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Mathematical Models of Diagnostic Information Granules Generated by Scaling Intuitionistic Fuzzy Sets

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Abstract: The paper presents a certain class of the mathematical models of diagnostic information granules describing the fuzzy symptoms-faults relationship. A certain fuzzy diagnostic information retrieval system is described as an application of an expert diagnostic system. Symptoms and faults are fuzzy, and with some scaling of the symptom-fault concept pair values. These value pairs can be considered as intuitionistic fuzzy sets for the space of diagnosed objects. In this article, for scaling intuitionistic fuzzy sets (n-ScIFS), the deductive theory is formulated. There the intuitionistic fuzzy sets (IFSs) and the Pythagorean fuzzy sets (PFSs) are generalized to the n-ScIFS objects. The membership and non-membership values, as standard, can be described by the 1:1 scale or the quadratic function scale. However, any power scale $n > 2$ can be used. In this paper, any n-Sc scaling functions retaining the IFSs properties are considered. The n-ScIFS theory introduces a conceptual apparatus analogous to the classical theory of Zadeh fuzzy sets and Yager PFSs, consistently striving, for the first time, to formulate the relational structure of n-ScIFSs as a model of a certain information granule system called here the diagnostic granule system. In addition, power- and linear-repeatable diagnostic granules are defined in the n-ScIFSs structure for serial or parallel diagnosis processes. The information granule base is determined and a diagnostic granule system model produced by this information granule base is shown. Certain algorithms have been given to establish the semantic language of diagnosis describing the system of diagnostic information granules.

Keywords: information granule database; intuitionistic fuzzy set; information granules; diagnostic information; n-scaling function; granule language

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1. Introduction

The fuzzy sets introduced by Zadeh [1] are widely used in many different disciplines of science. They can be found in many solutions in engineering, management, mathematics, statistics, social sciences, cybersecurity, medicine, and others. Using fuzzy set theory is also an important issue in medical and technical diagnostics. There are several some important fields and problems of diagnostics where fuzzy sets are used:

1. dynamic diagnosis: the issues of dynamic diagnosis, especially fault detection and isolation, are presented in many papers, books, publications, and monographs [2–11]. Commonly, when the complex industrial plants are diagnosed, there is a necessity to decompose this process into some smaller units. Then, these units can be simultaneously diagnosed [12,13];
2. static diagnosis: the problems of static diagnosis concern signal analysis [14]; isolation of faults and the fault-symptom relations [15]; application of fuzzy logic in diagnostics [16]; using the diagnostic matrix to generalized reasoning about faults [17]; and diagnostic test using fault-symptom relations [18];
3. mathematical models of information granules are being built [19,20]. They are models describing aggregated fuzzy diagnostic data [21,22] in the symptom-faults matrix. This type of information granules are also intuitionistic fuzzy sets presented in [23,24], as applications in management and engineering. The expert diagnostic system is

described as one that performs calculations on numerical data representing fuzzy diagnostic information. The application of this system are the procedures for searching for diagnostic information specified in the *diagnosis granule systems*. Thus, this paper presents some mathematical model of diagnostic information granules of the symptoms-faults relationship. Symptoms and faults are fuzzy. When some scale is used for values of the symptom-fault concept pair, then they can be considered as intuitionistic fuzzy sets in the space of diagnosed objects. In this purpose, the deductive theory of scaling intuitionistic fuzzy sets (*n-ScIFS*) is formulated.

Generally, diagnostic examines the symptoms of threats in obtaining information about the expected behavior of technical, medical, information-obtaining objects (in the web, in cyberspace), and others. One of the main goals of diagnostics is therefore to indicate the occurrence of threats f for the s states of the expected behavior of objects, i.e., to indicate their limitations and selection by occurrence of faults, defects, damage, or deficiencies that threaten the occurrence of the expected behavior of objects. The states s of the expected behavior of objects can be endangered in a real object and recognized on the basis of diagnostic signals generated by this object. Then these conditions are called *threat symptoms*. Of course, other external influences may also occur, however to simplify, they will be ignored in this study.

The feature of the state or symptom of an object is a component of the instantaneous state of the object, related to the property of the examined object. The *symptom* is some feature that characterize an object and the state of this object, which occur only when the object is not fully fit or damaged. Furthermore, the diagnostic symptom s defines the measure of the diagnostic signal and is called the signal vector component. This signal represents the specific type of fault or threat f .

The *parameter* is the feature that determines the condition of the examined object. Moreover, the diagnostic parameter indicates some diagnosed object, the observable description of this object, by some diagnostic signal or process. This signal indirectly represents some value of the state of the object.

In order to make a decision in the diagnostic system, the diagnostic inference is used. The response of this system, which is a decision, is based on the diagnostic symptom-state relationship. Symptoms are conditional attributes, while faults are decision attributes. Let the set $S = \{s_1, s_2, \dots, s_l, \dots, s_k\}$ be the set of symptoms attributes and let $F = \{f_1, f_2, \dots, f_l, \dots, f_n\}$ be the set of faults attributes.

Let the set $X = O \times S \times F$ be called the *diagnostic range* of objects o with the symptom s_i and faults f_j attributes. Some data about these objects are *fuzzy*, then the *diagnosis* of the object $x \in X$ with the attributes s_i, f_j , with some scaling, corresponds to the pair (μ_i, v_j) of fuzzy sets $\mu_i : X \rightarrow [0, 1], v_j : X \rightarrow [0, 1]$. Observations of symptoms and faults are disjoint sets. The values of functions μ_i and v_j satisfy the condition:

$$\mu_i(o) + v_j(o) \leq 1. \quad (1)$$

This means that the pair of fuzzy sets (μ_i, v_j) is the intuitionistic fuzzy set (IFSs) in the Atanassov sense [25].

In the Atanassov study [25], the intuitionistic fuzzy sets (IFSs) consider that there is the value μ of the membership function and additionally there is the value v of the non-membership function. In this research, the following constraint is taken into account: $0 \leq \mu + v \leq 1$. Moreover, in the Yager study [26,27], the Pythagorean fuzzy sets (PFSs) model is proposed in order to handle imprecise information. If we compare this model to the Atanassov one, the IFSs condition is relaxed and has the form: $0 \leq \mu^2 + v^2 \leq 1$. In the literature, the research of Zhang and Xu [28] can be found. They established the model of the Pythagorean fuzzy number (PFN). Furthermore, Garg's research [29,30] shows the applications of PFSs in decision-making problems.

The membership and non-membership values are usually measured for some **scale function**. Until now, they have been described on the scale 1:1 [31] or on the quadratic

function scale of PFSs [32]. In [33], scales for the proven power $n > 2$ are used. In [24], the function scale is the power function $Sc(x) = x^a, a \in (1, +\infty)$.

In addition, in the IFS articles [31,34–36] and those related to PFSs [26,27,32,37–40], a number of mathematical and logical inaccuracies are corrected and presented in [33]. The paper [33] presented the elements of the deductive theory of n-PFS, which determines the membership degree μ and the non-membership degree v not only for the square scale, but also for any power scale, i.e., $0 \leq \mu^n + v^n \leq 1$, for some $\mu, v \in [0, 1]$, if $\mu + v > 1$. This makes it possible to formulate any local deductions in terms of n-PFS.

In this context, the motivation for this paper was the use of the described model to create analogous models, however based on any selected functional scales, e.g.,

- The map scale $f_n(x) = x/n : 0 \leq \mu/n + v/n \leq 1$,
- Or the sequence of logarithmic scale functions $f_n(x) = 1 - \log_{(1-1/n)} x$, for $n > 1, x \in (0, 1]$, where $0 \leq \log_{(1-1/n)} \mu + \log_{(1-1/n)} v \leq 1$.

However, two research problems arose:

1. How to determine the sequences $\{Sc_n\}, \{Sc'_n\}$ of scaling functions, such that from a certain number n there is $0 \leq Sc_n(\mu) + Sc'_n(v) \leq 1$, for any results of diagnostic measurements $\mu, v \in [0, 1]$,
2. How to determine the number of n sequences $\{Sc_n\}, \{Sc'_n\}$ satisfying the condition 1.

In this paper, the solution to problems 1 and 2 is presented for the first time.

Furthermore, the conceptual apparatus describing n -PFSs has been generalized and clarified [33]. Furthermore, for the sequence $\{Sc_n\}$ of some scaling functions, for any natural number $n > 0$, deductive theories of n -scaling intuitionistic fuzzy sets, with the condition $0 \leq Sc_n(\mu) + Sc_n(v)' \leq 1$, are formulated. The constructions of these theories are presented below. Their formal description is called the *diagnostic granule system*.

The following are presented in the coming sections: the norm systems, the n -scaling intuitionistic fuzzy sets, Yager aggregation operators, and the triangular norms in n -ScIFN and n -ScIFN algebra. By intuition, it is assumed that this system can interpret a diagnostic descriptive language that describes the symptoms-faults properties. In the fifth chapter of this paper, it is shown that the n -scaling intuitionistic fuzzy sets theory is a model of the diagnostic granule system and that this system differs from the information granule system [20] in terms of including linear and power repetition operations on granules. The final chapter of this paper presents a simple example of a standard diagnostic granule system.

2. The Norm Systems

2.1. The Triangular Norms

Triangular norms are used to construct operations on fuzzy sets and Yager [32] fuzzy sets.

Definition 1. The triangular norm or the ***t*-norm** in $[0, 1]$ is the operation $\bullet_t : [0, 1] \times [0, 1] \rightarrow [0, 1]$, when for any numbers $x, y, z \in [0, 1]$ meets the following conditions:

- $0 \bullet_t y = 0, y \bullet_t 1 = y$ —boundary conditions,
- $x \bullet_t y \leq z \bullet_t y$, when $x \leq z$ —monotonicity,
- $x \bullet_t y = y \bullet_t x$ —commutativity,
- $x \bullet_t (y \bullet_t z) = (x \bullet_t y) \bullet_t z$ —associativity.

The triangular t -conorm or the ***s*-norm** in $[0, 1]$ is the operation $\bullet_s : [0, 1] \times [0, 1] \rightarrow [0, 1]$, when for any numbers $x, y, z \in [0, 1]$ it meets the following conditions:

- $0 \bullet_s y = y, y \bullet_s 1 = 1$ —boundary conditions,
- $x \bullet_s y \leq z \bullet_s y$, when $x \leq z$ —monotonicity,
- $x \bullet_s y = y \bullet_s x$ —commutativity,
- $x \bullet_s (y \bullet_s z) = (x \bullet_s y) \bullet_s z$ —associativity.

Theorem 1. Operation $\bullet : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is the s-norm if it is defined by the formula $x \bullet y = 1 - (1 - x) \bullet_t (1 - y)$, for any numbers $x, y \in [0, 1]$.

Example 1. The operation defined by the formula $x \bullet_t y = \min\{x, y\}$ for any $x, y \in [0, 1]$ is a t-norm.

The operation defined by the formula $x \bullet_s y = \max\{x, y\}$ for any $x, y \in [0, 1]$ is the s-norm.

Further in this paper, we consider only the continuous t-norms and s-norms. The general construction of triangular norms and the functional equations [41] leads to the Theorem:

Theorem 2. For each strictly decreasing continuous function $f_t : [0, 1] \rightarrow [0, +\infty)$ such that $f_t(1) = 0$ and for any $x, y \in [0, 1]$, the t-norm is the operation determined by the formula:

$$x \bullet_t y = \begin{cases} f_t^{-1}[f_t(x) + f_t(y)], & \text{for } f_t(x) + f_t(y) \in [0, f_t(0)], \\ 0, & \text{otherwise.} \end{cases} \quad (2)$$

For each strictly increasing continuous function $f_s : [0, 1] \rightarrow [0, +\infty)$ such that $f_s(0) = 0$ and for any $x, y \in [0, 1]$, the s-norm is the operation determined by the formula:

$$x \bullet_s y = \begin{cases} f_s^{-1}[f_s(x) + f_s(y)], & \text{for } f_s(x) + f_s(y) \in [0, f_s(1)], \\ 1, & \text{otherwise.} \end{cases} \quad (3)$$

Then the f_t, f_s functions are called t-norm and s-norm **generators**, respectively.

When the functions f_t, f_s satisfy the condition: $f_s(x) = 1 - f_t(1 - x)$ for every $x \in [0, 1]$, then they are called dual t-norm and s-norm generators.

Remark 1. Only dual standard generators will be described hereinafter.

Example 2. For any $x, y \in [0, 1]$, if the t-norm $x \bullet_t y = 1 - \min\{1, ((1 - x)^p + (1 - y)^p)^{1/p}, p \geq 1$ is considered, then the generator is $f_t(x) = 1 - x^p$.

For any $x, y \in [0, 1]$, if the s-norm $x \bullet_s y = \min\{1, (x^p + y^p)^{1/p}, p \geq 1$ is considered, then the generator is $f_s(x) = x^p$.

2.2. The Repeatability Norms

For any generators of triangular norms f_t, f_s , and any number $x \in [0, 1]$, assuming notation $x = x_1 = x_2 = \dots = x_k$:

$$y = x_1 \bullet_t x_2 \bullet_t \dots \bullet_t x_k = f_t^{-1}[kf_t(x)] \text{ for } kf_t(x) \in [0, f_t(0)],$$

$$z = x_1 \bullet_s x_2 \bullet_s \dots \bullet_s x_k = f_s^{-1}[kf_s(x)] \text{ for } kf_s(x) \in [0, f_s(1)].$$

Thus, the k -times execution of the triangular norm for the number x is a number which generator value is k -times the generator value for the number x : $f_t(x) = (1/k)f_t(y)$ and $f_s(x) = (1/k)f_s(z)$.

Hence $if_t(x) = (i/k)f_t(y)$ and $if_s(x) = (i/k)f_s(z)$. Moreover, for $i < k$:

$$x_1 \bullet_t x_2 \bullet_t \dots \bullet_t x_i = f_t^{-1}[(i/k)f_t(y)],$$

$$x_1 \bullet_s x_2 \bullet_s \dots \bullet_s x_i = f_s^{-1}[(i/k)f_s(z)].$$

Since for any $y \in [0, 1]$ there exists a number $x \in [0, 1]$ satisfying the above formulas, we can assume the notation:

$$(i/k) \bullet_p y =_{df} f_t^{-1}[(i/k)f_t(y)],$$

$$(i/k) \bullet_l z =_{df} f_s^{-1}[(i/k)f_s(z)].$$

If we assume that the procedures of repeating norms (calculating the values of the generators of these norms) for any x can be used only with a certain degree $\lambda \in [0, 1]$, it will be natural to adopt following Definitions of repeatability norms by applying these procedures.

Definition 2. The operation $\bullet_p : [0, 1] \times [0, 1] \rightarrow [0, 1]$ will be called the power repeatability standard of the t -norm $\bullet_t : [0, 1] \times [0, 1] \rightarrow [0, 1]$ (briefly: p -norm), if the following conditions are met, for any $\lambda_1, \lambda_2, x, y \in [0, 1]$:

- $1 \bullet_p x = x, 0 \bullet_p x = 1,$
- if $x \neq 0$ and $\lambda_1 \bullet_p x = \lambda_2 \bullet_p x$, then $\lambda_1 = \lambda_2,$
- if $x \neq 0, y \neq 0, \lambda_2 \neq 0$, and $\lambda_1 \bullet_p x = \lambda_2 \bullet_p y$, then $(\lambda_1 / \lambda_2) \bullet_p x = y,$
- $(\lambda_1 * \lambda_2) \bullet_p x = \lambda_1 \bullet_p (\lambda_2 \bullet_p x),$
- $(\lambda_1 + \lambda_2) \bullet_p x = (\lambda_1 \bullet_p x) \bullet_t (\lambda_2 \bullet_p x),$
- $\lambda_1 \bullet_p (x \bullet_t y) = (\lambda_1 \bullet_p x) \bullet_t (\lambda_1 \bullet_p y).$

The operation $\bullet_l : [0, 1] \times [0, 1] \rightarrow [0, 1]$ will be called the linear repeatability standard of the s -norm $\bullet_s : [0, 1] \times [0, 1] \rightarrow [0, 1]$ (briefly: l -norm), if the following conditions are met, for any $\lambda_1, \lambda_2, x, y \in [0, 1]$:

- $1 \bullet_l x = x, 0 \bullet_l x = 0,$
- if $x \neq 0$ and $\lambda_1 \bullet_l x = \lambda_2 \bullet_l x$, then $\lambda_1 = \lambda_2,$
- if $x \neq 0, y \neq 0, \lambda_2 \neq 0$, and $\lambda_1 \bullet_l x = \lambda_2 \bullet_l y$, then $(\lambda_1 / \lambda_2) \bullet_l x = y,$
- $(\lambda_1 * \lambda_2) \bullet_l x = \lambda_1 \bullet_l (\lambda_2 \bullet_l x),$
- $(\lambda_1 + \lambda_2) \bullet_l x = (\lambda_1 \bullet_l x) \bullet_s (\lambda_2 \bullet_l x),$
- $\lambda_1 \bullet_l (x \bullet_s y) = (\lambda_1 \bullet_l x) \bullet_s (\lambda_1 \bullet_l y).$

Definition 3. The system $S_{Yager} = \langle [0, 1], \bullet_t, \bullet_s, \bullet_p, \bullet_l, 0, 1 \rangle$ is called the Yager system of norms (triangular and repeatability).

Theorem 3. Let f_t be a generator of the triangular norm $\bullet_t$. Then there are operations $\bullet_p : [0, 1] \times [0, 1] \rightarrow [0, 1]$ defined by formula:

$$\lambda \bullet_p x = \begin{cases} f_t^{-1}[\lambda f_t(x)] & \text{for } \lambda f_t(x) \in [0, f_t(0)], \\ 0, & \text{otherwise.} \end{cases} \quad (4)$$

Let f_s be a generator of the triangular norm $\bullet_s$. Then there are operations $\bullet_l : [0, 1] \times [0, 1] \rightarrow [0, 1]$ defined by formula:

$$\lambda \bullet_l x = \begin{cases} f_s^{-1}[\lambda f_s(x)] & \text{for } \lambda f_s(x) \in [0, f_s(1)], \\ 1, & \text{otherwise.} \end{cases} \quad (5)$$

Proof. Since f_t is a strictly descending function, there is only one value of $f_t^{-1}[\lambda f_t(x)] \in [0, 1]$, when $\lambda f_t(x) \in [0, f_t(0)]$. Let us denote that $\lambda \bullet_p x =_{df} f_t^{-1}[\lambda f_t(x)]$, for $\lambda f_t(x) \in [0, f_t(0)]$, and $\lambda \bullet_p x =_{df} 0$, otherwise. We define the operation $\bullet_l$ in the same way. $\square$

The following notation agreement is assumed here:

$$\lambda \bullet_p x =_{df} x^\lambda \text{ and } \lambda \bullet_l x =_{df} \lambda x.$$

Fact 1. If $x^\lambda = f_t^{-1}[\lambda f_t(x)]$, for $\lambda f_t(x) \in [0, f_t(0)]$, then $f_t(x^\lambda) = \lambda f_t(x)$.
If $\lambda x = f_s^{-1}[\lambda f_s(x)]$, for $\lambda f_s(x) \in [0, f_s(1)]$, then $f_s(\lambda x) = \lambda f_s(x)$.

Theorem 4. The repetition operations $\bullet_p, \bullet_l$ for Theorem 3 satisfy the following conditions, for any $x, y, \lambda, \lambda_1, \lambda_2 \in [0, 1]$:

1. $\lambda(x \bullet_s y) = \lambda x \bullet_s \lambda y,$
2. $(x \bullet_t y)^\lambda = x^\lambda \bullet_t y^\lambda,$
3. $(\lambda_1 + \lambda_2)x = \lambda_1 x \bullet_s \lambda_2 x,$
4. $x^{\lambda_1 + \lambda_2} = x^{\lambda_1} \bullet_t x^{\lambda_2}.$

Thus, they are p -norm and l -norm.

Proof.

1. $\lambda(x \bullet_s y) = f_s^{-1}[\lambda f_s(x \bullet_s y)] = f_s^{-1}[\lambda f_s(x) + \lambda f_s(y)] = f_s^{-1}[f_s(\lambda x) + f_s(\lambda y)] = \lambda x \bullet_s \lambda y$, for $f_s(x \bullet_s y) \in [0, 1]$, and because $\lambda f_s(x \bullet_s y) \leq f_s(x \bullet_s y)$, so $\lambda f_s(x \bullet_s y) \in [0, 1]$.
2. $(x \bullet_t y)^\lambda = f_t^{-1}[\lambda f_t(x \bullet_t y)] = f_t^{-1}[\lambda f_t(x) + \lambda f_t(y)] = f_t^{-1}[f_t(x^\lambda) + f_t(y^\lambda)] = x^\lambda \bullet_t y^\lambda$, for $f_t(x \bullet_t y) \in [0, 1]$, and because $\lambda f_t(x \bullet_t y) \leq f_t(x \bullet_t y)$, so $\lambda f_t(x \bullet_t y) \in [0, 1]$.
3. $(\lambda_1 + \lambda_2)x = f_s^{-1}[(\lambda_1 + \lambda_2)f_s(x)] = f_s^{-1}[\lambda_1 f_s(x) + \lambda_2 f_s(x)] = f_s^{-1}[f_s(\lambda_1 x) + f_s(\lambda_2 x)] = \lambda_1 x \bullet_s \lambda_2 x$, for $(\lambda_1 + \lambda_2)f_s(x) = f_s(\lambda_1 x) + f_s(\lambda_2 x) \in [0, 1]$.
4. $x^{\lambda_1 + \lambda_2} = f_t^{-1}[(\lambda_1 + \lambda_2)f_t(x)] = f_t^{-1}[\lambda_1 f_t(x) + \lambda_2 f_t(x)] = f_t^{-1}[f_t(x^{\lambda_1}) + f_t(x^{\lambda_2})] = x^{\lambda_1} \bullet_t x^{\lambda_2}$, for $(\lambda_1 + \lambda_2)f_t(x) = f_t(x^{\lambda_1}) + f_t(x^{\lambda_2}) \in [0, 1]$.

□

3. The n-Scaling Intuitionistic Fuzzy Sets and Yager Aggregation Operators

In this section, a mathematical model of information granulation is presented. It is about the set X of the information instances obtained by measurements using a certain sequence of scaling functions $\{Sc_n\}$.

Definition 4. The sequence $\{Sc_n\}$ of the function $Sc_n : [0, 1] \rightarrow [0, 1]$ will be called the scale function sequence of the intuitionistic fuzzy sets when the following conditions are met:

1. Sc_1 is a function of identity, Sc_n is a continuous and strictly increasing function in the interval $[0, 1]$, for $n > 1$,
2. $Sc_n(0) = 0, Sc_n(1) = 1$, for any n ,
3. $Sc_{n+1}(x) \leq Sc_n(x)$, for any n and $x \in [0, 1]$,
4. For any numbers $u, v \in [0, 1]$, there exists such function Sc_k , that $0 \leq Sc_k(u) + Sc_k(v) \leq 1$.

Example 3. The map scale. A sequence of map functions $Sc_n(x) = x/n$, for $n > 0, x \in [0, 1]$ is a sequence of scaling functions, since:

1. $Sc_1(x) = x$ and functions Sc_n are strictly growing,
2. $Sc_n(0) = 0, Sc_n(1) = 1$, for any n ,
3. $x/(n+1) \leq x/n$ iff $Sc_{n+1}(x) \leq Sc_n(x)$, for any n and $x \in [0, 1]$,
4. For $k = 2, u, v \in [0, 1]: 0 \leq u/2 + v/2 \leq 1$,
5. $Sc_1^{-1}(x) = x$ and for $n > 1: Sc_n^{-1}(x) = nx$.

Example 4. The logarithmic scale. A sequence of logarithmic functions $Sc_n(x) = 1 - \log_{(1-1/n)} x$, for $n > 1, x \in (0, 1]$, when $Sc_1(x) = x$ and $Sc_n(0) = 0$, is a sequence of scaling functions, since:

1. $Sc_1(x) = x$ and functions Sc_n are strictly growing,
2. $Sc_n(0) = 0, Sc_n(1) = 1 - \log_{(1-1/n)} 1 = 1$, for any $n > 1$,
3. For any $x \in (0, 1], x \leq 1$ iff $1/2 \leq 1/2x$ iff $\log_{1/2} 1/2x \leq \log_{1/2} 1/2$ iff $1 - \log_{1/2} x \leq 1$ iff $Sc_2(x) \leq Sc_1(x)$,
for any $n > 1, x \in (0, 1], 1 - 1/n \leq 1 - 1/(n+1)$ iff $\log_{(1-1/n)} x \leq \log_{(1-1/(n+1))} x$ iff $1 - \log_{(1-1/(n+1))} x \leq 1 - \log_{(1-1/n)} x$ iff $Sc_{n+1}(x) \leq Sc_n(x)$,
4. For $u, v \in [0, 1]$, when $uv = 0$ or for $uv = 1$ there is $0 \leq Sc_1(u) + Sc_1(v) \leq 1$,
for $uv < 1$ there is such $k > 1$, that $uv \leq 1 - 1/k$ iff $\log_{(1-1/k)} (1 - 1/k) \leq \log_{(1-1/k)} uv$ iff $2 - \log_{(1-1/k)} uv \leq 1$ iff $(1 - \log_{(1-1/k)} u) + (1 - \log_{(1-1/k)} v) \leq 1$ iff $0 \leq Sc_k(u) + Sc_k(v) \leq 1$,
5. $Sc_1^{-1}(x) = x$ and for $n > 1, Sc_n^{-1}(x) = (1 - 1/n)^x$.

Example 5. The power scale A sequence of power functions $Sc_n(x) = x^n$, for $n > 0, x \in [0, 1]$ is a sequence of scaling functions, since:

1. $Sc_1(x) = x$ and functions Sc_n are strictly growing,
2. $Sc_n(0) = 0, Sc_n(1) = 1$, for any n ,
3. $x \leq 1$ iff $x^{(n+1)} \leq x^n$ iff $Sc_{n+1}(x) \leq Sc_n(x)$, for any n and $x \in [0, 1]$,

4. For $u, v \in [0, 1]$ exists such k_1, k_2 , that $u^{k_1} \leq 1/2$ and $v^{k_2} \leq 1/2$, and when $k = \max\{k_1, k_2\}$, then $0 \leq u^k + v^k \leq 1$, hence $0 \leq Sc_k(u) + Sc_k(v) \leq 1$,
5. For $n > 0$, $Sc_n^{-1}(x) = x^{1/n}$.

Definition 5. Let $F = \{f : X \rightarrow [0, 1]\}$ be the set of all fuzzy sets on non-empty space X .

Any function $p : X \rightarrow [0, 1] \times [0, 1]$, for any $\mu_p, v_p \in F$ and some natural number $n > 0$, is defined by formulas:

$$\begin{aligned}\mu_p &= \{ \langle x, u \rangle : \text{exists } v \in [0, 1] \text{ such that } p(x) = \langle u, v \rangle, x \in X \}, \\ v_p &= \{ \langle x, v \rangle : \text{exists } u \in [0, 1] \text{ such that } p(x) = \langle u, v \rangle, x \in X \}, \\ p &= \{ \langle x, \langle \mu_p(x), v_p(x) \rangle \rangle : x \in X \},\end{aligned}$$

will be called the ***n*-scaling intuitionistic fuzzy sets (n-ScIFS)** for sequences $\{Sc_n\}, \{Sc_n'\}$ of scaling functions, if the condition is met:

$$0 \leq Sc_n(\mu_p(x)) + Sc_n'(v_p(x)) \leq 1, \text{ for any } x \in X.$$

The ***n*-ScIFS** is a set of all *n*-scaling intuitionistic fuzzy sets for sequences $\{Sc_n\}, \{Sc_n'\}$ of scaling functions.

The fuzzy set μ_p indicates the **membership** and v_p indicates the **non-membership**.

Let $p =_{df} \langle \mu_p, v_p \rangle$ represent the *n*-fuzzy number *n*-ScIFN. Then $p(x) =_{df} \langle \mu_p(x), v_p(x) \rangle$ and $\langle \mu_p, v_p \rangle(x) =_{df} \langle \mu_p(x), v_p(x) \rangle$. For this reason, all operations and relations on the *n*-ScIFN will still be considered as defined on the set:

$$\mathbf{n\text{-ScIFN}} =_{df} \{ \langle \mu, v \rangle \in [0, 1] \times [0, 1] : 0 \leq Sc_n(\mu) + Sc_n(v) \leq 1 \}.$$

Moreover, there is:

Fact 2. $\mathbf{n\text{-ScIFN}} = \{p(x) : x \in X, p \in \mathbf{n\text{-ScIFS}}\}$,

Any function $p : X \rightarrow \mathbf{n\text{-ScIFN}}$ is the *n*-scaling intuitionistic fuzzy sets for sequences $\{Sc_n\}, \{Sc_n'\}$ of scaling functions.

When $n = 1$, then 1-ScIFSs are the intuitionistic fuzzy sets (IFS), which have been studied by Atanassow [25], and 2-ScIFSs for $Sc_2(x) = Sc_2'(x) = x^2$ are the PFS of Yager [26].

From Definition 4 and from a simple arithmetic property: $0 \leq Sc_{n+1}(\mu_p(x)) + Sc_{n+1}'(v_p(x)) \leq Sc_n(\mu_p(x)) + Sc_n'(v_p(x)) \leq 1$ for $\langle \mu_p, v_p \rangle \in \mathbf{n\text{-ScIFS}}$, there is:

Remark 2. In order to simplify the description of the diagnosis granule model, it will be assumed in the further part of the paper that the equality holds for sequences of scaling functions: $\{Sc_n\} = \{Sc_n'\}$.

Theorem 5. For any natural number $n > 1$:

1. $\mathbf{n\text{-ScIFN}} \subseteq \mathbf{(n+1)\text{-ScIFN}} \subseteq [0, 1] \times [0, 1]$,
2. $\mathbf{n\text{-ScIFN}} \subseteq \mathbf{(n+1)\text{-ScIFN}}$.

The membership and non-membership values $\langle \mu, v \rangle \in [0, 1] \times [0, 1]$ such that $\mu + v > 1$, are replaced with $\langle Sc_n(\mu), Sc_n(v) \rangle \in \mathbf{1\text{-ScIFN}}$, for some n , when the power scale is entered. Then, the aggregation operations on the IFS can be extended to the *n*-ScIFN.

Theorem 6. In any system, $\mathbf{S_{Yager}} = \langle [0, 1], \bullet_t, \bullet_s, \bullet_p, \bullet_l, 0, 1 \rangle$, for any $\langle \mu_1, v_1 \rangle, \langle \mu_2, v_2 \rangle \in \mathbf{1\text{-ScIFN}}$ and numer $\lambda \in [0, 1]$, the conditions are satisfied:

1. $\langle \mu_1 \bullet_s \mu_2, v_1 \bullet_t v_2 \rangle \in \mathbf{1\text{-ScIFN}}$,
2. $\langle \mu_1 \bullet_t \mu_2, v_1 \bullet_s v_2 \rangle \in \mathbf{1\text{-ScIFN}}$,

and in some systems of $\mathbf{S_{Yager}}$, the below conditions are also satisfied:

3. $\langle \lambda \bullet_l \mu_1, \lambda \bullet_p v_1 \rangle \in \mathbf{1\text{-ScIFN}}$,
4. $\langle \lambda \bullet_p \mu_1, \lambda \bullet_l v_1 \rangle \in \mathbf{1\text{-ScIFN}}$.

Proof.

1. $\langle \mu_1, v_1 \rangle, \langle \mu_2, v_2 \rangle \in \mathbf{1-ScIFN}$ iff $\mu_1 + v_1 \leq 1, \mu_2 + v_2 \leq 1$ iff $\mu_1 \leq 1 - v_1, \mu_2 \leq 1 - v_2$. Hence and from the monotonicity of the s-norm and its determination by the t-norms: $\mu_1 \bullet_s \mu_2 \leq (1 - v_1) \bullet_s (1 - v_2) = 1 - (1 - (1 - v_1) \bullet_s (1 - v_2)) = 1 - v_1 \bullet_t v_2$ iff $\mu_1 \bullet_s \mu_2 + v_1 \bullet_t v_2 \leq 1$ iff $\langle \mu_1 \bullet_s \mu_2, v_1 \bullet_t v_2 \rangle \in \mathbf{1-ScIFN}$.
2. $\langle \mu_1, v_1 \rangle, \langle \mu_2, v_2 \rangle \in \mathbf{1-ScIFN}$ iff $\langle v_1, \mu_1 \rangle, \langle v_2, \mu_2 \rangle \in \mathbf{1-ScIFN}$. Hence and from the point 1 it follows that: $\langle v_1 \bullet_s v_2, \mu_1 \bullet_t \mu_2 \rangle \in \mathbf{1-ScIFN}$. Which is equivalent to $\langle \mu_1 \bullet_t \mu_2, v_1 \bullet_s v_2 \rangle \in \mathbf{1-ScIFN}$.
3. Let $\mu \bullet_s v = \max\{\mu, v\}, \mu \bullet_t v = \min\{\mu, v\}$, then $f_s(x) = x, f_t(x) = 1 - x$ (see Example 1);
 $\langle \mu_1, v_1 \rangle \in \mathbf{1-ScIFN}$ iff $\mu_1 + v_1 \leq 1$;
 $\lambda \bullet_l \mu_1 = f_s^{-1}[\lambda f_s(\mu_1)] = \lambda \mu_1$ and $\lambda \bullet_p v_1 = f_s^{-1}[\lambda f_s(v_1)] = 1 - \lambda(1 - v_1)$;
 $\lambda \bullet_l \mu_1 + \lambda \bullet_p v_1 = \lambda \mu_1 + 1 - \lambda(1 - v_1) = 1 - \lambda + \lambda(\mu_1 + v_1)$.
Since $\lambda(\mu_1 + v_1) \leq \lambda$, so $0 \leq 1 - \lambda + \lambda(\mu_1 + v_1) \leq 1$.
4. Let $\mu \bullet_s v = \max\{\mu, v\}, \mu \bullet_t v = \min\{\mu, v\}$, then $f_s(x) = x, f_t(x) = 1 - x$;
 $\langle \mu_1, v_1 \rangle \in \mathbf{1-ScIFN}$ iff $\mu_1 + v_1 \leq 1$ iff $\langle v_1, \mu_1 \rangle \in \mathbf{1-ScIFN}$. Hence and from point 3, it follows that: $\langle \lambda \bullet_l v_1, \lambda \bullet_p \mu_1 \rangle \in \mathbf{1-ScIFN}$.
Which is equivalent to $\langle \lambda \bullet_p \mu_1, \lambda \bullet_l v_1 \rangle \in \mathbf{1-ScIFN}$.

□

Remark 3. Adopting the triangular norm generators from Example 2:

$\lambda \bullet_l \mu_1 = \lambda^{1/p} \mu_1, \lambda \bullet_p v_1 = 1 - \lambda^{1/p}(1 - v_1)$, for $\mu_1 = 1/2, v_1 = 7/8, \lambda = 1/4$ and $p = 2$ we get $\langle \lambda \bullet_l \mu_1, \lambda \bullet_p v_1 \rangle = \langle 1/8, 15/16 \rangle \notin \mathbf{1-ScIFN}$ (because $1 < 1/8 + 15/16$).

Theorem 7. Allow in the system $\mathbf{S}_{\text{Yager}}$, conditions 1–4 of Theorem 6 apply. Then for any natural number $n > 1$ for any $\langle \mu_1, v_1 \rangle, \langle \mu_2, v_2 \rangle \in \mathbf{n-ScIFN}$ and number $\lambda \in [0, 1]$, the following conditions are satisfied:

1. $\langle Sc_n^{-1}(Sc_n(\mu_1) \bullet_s Sc_n(\mu_2)), Sc_n^{-1}(Sc_n(v_1) \bullet_t Sc_n(v_2)) \rangle \in \mathbf{n-ScIFN}$,
2. $\langle Sc_n^{-1}(Sc_n(\mu_1) \bullet_t Sc_n(\mu_2)), Sc_n^{-1}(Sc_n(v_1) \bullet_s Sc_n(v_2)) \rangle \in \mathbf{n-ScIFN}$,
3. $\langle Sc_n^{-1}(\lambda \bullet_l Sc_n(\mu_1)), Sc_n^{-1}(\lambda \bullet_p Sc_n(v_1)) \rangle \in \mathbf{n-ScIFN}$,
4. $\langle Sc_n^{-1}(\lambda \bullet_p Sc_n(\mu_1)), Sc_n^{-1}(\lambda \bullet_l Sc_n(v_1)) \rangle \in \mathbf{n-ScIFN}$.

Proof. $\langle \mu_1, v_1 \rangle, \langle \mu_2, v_2 \rangle \in \mathbf{n-ScIFN}$ iff $\langle Sc_n(\mu_1), Sc_n(v_1) \rangle, \langle Sc_n(\mu_2), Sc_n(v_2) \rangle \in \mathbf{1-ScIFN}$.

Then the conditions of Theorem 6 equivalent to the above conditions 1–4 are satisfied. □

Definition 6. In the system $\mathbf{S}_{\text{Yager}}$, the following aggregation operators, called **Yager operators** on $\mathbf{n-ScIFN}$, are defined for any $\langle \mu_1, v_1 \rangle, \langle \mu_2, v_2 \rangle \in \mathbf{n-ScIFN}$ and the number $\lambda \in [0, 1]$:

1. $\langle \mu_1, v_1 \rangle \oplus \langle \mu_2, v_2 \rangle = \langle Sc_n^{-1}(Sc_n(\mu_1) \bullet_s Sc_n(\mu_2)), Sc_n^{-1}(Sc_n(v_1) \bullet_t Sc_n(v_2)) \rangle$,
2. $\langle \mu_1, v_1 \rangle \otimes \langle \mu_2, v_2 \rangle = \langle Sc_n^{-1}(Sc_n(\mu_1) \bullet_t Sc_n(\mu_2)), Sc_n^{-1}(Sc_n(v_1) \bullet_s Sc_n(v_2)) \rangle$,
3. $\lambda \langle \mu_1, v_1 \rangle = \langle Sc_n^{-1}(\lambda \bullet_l Sc_n(\mu_1)), Sc_n^{-1}(\lambda \bullet_p Sc_n(v_1)) \rangle$,
4. $\langle \mu_1, v_1 \rangle^\lambda = \langle Sc_n^{-1}(\lambda \bullet_p Sc_n(\mu_1)), Sc_n^{-1}(\lambda \bullet_l Sc_n(v_1)) \rangle$.

When $\langle Sc_n^{-1}(\lambda \bullet_l Sc_n(\mu_1)), Sc_n^{-1}(\lambda \bullet_p Sc_n(v_1)) \rangle \notin \mathbf{n-ScIFN}$, then:

5. $\lambda \langle \mu_1, v_1 \rangle = \langle 1, 0 \rangle$,

or when $\langle Sc_n^{-1}(\lambda \bullet_p Sc_n(\mu_1)), Sc_n^{-1}(\lambda \bullet_l Sc_n(v_1)) \rangle \notin \mathbf{n-ScIFN}$, then:

6. $\langle \mu_1, v_1 \rangle^\lambda = \langle 0, 1 \rangle$.

Hence, in any system $\mathbf{S}_{\text{Yager}} = \langle [0, 1], \bullet_t, \bullet_s, \bullet_p, \bullet_l, 0, 1 \rangle$, by Theorem 4, there is:

Theorem 8. For any $\langle \mu_1, v_1 \rangle, \langle \mu_2, v_2 \rangle, \langle \mu_3, v_3 \rangle \in \mathbf{n-ScIFN}$ and any number $\lambda \in [0, 1]$:

1. $\langle \mu_1, v_1 \rangle \oplus \langle 0, 1 \rangle = \langle \mu_1, v_1 \rangle, \langle \mu_1, v_1 \rangle \oplus \langle 1, 0 \rangle = \langle 1, 0 \rangle$,
2. $\langle \mu_1, v_1 \rangle \oplus \langle \mu_2, v_2 \rangle = \langle \mu_2, v_2 \rangle \oplus \langle \mu_1, v_1 \rangle$,
3. $(\langle \mu_1, v_1 \rangle \oplus \langle \mu_2, v_2 \rangle) \oplus \langle \mu_3, v_3 \rangle = \langle \mu_1, v_1 \rangle \oplus (\langle \mu_2, v_2 \rangle \oplus \langle \mu_3, v_3 \rangle)$,

4. $\langle \mu_1, v_1 \rangle \otimes \langle 1, 0 \rangle = \langle \mu_1, v_1 \rangle, \langle \mu_1, v_1 \rangle \oplus \langle 0, 1 \rangle = \langle 1, 0 \rangle,$
5. $\langle \mu_1, v_1 \rangle \otimes \langle \mu_2, v_2 \rangle = \langle \mu_2, v_2 \rangle \otimes \langle \mu_1, v_1 \rangle,$
6. $(\langle \mu_1, v_1 \rangle \otimes \langle \mu_2, v_2 \rangle) \otimes \langle \mu_3, v_3 \rangle = \langle \mu_1, v_1 \rangle \otimes (\langle \mu_2, v_2 \rangle \otimes \langle \mu_3, v_3 \rangle),$
7. $\lambda(\langle \mu_1, v_1 \rangle \oplus \langle \mu_2, v_2 \rangle) = \lambda \langle \mu_1, v_1 \rangle \oplus \lambda \langle \mu_2, v_2 \rangle,$
8. $(\langle \mu_1, v_1 \rangle \otimes \langle \mu_2, v_2 \rangle) \lambda = \langle \mu_1, v_1 \rangle \lambda \otimes \langle \mu_2, v_2 \rangle \lambda,$
9. $(\lambda_1 + \lambda_2) \langle \mu_1, v_1 \rangle = \lambda_1 \langle \mu_1, v_1 \rangle \oplus \lambda_2 \langle \mu_1, v_1 \rangle,$
10. $\langle \mu_1, v_1 \rangle^{\lambda_1 + \lambda_2} = \langle \mu_1, v_1 \rangle^{\lambda_1} \otimes \langle \mu_1, v_1 \rangle^{\lambda_2}.$

4. Triangular Norms in n-ScIFN and n-ScIFN Algebra

Any set *n-ScIFN* can be ordered by the relation $\leq_n$ defined as follows:

Definition 7. For any $\langle \mu_1, v_1 \rangle, \langle \mu_2, v_2 \rangle \in \mathbf{n-ScIFN}$:

$$\langle \mu_1, v_1 \rangle \leq_n \langle \mu_2, v_2 \rangle \text{ iff } \mu_1 \leq \mu_2, v_1 \geq v_2. \quad (6)$$

Fact 3. For any $x, y \in [0, 1]$,

1. $\langle 0, 1 \rangle \leq_n \langle 0, x \rangle \leq_n \langle 0, 0 \rangle \leq_n \langle y, 0 \rangle \leq_n \langle 1, 0 \rangle,$
2. $\langle 0, 1 \rangle \leq_n \langle x, y \rangle \leq_n \langle 1, 0 \rangle,$ when $\langle x, y \rangle \in \mathbf{n-ScIFN}.$
3. $\langle 0, 1 \rangle = \min_n \mathbf{n-ScIFN}, \langle 1, 0 \rangle = \max_n \mathbf{n-ScIFN}.$

Analogous properties of the order relation $\leq_n$ to the relation $\leq$ on real numbers enable the generalization of triangular norms into the set *n-ScIFN*.

Definition 8.

1. The operation $\circ_t : \mathbf{n-ScIFN} \times \mathbf{n-ScIFN} \rightarrow \mathbf{n-ScIFN}$ is called the *t-norm* in the set *n-ScIFN* ordered by the relation $\leq_n$, when for any $\langle \mu_1, v_1 \rangle, \langle \mu_2, v_2 \rangle, \langle \mu_3, v_3 \rangle \in \mathbf{n-ScIFN}$:
 - (a) $\langle 0, 1 \rangle \circ_t \langle \mu_1, v_1 \rangle = \langle 0, 1 \rangle, \langle \mu_1, v_1 \rangle \circ_t \langle 1, 0 \rangle = \langle \mu_1, v_1 \rangle$ —boundary conditions,
 - (b) $\langle \mu_1, v_1 \rangle \circ_t \langle \mu_2, v_2 \rangle \leq_n \langle \mu_3, v_3 \rangle \circ_t \langle \mu_2, v_2 \rangle,$ when $\langle \mu_1, v_1 \rangle \leq_n \langle \mu_3, v_3 \rangle$ —monotonicity,
 - (c) $\langle \mu_1, v_1 \rangle \circ_t \langle \mu_2, v_2 \rangle = \langle \mu_2, v_2 \rangle \circ_t \langle \mu_1, v_1 \rangle$ —commutativity,
 - (d) $\langle \mu_1, v_1 \rangle \circ_t (\langle \mu_2, v_2 \rangle \circ_t \langle \mu_3, v_3 \rangle) = (\langle \mu_1, v_1 \rangle \circ_t \langle \mu_2, v_2 \rangle) \circ_t \langle \mu_3, v_3 \rangle$ —associativity.
2. The operation $\circ_s : \mathbf{n-ScIFN} \times \mathbf{n-ScIFN} \rightarrow \mathbf{n-ScIFN}$ is called the *s-norm* in the set *n-ScIFN* ordered by the relation $\leq_n$, when for any $\langle \mu_1, v_1 \rangle, \langle \mu_2, v_2 \rangle, \langle \mu_3, v_3 \rangle \in \mathbf{n-ScIFN}$:
 - (a) $\langle 1, 0 \rangle \circ_s \langle \mu_1, v_1 \rangle = \langle 1, 0 \rangle, \langle \mu_1, v_1 \rangle \circ_s \langle 0, 1 \rangle = \langle \mu_1, v_1 \rangle$ —boundary conditions,
 - (b) $\langle \mu_1, v_1 \rangle \circ_s \langle \mu_2, v_2 \rangle \leq_n \langle \mu_3, v_3 \rangle \circ_s \langle \mu_2, v_2 \rangle,$ when $\langle \mu_1, v_1 \rangle \leq_n \langle \mu_3, v_3 \rangle$ —monotonicity,
 - (c) $\langle \mu_1, v_1 \rangle \circ_s \langle \mu_2, v_2 \rangle = \langle \mu_2, v_2 \rangle \circ_s \langle \mu_1, v_1 \rangle$ —commutativity,
 - (d) $\langle \mu_1, v_1 \rangle \circ_s (\langle \mu_2, v_2 \rangle \circ_s \langle \mu_3, v_3 \rangle) = (\langle \mu_1, v_1 \rangle \circ_s \langle \mu_2, v_2 \rangle) \circ_s \langle \mu_3, v_3 \rangle$ —associativity.

Theorem 9. The Yager operator $\otimes$ on the *n-ScIFN* is the *t-norm* in the set *n-ScIFN*. The Yager operator $\oplus$ in the *n-ScIFN* is the *s-norm*.

Proof. Conditions (4)–(6) of Theorem 8 prove that the conditions (a), (c), (d) of Definition 7 (1) are satisfied with the *t-norm* in the set *n-ScIFN* by the operator $\otimes$. It is enough to prove that monotonicity is satisfied by this operation.

Let for any $\langle \mu_1, v_1 \rangle, \langle \mu_2, v_2 \rangle, \langle \mu_3, v_3 \rangle \in \mathbf{n-ScIFN}$:

- $\langle \mu_1, v_1 \rangle \otimes \langle \mu_2, v_2 \rangle = \langle Sc_n^{-1}(Sc_n(\mu_1) \bullet_t Sc_n(\mu_2)), Sc_n^{-1}(Sc_n(v_1) \bullet_s Sc_n(v_2)) \rangle,$
- $\langle \mu_3, v_3 \rangle \otimes \langle \mu_2, v_2 \rangle = \langle Sc_n^{-1}(Sc_n(\mu_3) \bullet_t Sc_n(\mu_2)), Sc_n^{-1}(Sc_n(v_3) \bullet_s Sc_n(v_2)) \rangle,$

and let $\langle \mu_1, v_1 \rangle \leq_n \langle \mu_3, v_3 \rangle$. Then $\mu_1 \leq \mu_3, v_1 \geq v_3$. Hence and from the monotonicity of *t-norm* and *s-norm* as well as the scale function Sc_n , there are inequalities:

$$\begin{aligned} Sc_n(\mu_1) \bullet_t Sc_n(\mu_2) &\leq Sc_n(\mu_3) \bullet_t Sc_n(\mu_2), \\ Sc_n(v_3) \bullet_s Sc_n(v_2) &\leq Sc_n(v_1) \bullet_s Sc_n(v_2) \text{ iff} \\ Sc_n^{-1}(Sc_n(\mu_1) \bullet_t Sc_n(\mu_2)) &\leq Sc_n^{-1}(Sc_n(\mu_3) \bullet_t Sc_n(\mu_2)), \\ Sc_n^{-1}(Sc_n(v_3) \bullet_s Sc_n(v_2)) &\geq Sc_n^{-1}(Sc_n(v_1) \bullet_s Sc_n(v_2)) \text{ iff} \\ \langle Sc_n^{-1}(Sc_n(\mu_1) \bullet_t Sc_n(\mu_2)), Sc_n^{-1}(Sc_n(v_1) \bullet_s Sc_n(v_2)) \rangle &\leq_n \end{aligned}$$

$\langle Sc_n^{-1}(Sc_n(\mu_3) \bullet_t Sc_n(\mu_2)), Sc_n^{-1}(Sc_n(v_3) \bullet_s Sc_n(v_2)) \rangle$ iff $\langle \mu_1, v_1 \rangle \otimes \langle \mu_2, v_2 \rangle \leq_n \langle \mu_3, v_3 \rangle \otimes \langle \mu_2, v_2 \rangle$.

The fulfillment by the operator $\oplus$ of the conditions of Definition 7 (2) of the s-norm in the n -ScIFN set is proved analogously. $\square$

The monotonicity of the operators $\oplus, \otimes$ describes important Theorem:

Theorem 10. For any $\langle \mu_1, v_1 \rangle, \langle \mu_2, v_2 \rangle, \langle \mu_3, v_3 \rangle \in n$ -ScIFN:

1. $\langle \mu_1, v_1 \rangle \otimes \langle \mu_2, v_2 \rangle \leq_n \langle \mu_1, v_1 \rangle$ and $\langle \mu_1, v_1 \rangle \otimes \langle \mu_2, v_2 \rangle \leq_n \langle \mu_2, v_2 \rangle$,
2. $\langle \mu_1, v_1 \rangle \leq_n \langle \mu_1, v_1 \rangle \oplus \langle \mu_2, v_2 \rangle$ and $\langle \mu_2, v_2 \rangle \leq_n \langle \mu_1, v_1 \rangle \oplus \langle \mu_2, v_2 \rangle$,
3. $\langle \mu_1, v_1 \rangle \otimes \langle \mu_2, v_2 \rangle \leq_n \langle \mu_1, v_1 \rangle \oplus \langle \mu_2, v_2 \rangle$,
4. if $\langle \mu_1, v_1 \rangle \leq_n \langle \mu_3, v_3 \rangle$, then $\langle \mu_1, v_1 \rangle \otimes \langle \mu_2, v_2 \rangle \leq_n \langle \mu_3, v_3 \rangle \otimes \langle \mu_2, v_2 \rangle$ and $\langle \mu_1, v_1 \rangle \oplus \langle \mu_2, v_2 \rangle \leq_n \langle \mu_3, v_3 \rangle \oplus \langle \mu_2, v_2 \rangle$,
5. if $\langle \mu_1, v_1 \rangle \leq_n \langle \mu_2, v_2 \rangle$, then for any number $\lambda \in [0, 1]$, $\lambda \langle \mu_1, v_1 \rangle \otimes \leq_n \lambda \langle \mu_2, v_2 \rangle$ and $\langle \mu_1, v_1 \rangle^\lambda \otimes \leq_n \langle \mu_2, v_2 \rangle^\lambda$.

Proof. From fact 3 (2) it follows $\langle \mu_1, v_1 \rangle \leq_n \langle 1, 0 \rangle, \langle \mu_2, v_2 \rangle \leq_n \langle 1, 0 \rangle$. Hence, from monotonicity of $\otimes$, by Definition 8 (1), there is a proof of the point (1):

$$\langle \mu_1, v_1 \rangle \otimes \langle \mu_2, v_2 \rangle \leq_n \langle 1, 0 \rangle \otimes \langle \mu_2, v_2 \rangle = \langle \mu_2, v_2 \rangle,$$

$$\langle \mu_1, v_1 \rangle \otimes \langle \mu_2, v_2 \rangle \leq_n \langle \mu_1, v_1 \rangle \otimes \langle 1, 0 \rangle = \langle \mu_1, v_1 \rangle.$$

From fact 3 (2) it follows $\langle 0, 1 \rangle \leq_n \langle \mu_1, v_1 \rangle, \langle 0, 1 \rangle \leq_n \langle \mu_2, v_2 \rangle$. Hence, from monotonicity of $\oplus$, by Definition 8 (1), there is a proof of point (2):

$$\langle \mu_1, v_1 \rangle = \langle 0, 1 \rangle \otimes \langle \mu_1, v_1 \rangle \leq_n \langle \mu_1, v_1 \rangle \otimes \langle \mu_2, v_2 \rangle,$$

$$\langle \mu_2, v_2 \rangle = \langle 0, 1 \rangle \otimes \langle \mu_2, v_2 \rangle \leq_n \langle \mu_1, v_1 \rangle \otimes \langle \mu_2, v_2 \rangle.$$

Point (3) of Theorem follows the transitivity of the relations $\leq_n$, point (4) is obtained based on Theorem 9 and point (5) follows the monotonicity of linear and power repetition operations. $\square$

For relation $\leq_n$ and any $A \subseteq n$ -ScIFN, it is not always possible to define supremum and infinitum operations in terms of mathematical logic. Therefore it is accepted:

Definition 9. For any $A \subseteq n$ -ScIFN and relations $\leq_n$ the n -supremum A is $\sup_n A$:

- For $A = \emptyset, \sup_n A = \langle 0, 1 \rangle$,
- For $A \neq \emptyset: \sup_n A =_{df} \langle \sup\{u : \text{exists } v \in [0, 1] \text{ such that } \langle u, v \rangle \in A\}, \inf\{v : \text{exists } u \in [0, 1] \text{ such that } \langle u, v \rangle \in A\} \rangle$
and the n -infimum A is $\inf_n A$:
- For $A = \emptyset, \inf_n A = \langle 0, 1 \rangle$,
- For $A \neq \emptyset: \inf_n A =_{df} \langle \inf\{u : \text{exists } v \in [0, 1] \text{ such that } \langle u, v \rangle \in A\}, \sup\{v : \text{exists } u \in [0, 1] \text{ such that } \langle u, v \rangle \in A\} \rangle$.

From Definitions 7 and 9 there is:

Fact 4. For any $A \subseteq n$ -ScIFN and $\langle u, v \rangle \in A$, $\inf_n A \leq_n \langle u, v \rangle \leq_n \sup_n A$.

5. Diagnostic Information Granules

The concept of the information granule system presented in [20] has models in the theories of: sets, probability, possible data sets in evidence systems, fuzzy sets, rough sets, and shadowed sets [20,42,43]. Here this concept is expanded and a broader model of information granules is formulated. It is called the system of diagnostic information granules.

The **diagnosis** of the object $x \in X$ is a pair of fuzzy data symptoms $u \in [0, 1]$ and faults $v \in [0, 1]$ of the behavior of the diagnosed object in the diagnostic process. A **diagnosis instance** is understood as a triple $\langle x, \langle u, v \rangle \rangle$, such that if symptoms $u = 0$, then faults $v \neq 1$ (if there is no symptom, the error cannot be certain). Let U denote the set of diagnosis instances. The symptom-faults two-dimensional granule system was extended to

include operations of the degree of repetition of diagnostic procedures: linear and power, for parallel and serial measurements, respectively. In order to determine the **diagnostic information granules** (shortly: diagnostic granules or granules), a first distinction is made between the **set of the granule instances** G_{inst} and the **set of the granule of instances sets** G_{set} .

5.1. The Diagnostic Granule System: Its Language and Interpretation

The set granules is defined by the set of instance granules. In other words, elements of the instance set granules are instance granules. In summary, the knowledge about operations and relations in *n-ScIFN* introduced above can, for the fuzzy sets of *n-ScIFS* in the X space of the information instance, be defined as a diagnostic information granule system as follows:

Definition 10. *Relational structure:*

$$\mathbf{G}_{diag} = \langle G, \cup_G, \cap_G, \setminus_G, \iota^G, \{\}_G, \{\Sigma_G^\lambda\}_{\lambda \in [0,1]}, \{\Pi_G^\lambda\}_{\lambda \in [0,1]}, \in_G, \subseteq_G, =_G, 0_G, 1_G, G_{inst}, G_{set}, G_0 \rangle, \quad (7)$$

will be called the **system of diagnostic granules**, where:

- The **set of diagnostic granules** $G = G_{inst} \cup G_{set}$ is the sum of the sets: G_{inst} —**instance granules** and G_{set} —**set granules**,
- Operations $\cup_G, \cap_G$ are the **sum** and **product** of two granules, as well as the generalized sum and product on the granule sets,
- $\setminus_G$ is an operation of **granule difference**,
- ι^G is an operation of **granule closure**,
- $\{\}_G$ is an operation of **singleton**,
- Σ_G^λ is the operation of the **linear repetition of granules** with the degree λ ,
- Π_G^λ is the operation of the **power repetition of granules** with the degree λ ,
- 0_G is an **empty granule** for an empty set and 1_G is a **full granule** for a set G ,
- G_0 is the set of distinguished granules called **instance singleton granules**.

For granules of instance sets, the following relations are defined:

- $\in_G$ is the relation of **being an element of the granule**,
- $\subseteq_G$ is the **granule inclusion relation**,
- $=_G$ is the **granule closeness relation**.

The symbols of the relational structure $\mathbf{G}_{diag}$ and the language expressions of the diagnostic granule system are interpreted as follows:

1. $G_{set} = n\text{-ScIFN}$,
2. $G_{inst} = \{ \langle a, \langle \mu_p(a), v_p(a) \rangle \rangle : p \in G_{set}, a \in X, \langle \mu_p(a), v_p(a) \rangle \neq \langle 0, 1 \rangle \text{ and for } x \in X, x \neq a, \langle \mu_p(x), v_p(x) \rangle = \langle 0, 1 \rangle \}$,
3. $G_0 = \{ p \in G_{set} : \text{exists } a \in X, \langle \mu_p(a), v_p(a) \rangle \neq \langle 0, 1 \rangle \text{ and for } x \in X, x \neq a, \langle \mu_p(x), v_p(x) \rangle = \langle 0, 1 \rangle \}$.

For any $A \subseteq G_{set}$:

4. $\cup_G A = \{ \langle \mu_p(x), v_p(x) \rangle : \langle \mu_p(x), v_p(x) \rangle = \sup_n \{ \langle \mu_e(x), v_e(x) \rangle : e \in A \} \text{ and } x \in X \}$,
5. The operation $\{\}_G : G_{inst} \rightarrow G_0$, **determining a singleton instance**, is given by the formula: if for $p \in G_0$ and for $a \in X, \langle a, \langle \mu_p(a), v_p(a) \rangle \rangle \in G_{inst}$, then $\{ \langle a, \langle \mu_p(a), v_p(a) \rangle \rangle \}_G = p$.

G_0 is a set of granules called **instance singleton granules** such that a function is given $\{\}_G : G_{inst} \rightarrow G_0$ and $G_0 = \{ x \in G_{set} : \exists z \in G_{inst} (x = \{z\}_G) \}$.

For any $p_1, p_2 \in G_{set}$ and any number $\lambda \in [0, 1]$:

6. $p_1 \cup_G p_2 = \{ \langle x, \langle \mu_{p_1}(x), v_{p_1}(x) \rangle \oplus \langle \mu_{p_2}(x), v_{p_2}(x) \rangle \rangle : x \in X \}$,
7. $p_1 \cap_G p_2 = \{ \langle x, \langle \mu_{p_1}(x), v_{p_1}(x) \rangle \otimes \langle \mu_{p_2}(x), v_{p_2}(x) \rangle \rangle : x \in X \}$,
8. $\Sigma_G^\lambda p_1 = \{ \langle x, \lambda \langle \mu_{p_1}(x), v_{p_1}(x) \rangle \rangle : x \in X \}$,

9. $\Pi_G^\lambda p_1 = \{\langle x, \langle \mu_{p_1}(x), v_{p_1}(x) \rangle^\lambda \rangle : x \in X\}$,
10. $p_1 \subseteq_G p_2$ iff for any $x \in X$, $\langle \mu_{p_1}(x), v_{p_1}(x) \rangle \leq_n \langle \mu_{p_2}(x), v_{p_2}(x) \rangle$, i.e., $\mu_{p_1}(x) \leq \mu_{p_2}(x)$, and $v_{p_1}(x) \geq v_{p_2}(x)$,
11. $p_1 =_G p_2$ iff $\bigcup_G \{z \in G_0 : (z \subseteq_G p_1) \Leftrightarrow (z \subseteq_G p_2)\} = 1_G$.

For any $e \in G_{inst}$, $p \in G_{set}$:

12. $e \in_G p$ iff $\{e\}_G \subseteq_G p$.

Symbols $0_G, 1_G$, are interpreted by formulas:

13. $p = 0_G$ iff for any $x \in X$, $\langle \mu_p(x), v_p(x) \rangle = \langle 0, 1 \rangle$,
14. $p = 1_G$ iff for any $x \in X$, $\langle \mu_p(x), v_p(x) \rangle = \langle 1, 0 \rangle$.

Similarly to the set theory, it can be assumed that:

15. $y'^G = \bigcup_G \{t \in G_0 : \neg t \subseteq_G y\}$,
16. $y \setminus_G z = \bigcup_G \{t \in G_0 : t \subseteq_G y \wedge \neg t \subseteq_G z\}$.

By applying the above conditions (1)–(16), and simple calculus, Definition 9, Fact 5, and Theorem 11 are proved.

Fact 5.

1. $\bigcup_G \emptyset = 0_G$,
2. $\bigcup_G G_0 = 1_G$,
3. $y'^G = 1_G$,
4. $y \setminus_G 1_G = 0_G$,
5. $y \setminus_G z \subseteq_G y \cap_G z'^G = y$.

Theorem 11. For any $p_1, p_2 \in G_{set}$ and $A, B \subseteq G_{set}$:

1. $p_1 \subseteq_G p_2$ iff for any $t \in G_0$, if $t \subseteq_G p_1$, then $t \subseteq_G p_2$,
2. $p_1 \subseteq_G p_2$ iff for any $e \in G_{inst}$, if $e \in_G p_1$, then $e \in_G p_2$,
3. $p_1 =_G p_2$ iff $p_1 \subseteq_G p_2$ and $p_2 \subseteq_G p_1$,
4. $p_1 =_G p_2$ iff $p_1 = p_2$,
5. $p_1 = \bigcup_G \{t \in G_0 : t \subseteq_G p_1\}$,
6. if $A \subseteq B$, then $\bigcup_G A \subseteq_G \bigcup_G B$,
7. for any $e \in G_0$ exists $p \in A$ such that $e \subseteq_G p$ iff $e \subseteq_G \bigcup_G A$.

Proof. For example, condition (5) of Theorem 11 is proven.

Let $e \in G_0$. If $e \subseteq_G p_1$, then $e \subseteq_G \bigcup_G \{t \in G_0 : t \subseteq_G p_1\}$.

If $e \subseteq_G p = \bigcup_G \{t \in G_0 : t \subseteq_G p_1\}$, then for any $x \in X$, $e(x) = \langle u_e(x), v_e(x) \rangle \leq_n \langle \mu_p(x), v_p(x) \rangle = \sup_n \{\langle \mu_t(x), v_t(x) \rangle : t \in G_0, t \subseteq_G p_1\}$.

Since $\langle u_e(a), v_e(a) \rangle \leq_n \langle \mu_p(x), v_p(x) \rangle \leq_n \langle \mu_{p_1}(x), v_{p_1}(x) \rangle$, so $e \subseteq_G p_1$.

Hence, based on condition (3) of Theorem 11, there is $p_1 = \bigcup_G \{t \in G_0 : t \subseteq_G p_1\}$. $\square$

Hence and based on Theorem 8, introducing a notation agreement: $\Sigma_G^\lambda p =_{df} \lambda p$, $\Pi_G^\lambda p =_{df} p^\lambda$, we get:

Theorem 12. For any $p_1, p_2, p_3 \in n\text{-ScIFN}$ and any number $\lambda, \lambda_1, \lambda_2 \in [0, 1]$:

1. $p_1 \cup_G 0_G = p_1, p_1 \cup_G 1_G = 1_G$,
2. $p_1 \cup_G p_2 = p_2 \cup_G p_1$,
3. $(p_1 \cup_G p_2) \cup_G p_3 = p_1 \cup_G (p_2 \cup_G p_3)$,
4. $p_1 \cap_G 1_G = p_1, p_1 \cap_G 0_G = 0_G$,
5. $p_1 \cap_G p_2 = p_2 \cap_G p_1$,
6. $(p_1 \cap_G p_2) \cap_G p_3 = p_1 \cap_G (p_2 \cap_G p_3)$,
7. $\lambda(p_1 \cup_G p_2) = \lambda p_1 \cup_G \lambda p_2$,
8. $(p_1 \cap_G p_2)^\lambda = p_1^\lambda \cap_G p_2^\lambda$,
9. $(\lambda_1 + \lambda_2)p_1 = \lambda_1 p_1 \cup_G \lambda_2 p_1$,
10. $p_1^{\lambda_1 + \lambda_2} = p_1^{\lambda_1} \cap_G p_1^{\lambda_2}$.

Definition 11. Conditions (1)–(10) in Theorem 12 define the *granular algebra of diagnosis* defined on the *n*-ScIFS set.

5.2. Information Granule System Database

Using the constructions on rough sets given by E. Bryniarski in the papers [42,44], it is possible to define the simplest sub-system of the granules system, in which from the set of instances G_{inst} using the operation $\{\}_G$, transforming the instances as elementary granules (granular elements) into granular singletons and operation of singleton summation $\cup_G$, any information granule is obtained.

Definition 12. Any granule system $\mathbf{G}$:

$$\mathbf{G}_{base} = \langle G, \{\}_G, \cup_G, \subseteq_G, =_G, 0_G, 1_G, G_{inst}, G_{set}, G_0 \rangle, \quad (8)$$

is called the *information granule database*, when the elements of the G set are called granules, operations $\{\}_G$ is the singleton, $\cup_G$ is the granule sum, $\subseteq_G$ is the inclusion relation, $=_G$ is the closeness relation, 0_G is the empty granule and 1_G is the full granule, and G_{inst} is the set of instance granules, G_{set} is the set of set granules, and G_0 is the singleton set of instance granules. The system $\mathbf{G}_{base}$ satisfies the following axioms:

1. $G_0 \subseteq G_{set}$,
2. $G = G_{inst} \cup G_{set}$,
3. $G_{inst} \cap G_{set} = \emptyset$.

For any $x \in G_{set}$:

4. $x \subseteq_G x$,
5. $0_G \subseteq_G x$,
6. If $x \neq 0_G$, then it is not true that $x \subseteq_G 0_G$.

There is some function $\{\}_G : G_{inst} \rightarrow G_0$, such that:

7. For any $x \in G_{inst}$, $\{x\}_G \subseteq_G 1_G$,
8. $G_0 = \{x \in G_0 : \exists z \in G_{inst} (x = \{z\}_G)\}$.

There is some function $\cup_G : \varphi(G_0) \rightarrow G_{set}$, such that:

9. $\cup_G \emptyset = 0_G$,
10. $\cup_G G_0 = 1_G$,
11. for any $y \in G_{set}$, $y = \cup_G \{x \in G_0 : x \subseteq_G y\}$,
12. for any $x, y \in G_{set}$, $(x =_G y) \Leftrightarrow_{df} \cup_G \{z \in G_0 : (z \subseteq_G x) \Leftrightarrow (z \subseteq_G y)\} = 1_G$.

The information granule database $\mathbf{G}_{base}$ is a diagnostic granule subsystem. When we join to the conditions (1)–(12), a Definition: for any $x \in G_{inst}$ and $y \in G_{set}$, $x \in_G y$ iff $\{x\}_G \subseteq_G y$.

Then we get the following Theorem.

Theorem 13.

1. $\{x \in G_{inst} : x \in_G 0_G\} = \emptyset$,
2. $\{x \in G_{inst} : \exists z \in G_0 (x \in_G z)\} = G_{inst}$,
3. $\{z \in G_{set} : \exists x \in G_{inst} (x \in_G z)\} = G_{set}$,
4. $y = \cup_G \{\{x\}_G : x \in_G y\}$, for any $y \in G_{set}$.

Theorem 7 describes the set granules analogously to the classical set theory. The following theorem answers question about the analogy of operation on diagnostic granules to set operations.

Theorem 14. In the granule database $\mathbf{G}_{diag}$, the diagnostic granules $y'^G, y \cap_G z, y \cup_G z, y \setminus_G z$, are described by formulas:

1. $y'^G = \cup_G \{z \in G_0 : \exists x \in G_{inst} (x \in_G z \wedge \neg x \in_G y)\}$,

2. $y \cap_G z \subseteq_G \cup_G \{t \in G_0 : \exists x \in G_{inst}(x \in_G t \wedge x \in_G y \wedge x \in_G z)\},$
3. $\cup_G \{t \in G_0 : \exists x \in G_{inst}(x \in_G t \wedge (x \in_G y \vee x \in_G z))\} \subseteq_G y \cup_G z,$
4. $y \setminus_G z = \cup_G \{t \in G_0 : \exists x \in G_{inst}(x \in_G t \wedge x \in_G y \wedge \neg x \in_G z)\}.$

Proof. Conditions (1) and (4) are fulfilled by axioms (15) and (16) of Definition 10 of the diagnostic granule system and of Fact 5. Conditions (2) and (3) are proofed based on Theorems 10 and 11 and Definition 10.

Proof of condition (2).

$$y \cap_G z \subseteq_G y \text{ and } y \cap_G z \subseteq_G z.$$

$$\text{Hence } \{t \in G_0 : t \subseteq_G y \cap_G z\} \subseteq \{t \in G_0 : t \subseteq_G y \wedge t \subseteq_G z\} = \{t \in G_0 : \exists x \in G_{inst}(x \in_G t \wedge x \in_G y \wedge x \in_G z)\}.$$

$$\text{Moreover, } y \cap_G z \subseteq_G \cup_G \{t \in G_0 : t \subseteq_G y \cap_G z\} \subseteq_G \cup_G \{t \in G_0 : t \subseteq_G y \wedge t \subseteq_G z\} = \cup_G \{t \in G_0 : \exists x \in G_{inst}(x \in_G t \wedge x \in_G y \wedge x \in_G z)\}.$$

Proof of condition (3).

$$y \subseteq_G y \cup_G z \text{ and } z \subseteq_G y \cup_G z.$$

$$\text{Hence } \{t \in G_0 : \exists x \in G_{inst}(x \in_G t \wedge (x \in_G y \vee x \in_G z))\} = \{t \in G_0 : t \subseteq_G y \vee t \subseteq_G z\} \subseteq \{t \in G_0 : t \subseteq_G y \cup_G z\}.$$

$$\text{Moreover } \cup_G \{t \in G_0 : \exists x \in G_{inst}(x \in_G t \wedge (x \in_G y \vee x \in_G z))\} \subseteq_G \cup_G \{t \in G_0 : t \subseteq_G y \cup_G z\} = y \cup_G z. \quad \square$$

6. An Example of a Standard Diagnostic Granule System

6.1. Standard Diagnostic Granule System

For any $\langle \mu_1, v_1 \rangle, \langle \mu_2, v_2 \rangle \in n\text{-ScIFS}, \mu \in [0, 1]$ there are the following formulas:

$$\text{S1. } \mu_1 \bullet_s v_1 = \max\{\mu_1, v_1\},$$

$$\mu_1 \bullet_t v_1 = \min\{\mu_1, v_1\},$$

$$\text{and then } f_s(x) = x, f_t(x) = 1 - x;$$

$$\lambda \bullet_l \mu = f_s^{-1}[\lambda f_s(\mu)] = \lambda \mu \text{ and}$$

$$\lambda \bullet_p \mu = f_t^{-1}[\lambda f_t(\mu)] = 1 - \lambda(1 - \mu);$$

$$\begin{aligned} \text{S2. } \langle \mu_1, v_1 \rangle \oplus \langle \mu_2, v_2 \rangle &= \\ \langle Sc_n^{-1}(Sc_n(\mu_1) \bullet_s Sc_n(\mu_2)), Sc_n^{-1}(Sc_n(v_1) \bullet_t Sc_n(v_2)) \rangle &= \\ \langle \max\{\mu_1, \mu_2\}, \min\{v_1, v_2\} \rangle, \end{aligned}$$

$$\begin{aligned} \text{S3. } \langle \mu_1, v_1 \rangle \otimes \langle \mu_2, v_2 \rangle &= \\ \langle Sc_n^{-1}(Sc_n(\mu_1) \bullet_t Sc_n(\mu_2)), Sc_n^{-1}(Sc_n(v_1) \bullet_s Sc_n(v_2)) \rangle &= \\ \langle \min\{\mu_1, \mu_2\}, \max\{v_1, v_2\} \rangle, \end{aligned}$$

$$\begin{aligned} \text{S4. } \lambda \langle \mu_1, v_1 \rangle &= \\ \langle Sc_n^{-1}(\lambda \bullet_l Sc_n(\mu_1)), Sc_n^{-1}(\lambda \bullet_p Sc_n(v_1)) \rangle &= \\ \langle Sc_n^{-1}(\lambda Sc_n(\mu_1)), Sc_n^{-1}(1 - \lambda(1 - Sc_n(v_1))) \rangle, \end{aligned}$$

$$\begin{aligned} \text{S5. } \langle \mu_1, v_1 \rangle \lambda &= \\ \langle Sc_n^{-1}(\lambda \bullet_p Sc_n(\mu_1)), Sc_n^{-1}(\lambda \bullet_l Sc_n(v_1)) \rangle &= \\ \langle Sc_n^{-1}(1 - \lambda(1 - Sc_n(\mu_1))), Sc_n^{-1}(\lambda Sc_n(v_1)) \rangle. \end{aligned}$$

It is assumed that:

- Symptom-fault diagnosis (s, f) corresponds to the diagnostic granule $p = (\mu_p, v_p) \in n\text{-ScIFS}$ defined by fuzzy sets $\mu_p : X \rightarrow [0, 1], v_p : X \rightarrow [0, 1]$ and the values of the function μ_i, v_j in the scale Sc_n satisfy the condition for any $x \in X$:

$$0 \leq Sc_n(\mu_p(x)) + Sc_n(v_p(x)) \leq 1. \quad (9)$$

- For any diagnostic granule $p = (\mu_p, v_p)$, granules λp is a result of parallel repetition of an alternative of the diagnosis p , and p^λ is the result of linear repetition of the conjunction of diagnosis p . This is due to the inaccuracy, uncertainty, or imprecision of the diagnosis results. The degree of diagnosis that requires repetitions is the number $\lambda \in (0, 1]$. If the diagnosis does not need to be repeated and is certain, then $\lambda = 1$. The diagnosis can be repeated multiple times until it is certain. For example, if we

want to repeat diagnosis maximally 5 times, then $\lambda = 1 - i/5$, where i is the number of repetitions, then $\lambda = 1, 4/5, 3/5, 2/5, 1/5, 0$.

Definition 13. The diagnostic granule system determined by the assumptions described above will be called the **standard diagnostic granule system**.

Theorem 15. In the standard diagnostic granules system with the scaling function Sc_n , for any granules $y, z \in n\text{-ScIFS}$, the following conditions apply:

1. $y \cap_G z =_{df} \cup_G \{t \in G_0 : \exists x \in G_{inst}(x \in_G t \wedge x \in_G y \wedge x \in_G z)\}$,
2. $y \cup_G z =_{df} \cup_G \{t \in G_0 : \exists x \in G_{inst}(x \in_G t \wedge (x \in_G y \vee x \in_G z))\}$.

Proof. The proof of the point (1):

$x = \langle a, \langle \mu_p(a), v_{p1}(a) \rangle \rangle \in_G \cup_G \{t \in G_0 : \exists x \in G_{inst}(x \in_G t \wedge x \in_G y \wedge x \in_G z)\}$
iff $\langle \mu_p(a), v_p(a) \rangle \leq_n \langle \mu_y(a), v_y(a) \rangle \wedge \langle \mu_p(a), v_p(a) \rangle \leq_n \langle \mu_z(a), v_z(a) \rangle$ iff $\mu_p(a) \leq \mu_y(a) \wedge v_y(a) \leq v_p(a) \wedge \mu_p(a) \leq \mu_z(a) \wedge v_z(a) \leq v_p(a)$.

Hence $\mu_p(a) \leq \inf\{\mu_y(a), \mu_z(a)\} \wedge \max\{v_y(a), v_z(a)\} \leq v_p(a)$ iff $\langle \mu_p(a), v_p(a) \rangle \leq_n \langle \inf\{\mu_y(a), \mu_z(a)\}, \max\{v_y(a), v_z(a)\} \rangle = \langle \mu_y(a), v_y(a) \rangle \otimes \langle \mu_z(a), v_z(a) \rangle$
iff $x = \langle a, \langle \mu_p(a), v_{p1}(a) \rangle \rangle \in_G y \cap_G z$.

Therefore $\cup_G \{t \in G_0 : \exists x \in G_{inst}(x \in_G t \wedge x \in_G y \wedge x \in_G z)\} \subseteq_G y \cap_G z$.

From Theorem 14: $y \cap_G z \subseteq_G \cup_G \{t \in G_0 : \exists x \in G_{inst}(x \in_G t \wedge x \in_G y \wedge x \in_G z)\}$.

Hence and from Theorem 11: $y \cap_G z = \cup_G \{t \in G_0 : \exists x \in G_{inst}(x \in_G t \wedge x \in_G y \wedge x \in_G z)\}$.

The proof of the point (2) is analogous. $\square$

6.2. The First Element of the Sequence Sc_n that Satisfies the $n\text{-ScIFS}$

Table 1 shows two **diagnostic generators** p_1, p_2 which define all considered diagnostic granules $p = \langle s, f \rangle$ belonging to $n\text{-ScIFS}$ using the operations described in Formulas (S1)–(S5).

Table 1. Symptoms, faults, fuzzy degree, and determination of the number n for $n\text{-ScIFS}$.

Diagnosis p	Object o	Symptom s	Fault f	Number n for $Sc_n(x) = x/n$	Number n for $Sc_n(x) = 1 - \log_{(1-1/n)} x$	Number n for $Sc_n(x) = x^n$
1	1	0.5	0.4	1	1	1
	2	0.8	0.8	2	3	4
	3	0.8	0.4	2	2	2
	4	0.5	0.9	2	2	3
	5	0.4	0.9	2	2	2
2	1	0.7	0.4	2	2	2
	2	0.9	0.4	2	2	2
	3	0.9	0.4	2	2	2
	4	0.8	0.6	2	2	3
	5	0.8	0.8	2	3	4

The number n satisfying the Definition of $n\text{-ScIFS}$ is searched for as follows.

For the diagnostic generator p , each pair (μ, v) has function values μ_p, v_p . Then, the *smallest* natural number n is searched, such that for each $x \in X$:

$$0 \leq Sc_n^{-1}(\mu_p(x)) + Sc_n^{-1}(v_p(x)) \leq 1, \quad (10)$$

$$0 \leq Sc_n(\mu_p(x)) + Sc_n(v_p(x)) \leq 1. \quad (11)$$

Then, $(\mu, v) \in n\text{-ScIFS}$.

From among the numbers determined in this way, n is selected as the largest, for all diagnosed objects $o \in X$. The largest number n for the map scale is 2, for the logarithmic scale it is 3, and for the power scale it is 4. According to Theorem 5, for the map scale $p \in n\text{-ScIFS}$, when $n > 1$, for the logarithmic scale $p \in n\text{-ScIFS}$ when $n > 2$, and for the power scale $p \in n\text{-ScIFS}$, when $n > 3$.

6.3. Examples of Calculations on Granules, Their Order, and Elements

On the basis of axioms (6) and (7) of Definition 10 and conditions S2 and S3 in the standard diagnostic granules system, the following calculation formulas are adopted for the diagnostic generators p_1, p_2 :

$$\text{S2a. } (p_1 \cup_G p_2)(o) = (p_1 \oplus p_2)(o) = \langle \mu_{p_1}(o), v_{p_1}(o) \rangle \oplus \langle \mu_{p_2}(o), v_{p_2}(o) \rangle = \langle \max\{\mu_{p_1}(o), \mu_{p_2}(o)\}, \min\{v_{p_1}(o), v_{p_2}(o)\} \rangle,$$

$$\text{S3a. } (p_1 \cap_G p_2)(o) = (p_1 \otimes p_2)(o) = \langle \mu_{p_1}(o), v_{p_1}(o) \rangle \otimes \langle \mu_{p_2}(o), v_{p_2}(o) \rangle = \langle \min\{\mu_{p_1}(o), \mu_{p_2}(o)\}, \max\{v_{p_1}(o), v_{p_2}(o)\} \rangle.$$

The calculation results are presented in Table 2.

Table 2. Calculation of $(p_1 \oplus p_2)(o) = \langle s, f \rangle$ and $(p_1 \otimes p_2)(o) = \langle s, f \rangle$ for the diagnostic generators $p_1, p_2 = \langle s, f \rangle$.

Object	$p_1(o)$		$p_2(o)$		$(p_1 \oplus p_2)(o)$		$(p_1 \otimes p_2)(o)$	
O	s	f	s	f	s	f	s	f
1	0.5	0.4	0.7	0.4	0.7	0.4	0.5	0.4
2	0.8	0.8	0.9	0.4	0.9	0.4	0.8	0.8
3	0.8	0.4	0.9	0.4	0.9	0.4	0.8	0.4
4	0.5	0.9	0.8	0.6	0.8	0.6	0.5	0.9
5	0.4	0.9	0.8	0.8	0.8	0.8	0.4	0.9

According to Table 2, $p_1 \cup_G p_2 = p_2$ and $p_1 \cap_G p_2 = p_1$. This is also due to the fact that $p_1 \subseteq_G p_2$, i.e., $p_1(o) \leq_n p_2(o)$ iff $\langle \mu_{p_1}(o), v_{p_1}(o) \rangle \leq_n \langle \mu_{p_2}(o), v_{p_2}(o) \rangle$ iff $\mu_{p_1}(o) \leq \mu_{p_2}(o)$ and $v_{p_2}(o) \leq v_{p_1}(o)$.

For example, $p_1(4) \leq_n p_2(4)$ iff $\langle \mu_{p_1}(4), v_{p_1}(4) \rangle \leq_n \langle \mu_{p_2}(4), v_{p_2}(4) \rangle$ iff $\langle 0.5, 0.9 \rangle \leq_n \langle 0.8, 0.6 \rangle$ iff $0.5 \leq 0.8$ and $0.6 \leq 0.9$.

Indeed, if $\mu_{p_1}(o) \leq \mu_{p_2}(o)$ and $v_{p_2}(o) \leq v_{p_1}(o)$, then from Formulas (S2a) and (S3a) we get:

- $(p_1 \cup_G p_2)(o) = \langle \max\{\mu_{p_1}(o), \mu_{p_2}(o)\}, \min\{v_{p_1}(o), v_{p_2}(o)\} \rangle = \langle \mu_{p_2}(o), v_{p_2}(o) \rangle = p_2(o),$
- $(p_1 \cap_G p_2)(o) = \langle \min\{\mu_{p_1}(o), \mu_{p_2}(o)\}, \max\{v_{p_1}(o), v_{p_2}(o)\} \rangle = \langle \mu_{p_1}(o), v_{p_1}(o) \rangle = p_1(o).$

The above formulas can also be analyzed in the context of determining the singletons of granules contained in the generators of granules or by examining the elements of these generators; Table 3 shows selected singletons contained in generator p_1 .

Table 3. Example singletons $e_1, e_2, e_3 = \langle s, f \rangle$ contained in the diagnostic generators $p_1 = \langle s, f \rangle$.

Object	$p_1(o)$		$e_1(o)$		$e_2(o)$		$e_3(o)$	
O	s	f	s	f	s	f	s	f
1	0.5	0.4	0	1	0	1	0.5	0.6
2	0.8	0.8	0	1	0	1	0	1
3	0.8	0.4	0.8	0.4	0.7	0.7	0	1
4	0.5	0.9	0	1	0	1	0	1
5	0.4	0.9	0	1	0	1	0	1

If $p_1(o_i) = \langle \mu_{p_1}(o_i), v_{p_1}(o_i) \rangle \neq \langle 0, 1 \rangle$, then $e(o_i) = \langle \mu_e(o_i), v_e(o_i) \rangle \leq_n \langle \mu_{p_1}(o_i), v_{p_1}(o_i) \rangle$ and $e(o_i) = \langle 0, 1 \rangle$ for $o \neq o_i$.

Thus e is a singleton contained in p : $e \in G_0, e \subseteq_G p_1$.

Hence $\langle o_i, \langle \mu_e(o_i), v_e(o_i) \rangle \rangle \in_G p_1$.

For example: $\langle 3, \langle \mu_{e_2}(3), v_{e_2}(3) \rangle \rangle = \langle 3, \langle 0.7, 0.7 \rangle \rangle \in_G p_1(3) = \langle 0.8, 0.4 \rangle$. Moreover $e_2 \subseteq_G e_1$.

6.4. Calculation of the Repeatability of Diagnostic Granules

Repeatability of diagnostic granules in the map scale

$Sc_n(x) = x/n, Sc_n^{-1}(x) = nx$, for $n > 0, x \in [0, 1], \langle s, f \rangle \in n\text{-ScIFS}$ (Example 3).

$$\lambda \langle s, f \rangle = \langle Sc_n^{-1}(\lambda Sc_n(s)), Sc_n^{-1}(1 - \lambda(1 - Sc_n(f))) \rangle = \langle n\lambda s/n, n(1 - \lambda(1 - f/n)) \rangle = \langle \lambda s, n - n\lambda + \lambda f \rangle,$$

$$\langle s, f \rangle^\lambda = \langle Sc_n^{-1}(1 - \lambda(1 - Sc_n(s))), Sc_n^{-1}(\lambda Sc_n(f)) \rangle = \langle n(1 - \lambda(1 - fs/n)), n\lambda s/n \rangle = \langle n - n\lambda + \lambda s, \lambda f \rangle.$$

Repeatability of diagnostic granules in the logarithmic scale.

$Sc_n(x) = 1 - \log_{(1-1/n)} x, Sc_1^{-1}(x) = x$ and for $n > 1, Sc_n^{-1}(x) = (1 - 1/n) \sim x$, for $n > 1, x \in (0, 1]$, when $Sc_1(x) = x$ and $Sc_n(0) = 0, \langle s, f \rangle \in n\text{-ScIFS}$ (Example 4), where $a \sim x =_df a^x$,

$$\lambda \langle s, f \rangle = \langle Sc_n^{-1}(\lambda Sc_n(s)), Sc_n^{-1}(1 - \lambda(1 - Sc_n(f))) \rangle = \langle (1 - 1/n) \sim \lambda(1 - \log_{(1-1/n)} s), (1 - 1/n) \sim (1 - \lambda(1 + \log_{(1-1/n)} f)) \rangle = \langle ((1 - 1/n) \sim \lambda)/s \sim \lambda, (1 - 1/n) \sim (1 - \log_{(1-1/n)}(f \sim \lambda)) \rangle = \langle ((1 - 1/n)/s) \sim \lambda, (1 - 1/n)/(f \sim \lambda) \rangle$$

$$\langle s, f \rangle^\lambda = \langle Sc_n^{-1}(1 - \lambda(1 - Sc_n(s))), Sc_n^{-1}(\lambda Sc_n(f)) \rangle = \langle (1 - 1/n)/(s \sim \lambda), ((1 - 1/n)/f) \sim \lambda \rangle.$$

Repeatability of diagnostic granules in the power scale

$Sc_n(x) = x^n, Sc_n^{-1}(x) = x^{1/n}$, for $n > 0, x \in [0, 1], \langle s, f \rangle \in n\text{-ScIFS}$ (Example 5).

$$\lambda \langle s, f \rangle = \langle Sc_n^{-1}(\lambda Sc_n(s)), Sc_n^{-1}(1 - \lambda(1 - Sc_n(f))) \rangle = \langle (\lambda s^n)^{1/n}, (1 - \lambda(1 - f^n))^{1/n} \rangle = \langle \lambda^{1/n} s, (1 - \lambda(1 - f^n))^{1/n} \rangle,$$

$$\langle s, f \rangle^\lambda = \langle Sc_n^{-1}(1 - \lambda(1 - Sc_n(s))), Sc_n^{-1}(\lambda Sc_n(f)) \rangle = \langle (1 - \lambda(1 - s^n))^{1/n}, \lambda^{1/n} f \rangle.$$

It is easy to see that for $\lambda = 1, \lambda \langle s, f \rangle = \langle s, f \rangle^\lambda = \langle s, f \rangle$ and for $\lambda = 0, \lambda \langle s, f \rangle = \langle 0, 1 \rangle, \langle s, f \rangle^\lambda = \langle 1, 0 \rangle$.

Furthermore, in Tables 4–7, it is assumed that the diagnosis can be repeated a maximum of 5 times until certainty is achieved, i.e., $\lambda = 1 - i/5$, where i is the number of repetitions, i.e., $\lambda = 1, 4/5, 3/5, 2/5, 1/5, 0$.

Table 4. Calculations of linear repeatability of granules in the power scale: λp_1 .

λ														
Object	p_1		1		0.8		0.6		0.4		0.2		0	
o	s	f	s	f	s	f	s	f	s	f	s	f	s	f
1	0.50	0.40	0.50	0.40	0.47	0.69	0.44	0.80	0.40	0.88	0.33	0.95	0	1
2	0.80	0.80	0.80	0.80	0.76	0.85	0.70	0.90	0.64	0.93	0.53	0.97	0	1
3	0.80	0.40	0.80	0.40	0.76	0.69	0.70	0.80	0.64	0.88	0.53	0.95	0	1
4	0.50	0.90	0.50	0.90	0.47	0.92	0.44	0.94	0.40	0.96	0.33	0.98	0	1
5	0.40	0.90	0.90	0.40	0.38	0.92	0.35	0.94	0.32	0.96	0.27	0.98	0	1

Table 5. Calculations of linear repeatability of granules in the power scale: λp_2 .

λ														
Object	p_2		1		0.8		0.6		0.4		0.2		0	
o	s	f	s	f	s	f	s	f	s	f	s	f	s	f
1	0.70	0.40	0.70	0.40	0.66	0.69	0.62	0.80	0.56	0.88	0.47	0.95	0	1
2	0.90	0.40	0.90	0.40	0.85	0.69	0.79	0.80	0.72	0.88	0.60	0.95	0	1
3	0.90	0.40	0.90	0.40	0.85	0.69	0.79	0.80	0.72	0.88	0.60	0.95	0	1
4	0.80	0.60	0.80	0.60	0.76	0.74	0.70	0.83	0.64	0.90	0.53	0.95	0	1
5	0.80	0.80	0.80	0.80	0.76	0.85	0.70	0.90	0.64	0.93	0.53	0.97	0	1

Tables 4 and 5 present the results of the calculation of the linear repeatability $\lambda p_1, \lambda p_2$ in the power scale $Sc_n(x) = x^n$:

- A decrease in the repeatability degree λ leads to a decrease of the symptom degree s ,
- A decrease in the repeatability degree λ leads to an increase of the faults degree f ,
- $\lambda p_1 \subseteq_G \lambda p_2$ according to the condition (5) of Theorem 10, because $p_1 \subseteq_G p_2$.

Table 6. Calculations of the power repeatability of granules in the power scale: p_1^λ .

λ														
Object	p_1		1		0.8		0.6		0.4		0.2		0	
o	s	f	s	f	s	f	s	f	s	f	s	f	s	f
1	0.50	0.40	0.50	0.40	0.71	0.38	0.81	0.35	0.89	0.32	0.95	0.27	1	0
2	0.80	0.80	0.80	0.80	0.85	0.76	0.90	0.70	0.93	0.64	0.97	0.53	1	0
3	0.80	0.40	0.80	0.40	0.85	0.38	0.90	0.35	0.93	0.32	0.95	0.27	1	0
4	0.50	0.90	0.50	0.90	0.71	0.85	0.81	0.79	0.89	0.72	0.98	0.60	1	0
5	0.40	0.90	0.40	0.90	0.69	0.85	0.80	0.79	0.88	0.72	0.98	0.60	1	0

Table 7. Calculations of the power repeatability of granules in the power scale: p_2^λ .

λ														
Object	p_2		1		0.8		0.6		0.4		0.2		0	
o	s	f	s	f	s	f	s	f	s	f	s	f	s	f
1	0.70	0.40	0.70	0.40	0.79	0.38	0.86	0.35	0.91	0.32	0.95	0.27	1	0
2	0.90	0.40	0.90	0.40	0.92	0.38	0.94	0.35	0.96	0.32	0.95	0.27	1	0
3	0.90	0.40	0.90	0.40	0.92	0.38	0.94	0.35	0.96	0.32	0.95	0.27	1	0
4	0.80	0.60	0.80	0.60	0.85	0.57	0.90	0.53	0.93	0.48	0.95	0.40	1	0
5	0.80	0.80	0.80	0.80	0.85	0.76	0.90	0.71	0.93	0.64	0.97	0.53	1	0

Tables 6 and 7 present the results of the calculations of the power repeatability of p_1^λ, p_2^λ on the power scale $Sc_n(x) = x^n$:

- A decrease in the repeatability degree λ leads to an increase of the symptom s degree,
- A decrease in the repeatability degree λ leads to a decrease of the faults f degree,
- $p_1^\lambda \subseteq_G p_2^\lambda$ according to the condition (5) of Theorem 10, because $p_1 \subseteq_G p_2$.

7. Conclusions

The paper formulates the following procedures:

1. Setting the sequences $\{Sc_n\}, \{Sc'_n\}$ of functions scaling the values of symptom attributes and faults, such that from a certain number n there is $0 \leq Sc_n(\mu) + Sc'_n(v) \leq 1$, for any results of diagnostic measurements $\mu, v \in [0, 1]$,
2. Determining the number n of the sequences $\{Sc_n\}, \{Sc'_n\}$ which satisfy the condition specified in the above procedure 1.

The language of diagnosis was defined as the language of the diagnosis granule system.

Moreover, it was shown that the diagnosis granule system (Definition 13) is induced by the diagnosis granule base (Definition 12). The model of such a system is the standard diagnostic granule system described in this paper. Models of other granule systems have been described in [20,42,43,45,46]. These papers also use many ideas contained in the information granulation literature on the granulation of information.

After years of theoretical work on information granulation, it is correct to assume that information granules are defined by some general axioms of the information granule database (Definition 12). Research on the granulation of self-diagnosis of information retrieval on the Web is also being developed and detailed, as well as in the field of heuristic methods of identifying information, such as analysis, synthesis, reduction, and induction [47]. Furthermore, the application of the theory of the diagnostic granules presented

in this paper, implemented as part of the project for the diagnosis of the reliability of the technical objects used in everyday life, will allow for the proper selection of scaling functions and triangular norms as well as standards of repetition of the diagnosis of these objects. This paper is both an elaboration of the syntax and semantics of the descriptive language of diagnosis, i.e., the system of diagnosis granules. For example, this language will make it possible to represent knowledge about cybersecurity of information retrieval on the web.

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