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The Approximate Solution of the Nonlinear Exact Equation of Deflection of an Elastic Beam with the Galerkin Method

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Abstract: Calculating the large deflection of a cantilever beam is one of the common problems in engineering. The differential equation of a beam under large deformation, or the typical elastica problem, is hard to approximate and solve with the known solutions and techniques in Cartesian coordinates. The exact solutions in elliptic functions are available, but not the explicit expressions in elementary functions in expectation. This paper attempts to solve the nonlinear differential equation of deflection of an elastic beam with the Galerkin method by successfully solving a series of nonlinear algebraic equations as a novel approach. The approximate solution based on the trigonometric function is assumed, and the coefficients of the trigonometric series solution are fitted with Chebyshev polynomials. The numerical results of solving the nonlinear algebraic equations show that the third-order approximate solution is highly consistent with the exact solution of the elliptic function. The effectiveness and advantages of the Galerkin method in solving nonlinear differential equations are further demonstrated.

Keywords: nonlinear; beam; elastica; Galerkin; large; deflection

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1. Introduction

Elastic beams are popular structural elements with broad applications and long history of in-depth study [1–3]. Interestingly, long before the theory of elasticity was established and used, the beam as an elastic element has been extensively studied by many well-known scholars for the flexure and the curve, known as a problem of elastica [4–10]. It is always desirable to obtain the curve and amplitude of the flexure of an elastic beam, particularly under a large loading and deformation, as a typical nonlinear problem. This is a complicated problem in analyzing even a simple structural element like a beam. However, such analyses are also critically needed for other structural elements such as a plate and truss, frequently encountered in engineering and daily life. Without considering the restrictions due to material properties or the strength and constitutive relations, the known nonlinear flexural equation of an elastic beam is hard to solve due to the nonlinear nature of the differential equation and large parameters which exhibit the strong nonlinearity otherwise unheard in most practical problems. The study of elastic deformation and subsequent solution techniques have been a challenging problem for over two hundred years, and there are many solutions techniques developed along the way through the continued study by many outstanding scientists. Galileo, in 1638, posed a fundamental problem as the founding of the mathematical study of elasticity [11], then lately many other researchers worked on the problem, including Leonhard Euler [12], Daniel Bernoulli [13], and many others [14]. In addition to many approximate methods, the problem was eventually solved with the definition of elliptic functions by Bisshopp and Drucker [15]. Since then, the exact solutions have been used to compare and validate many approximate solutions [16–20]. The search for reasonable approximate solutions has never stopped, and there are many attempts for simple and accurate solutions as part of the study of nonlinear differential equations [21–26]. One widely accepted approximate technique is the homotopy analysis method (HAM) by

Liao and his collaborators [27–30], which has been widely adopted for the solution of the elastic problem by many authors [31–34].

Recently, as part of the efforts to solve nonlinear differential equations arising from vibrations and wave propagation of elastic structures and solids with the nonlinear feature, several approximate methods have been examined and revisited for possible utilization [35]. In this process, it has been found that traditional methods for typical solid mechanics problems can also be utilized for the elastic problem. Specifically, it is found that the Galerkin method, which is one of the essential and popular methods in the late development of numerical methods, including the finite element method for problems of solid mechanics and differential equations in general, can be used for nonlinear problems by just solving the resulting nonlinear algebraic equations for a good approximation. The procedure and effectiveness have recently been shown in a few papers [36,37]. Of course, for nonlinear vibrations, the time factor is considered through the weighted integration over one period with the periodic solutions by introducing the extended Galerkin method [35]. The procedure works well with the examples demonstrated [38]. As the equivalent of the Galerkin method, the Rayleigh-Ritz method is also extended to solve some nonlinear vibration problems similarly with promising results [39,40]. Such efforts on these typical problems are essential in demonstrating the method's effectiveness in a detailed procedure for potential applications to other problems. Furthermore, there is great potential that this approximate method and technique will be adopted in other computational procedures, such as finite element analysis, for improvements and innovation.

In this paper, the exact differential equation of a beam is treated with the Galerkin method as a new procedure for approximate solutions. This is a relatively complicated form of beam equation, which has to be simplified with the assumption of small deformation for the Euler-Bernoulli beam equation which is the main subject of beam analysis in textbooks [41]. The treatment of the exact equation, as shown in this study, was also investigated before for exact solutions as part of continuing efforts for accurate approximate solutions. Not surprisingly, the solutions have to be given in special functions precisely for this equation, or the elliptic functions, which are also applicable to similar equations encountered in many other fields. Other than this, the solution to the exact equation is a challenge, and applicable methods and approximate solutions are scarce, even with many efforts, as seen in the literature. Additional efforts to solve such a problem are worth developing new techniques or methods to treat similar or even more challenging problems. Besides, the method detailed here will also be a continuing effort for further improvement and optimization of this procedure based on the Galerkin method.

2. Exact Differential Equation of the Deflection of an Elastic Beam

As appeared in textbooks [42], the exact differential equation and boundary conditions of deflection, also known as flexure, of an elastic beam, as shown with a cantilever beam in Figure 1, is

$$\begin{aligned} \frac{w''(x)}{\sqrt{[1+w'^2(x)]^3}} &= \frac{P}{EI}(l_1 - x), \\ w(0) = w'(0) &= w''(l) = 0, \end{aligned} \tag{1}$$

where w, P, EI, and l_1 are deflection, force, bending stiffness, and horizontal projection of the beam, respectively. As can be found in textbooks [43], a differential equation in this form is hard to solve. In most cases, the exact solutions are not needed because of the significant expense. For approximation, the deflection w also implies an infinitesimal quantity, and the equation is then reduced to the standard Euler-Bernoulli beam equation as

$$EIw'' = M(x), \tag{2}$$

where $M(x)$ is the moment acting on the cross-section of the beam. As it is well known, such an equation of flexure of the beam is relatively easy to solve with elementary functions

for the flexural deformation. There are extensive studies on the solution and properties of the solution from the Euler-Bernoulli beam equation in Equation (2) [44].

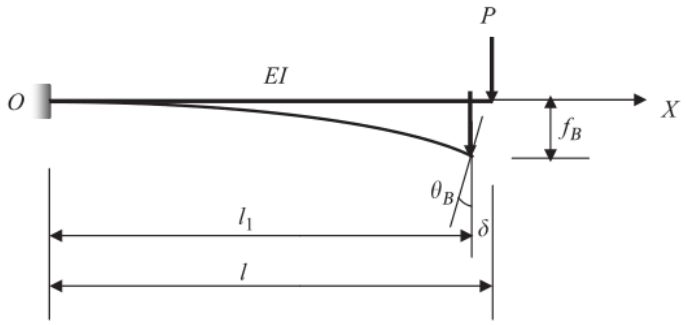


Figure 1. An elastic cantilever beam with essential properties and deformation variables under a point load.

In case of relatively large deformation that the derivative of deflection $w'(x)$ can no longer be neglected, the exact equation in Equation (1) has to be used for the analysis. The solutions of such an equation are hard to obtain, and the exact solution is usually given in elliptic functions, as shown in many papers [15,18]. Since the elliptic equations are well studied and tabulated, the evaluation of deformation from the exact beam equation is not a problem with the rich resources of symbolic mathematical tools and libraries such as Mathematica®, Matlab®, and Maple®, among others. Alternatively, the equation can be solved with many approximate techniques for asymptotic solutions with the Frobenius method [45] and the homotopy analysis method [24,27–30]. Naturally, novel methods and techniques for better approximate solutions are always seeking to refine and improve these existing solutions and methods.

With such an objective in recent dealings with nonlinear differential equations, it is realized that the past efforts on solution techniques have been focused on linear approximation, centering on obtaining the best approximate equations in a linear form and their combinations. The advantage, needless to say, is the easy evaluation of the resulting expressions. However, with the fast improvement of numerical and symbolic computational tools, nonlinear equations are also easy to solve for approximate solutions. This implies that the computational cost can be neglected because fewer terms can be evaluated if proper nonlinear solutions are tried. This is clearly shown in our recent papers dealing with the elastic beam problem with the new method [22,36] by selecting the trial solution satisfying the boundary conditions and the numerical solutions with the undetermined constants like coefficients of a polynomial or trigonometric function. Focusing on the evaluation of the coefficients from nonlinear algebraic equations and the combination of series expansions, the approximate solutions can be obtained from the nonlinear equation to enhance the accuracy with fewer terms. This has been shown in our recent studies of nonlinear problems [36]. This path for better approximation will be taken in this study to demonstrate the new strategy and procedure of solving nonlinear differential equations by solving the nonlinear algebraic equations, in strong contrast to traditional approximate techniques aimed at obtaining much simpler linear equations from continuing linearization process. Elastic beams of other boundary conditions and loadings can also be analyzed with the same principle and procedure outlined below, except that the calculation process can be lengthy and more complicated.

To make the equation more solvable with a simple form, a differentiation of Equation (1) with respect to x will change it to

$$\frac{w'''(x)}{\sqrt{[1+w'^2(x)]^3}} - 3\frac{w''^2(x)w'(x)}{\sqrt{[1+w'^2(x)]^5}} = -\frac{P}{EI}. \tag{3}$$

With a substitution of

$$\frac{dw(x)}{dx} = g(x), \tag{4}$$

Equation (3) will be changed to a new form as

$$\frac{g''(x)}{\sqrt{[1+g^2(x)]^3}} - 3\frac{g'^2(x)g(x)}{\sqrt{[1+g^2(x)]^5}} = -\frac{P}{EI}. \tag{5}$$

By further introducing a dimensionless variable $\xi = x/l$, Equation (5) will be changed to

$$\frac{g''(\xi)}{\sqrt{[1+g^2(\xi)]^3}} - 3\frac{g'^2(\xi)g(\xi)}{\sqrt{[1+g^2(\xi)]^5}} + \alpha = 0, \tag{6}$$

where $\alpha = \frac{Pl^2}{EI}$ is a parameter used in the differential equation of an elastica with the primary variable of the rotation of the cross-section [46].

Now let the solution of Equation (6) is a trigonometric series in the form of

$$g(\xi) = \sum_{n=1}^{\infty} A_n \sin n\xi, \tag{7}$$

satisfying boundary conditions $g(0) = 0$ and $g'(1) = 0$. Apparently, by letting the series as

$$g(\xi) = \sum_{n=1}^{N} A_n \sin \frac{(2n-1)\pi}{2}\xi, n = 1,\ 2,\ 3, \cdots \tag{8}$$

it will be a possible combination for the optimal solution with fewer terms.

With the solution assumption of Equation (8), applying the Galerkin method to Equation (6) as

$$\int_0^1 \left\{ \frac{g''(\xi)}{\sqrt{[1+g^2(\xi)]^3}} - 3\frac{g'^2(\xi)g(\xi)}{\sqrt{[1+g^2(\xi)]^5}} + \alpha \right\} \delta g dz = 0 \tag{9}$$

with the approximate expansion of $\frac{1}{\sqrt{[1+g^2(\xi)]^3}} \approx 1$, $\frac{1}{\sqrt{[1+g^2(\xi)]^5}} \approx 1 - \frac{5}{2}g^2(\xi) + \frac{35}{8}g^4(\xi)$ and Equation (8), Equation (9) will be written as

$$\begin{aligned}
&\int_0^1 \Bigg\{ \alpha + \Bigg\{ \left[\sum_{n=1}^{N} -\frac{A_n(2n-1)^2\pi^2}{4} \sin \frac{(2n-1)\pi}{2}\xi \right] \\
&\quad -3\left[\sum_{n=1}^{N} \frac{A_n(2n-1)\pi}{2} \cos\left(\frac{(2n-1)\pi}{2}\xi \right) \right]^2 \\
&\quad \times \left[\sum_{n=1}^{N} A_n \sin \frac{(2n-1)\pi}{2}\xi \right] \\
&\quad \times \Bigg\{ 1 - \frac{5}{2}\left[\sum_{n=1}^{N} A_n \sin \frac{(2n-1)\pi}{2}\xi \right]^2 \\
&\quad + \frac{35}{8}\left[\sum_{n=1}^{N} A_n \sin \frac{(2n-1)\pi}{2}\xi \right]^4 \Bigg\} \Bigg\} \Bigg\} \sin \frac{\pi}{2}\xi d\xi = 0.\ N = 1.
\end{aligned} \tag{10}$$

As demonstrated in an earlier paper, the approximate solution is obtained with the determination of coefficients A_1 in Equation (10) [36].

From here, there are two approaches for the determination of coefficients. The simple one assumes there is only one solution by the first term of the series; then, there is only one unknown or the coefficient A_1. Consequently, there is only one equation for the unknown, and it will be a nonlinear algebraic equation. By taking this procedure, with only one

equation for the coefficient from Equation (8), it is easily obtained that the coefficient A_1 is the solution from the nonlinear equation

$$\frac{2\,\alpha}{\pi} - \frac{A_1\left(512 + 384\,A_1^2 - 480\,A_1^4 + 525\,A_1^6\right)\pi}{4096} = 0. \tag{11}$$

The equation above has no analytical solution with parameter α, and the implicit solution can be obtained with the aid of a Chebyshev series using symbolic mathematical software such as Matlab® or Mathematica®,

$$\begin{gathered} A_n(\alpha) = \sum_{k=0}^{\infty} C_k T_k\left(\tfrac{\alpha}{5} - 1\right), \\ C_k = \tfrac{2}{5\pi}\int_0^{10} A_n(\alpha)\frac{T_k\left(\frac{\alpha}{5}-1\right)}{\sqrt{1-\left(\frac{\alpha}{5}-1\right)^2}}d\alpha,\ (k = 1,\ 2,\ 3,\ \cdots,\infty), \\ C_0 = \tfrac{1}{5\pi}\int_0^{10} A_0(\alpha)\frac{T_0\left(\frac{\alpha}{5}-1\right)}{\sqrt{1-\left(\frac{\alpha}{5}-1\right)^2}}d\alpha. \end{gathered} \tag{12}$$

By evaluating with symbolic software tools, it is easily obtained that the coefficient as a polynomial

$$\begin{aligned} A_1 \quad &= -0.00045679290254130 \\ &+0.5457933917923579\alpha - 0.14674177515953207\alpha^2 \\ &+0.13351098076023404\alpha^3 - 0.11194284067204267\alpha^4 \\ &+0.051839004053003124\alpha^5 - 0.01485367508499742\alpha^6 \\ &+0.0028596124004239727\alpha^7 - 0.0003868993474699787\alpha^8 \\ &+0.00003761922105859693\alpha^9 - 0.000002645845880748856\alpha^{10} \\ &+1.335599198474874 \times 10^{-7}\alpha^{11} - 4.719951997798395 \times 10^{-9}\alpha^{12} \\ &+1.108925046190397 \times 10^{-10}\alpha^{13} - 1.556059378808374 \times 10^{-12}\alpha^{14} \\ &+9.868406118277864 \times 10^{-15}\alpha^{15}. \end{aligned} \tag{13}$$

Then, the rotation angle of the beam is obtained and should be convergent for a reasonable parameter α. Since the function $g(z)$ is defined as

$$g(\xi) = \frac{dw(\xi)}{dx} = \frac{dw(\xi)}{d\xi}\frac{d\xi}{dx} = \frac{1}{l}\frac{dw(\xi)}{d\xi}, \tag{14}$$

Letting the normalized deflection $\frac{w(l)}{l} = \frac{f_B}{l}$, with the consideration of boundary condition $w(0) = 0$, the deflection of the end of the free tip of the cantilever beam in Figure 1 is

$$\frac{f_B}{l} = \int_0^1 A_1 \sin\frac{\pi}{2}\xi d\xi = \frac{2}{\pi}A_1. \tag{15}$$

By substituting Equation (13) into Equation (15), the calculated deflection of the end of the beam is compared with the exact solution in Figure 2. Clearly, the first-order approximation with only one term can obtain the coefficients of a polynomial in the parameter of the beam and the accuracy is surprisingly good as compared with other approximate techniques which may require extensive numerical calculations. The procedure above is not widely used before for the nonlinear solution of coefficients, but the advantage of such an approach is remarkable and straightforward. It is recommendable to use this procedure for similar nonlinear differential equations for simple and accurate solutions. Remarkably, the coefficients as a function of parameters are an approach that can significantly optimize the solution and procedure. Such results from the solutions of nonlinear algebraic equations are novel, and the computational cost is much lower than other techniques.

Next, for the second-order approximation, the solution is now expanded as

$$g(\xi) = A_1 \sin\frac{\pi}{2}\xi + A_2 \sin\frac{3\pi}{2}\xi. \tag{16}$$

Again, by applying the Galerkin method in Equation (9) with $\frac{1}{\sqrt{[1+g^2(\xi)]^3}} \approx 1$, $\frac{1}{\sqrt{[1+g^2(\xi)]^5}} \approx 1 - \frac{5}{2}g^2(\xi) + \frac{35}{8}g^4(\xi)$ and Equation (8), Equation (9) will be rewritten as

$$\begin{aligned}
&\int_0^1 \Bigg\{ \alpha + \Bigg\{ \Bigg[\sum_{n=1}^{N} -\frac{A_n(2n-1)^2\pi^2}{4} \sin\frac{(2n-1)\pi}{2}\xi \Bigg] \\
&\quad -3\Bigg[\sum_{n=1}^{N} \frac{A_n(2n-1)\pi}{2} \cos\Big(\frac{(2n-1)\pi}{2}\xi\Big) \Bigg]^2 \\
&\quad \times \Bigg[\sum_{n=1}^{N} A_n \sin\frac{(2n-1)\pi}{2}\xi \Bigg] \\
&\quad \times \Bigg\{ 1 - \frac{5}{2}\Bigg[\sum_{n=1}^{N} A_n \sin\frac{(2n-1)\pi}{2}\xi \Bigg]^2 \\
&\quad + \frac{35}{8}\Bigg[\sum_{n=1}^{N} A_n \sin\frac{(2n-1)\pi}{2}\xi \Bigg]^4 \Bigg\} \Bigg\} \Bigg\} \sin\tfrac{3\pi}{2}\xi d\xi = 0.\ N = 2.
\end{aligned} \tag{17}$$

with the known coefficient A_1 in Equation (13). For simplicity, the linear equation for coefficient A_2 is obtained from Equation (17) after the integration and neglecting the high-order terms of A_2,

$$\frac{2a}{3\pi} - \frac{3\big(128A_1^3 - 80A_1^5 + 35A_1^7 + 1536A_2 + 256A_1^2A_2 - 480A_1^4A_2 + 735A_1^6A_2\big)\pi^2}{4096} = 0. \tag{18}$$

with the known solution of A_1, A_2 is expressed in the beam parameter after performing the procedure in Equation (12) as

$$\begin{aligned}
A_2 \quad &= 0.0000010872452629422 + 0.018955820413029408\alpha \\
&+0.0016728394076443503\alpha^2 - 0.01788402739977677\alpha^3 \\
&+0.0027846152929407\alpha^4 + 0.03432400332974922\alpha^5 \\
&-0.06708458935755224\alpha^6 + 0.07064031279329828\alpha^7 \\
&-0.04889186796499801\alpha^8 + 0.023910157311563908\alpha^9 \\
&-0.008601263799217123\alpha^{10} + 0.0023329168414211857\alpha^{11} \\
&-0.0004840398532387501\alpha^{12} + 0.00007732755268991636\alpha^{13} \\
&-0.000009504836042077459\alpha^{14} + 8.913665344962865 \times 10^{-7}\alpha^{15} \\
&-6.263562474103492 \times 10^{-8}\alpha^{16} + 3.193023389737545 \times 10^{-9}\alpha^{17} \\
&-1.115113423717088 \times 10^{-10}\alpha^{18} + 2.386379163849583 \times 10^{-12}\alpha^{19} \\
&-2.359922673050108 \times 10^{-14}\alpha^{20}.
\end{aligned} \tag{19}$$

Now the deflection at the end of the cantilever beam is

$$\frac{f_B}{l} = \int_0^1 \Big(A_1 \sin\frac{\pi}{2}\xi + A_2 \sin\frac{3\pi}{2}\xi \Big) d\xi = \frac{2A_1}{\pi} + \frac{2A_2}{3\pi}, \tag{20}$$

and the comparison of solutions is shown in Figure 3.

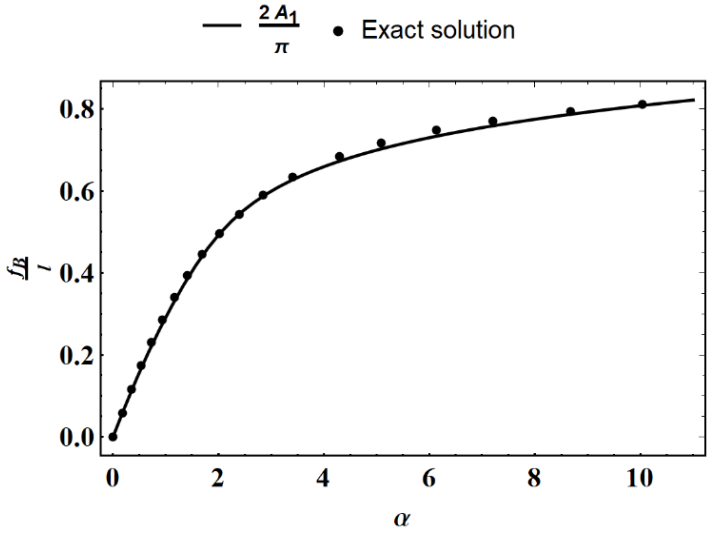


Figure 2. A comparison of deflection of the first-order approximation and exact solution of deflection at the end of the beam.

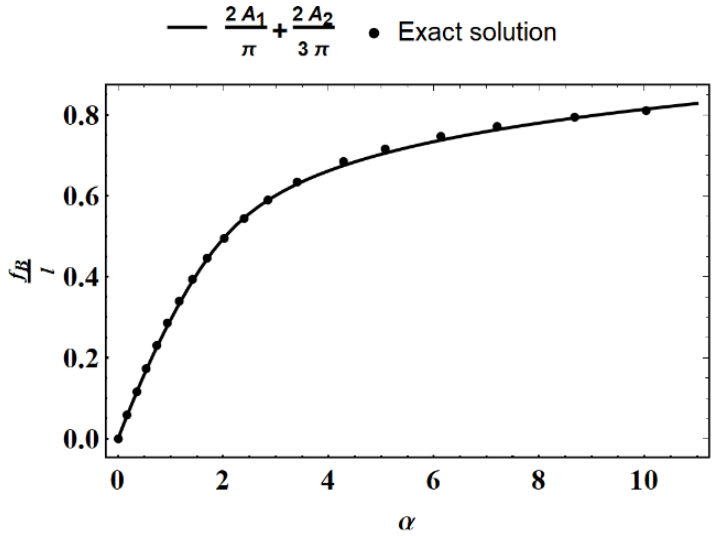


Figure 3. A comparison of deflection of the second-order approximation and exact solution of deflection at the end of the beam.

The accuracy of the solution with the second-order approximation is improved slightly, and the advantage of the solution procedure is shown. Naturally, higher-order solutions can be obtained in such a systematic procedure.

For the third-order approximation, the solution is assumed as

$$g(\xi) = A_1 \sin\frac{\pi}{2}\xi + A_2 \sin\frac{3\pi}{2}\xi + A_3 \sin\frac{5\pi}{2}\xi. \tag{21}$$

By following the procedure before, applying the Galerkin method in Equation (9) with $\frac{1}{\sqrt{[1+g^2(\xi)]^3}} \approx 1 - \frac{3}{2}g^2(\xi) + \frac{15}{8}g^4(\xi)$, $\frac{1}{\sqrt{[1+g^2(\xi)]^5}} \approx 1 - \frac{5}{2}g^2(\xi) + \frac{35}{8}g^4(\xi)$ and Equation (8), Equation (9) will be taken a new form as

$$\begin{aligned}
&\int_0^1 \Bigg\{ \alpha + \Bigg\{ \Bigg[\sum_{n=1}^{N} -\frac{A_n(2n-1)^2\pi^2}{4} \sin\frac{(2n-1)\pi}{2}\xi \Bigg] \\
&-3\Bigg[\sum_{n=1}^{N} \frac{A_n(2n-1)\pi}{2} \cos\Big(\frac{(2n-1)\pi}{2}\xi\Big) \Bigg]^2 \\
&\times \Bigg[\sum_{n=1}^{N} A_n \sin\frac{(2n-1)\pi}{2}\xi \Bigg] \\
&\times \Bigg\{ 1 - \frac{5}{2}\Bigg[\sum_{n=1}^{N} A_n \sin\frac{(2n-1)\pi}{2}\xi \Bigg]^2 \\
&+ \frac{35}{8}\Bigg[\sum_{n=1}^{N} A_n \sin\frac{(2n-1)\pi}{2}\xi \Bigg]^4 \Bigg\} \Bigg\} \Bigg\} \sin\frac{5\pi}{2}\xi d\xi = 0.\ N = 3.
\end{aligned} \tag{22}$$

Furthermore, with known coefficients A_1 and A_2, the solution of coefficient A_3 can be obtained from Equation (22) after the integration and neglecting the nonlinear terms of A_3 is

$$\begin{aligned}
&\frac{2a}{5\pi} - \frac{75\pi^2 A_1^5}{1024} + \frac{315\pi^2 A_1^7}{4096} - \frac{75\pi^2 A_1^2 A_2}{64} + \frac{375\pi^2 A_1^4 A_2}{256} \\
&-\frac{2625\pi^2 A_1^6 A_2}{4096} + \frac{75\pi^2 A_1 A_2^2}{64} - \frac{1125\pi^2 A_1^3 A_2^2}{512} \\
&+\frac{315\pi^2 A_1^5 A_2^2}{1024} + \frac{1125\pi^2 A_1^2 A_2^3}{512} + \frac{6825\pi^2 A_1^4 A_2^3}{4096} \\
&-\frac{375\pi^2 A_1 A_2^4}{256} - \frac{3675\pi^2 A_1^3 A_2^4}{1024} + \frac{5775\pi^2 A_1^2 A_2^5}{2048} - \frac{7875\pi^2 A_1 A_2^6}{4096} \\
&+\mathrm{A}_3\Bigg(-\frac{25\pi^2}{8} + \frac{75\pi^2 A_1^2}{32} - \frac{1125\pi^2 A_1^4}{512} - \frac{525\pi^2 A_1^6}{1024} \\
&+\frac{375\pi^2 A_1^3 A_2}{128} + \frac{3675\pi^2 A_1^5 A_2}{1024} + \frac{75\pi^2 A_2^2}{32} \\
&-\frac{1125\pi^2 A_1^2 A_2^2}{128} - \frac{75075\pi^2 A_1^4 A_2^2}{4096} + \frac{375\pi^2 A_1 A_2^3}{256} \\
&+\frac{23625\pi^2 A_1^3 A_2^3}{1024} - \frac{1125\pi^2 A_2^4}{512} - \frac{129675\pi^2 A_1^2 A_2^4}{4096} \\
&+\frac{12075\pi^2 A_1 A_2^5}{2048} - \frac{4725\pi^2 A_2^6}{1024} \Bigg) = 0.
\end{aligned} \tag{23}$$

with the known solutions of A_1 and A_2, A_3 as a polynomial is obtained from the evaluation procedure in Equation (12), as

$$\begin{aligned}
A_3 = \ & 1.786990256908465 \times 10^{-7} + 0.0041163549103228475\alpha \\
&-0.00013491433751838984\alpha^2 \\
&+0.0016068782454554346\alpha^3 - 0.013505218112915465\alpha^4 \\
&+0.03545582563751064\alpha^5 - 0.0523877606921864\alpha^6 \\
&+0.04975775899223047\alpha^7 - 0.03251839878058734\alpha^8 \\
&+0.01530721628567603\alpha^9 - 0.0053555166782833325\alpha^{10} \\
&+0.0014218503130418087\alpha^{11} - 0.00029001852879782645\alpha^{12} \\
&+0.00004568762823808967\alpha^{13} - 0.000005555022633345154\alpha^{14} \\
&+5.153148159172351 \times 10^{-7}\alpha^{15} - 3.589826452006164 \times 10^{-8}\alpha^{16} \\
&+1.8161755593466 \times 10^{-9}\alpha^{17} - 6.300272696679216 \times 10^{-11}\alpha^{18} \\
&+1.340229661870947 \times 10^{-12}\alpha^{19} \\
&-1.31825942377728 \times 10^{-14}\alpha^{20}.
\end{aligned} \tag{24}$$

Again, the deflection at the free end is

$$\frac{f_B}{l} = \int_0^1 A_1 \sin\frac{\pi}{2}\xi + A_2 \sin\frac{3\pi}{2}\xi + A_3 \sin\frac{5\pi}{2}\xi d\xi = \frac{2A_1}{\pi} + \frac{2A_2}{3\pi} + \frac{2A_3}{5\pi}. \tag{25}$$

Finally, the comparison of solutions is shown in Figure 4.

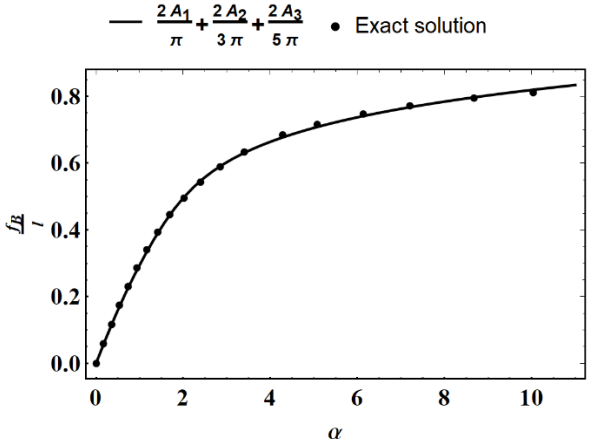


Figure 4. A comparison of deflection of the third-order approximation and exact solution of deflection at the end of the beam.

3. Conclusions and Discussion

Applying the popular Galerkin method to the nonlinear differential equation of an elastica, a nonlinear algebraic equation for the coefficients of asymptotic solutions is obtained and solved with powerful symbolic mathematical software tools, and accurate solutions are obtained in a systematic manner. In practice, the Galerkin method is rarely used with nonlinear differential equations due to difficulties in solving the resulting nonlinear algebraic equations, although the linear equations are frequently treated with the method as shown in textbooks. It is not surprising that the Galerkin method has led to the emergence of the most popular and influential method in dealing with approximate numerical solutions of differential equations with possible applications in the finite element method. The nonlinear differential equations, as widely encountered in the analysis of physical and natural processes and phenomena today, can also be solved with the finite element method through approximate techniques such as the popular linearization of equations. Such solutions are generally acceptable, and the method is widely accepted and used for practical problems. On the other hand, asymptotic solutions with a series and undetermined coefficients are the primary technique in dealing with solutions of differential equations. It should also be used for the solutions of nonlinear differential equations with the conservation of nonlinear features of the equation. The procedure should be much more straightforward and less expensive in computing the coefficients for comparisons with other methods, as demonstrated in a series of recent papers by our earlier work. These solutions and techniques, as it is shown, are obtained from some problems and difficult nonlinear differential equations in structural analysis and have been studied by many pioneers in mathematics and mechanics for more straightforward solutions with various solution techniques before. The procedure can be used in an iterative process to improve the solution accuracy through the increase of the number of terms of linearly independent solutions. The simultaneous solutions from resulting nonlinear algebraic equations can also be accurate and obtainable, but the solution procedure of a system of nonlinear algebraic equations can be much more time-consuming and resources demanding. For typical nonlinear equations, the iterative procedure is recommended for asymptotic solutions, and the procedure based on the finite element method is also suggested for future study. In this study, a new solution procedure is established for the more straightforward solutions of nonlinear differential equations to enrich the toolbox of approximate methods and techniques based on the Galerkin method,

which is not widely utilized for nonlinear problems. Furthermore, this study leads to the emergence of nonlinear eigenvalue problems associated with nonlinear differential equations as a significant challenge for the attention of efficient algorithms.

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