. *Inf. Sci.* **1988**, *45*, 379–389. [[CrossRef](#)]

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.