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Sub-Harmonic Response Analysis of Nonlinear Dynamic Behaviors Induced by Piecewise-Type Nonlinearities in a Torsional Vibratory System

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Abstract: Piecewise-type nonlinearities, such as clutch dampers in a torsional system, induce complex nonlinear dynamic behaviors that resemble super- and sub-harmonic responses. This study focuses on investigating the sub-harmonic responses induced by piecewise-type nonlinearities in the middle of various dynamic behaviors in a torsional vibratory system. To examine the dynamic characteristics in a sub-harmonic regime, the harmonic balance method (HBM) was implemented. Its results were compared with the numerical simulation (NS). To reveal the sub-harmonic responses, the input conditions of the HBM were modified with a small number of input values. In addition, bifurcation diagrams were numerically determined and projected onto stable and unstable solutions of the HBM to examine the effective dynamic behaviors within the unstable regimes. The results of the HBM with the modified input conditions reveal the sub-harmonic effects well, and the comparisons of bifurcation diagrams under unstable conditions lead to an understanding of the complex dynamic behaviors. Overall, this study suggests the first analytical technique to determine the sub-harmonic responses with the HBM, and second investigates the complex dynamic behaviors in a practical vibratory system by considering the bifurcations in the unstable regimes.

Keywords: harmonic balance method; piecewise-type nonlinearities; sub-harmonic response; bifurcation; multi-staged clutch damper; torsional system

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1. Introduction

Piecewise-type nonlinearities such as multi-staged clutch dampers used in a practical torsional system induce highly complex dynamic responses. In the middle of these nonlinear dynamic behaviors, sub-harmonic responses are relatively difficult to detect by employing the harmonic balance method (HBM), the basic matrix of which is constructed by the integer-based, incremental formulations [1–19]. In addition, various modifications have been implemented to calculate sub-harmonic responses. For example, nonlinear frequency response calculations of a torsional system with clearance-type nonlinearity have been developed by employing a multiterm HBM [1,2,5,8,10,12]. Our study suggested adapting the relevant time period for the sub-harmonic frequency values. Duan et al. suggested an excitation perturbation method to investigate the sub-harmonic resonance [10]. To capture the sub-harmonic effects, the authors modified the input conditions. For example, the relevant sub-harmonic input terms were artificially included, which triggered relevant sub-harmonic responses. In addition, various prior studies have discussed nonlinear problems by employing the HBM with respect to super- and sub-harmonic responses [2,7,10,12].

Despite the successful determination of the sub-harmonic responses, the stability conditions of the HBM often cannot adequately explain the practical dynamic conditions.

To understand the dynamic behaviors in complex stable and unstable conditions, bifurcation diagrams have been presented based on time-domain solutions. With regard to bifurcations, various approaches and results have been discussed. For instance, Al-shyyab and Kahraman investigated sub-harmonic and chaotic motions in a multi-mesh gear train by using a nonlinear time-varying dynamic model [5]. Detroux et al. examined bifurcation phenomena in a codimension-2 parameter space by using the HBM [14]. Xie et al. suggested the original extended system model for the detection and tracking of bifurcations by using the HBM with an arc-length continuation technique [15].

To the best of our knowledge based on the HBM and its relevant techniques, this study investigates the dynamic characteristics that occur in the sub-harmonic regimes, which are primarily concerned with a practical multi-staged clutch dampers that are employed in the conventional vehicle system. Thus, there are two specific objectives of this study. The first objective is to investigate the sub-harmonic regime which is just determined as unstable conditions based on the HBM. To capture the sub-harmonic resonances, the fictitious sub-harmonic input conditions will be employed numerically, which will advance the prior study [2]. The second objective is to examine the complex dynamic characteristics caused by a physical multi-staged clutch dampers by investigating the bifurcation diagrams that generally occur in the sub-harmonic unstable regimes. This will result in an increased understanding regarding the practical dynamic motions in a vehicle driveline system. In addition, this study focuses on one specific multi-staged clutch damper model in a torsional system with a single-degree-of-freedom (1 DOF) by limiting the sub-harmonic resonance areas around $\omega = 1$. In addition, Broader range of dynamic responses and their analysis of the bifurcation characteristics in the same system as described in Figure 1 can be referred [20].

Figure 1. A single-degree-of-freedom system with piecewise type nonlinearities: (a) a nonlinear torsional system model with 1DOF; (b) Torque $T_C(\delta_1)$ profile for a multi-staged clutch damper [1,2].

2. Problem Formulation with Multi-Staged Clutch Dampers

Figure 1a shows the nonlinear vibratory system with 1 DOF affected by piecewise-type nonlinearities based on multi-staged clutch dampers. The nonlinear model for piecewise-type nonlinearities is depicted in Figure 1b.

To investigate the dynamic characteristics, the employed parameters for the torsional system shown in Figure 1a are as follows [1,2]: inertia of flywheel, $I_f = 1.38 \times 10^{-1} \text{ kg}\cdot\text{m}^2$; viscous damping, $c_f = 1.59 \text{ N}\cdot\text{m}\cdot\text{s}/\text{rad}$. Here, the employed natural frequency ω_n is 115.6 (rad/s) from I_f and 4th stage of stiffness value listed in Table 1. In addition, θ_f is the absolute displacement of the flywheel (subscript f), as indicated in Figure 1a. The profiles of the clutch torque T_C are listed in Table 1, where k_{Ci} and H_i ($i = 1, 2, 3$, and 4) are the torsional stiffness and hysteresis at the i^{th} stage, respectively, and ϕ_{pi} (or ϕ_{ni}) is the transition angle at the positive (or negative) side [1,2]. Figure 1b illustrates the 3rd and

4th stages of stiffness and hysteresis areas. Based on the 1 DOF shown in Figure 1a, the equation of motion is derived as follows:

$$I_f \ddot{\theta}_f(t) + c \dot{\theta}_f(t) + f_n(\theta_f, \dot{\theta}_f) = T_E(t) - T_D. \quad (1)$$

Table 1. Properties of the real-life multi-staged clutch damper [1,2].

Property	Stage	Value
Torsional stiffness, k_{Ci} (linearized in a piecewise manner) (Nm/rad)	1	10.1
	2	61.8
	3	595.8
	4	1838.0
Hysteresis, H_i (Nm)	1	0.98
	2	1.96
	3	19.6
	4	26.5
Transition angle at positive side ($\theta_f > 0$), ϕ_{pi} (rad)	1	0.05
	2	0.16
	3	0.30
	4	0.39
Transition angle at negative side ($\theta_f < 0$), ϕ_{ni} (rad)	1	−0.04
	2	−0.05
	3	−0.09
	4	−0.15

Here, $f_n(\theta_f, \dot{\theta}_f)$ is a nonlinear function that describes the nonlinear clutch forces affected by piecewise-type nonlinearities. $T_E(t)$ and T_D are the sinusoidal input and drag torque, respectively.

In general, the input torque can be calculated using the Fourier coefficients based on the measured data, as follows:

$$T_E(t) = T_m + \sum_{i=1}^{N_{\max}} T_{pi} \cos(i\omega_p t + \phi_{pi}). \quad (2)$$

Here, T_m and T_{pi} are the mean and alternating parts of the input torque, respectively; ω_p and ϕ_{pi} are the excitation frequency and phase angle, respectively; and $N_{\max}$ is the maximum number of harmonics correlated with the harmonic index of the HBM. The input torque profiles employed are listed in Table 2. In this study, the drag torque is assumed as $T_D = T_m$ under steady-state conditions.

Table 2. List of the input torque profiles by assuming the WOT condition on the vehicle [1,2].

Torque Component		
Magnitude (Nm)		Phase (rad)
T_M	T_{p1}	
168.9	251.5	−1.93

In addition to the input conditions, $f_n(\theta_f, \dot{\theta}_f)$ as illustrated in Figure 1, is the main nonlinearity affected by multiple components, such as piecewise linear stiffnesses, hysteresis, and pre-load. Here, T_{SPr} is the total clutch torque induced by the pre-load, T_{pr1} (or T_{pr2}) is the positive (or negative) torque induced by the pre-load, and ϕ_{pr} is the angle located at the pre-load. Based on prior studies, the nonlinear function $f_n(\theta_f, \dot{\theta}_f)$ (or T_C) is derived as

follows [1,2]: First, the clutch torque induced by the stiffness $T_S(\theta_f)$ was defined using the smoothing factor σ_C .

$$T_S(\theta_f) = k_{C1}\theta_f + \frac{1}{2} \sum_{i=2}^N (k_{C(i)} - k_{C(i-1)}) (T_{sp(i-1)} - T_{sn(i-1)}), \quad (3a)$$

$$T_{sp(i)} = (\theta_f - \phi_{p(i)}) \left[\tanh\{\sigma_C(\theta_f - \phi_{p(i)})\} + 1 \right], \quad (3b)$$

$$T_{sn(i)} = (\theta_f + \phi_{n(i)}) \left[\tanh\{\sigma_C(\theta_f + \phi_{n(i)})\} - 1 \right]. \quad (3c)$$

Here, $k_{C(N)}$ (or $k_{C(i)}$) is the N^{th} (or i^{th}) stage of the clutch stiffness (with subscript N or i), $T_{sp(i)}$ (or $T_{sn(i)}$) is the positive (or negative) direction of the clutch torque induced by the stiffness at the i^{th} stage (with subscript p or n), and $\phi_{p(i)}$ (or $-\phi_{n(i)}$) is the i^{th} transition angle of the positive (or negative) side [3]. Second, T_H induced by hysteresis was derived using the smoothing factor σ_H [1,2].

$$T_H(\theta_f, \dot{\theta}_f) = \frac{H(N)}{2} \tanh(\sigma_H \dot{\theta}_f) + \sum_{i=2}^N \left(\frac{H(i)}{4} - \frac{H(i-1)}{4} \right) [T_{Hp(i-1)} + T_{Hn(i-1)}], \quad (4a)$$

$$T_{Hp(i)} = \tanh\{\sigma_C(\theta_f - \phi_{p(i)})\} [1 + \tanh(\sigma_H \dot{\theta}_f)], \quad (4b)$$

$$T_{Hn(i)} = \tanh\{\sigma_C(\theta_f + \phi_{n(i)})\} [1 - \tanh(\sigma_H \dot{\theta}_f)]. \quad (4c)$$

Here, H_N (or $H(i)$) is the N^{th} (or i^{th}) stage of hysteresis (with subscript N or i), and $T_{Hp(i)}$ (or $T_{Hn(i)}$) is the positive (or negative) side of the clutch torque induced by hysteresis at the i^{th} stage (with subscript p or n). The pre-load T_{SPr} was calculated as a function of θ_{1pr} .

$$T_{SPr}(\theta_{1pr}) = \frac{1}{2} T_{Pr1} [\tanh(\sigma_C \theta_{1pr}) + 1] + \frac{1}{2} T_{Pr2} [-\tanh(\sigma_C \theta_{1pr}) + 1], \quad (5a)$$

$$\theta_{1pr} = \theta_f - \phi_{Pr}. \quad (5b)$$

Overall, the total clutch torque is estimated by the summation of, $T_S(\theta_f)$, $T_H(\theta_f, \dot{\theta}_f)$, and $T_{SPr}(\theta_{1pr})$ from Equations (3)–(5), as follows:

$$f_n(\theta_f, \dot{\theta}_f) = T_C(\theta_{1pr}, \dot{\theta}_{1pr}) = T_S(\theta_{1pr}) + T_H(\theta_{1pr}, \dot{\theta}_{1pr}) + T_{SPr}(\theta_{1pr}). \quad (6)$$

The employed values for σ_C and σ_H are 1×10^3 and 0.1, respectively.

3. HBM with Modified Input Conditions

3.1. Basic Formulation of the HBM

From Equation (1), the Galerkin scheme of the governing equations is expressed as follows [2,12]:

$$-\omega^2 m \underline{\mathbf{H}} \underline{\mathbf{P}}'' \underline{\theta}_c + \omega c \underline{\mathbf{H}} \underline{\mathbf{P}}' \underline{\theta}_c + \underline{f}_n(\underline{\theta}_f, \dot{\underline{\theta}}_f) - \underline{F}_E(t) = \underline{0}. \quad (7)$$

Here, its corresponding formulas are defined as follows:

$$\underline{\theta}_f(t) = \underline{\mathbf{H}} \underline{\theta}_c, \quad (8a)$$

$$\underline{\theta}_f(t) = [\theta_f(t_0) \quad \theta_f(t_1) \quad \cdots \quad \theta_f(t_{m-2}) \quad \theta_f(t_{m-1})]^T, \quad (8b)$$

$$\underline{\theta}_c = [\theta_m \quad \theta_{a(1)} \quad \theta_{b(1)} \quad \cdots \quad \theta_{a(k)} \quad \theta_{b(k)} \quad \cdots \quad \theta_{a(\eta N_{\max})} \quad \theta_{b(\eta N_{\max})}]^T. \quad (8c)$$

$$\underline{\underline{\mathbf{H}}} = \begin{bmatrix} 1 & \cdots & \cos(k\Psi_0) & \sin(k\Psi_0) & \cdots \\ 1 & \cdots & \cos(k\Psi_1) & \sin(k\Psi_1) & \cdots \\ & \ddots & & & \ddots \\ 1 & \cdots & \cos(k\Psi_{N-2}) & \sin(k\Psi_{N-2}) & \cdots \\ 1 & \cdots & \cos(k\Psi_{N-1}) & \sin(k\Psi_{N-1}) & \cdots \end{bmatrix}, \underline{\underline{\mathbf{H}'}} = \omega \underline{\underline{\mathbf{H}}}\underline{\underline{\mathbf{P}'}}', \underline{\underline{\mathbf{H}''}} = -\omega^2 \underline{\underline{\mathbf{H}}}\underline{\underline{\mathbf{P}''}}', \quad (8d)$$

$$\underline{\underline{\mathbf{P}'}} = \begin{bmatrix} 0 & & & & \\ & \ddots & & & \\ & & \begin{bmatrix} 0 & k \\ -k & 0 \end{bmatrix} & & \\ & & & \ddots & \end{bmatrix}, \quad (8e)$$

$$\underline{\underline{\mathbf{P}''}} = \begin{bmatrix} 0 & & & & \\ & \ddots & & & \\ & & \begin{bmatrix} k^2 & 0 \\ 0 & k^2 \end{bmatrix} & & \\ & & & \ddots & \end{bmatrix}. \quad (8f)$$

Likewise, its nonlinear and input functions are expressed using the same schemes as follows:

$$\underline{\underline{f_n}}(\underline{\underline{\theta_f}}, \dot{\underline{\underline{\theta_f}}}) = \underline{\underline{\mathbf{H}}}\underline{\underline{f_{nc}}}, \quad (9a)$$

$$\underline{\underline{F_E}}(t) = \underline{\underline{\mathbf{H}}}\underline{\underline{F_{Ec}}}, \quad (9b)$$

$$\underline{\underline{f_{nc}}} = \begin{bmatrix} f_m & f_{a(1)} & f_{b(1)} & \cdots & f_{a(k)} & f_{b(k)} & \cdots & f_{a(\eta N_{\max})} & f_{b(\eta N_{\max})} \end{bmatrix}^T, \quad (9c)$$

$$\underline{\underline{F_{Ec}}} = \begin{bmatrix} F_m & F_{a(1)} & F_{b(1)} & \cdots & F_{a(k)} & F_{b(k)} & \cdots & F_{a(\eta N_{\max})} & F_{b(\eta N_{\max})} \end{bmatrix}^T. \quad (9d)$$

Here, let $\omega t = \Psi$ and

non-dimensionalize the time scale and normalize the frequency values. Based on this relationship and by considering the effective time period with the sub-harmonic effect, the relevant time ranges are defined as $0 \leq t < T \rightarrow$, where $0 \leq \Psi < \frac{2\pi}{\omega_n} T = \eta \tau$, where η is a sub-harmonic index and τ is the fundamental frequency. Thus, the index k in Equation (8) is incremented by $k = \omega_n, 2\omega_n, 3\omega_n \cdots$ which includes the non-dimensionalized factor ω_n . From this relationship, $\dot{\theta}(t) = \frac{d\theta}{dt} = \omega \frac{d\theta}{d\Psi} = \omega \theta'$ because $\omega dt = d\Psi$. Likewise, $\ddot{\theta}(t) = \omega^2 \theta''$. The matrices and coefficient vectors are constructed along with the incremental indices k and η . In addition, the range of the normalized time period should be adapted based on η . By substituting Equations (8) and (9) into Equation (7), the overall Galerkin scheme of the basic equation is defined as follows.

$$-\omega^2 m \underline{\underline{\mathbf{H}}}\underline{\underline{\mathbf{P}''}}' \underline{\underline{\theta_c}} + \omega c \underline{\underline{\mathbf{H}}}\underline{\underline{\mathbf{P}'}}' \underline{\underline{\theta_c}} + \underline{\underline{\mathbf{H}}}\underline{\underline{f_{nc}}} - \underline{\underline{\mathbf{H}}}\underline{\underline{F_{Ec}}} = 0. \quad (10)$$

Thus,

$$\underline{\underline{\mathbf{H}}}\underline{\underline{\Psi}} = 0, \quad (11a)$$

$$\underline{\underline{\Psi}} = -\omega^2 m \underline{\underline{\mathbf{P}''}}' \underline{\underline{\theta_c}} + \omega c \underline{\underline{\mathbf{P}'}}' \underline{\underline{\theta_c}} + \underline{\underline{f_{nc}}} - \underline{\underline{F_{Ec}}} = 0. \quad (11b)$$

To determine the solutions for $\underline{\underline{\theta_c}}$ in Equations (10) and (11) with their relevant values ω , the Newton-Raphson method is used by considering the condition $\underline{\underline{\Psi}} \rightarrow 0$ where $\underline{\underline{\Psi}}$ is considered as a function of $\underline{\underline{\theta_c}}$ and ω such as $\underline{\underline{\Psi}}(\underline{\underline{\theta_c}}, \omega)$. With respect to the implementation of the Newton-Raphson method, the Jacobian matrices for $\underline{\underline{\theta_c}}$ and ω must be calculated for

the condition $\underline{\Psi} \cong 0$. First, the Jacobian matrix $\underline{J}_c$ is considered $\underline{\Psi}$ as a function of $\underline{\theta}_c$, and its equation is as follows:

$$\underline{J}_c = \frac{\partial \underline{\Psi}}{\partial \underline{\theta}_c} = -\omega^2 m \underline{\mathbf{P}}'' + \omega c \underline{\mathbf{P}}' + \frac{\partial f_{nc}}{\partial \underline{\theta}_c}. \quad (12)$$

Here, to calculate $\frac{\partial f_{nc}}{\partial \underline{\theta}_c}$, the derivatives of $\underline{\theta}_f(t)$ and $\dot{\underline{\theta}}_f(t)$ must be implemented as follows [2]:

$$\frac{\partial f_{nc}}{\partial \underline{\theta}_c} = \underline{\mathbf{H}}^+ \frac{\partial f_n(\underline{\theta}, \dot{\underline{\theta}})}{\partial \underline{\theta}(t)} \underline{\mathbf{H}} + \omega \underline{\mathbf{H}}^+ \frac{\partial f_n(\underline{\theta}, \dot{\underline{\theta}})}{\partial \dot{\underline{\theta}}(t)} \underline{\mathbf{H}} \underline{\mathbf{P}}', \quad (13a)$$

$$\underline{\mathbf{H}}^+ = (\underline{\mathbf{H}}^T \underline{\mathbf{H}})^{-1} \underline{\mathbf{H}}^T. \quad (13b)$$

where $\underline{\mathbf{H}}^+$ is the pseudo-inverse matrix. In addition, $\frac{\partial f_n(\underline{\theta}_f, \dot{\underline{\theta}}_f)}{\partial \underline{\theta}_f(t)}$ and $\frac{\partial f_n(\underline{\theta}_f, \dot{\underline{\theta}}_f)}{\partial \dot{\underline{\theta}}_f(t)}$ are diagonal matrices defined as follows:

$$\frac{\partial f_n(\underline{\theta}_f, \dot{\underline{\theta}}_f)}{\partial \underline{\theta}_f(t)} = \text{diag} \left[\frac{\partial f_n(\underline{\theta}_f, \dot{\underline{\theta}}_f)}{\partial \underline{\theta}_f(t_0)} \frac{\partial f_n(\underline{\theta}_f, \dot{\underline{\theta}}_f)}{\partial \underline{\theta}_f(t_1)} \dots \frac{\partial f_n(\underline{\theta}_f, \dot{\underline{\theta}}_f)}{\partial \underline{\theta}_f(t_{m-2})} \frac{\partial f_n(\underline{\theta}_f, \dot{\underline{\theta}}_f)}{\partial \underline{\theta}_f(t_{m-1})} \right], \quad (14a)$$

$$\frac{\partial f_n(\underline{\theta}_f, \dot{\underline{\theta}}_f)}{\partial \dot{\underline{\theta}}_f(t)} = \text{diag} \left[\frac{\partial f_n(\underline{\theta}_f, \dot{\underline{\theta}}_f)}{\partial \dot{\underline{\theta}}_f(t_0)} \frac{\partial f_n(\underline{\theta}_f, \dot{\underline{\theta}}_f)}{\partial \dot{\underline{\theta}}_f(t_1)} \dots \frac{\partial f_n(\underline{\theta}_f, \dot{\underline{\theta}}_f)}{\partial \dot{\underline{\theta}}_f(t_{m-2})} \frac{\partial f_n(\underline{\theta}_f, \dot{\underline{\theta}}_f)}{\partial \dot{\underline{\theta}}_f(t_{m-1})} \right]. \quad (14b)$$

Second, the Jacobian matrix $\underline{J}_\omega$ by differentiating with respect to ω can be obtained from Equation (11), as follows:

$$\underline{J}_\omega = \frac{\partial \underline{\Psi}}{\partial \omega} = \left(-2\omega m \underline{\mathbf{P}}'' + c \underline{\mathbf{P}}' + \underline{\mathbf{H}}^+ \frac{\partial f_n(\underline{\theta}_f, \dot{\underline{\theta}}_f)}{\partial \underline{\theta}_f(t)} \underline{\mathbf{H}} \underline{\mathbf{P}}' \right) \underline{\theta}_c. \quad (15)$$

Overall, the Jacobian matrix with respect to the two parameters ($\underline{\theta}_c$ and ω) and the augmented vector parameterized by ω are as follows:

$$\underline{J} = \begin{bmatrix} \underline{J}_c & \underline{J}_\omega \end{bmatrix}, \quad (16a)$$

$$\underline{\theta}_a = \begin{bmatrix} \underline{\theta}_c \\ \omega \end{bmatrix}. \quad (16b)$$

To obtain the solutions for each step based on the arc-length continuation scheme, the Newton–Raphson technique is conducted as follows:

$$\underline{\theta}_a^{(k+1)} = \underline{\theta}_a^{(k)} - [\underline{J}^+ \underline{\Psi}(\underline{\theta}_c, \omega)]^{(k)}. \quad (17)$$

Here, the pseudo-inverse matrix of the Jacobian is estimated by $\underline{J}^+ = \left[(\underline{J}^T \underline{J})^{-1} \underline{J}^T \right]$. In addition, more detailed derivations and techniques for this system in Figure 1 can be found in [2].

3.2. Initial Results of the Basic HBM and Modification of the Input Conditions

Figure 2 as initial results shows a comparison of the two results obtained by HBM and NS, where the HBM is conducted with $\eta = 2$ and $N_{\max} = 12$. To obtain the NS solutions, the modified Runge-Kutta method was employed in this study [1,21]. However, the HBM cannot reveal the sub-harmonic responses even though the solutions of the HBM follow

the NS results well. This indicates that the sub-harmonic index η does not work properly. As for the detailed employment of the NS, it concerned research that can be referred [1,21].

Figure 2. Nonlinear frequency responses with RMS values: (a) comparison of HBM with NS; (b) super-harmonic regions; (c) sub-harmonic region. Key: —, HBM result with $\eta = 2$ and $N_{max} = 12$; $\circ$, NS result with frequency up-sweeping; +, NS result with frequency down-sweeping.

For example, the super-harmonic responses from both the HBM and NS are well-correlated, as shown in Figure 2b. However, the sub-harmonic regime in Figure 2c shows that the two results of HBM and NS are slightly different because the HBM does not capture the sub-harmonic responses satisfactorily, and its solutions generally reflect the unstable conditions based on Hill's method [2,12,17]. However, the NS still projects sub-harmonic responses satisfactorily since the resonance from the calculated results are clearly obtained in Figure 2. Thus, to overcome these discrepancies in the HBM, techniques to trigger the sub-harmonic effects should be implemented [10]. From Equation (9) and its employed values in Table 2, the valid components of the input torque vector with $\eta = 2$ and $N_{max} = 12$ are given as, $F_m = 168.9$, $F_{a(2)} = -87.97$, and $F_{b(2)} = 235.65$. To trigger the sub-harmonic responses, the components of the input torque pertaining to the sub-harmonic locations, such as $F_{a(1)}$ and $F_{b(1)}$ can be assigned with small values. In this study, $F_{a(1)} = \varepsilon F_{a(2)}$ and $F_{b(1)} = \varepsilon F_{b(2)}$ were used with $\varepsilon = 1 \times 10^{-5}$. With a small range of values multiplied by the fundamental input torque components, the sub-harmonic responses are effectively triggered, as shown in Figure 3. The results in Figure 3a,b successfully capture the sub-harmonic effects with the modified input conditions. Here, regardless of the number of N_{max} employed, such as $N_{max} = 1$ and $N_{max} = 12$, the sub-harmonic effects are clearly detectable.

Figure 3. Nonlinear frequency responses of HBM reflecting sub-harmonic effects with maximum, mean and minimum values of $\theta_f(t)$: (a) frequency responses of max, mean and min values with $\eta = 2$ and $N_{max} = 1$; (b) frequency responses of max, mean and min values with $\eta = 2$ and $N_{max} = 12$.

4. Comparison of the Sub-Harmonic Responses from HBM and NS

Figure 4 compares the results of the HBM under modified input conditions with the NS solutions. When the sub-harmonic responses are examined carefully, as illustrated in Figure 4b, the HBM still reflects the discrepancies. As marked with red lines and characters (A) and (B) in Figure 4b, the calculated results of the HBM at regimes less than (A) or greater than (B) correlate well with those from NS. However, the HBM solutions in the regimes between (A) and (B) are under unstable conditions, and their results do not show good agreement with the NS solutions. To examine the dynamic differences between both methods, the time histories and FFT results can be compared, as shown in Figures 5–7. Figure 5 compares the time histories of the HBM and NS at two different frequencies. Figure 5a shows the time histories based on both HBM and NS at $\omega = 111.2 \left(\frac{\text{rad}}{\text{s}} \right)$ (or 17.7 Hz) that are observed in the regimes below (A) marked in Figure 4b.

Figure 4. Cont.

Figure 4. Comparison of nonlinear frequency responses with RMS values: (a) HBM vs. NS in a whole range of frequency value; (b) HBM vs. NS focused on a sub-harmonic range. Key: —, stable solution from HBM with $N_{\max} = 12$ and $\rho = 2$; ·····, unstable solution from HBM with $N_{\max} = 12$ and $\rho = 2$; ○, NS result with frequency up-sweeping; +, NS result with frequency down-sweeping.

Figure 5. Comparison of time histories between HBM ($N_{\max} = 12$ and $\rho = 2$) and NS at different excitation conditions: (a) time histories at 17.7 Hz; (b) time histories at 18.8 Hz. Key: ○, HBM; —, NS.

As shown in Figure 4, the two calculated time histories coincide with each other. However, the time histories of the HBM at $\omega = 118.1 \left(\frac{\text{rad}}{\text{s}} \right)$ (or 18.8 Hz) do not show good agreement with the NS results, as shown in Figure 5b, because the HBM generally obtains solutions based on integer-based incremental harmonics. These differences between the HBM and NS are clearly observed in Figures 4b and 5b.

In addition, the FFT results reflect the discrepancies between the two methods in unstable regimes. Figure 6a shows that the two FFT results at $\omega = 111.2 \left(\frac{\text{rad}}{\text{s}} \right)$ (or 17.7 Hz) are correlated with each other. However, the FFT results in the unstable regime with $\omega = 118.1 \left(\frac{\text{rad}}{\text{s}} \right)$ (or 18.8 Hz) show discrepancies between the HBM and NS results. The two results still include the sub-harmonic terms clearly identifiable at $\omega = 59.1 \left(\frac{\text{rad}}{\text{s}} \right)$ (or 9.4 Hz), as shown in Figure 6b. Thus, the unstable conditions estimated by the HBM should have a practical meaning that the dynamic behaviors are affected by more complex factors, even

though the HBM has limitations owing to the integer-based harmonic term simulation. These phenomena result in quasi-harmonic or chaotic responses [22].

Figure 6. Comparison of FFT results between HBM ($N_{\max} = 12$ and $\rho = 2$) and NS at different excitation conditions: (a) FFT results at 17.7 Hz; (b) FFT results at 18.8 Hz. Key: $\square$, HBM; $\circ$, NS.

Figure 7. Comparisons of phase diagrams calculated by HBM ($N_{\max} = 12$ and $\rho = 2$) and NS at different excitation conditions: (a) phase diagrams at 17.7 Hz; (b) phase diagrams at 18.8 Hz. Key: $\cdots$, HBM; — , NS.

Figure 7 compares the phase-plane diagrams for two different frequency regimes with θ_f and $\dot{\theta}_f$. As seen previously, the phase planes at $\omega = 111.2 \left(\frac{\text{rad}}{\text{s}} \right)$ (or 17.7 Hz) from the two methods are correlated with each other. In Figure 7b, both phase-plane diagrams reveal two different effective cycles caused by the fundamental and sub-harmonic responses. In particular, the NS shows more complicated dynamic behaviors that contain many more harmonic terms, which is also confirmed by the number of harmonic components, as shown in Figure 6b.

5. Dynamic Characteristics with Bifurcation Diagrams in Sub-Harmonic Regimes

To understand the nonlinear dynamic behaviors, the bifurcation diagrams should be efficient, specifically under the frequency sweeping conditions based on the time-domain solutions. Figures 8a and 9a show the maximum θ_{rms} , mean, and minimum values of $\theta_f(t)$ with stable conditions from the HBM. Along with these results, 8b and 9b reflect the bifurcations that occur in the sub-harmonic regimes. To obtain results from the bifurcation phenomena based on the NS, the solutions for the steady state responses were captured by assuming that the transient responses are completely removed after 100 cycles while frequency sweeping was conducted with 5×10^{-3} a frequency step such as $\Delta\omega = 5 \times 10^{-3}$. Since the HBM for this study is parameterized by including, ω as described in Equations (15)–(17), the frequency responses shown in Figures 8a and 9a already include the information of bifurcation areas where unstable conditions are generally observed. Here, to obtain the bifurcation phenomena from the NS, the max, mean, min and rms values for each period are captured repeatedly. In addition, the bifurcations in Figures 8b and 9b show the common dynamic characteristics in terms of θ_{rms} , max, mean, and min values of $\theta_f(t)$ within the sub-harmonic regimes.

To investigate the dynamic behaviors in the sub-harmonic regimes in more detail, finer temporal steps of NS were conducted and compared with the HBM solutions, as shown in Figures 10 and 11. For instance, bifurcations from NS were obtained with 300 cycles and $\Delta\omega = 5 \times 10^{-4}$. As shown in Figures 10 and 11, nonlinear dynamic behaviors are clearly observed with respect to the stable and unstable conditions in the sub-harmonic areas. The various bifurcation characteristics are recognized with locations (1), (2), (3), and (4) locations, as illustrated in Figure 10. First, the comparisons between the HBM stable regimes and NS solutions are perfectly matched with each other in the areas prior to (1), and the period doubling occurs at location (1), where the unstable conditions start based on the HBM. Second, another period doubling effect is observed in regime (2). Third, more period doubling and complicated behaviors are clearly seen along with the upper and lower branches of solutions between regimes (2) and (3). Finally, the bifurcations still exist between regimes (3) and (4) even though the HBM indicates stable conditions, and finally the solutions from NS comply with the HBM solutions. The maximum, mean, and minimum values of $\theta_f(t)$ also follow the same dynamic characteristics, as shown in Figure 11. Thus, the nonlinear dynamic behaviors can be examined and understood more efficiently by employing both stability and bifurcation analyses.

Figure 8. Stability conditions from HBM with RMS and its bifurcation diagram at the sub-harmonic region: (a) stable and unstable conditions in a frequency response with RMS; (b) bifurcation diagram of the sub-harmonic responses. Key: $\bigcirc$, stable solutions of HBM; $+$, unstable solutions of HBM; $\bullet$, numerical results of the bifurcation.

Figure 9. Stability conditions from HBM with max, mean and min values, and its bifurcation diagrams at the sub-harmonic region: (a) stable and unstable conditions in a frequency response with max, mean and min values; (b) bifurcation diagram of the sub-harmonic responses. Key: $\bigcirc$, stable solutions of HBM; $+$, unstable solutions of HBM; $\bullet$, numerical results of the bifurcation.

Figure 10. Bifurcation diagram projected in the sub-harmonic region calculated by HBM with RMS. Key: —, stable solutions of HBM; +, unstable solutions of HBM; ●, numerical results of the bifurcations.

Figure 11. Bifurcation diagrams projected in the sub-harmonic region calculated by HBM with max, mean and min values. Key: —, stable solutions of HBM; +, unstable solutions of HBM; ●, numerical results of the bifurcations.

6. Conclusions

This study examined nonlinear dynamic characteristics by focusing on sub-harmonic regimes. To examine the nonlinear system responses, both the HBM and NS were employed, and their results were compared. For example, the overall system responses were examined based on the HBM by including the super- and sub-harmonic responses. In order to reveal the sub-harmonic responses, the additional numerical techniques were implemented. Then, the stable and unstable responses have been investigated by comparing the NS results which could lead us to understanding the relationships of stability conditions to the bifurcations well. For the contributions of this article, we first investigated the sub-harmonic responses which are determined as unstable conditions based on the HBM. In addition, numerical techniques have been suggested to reveal sub-harmonic responses regarding HBM. Second, to understand the nonlinear dynamic behaviors which are normally determined as unstable conditions, bifurcation analysis was implemented and compared with the frequency responses by focusing on the sub-harmonic regimes. This could result in an

improved understanding of the nonlinear dynamic behaviors of a practical system and suggest a complementary analysis with both HBM and NS.

This work was conducted to examine mostly the sub-harmonic effects with respect to torsion vibratory motions. However, there are still various deviations and complex behaviors in the course of frequency sweeping conditions affected by super-harmonics. In addition, the various conditions for the broader range of sub-harmonic responses will be investigated as for the next stage of research.

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