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A Single Equation to Depict Bottomhole Pressure Behavior for a Uniform Flux Hydraulic Fractured Well

Jang Hyun Lee * and Juhairi Aris Bin Muhamad Shuhili

Petroleum Engineering Department, Universiti Teknologi PETRONAS, Seri Iskandar 32610, Perak, Malaysia; juhairi.shuhli@utp.edu.my

* Correspondence: lee.janghyun@utp.edu.my

Abstract: Pressure transient analysis for a vertically hydraulically fractured well is evaluated using two different equations, which cater for linear flow at the early stage and radial flow in the later stage. However, there are three different stages that take place for an analysis of pressure transient, namely linear, transition and pseudo-radial flow. The transition flow regime is usually studied by numerical, inclusive methods or approximated analytically, for which no specific equation has been built, using the linear and radial equations. Neither of the approaches are fully analytical. The numerical, inclusive approach results in separate calculations for the different flow regimes because the equation cannot cater for all of the regimes, while the analytical approach results in a difficult inversion process to compute well test-derived properties such as permeability. There are two types of flow patterns in the fracture, which are uniform and non-uniform, called infinite conductivity in a high conductivity fracture. The study was conducted by utilizing an analogous study of linear flow equations. Instead of using the conventional error function, the exponential integral with an infinite number of wells was used. The results obtained from the developed analytical solution matched the numerical results, which proved that the equation was representative of the case. In conclusion, the generated analytical equation can be directly used as a substitute for current methods of analyzing uniform flow in a hydraulically fractured well.

Keywords: pressure transient analysis; hydraulically fractured well; uniform flow

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1. Background Study

Fluid flow by a vertical hydraulic fractured well starts with linear flow through an elliptical flow. Neither of the approaches are fully analytical. The numerical, inclusive approach results in separate calculations for the [1] different flow regimes because the equation cannot cater for all of the regimes, while the analytical approach results in a difficult inversion process to compute well test-derived properties such as permeability. The original linear flow equation to define uniform flow into a vertical hydraulically fractured well was generated by Gringarten [2]. The equation was generated because the Everdingen–Hurst equation, whereby it stimulates a well test-described negative skin effect, has a different early time pattern. In the past, the finite-difference or elements method could not be used to compute the transition period due to limited resources of computation power, which is an obsolete issue in today's world. However, even with the capability of the numerical method, this results in a difficult inversion process to compute for well test-derived properties such as permeability. In this study, instead of relying on a numerically-derived equation, which is not inversion friendly, an analytical equation had been constructed using the concept of an exponential integral with an infinite number of imaginary wells. This analytical equation did not only provide a more accurate solution but also made an equation that is completely inversible. This inversion process is very important to derive properties such as permeability, which will be used later in many reservoir and production engineering applications such as reservoir simulation and the

computation of productivity index (depending on flow regime, i.e., transient, pseudo steady state and steady state. The equation for uniform flux into a vertical fracture is described in Gringarten's study. Equation (1) is the equation used to relate the pressure changes with respect to time in a fractured reservoir. However, the equation is expressed in a dimensionless form and is very specifically for linear flow only. We managed to build an equation that works in all of the flow regimes including after the linear flow when the external boundary presence is greater than the well or fracture boundary due to the well-travelled radius of investigation. The equation built also managed to compute pressure changes at any time frame since the equation built is independent of flow regimes.

$$P_D(x_D, y_D, t_D) = \int_0^{t_D} \exp\left(-\frac{y_D^2}{4t'_D}\right).\left[erf\frac{1-x_D}{2\sqrt{t'_D}} + erf\frac{1+x_D}{2\sqrt{t'_D}}\right]\left(\frac{dt'_D}{4\left(t'_D/\pi\right)^{\frac{1}{2}}}\right) \quad (1)$$

Equation (1). Uniform Flux Vertical Fracture Equation.

The equation cannot be directly evaluated and, in a plane of fracture, the equation became $y_D = 0$. This equation differs from the uniform conductivity equation because the pressure varies along the fracture (except for early stages), whereas in an infinite equation, the pressure does not drop. However, the pressure drop along the fracture is considered negligible for uniform flow and the fracture conductivity is still high. Some fields matched the uniform flow pattern better than the infinite fracture conductivity, as mentioned in Gringarten's study [2]. Uniform flux assumption for fractured well pressure transient analysis was also used in quantifying the effect of the threshold pressure gradient in initiating flow. In a few instances, oil in a reservoir behaves in the same way as the Bingham fluid model in which a minimum pressure difference is required to initiate the flow, which is known as the threshold pressure gradient that was studied in a well test interpretation under the assumption of uniform flux, similar to the assumption that we used in our study [3]. Uniform flux assumption was also used in a study to evaluate the pressure behavior of a fractured horizontal well with the threshold pressure gradient consideration. In this study, the global flow or transient pressure progressed through the network of pressure indicating a constant rate or flux with changes in pressure only. The results obtained from the study were validated using a numerical method and the equation developed matched the response of a fractured well in Xinjiang Oil Field in China [4]. Thus, the assumption used in our study, which considered uniform flux assumption instead of infinite fracture conductivity, is well studied in the literature as well.

A study conducted on a hydraulically fractured well created a model that was hybrid in nature and that modelled the well's pressure distribution numerically but analytically at the reservoir condition. The reason behind the numerical approach being used to compute the fracture performance close to the wells is because it can detail the properties of the fracture, such as the non-uniform shape of the fractures being considered. Another reason for the numerical modelling in the form of the flow into the wellbore was because the flow was non-conformant to Darcy's flow, which actually created a non-Darcy flow. This non-Darcy flow created an additional pressure drop due to flow convergence around the wellbore, which actually necessitates the numerical model [1]. This model also predicted the performance of a hydraulically fractured well but resorted to a numerical approach for the performance in the vicinity of the wellbore. The drawback of this study was that a numerical approach was utilized, which is less accurate compared to the analytical approach used in our study.

A method was developed to study the pressure transient analysis in fractured horizontal wells with fracture networks. In this work, line source function was used to create the model. The flow at the intersection between fractures was modelled using nodal analysis

and diffusivity equation. Finally, the numerical inversion was conducted using Laplace Transformation. There were a few flow regimes that formed in this particular study, namely first bilinear flow, 'MF-HF' support, the second bilinear flow, formation linear flow, cross flow and pseudo-radial flow. This method has more consideration such as the inclusion of the fracture's intersection but still requires the numerical method to perform the inversion. Any inclusion of the numerical method results in less accuracy. On top of that, the equation that we built has a more general application whereby a single equation can cater for all flow regimes [5].

A study was conducted in an infinite slab reservoir by considering infinite and finite conductivity fractures by assuming uniform flux with a constant drawdown rate. The variables in this study were properties of fractures such as the half-length, conductivity and angles. An analytical solution was used to analyze pressure transient testing for the mid time stage while the early and late time stages were analyzed by using the semi-analytical model. This approach was very different from our equation, which actually catered for all of the time regions [6]. The equation built by us also showed some improvement during a study conducted in a horizontal well, which tested the performance of a single fracture and multiple fractures. In that study, the performance of both scenarios was tested using the numerical approach, which was later transformed in the Laplace space. The analytical approach always has a superiority over the numerical approach due to the accuracy presented. In our case, all of the flow regimes were evaluated using a single equation instead of having different equations designated for each flow regime or without resorting to the numerical approach [7].

A study was conducted modelling a hydraulic fracture in a tight formation such as a shale gas reservoir with pre-existing natural fractures. Unlike other approaches, the authors of the study relied on modelling the fractures from a close range instead of observing the model as a dual porosity model. The source function was solved analytically using the superposition principle, but flows from in between the fractures were modelled using the numerical approach rendering the works less accurate in comparison to the work we conducted, which completely relied on a single analytical equation from the beginning to the end [8]. Individual flow regimes were evaluated using different equations in a study conducted on pressure transient analysis of multi-fracture horizontal wells whereby at least four different flow regimes were expected to occur during the entire flow period, namely fracture radial, radial-linear, formation linear and pseudo-radial. In our study, all of the flow regimes to occur were modelled using a single analytical equation, which also showed a good match with the numerical study. The equation we created removes the hassle of turning to a new equation whenever a new flow regime comes in [9].

Instead of separating the radial flow from the fracture to the wellbore and the linear flow from the reservoir to the fracture, a new method called trilinear flow was established in a study to find an approximation for the analytic solution of finite-conductivity vertical fractures. However, this solution was only applicable for a short-time analysis, which definitely could not predict the pseudo-radial flow that appeared after the linear flow into the fractures. Unlike the trilinear approach, the equation we built can cater for all of the flow regimes regardless of the time span. The equation that we built is equally as accurate too since the equation built is an analytical equation [10]. A study was conducted to investigate the performance in terms of the productivity index and drainage area for a fractured horizontal well in a tight reservoir where the equation used was similar to well testing for a gas reservoir and the pseudo-pressure function. However, the drawback of this pseudo-function is that it is a semi-analytical model, which is subjected to a lack of accuracy that can contribute to false results for the overall analysis. The equation built by us is completely an analytical equation, which cannot be affected by the same fallacy. However, to compute such parameters, namely productivity index and drainage area, as there are no direct outputs from our equation there will be a need for some additional steps before arriving at the required outputs but with no reduction in accuracy [11]. A study was conducted to investigate the performance of hydraulic fracturing in a tight reservoir

and shale gas reservoir. Despite both fields being known for being unconventional, the contrast in permeability between the two fields was still high in disparity whereby the tight reservoir has permeability in the range of millidarcy while the latter reservoir has permeability in the range of nanodarcy. The investigation was performed using a trilinear model, which could only provide an accurate estimation at the earlier time, as discussed before. The tendency of the authors of the study to resort to the trilinear method showed that there was not a better prediction simulator that was as good as the equation that we constructed, which gave us more stance to come up with such an equation [12].

A similar study was conducted to investigate the pressure transient response of a hydraulically fractured horizontal well with natural micro-fracture. In this study, Laplace Transformation, source function and superposition principle were used to generate a semi-analytical model that matched a numerical solution, thus, confirming the reliability of the model. Further works were performed by conducting sensitivity analysis, which indicated the sensitive parameters towards the pressure response. Upon verification of the model, the model was then used as a type curve to evaluate the performance of a well stimulation technique from a field called Jimusar Sag [13]. However, the equation built by us did not require Laplace Transformation or inversion but simple leveraging on a single equation that can be used to compute pressure change with respect to the location and time without restriction of the type of flow. The origin of the fractured model study in well testing begun in 1960 when a study was conducted on the seepage of reservoir fluid along the fractured strata from which the term dual porosity was derived [14]. The work on the dual porosity model continued in 1963 when an ideal homogeneous model was built that comprised of two components, the homogeneous reservoir model and the network of fractures. In reservoir simulation, to describe a hydraulic fractured well, a dual porosity grid that contains the well and the hydraulically fractured area is used because an infinite conductive grid with small volume is not easy to be actualized by a normal single porosity. The introduction of terms such as storage coefficient and inter-porosity coefficient took place, which governed the fluid flow between the dual porosity medium, thus establishing a solid foundation for the dual porosity model. Our main contribution is to define a new equation that has universal application to any flow regimes. This means that, unlike other studies that generated several equations specific for flow through different mediums and different regimes, we managed to construct a single equation that works universally in all of the flow regimes that will occur for a well that is hydraulically fractured. This means that the inclusion of the dual porosity element can be eliminated using the equation that we built. Figure 1 shows dual porosity model from a simulator where the darker line indicated fractures originated from hydraulic fractures. Figure 2 shows the early time pressure response around hydraulically fractured well. The intensity on Figure 2 indicates the magnitude of pressure whereby around the hydraulically fractured well, a stark contrast can be seen. These outcomes take place because the flow is dominant from fracture to matrix instead of matrix to matrix. Thus, the pressure drops is more intense across the fracture.



Figure 1. Dual porosity Assigned Grid (Dark blue) for Hydraulic Fracture.

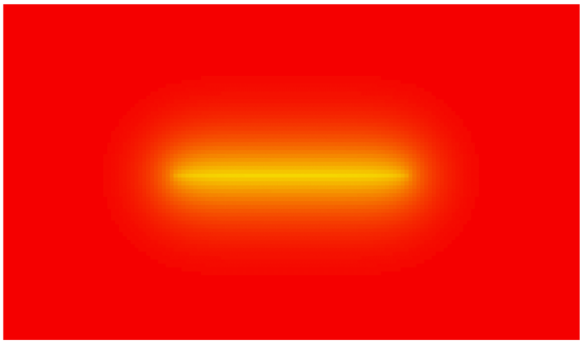


Figure 2. Early Time Pressure around Hydraulic Fractured Well.

The equation of shale matrix and fracture networks were coupled and an equation was generated. The source function used in this study was Bessel's function, which was modified to be used in Laplace Transformation [13].

$$\overline{P}_D = \overline{q}_{fd} K_o \left(\sqrt{x_D^2 + y_D^2} \sqrt{f(s)} \right) \tag{2}$$

Equation (2). Shale Matrix and Hydraulic Fracture Coupling.

$$f(s) = s \frac{\omega S(1-\omega) + \lambda}{S(1-\omega) + \lambda} \tag{3}$$

Equation (3). Bessel Function with Laplace Transformation (Source Function).

The final equation used in this study is the superposition equation as expressed in

$$\overline{P}_{FDi} = \sum_{k=1}^{n_p} \int_{y_{wDk,dw}}^{y_{wDk,up}} \overline{q}_{FDk} \mathrm{K}_0 \left[f(s) \sqrt{(x_{Di} - x_{wD})^2 + (y_{Di} - u)^2} \right] du \tag{4}$$

Equation (4). Superposition Equation.

This equation can be used to compute and model the pressure in both the shale matrix and in fracture networks, which actually caters for the linear flow during the flow into the fracture. However, the equation constructed cannot be used to model the subsequent flows that occurred after the linear flow such as the elliptical flow (around the well and fracture networks) and the radial flow (reservoir boundary dominated flow). In order to simplify the case, we managed to build an equation that can govern all of the flow regimes, and the equation that we built does not require Laplace transformation in order to be used.

Another study conducted was on constructing a semi-analytical model for pressure transient analysis of a hydraulically fractured well with fracture orientation in an isotropic reservoir. A set of diffusivity equations was developed using a nodal analysis technique, which describes the flow in the hydraulic fracture. The coupling of the source solution and discrete fracture solution generated the semi-analytical solution. The generated semi-analytical solution was then validated using the numerical solution [15]. There were two equations generated in the paper which were for the flow from one fracture node to another and the flow from the fracture to the vertical well, as shown in Equations (5) and (6).

$$\widetilde{p}_{fDk+1} - \widetilde{p}_{fDk} = \frac{2\pi}{F_{cD,i}} \int_{I_{Dk}}^{I_{Dk+1}} \left[\widetilde{q}_{fwDk} + \widetilde{q}_{fDk} (l_D - l_{Dk}) \right] dl_D \tag{5}$$

Equation (5). Flow between Fracture Nodes.

$$\widetilde{p}_{Dk,i}(x_{Dk} y_{Dk}, x_{Di} y_{Di}, s) = \sum_{k=1}^{N_I+1} \sum_{i=1}^{N_I} \widetilde{q}_{D,i} R_{Dk,i}(x_{Dk}, y_{Dk}, x_{Di}, y_{Di}, s) \tag{6}$$

Equation (6). Flow from Fracture to Vertical Well.

In this study, the equation is discretized where the pressure drop is computed by the fluid flow from every interconnecting fracture and matrix surface, as shown in Equation (6). This approach differed from our work where a single equation was constructed to compute the pressure response in the linear flow regime analytically. The analytical approach always yields a more accurate solution than any numerical approach; thus, our work will deliver a more accurate result. The boundary condition in this study and our study was treated differently. In this study, since the flow rate along the fracture plane was added individually, the sum that was used to compute the total pressure drop indicated that the pressure drop was considered uniform instead of a uniform flow assumption.

Another study was conducted on building semi-analytical solutions for multi-fractured horizontal wells in box-shaped reservoirs. In this study, Laplace and Fourier Transforms were used to generate the semi-analytical solution. Prior to undergoing the transformations, two main models were predefined such as the reservoir and fracture model, which were used to represent the models, respectively. In this study, the semi-analytical solution yielded a higher flowrate in the outermost part of the fracture, which resembles the assumption of a constant pressure drop over the fracture interval. This underlying assumption was different from our work, which assumed a uniform flux into the fracture, thus creating a gap in the literature. In this study, there were six flow regimes established altogether, namely bilinear flow, first linear flow, first radial flow, second linear flow, second radial flow and, finally, boundaries dominated flow. This finding contradicted the common flow regimes or patterns often established in a fractured reservoir, which begins with a linear flow from the matrix to the fracture, a pseudo-radial flow that is established as the flow from the matrix enters the side and at the tip of the fractures creating an elliptical behavior, and, finally, the radial flow, which is a boundary dominated flow that is established at late stage [16]. Basically, we defined three different flow regimes, unlike this study, [6].

The main contributor to the difference between our study and the author study is the orientation of the well. In our study, all of the well was vertical with a single hydraulic fracture, while in the literature study, the well was horizontal and had multiple fractures. Apart from these differences, the flow nature was of an infinite conductivity in the literature, while in our study it was a uniform flux. In a study conducted on transient pressure behavior for a well with a finite conductivity vertical fracture, a mathematical model was developed. There were a few underlying assumptions used to develop the model. The first assumption was that it was an isotropic, homogeneous, horizontal, infinite and slab reservoir. The fluid in the reservoir was slightly compressible, which means the viscosity and compressibility were both constant. Fluid entered the wellbore only from the fracture, which neglected the direct flow from the formation to the wellbore. There were two equations generated in this study for the fracture and reservoir models. There were 6 flow regimes mentioned in this study as shown in Figure 3. However, these flow regimes only exist for horizontal wells. In this study, it was mentioned that infinite conductivity does not work in all cases where, for some cases such as for large or very low-capacity fractures, finite conductivity or uniform flow should be used instead. The author of this study mainly wanted to produce a general solution for the pressure transient of a well intersecting a vertical fracture with finite conductivity. The following equations were derived to be used as the solution [17]. Despite a similarity between the study cited here and our work, the cited work experienced some setbacks in matching extreme cases where the conductivity was extremely low or high. This probably happened because the transient equation was used generally for all of the flow regimes where the inaccuracies may have taken place once the radius of investigation had reached the reservoir boundary.

$$\frac{\partial^2 P_f}{\partial x^2} + \frac{u}{k_f}\frac{q_f(x,t)}{wh} = \frac{\phi_f \mu c_{ft}}{k_f}\frac{\partial p_f}{\partial t} \tag{7}$$

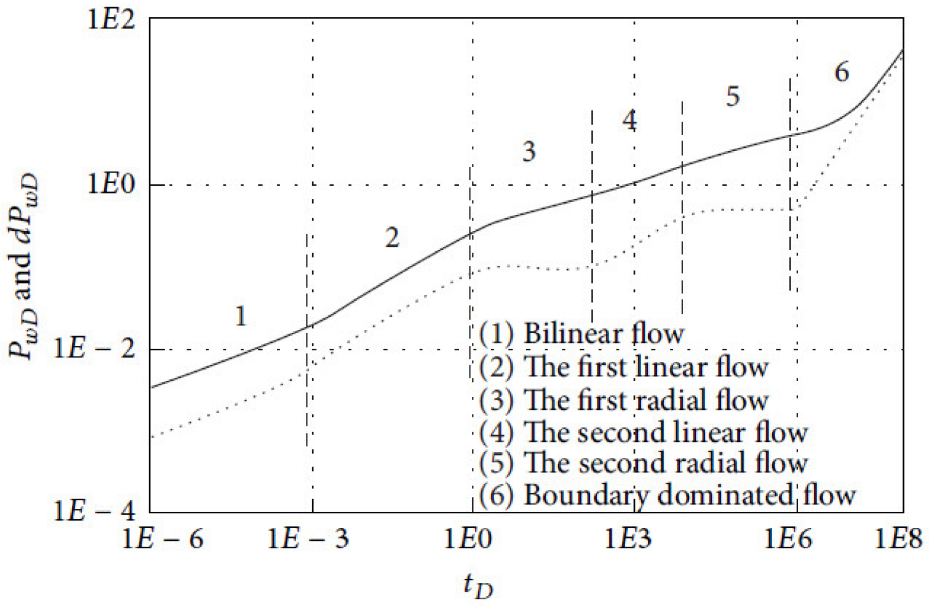


Figure 3. 6 Flow Regimes for a Multi-fractured Horizontal Well.

Equation (7). Partial Differential Equation for Fracture (Unsteady State).

$$P_{fD}(x_D, t_D) = \frac{x_f}{w}\sqrt{\frac{k\phi c_t \pi}{k_f \phi_f c_{ft}}} \sum_{n=-}^{\infty} \int_0^{t_D} \left\{ \frac{e^{-\left[\frac{(x_D-2n)^2}{4\tau\left(k_f\phi c_t / k\phi_f c_{ft}\right)}\right]}}{\sqrt{\tau}} - \int_{2n-1}^{2n+1} q_{fD}(x', \tau) \frac{e^{-\left[\frac{(x_D-x')^2}{4\tau\left(k_f\phi c_t / k\phi_f c_{ft}\right)}\right]}}{2\sqrt{t_D - \tau}} dx' \right\} d\tau \quad (8)$$

Equation (8). Fracture Model Solution for Finite Conductivity.

The developed solution for the fracture model in a finite conductivity case only works for the linear flow that happens as the fluid from the matrix starts flowing into the fracture where the previous fracture dominated the flow or the post linear flows cannot be modeled by the solution. This rendered the equation totally inaccurate because the main flow occurred from the fracture into the well, which was not modelled in the literature. In our work, not only did we model this flow but also any subsequent flows that would take place in a hydraulically fractured vertical well. We managed to produce a single equation that provided the solution to all of the comprised stages in fractured well testing.

An analytical model was developed for pressure transient analysis for multi-fractured horizontal wells in heterogenous shale reservoirs. This analytical model catered for many assumptions, which were neglected in many established models, such as reservoir heterogeneity, gas slippage, adsorption/desorption and diffusion effect. The model was divided into four main divisions, namely upper/lower regions, outer-reservoir regions, inner reservoir regions and fracture regions. The model created in this study is different from the author's model, which has a single equation coupling the fracture and reservoir models. In their model, all of the flow regimes have their own equations, which denied the general application of a single equation to all cases, as in our study [18].

2. Methodology

Fracture Length Division

Figures 4 and 5 show a multiple number of source or sink points that are able to develop early stage linear and elliptic flows. The gradual increase in the number of source or sink points results in the flow turning into a linear flow eventually. The first step in this study was to divide the fractures into equal infinite small lengths in which a constant flow enters the fracture, which defines the uniform flux. The uniform flux is defined as the constant flow into the fracture with fluctuating pressure difference, while non-uniform flow is defined vice versa. Uniform flow is described by the equal amount of flow into the fracture from the formation, whereas non-uniform flow is distributed flow between the cusp (tip of fracture) where the flow is more dominant and the center where there is the least flow. In this study, the scope is limited to uniform flow only. The segmentation of fracture length is shown in Figure 6.

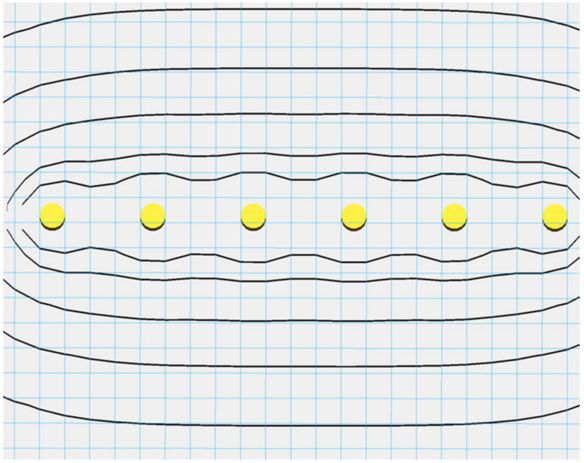


Figure 4. Pressure Iso-Line for Multiple Number of Source Points in Parallel.

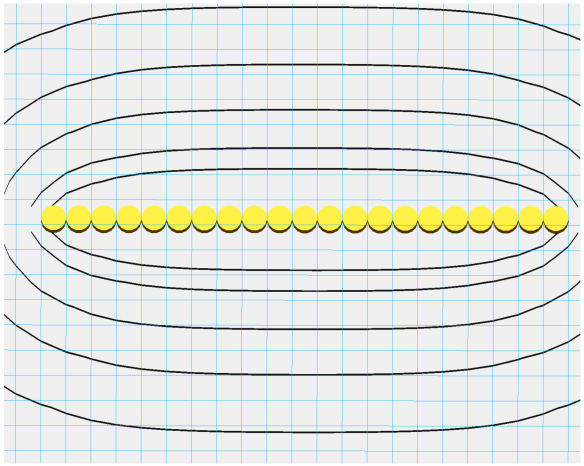


Figure 5. Pressure Iso-Line for Infinite Number of Source Points in Parallel.

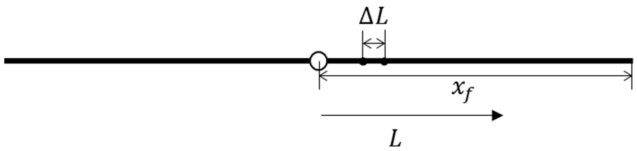


Figure 6. The Segmentation of Fracture Length.

The following Equation (9) defined the fracture length segmentation

$$\begin{gathered} \Delta L = \frac{x_f - r_w}{n} \\ L_n = r_w + n\Delta L \\ f(L_n) = f(r_w + n\Delta L) \end{gathered} \tag{9}$$

Equation (9): Fracture Length Segmentation.
The flow rate per unit area $u(L)$ is defined by Equation (10).

$$2\int_0^{x_f} u(L)dL = q \tag{10}$$

Equation (10): Flow Rate per Unit Area
When the flowrate is uniform, Equation (10) is modified to Equation (11).

$$\begin{gathered} u(L) = \frac{q}{2x_f} \\ 2\int_{r_w}^{x_f} \frac{q}{2x_f} dL = \frac{x_f - r_w}{x_f} q \\ 2\int_0^{r_w} \frac{q}{2x_f} dL = \frac{r_w}{x_f} q \end{gathered} \tag{11}$$

Equation (11): Flow Rate per Unit Area (Uniform).
Pressure Drop Calculation: Solution to Analytical Model.
The final analytical solutions used to compute the pressure drop at the wellbore and along the fracture are defined by Equations (12) and (13). The derivation of the solution is shown as well.

$$\begin{gathered} p_{flowing}(r = r_w, t) = p_i - \frac{70.6B\mu}{kh} \frac{r_w}{x_f} q \left[-Ei\left(-\frac{948.1\phi\mu c_t r_w^2}{kt} \right) \right] \\ -\frac{70.6B\mu}{kh} \frac{q}{x_f} \int_{r_w}^{x_f} \left[-Ei\left(-\frac{948.1\phi\mu c_t L^2}{kt} \right) \right] dL \end{gathered} \tag{12}$$

Equation (12): Pressure Drop at Wellbore.

$$p_{flowing}(r,t) = p_i - \frac{70.6B\mu}{kh} \frac{r_w}{x_f} q \int_{\frac{r_w^2}{t}}^{\infty} \left(\frac{1}{u} \exp\left(-\frac{948.1\phi\mu c_t}{k} u \right) \right) du \; \frac{70.6qB\mu}{kh} \frac{q}{x_f} \int_{r_w}^{x_f} \int_{\frac{L^2}{t}}^{\infty} \left(\frac{1}{u} \exp\left(-\frac{948.1\phi\mu c_t}{k} u \right) \right) du \, dL \tag{13}$$

Equation (13): Pressure Drop Solution along Fracture.

3. Results and Discussion

3.1. Comparison with Numerical Solution

The solution obtained from the analytical equation was compared with the solution from the numerically calculated method. The exact same properties of reservoir and fracture were used to compare the results. The plots were shown in Figures 7 and 8.

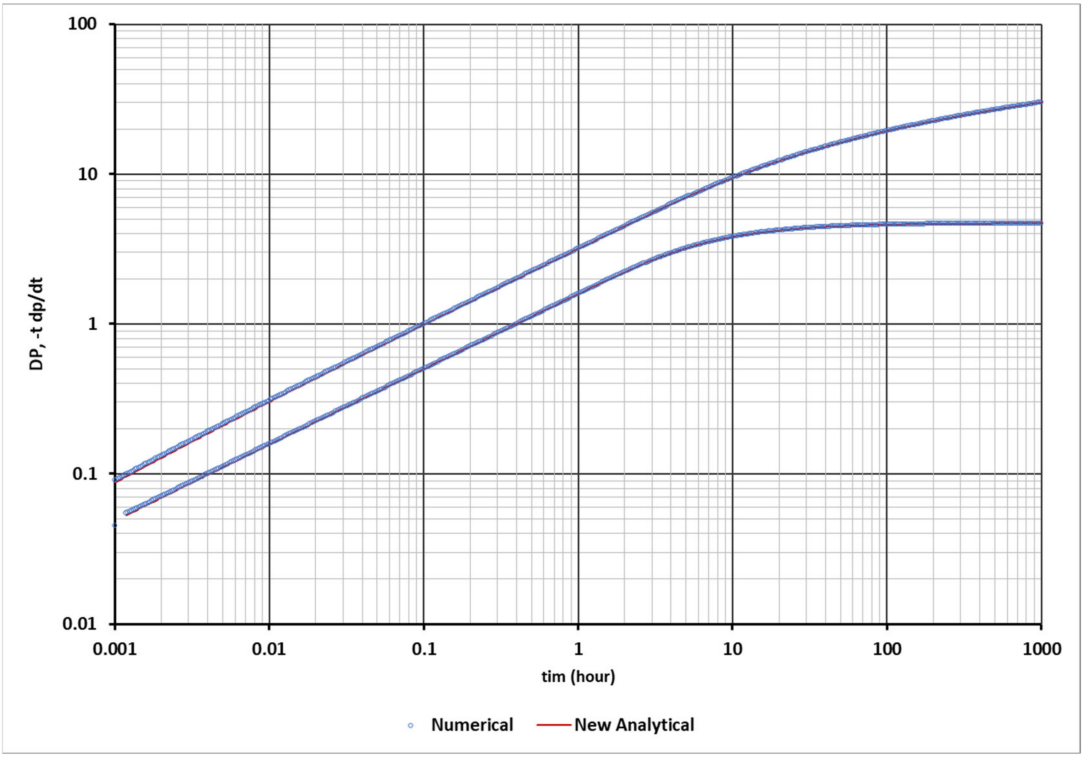


Figure 7. Results Comparison between Analytical Model and Numerical Calculation in Log-Log scale.

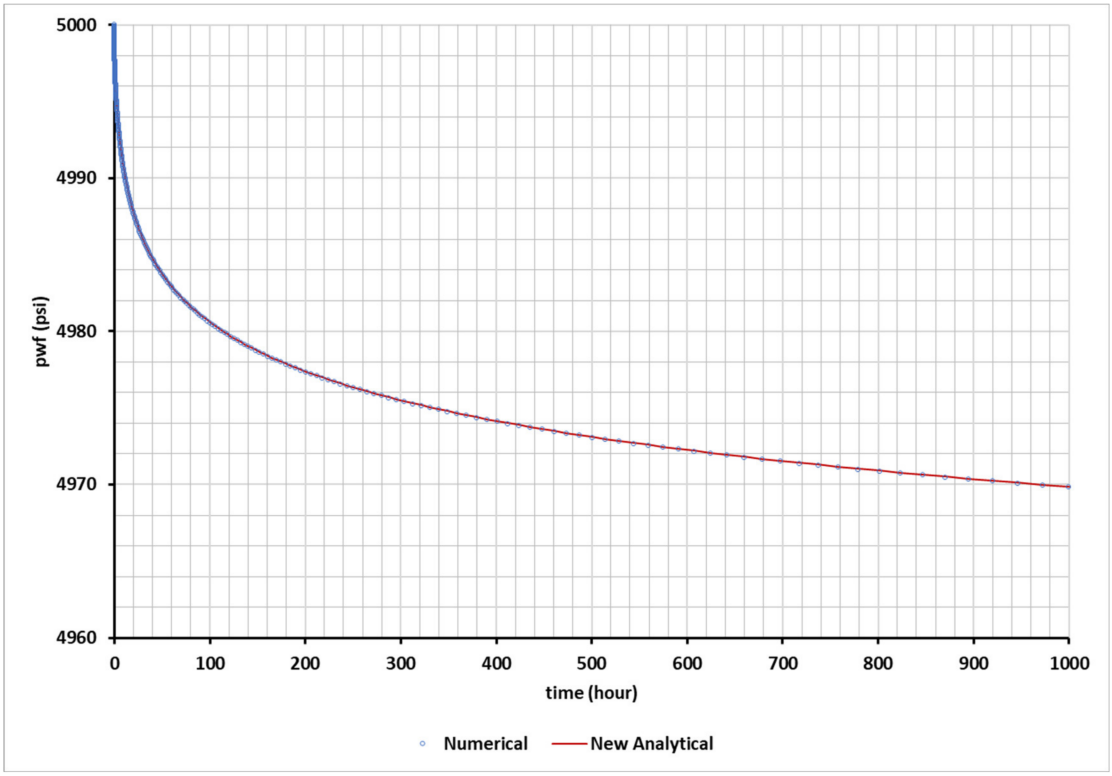


Figure 8. Results Comparison between Analytical Model and Numerical Calculation.

3.2. Discrete Function to Continuous Function

Based on the constant flow rate, the pressure drop is calculated using Equation (14). The pressure drop calculation is divided into wellbore and along the pressure.

$$\lim_{n\to\infty}\sum_{j=1}^{n} p(L_j)\Delta L = \lim_{n\to\infty}\sum_{j=1}^{n} p(r_w + j\Delta L)\cdot\Delta L = \lim_{n\to\infty}\sum_{j=1}^{n} p\left(r_w + \frac{x_f - r_w}{n}j\right)\cdot\frac{x_f - r_w}{n} \tag{14}$$

Equation (14): Pressure Drop Calculation.

3.3. Conversion from Discrete to Continuous Model

The discrete equation used to compute the pressure drop along the fracture and at the wellbore is then transformed from discrete to continuous. The continuous form for the analytical model is

$$\lim_{n\to\infty}\sum_{j=1}^{n} p\left(r_w + \frac{x_f - r_w}{n}j\right)\cdot\frac{x_f - r_w}{n} = \int_{r_w}^{x_f} p(L)dL\cdot \tag{15}$$

Equation (15): Pressure Drop Calculation (Continuous).

3.4. Single Source Pressure Drop

Equation (16): Pressure Drop Solution for single source.

$$p(L_j) = Ei\left(-\frac{948.1\phi\mu c_t(r_w + j\Delta L)^2}{kt}\right) = Ei\left(-\frac{948.1\phi\mu c_t L_j^2}{kt}\right) \tag{16}$$

3.5. Discrete Solution

Equation (17): Pressure Drop Solution at Wellbore.

$$\begin{aligned} p_{flowing}(r = r_w, t) = p_i - \frac{70.6B\mu}{kh}\frac{r_w q}{x_f}\left[-Ei\left(-\frac{948.1\phi\mu c_t r_w^2}{kt}\right)\right] - \\ \lim_{n\to\infty}\sum_{j=1}^{n}\frac{70.6B\mu}{kh}\frac{q}{x_f}\left[-Ei\left(-\frac{948.1\phi\mu c_t\left(r_w + \frac{x_f - r_w}{n}j\right)^2}{kt}\right)\right]\frac{x_f - r_w}{n} \end{aligned} \tag{17}$$

The developed equation showed a perfect superimposition. The developed analytical model managed to provide a smooth computation of all of the three phases, linear, transition and pseudo-radial, using a single equation. Moreover, the final stage is real pseudo-radial flow, not conventional radial.

The developed analytical model matched with the numerical calculation. Thus, the analytical model again managed to depict all of the stages of flow using a single relatively easy inversible equation, which is unlike the numerical solution. All of the three stages of flow, the linear, transition and pseudo-radial flow schematics are also shown in Figures 9–11, respectively.

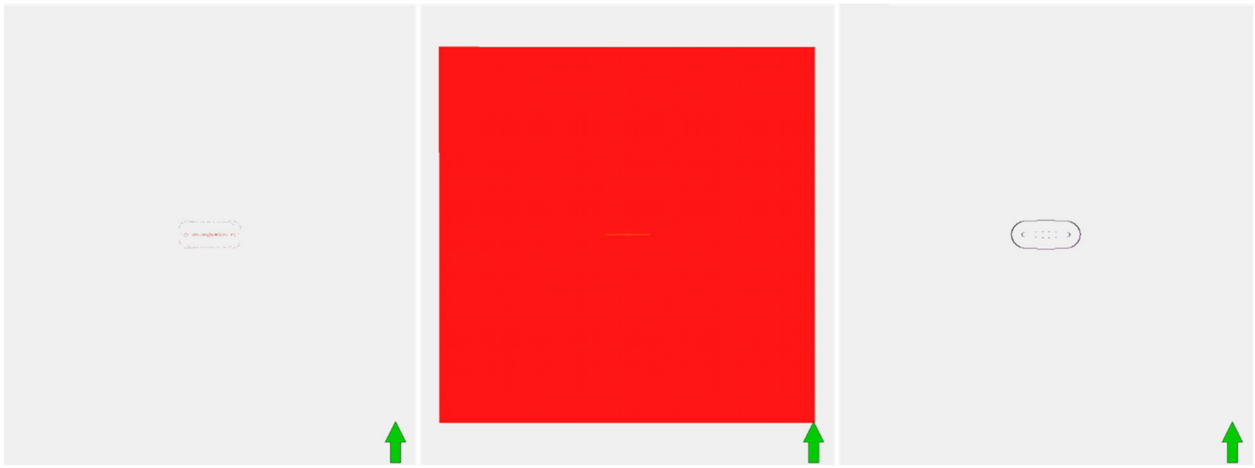


Figure 9. Early Time Linear Flow.

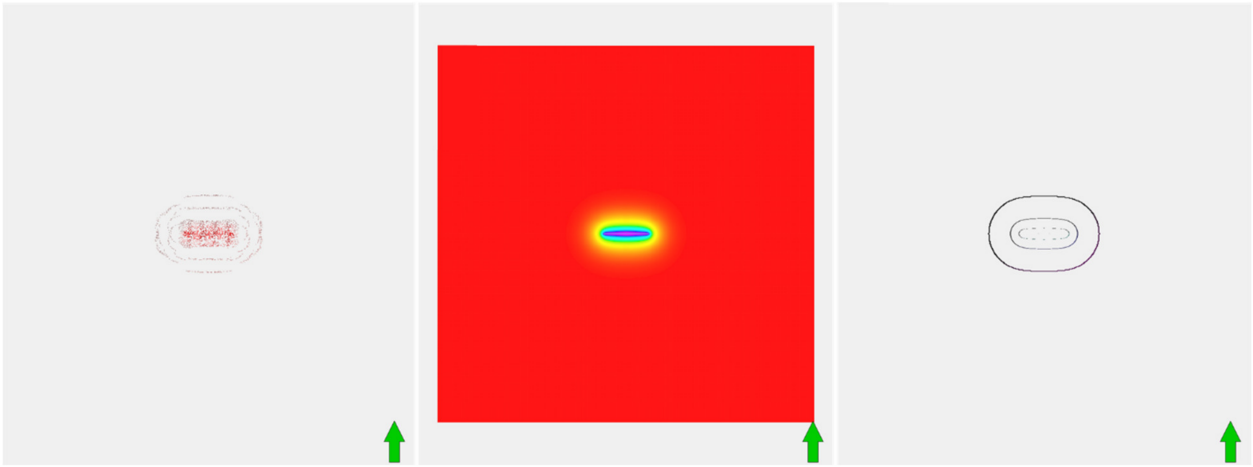


Figure 10. Transition from Linear to Pseudo-Radial.

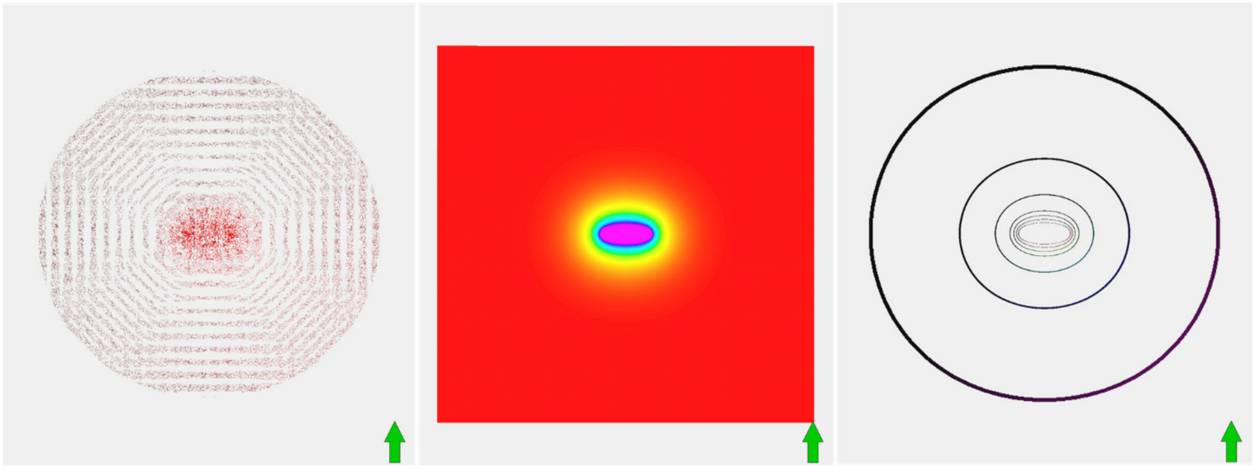


Figure 11. Pseudo-Radial Flow in Late time.

In a study conducted on pressure transient analysis in an elongated linear flow system, the applicability of the linear equation for interference, drawdown and build-up tests was thoroughly studied. In this study, an equation was built and the equation was utilized for gas, oil and steam geothermal wells. The built equation is shown in Equation (14).

However, the drawback of this equation is that there is no consideration of pseudo-radial flow in a later time. Although the surface area at the tip is less, there is still flow into the section, which is not captured by the study. However, the newly developed equation by the author accounts for the flow from both tips of the fracture [19]. This can be seen in Figures 12 and 13, which clearly demonstrate the difference between the outcomes of the two studies. The flow into the tip is the reason why, eventually, the flow turns elliptical. In the previous study, there was no consideration of the pseudo-radial flow and the transition period [19].

$$\mathbf{D = 0.00026367}\frac{\boldsymbol{k}}{\boldsymbol{\phi\mu c_t}}$$
$$p = p_i - \frac{141.2qB\mu}{khx_f}\left[\sqrt{\pi 0.0002637\frac{kt}{\phi\mu c_t}} - \pi x\right] \quad (18)$$

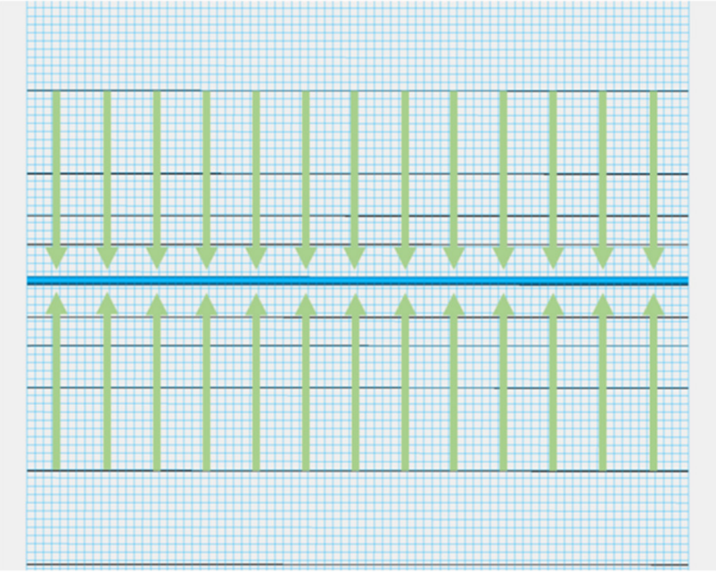


Figure 12. Pressure Iso-line (Conventional Analytical for early time linear flow Model).

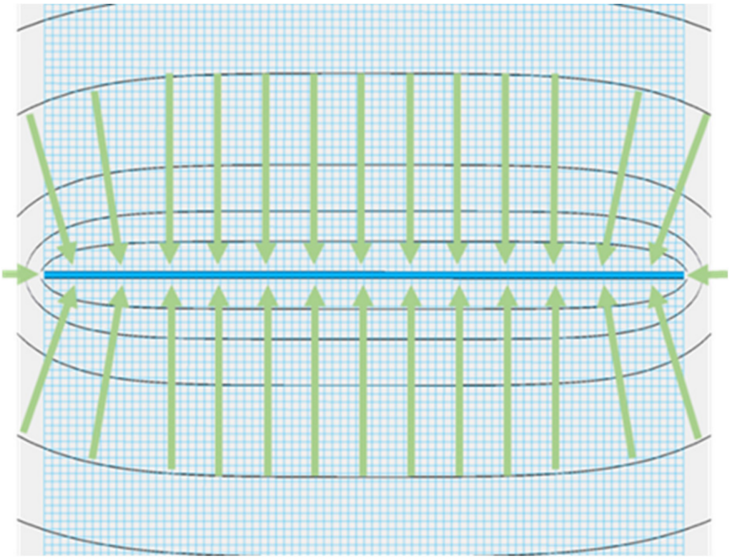


Figure 13. Pressure Iso-line Assumption in Developed Analytical Model for early time semi-linear flow.

Equation (18): Elongated Linear Flow Equation.

Another similar study was conducted focused on building an analytical solution for transient flow in a hydraulically vertically fractured well in an elliptical shaped and an isotropic reservoir producing at a constant rate and pressure. An elliptical model was suggested because the radial model is only applicable for a homogeneous and isotropic reservoir. There were a few underlying assumptions used in developing the equation for the transient model linear flow such as it being an isotropic, homogeneous, horizontal, infinite reservoir with constant thickness, permeability and porosity. The reservoir fluid was assumed to be slightly compressible with constant viscosity. All of the properties were assumed to be independent of pressure. Laminar flow was considered with negligible gravity effect [20]. The analytical model built via this study is an approximated model unlike the model built by the author. The model built by the author of this study is an exact analytical solution for this behavior. On top of that, the mode built in the previous study is an equation for transient flow only, not pseudo-radial and linear. In order to cater for elliptical flow, a different permeability in the x- and y-direction was introduced, which brings in the term permeability anisotropy [20].

Figure 14 shows the pressure iso-line assumption in the developed analytical model for transient elliptical flow, which describes the pressure distribution exactly for transient flow without any approximation. Linear flow is solved using two equations, which are error function and ei-function by considering an infinite number of wells. The error function analysis is limited to use only for the linear flow, whereas the ei-function can cater for radial and linear flow together. However, the order of the flow regimes in an infinite number of wells is from radial to linear, which is an inverted order in this study: from linear to radial. The pressure iso-line assumption in late time radial flow for the conventional analytical solution in the top view, and the pressure iso-line assumption in late time pseudo-radial flow for the developed analytical model are illustrated in, respectively, Figures 15 and 16.

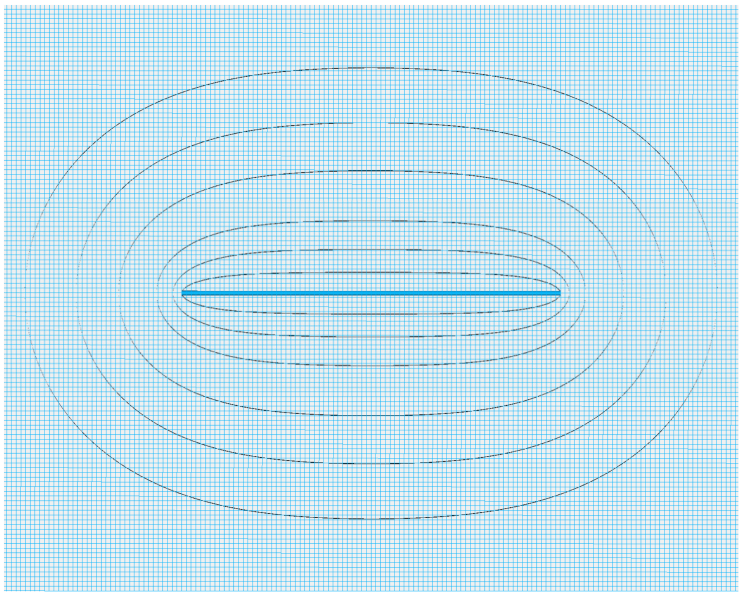


Figure 14. Pressure Iso-line Assumption in Developed Analytical Model for Transient Elliptical Flow.

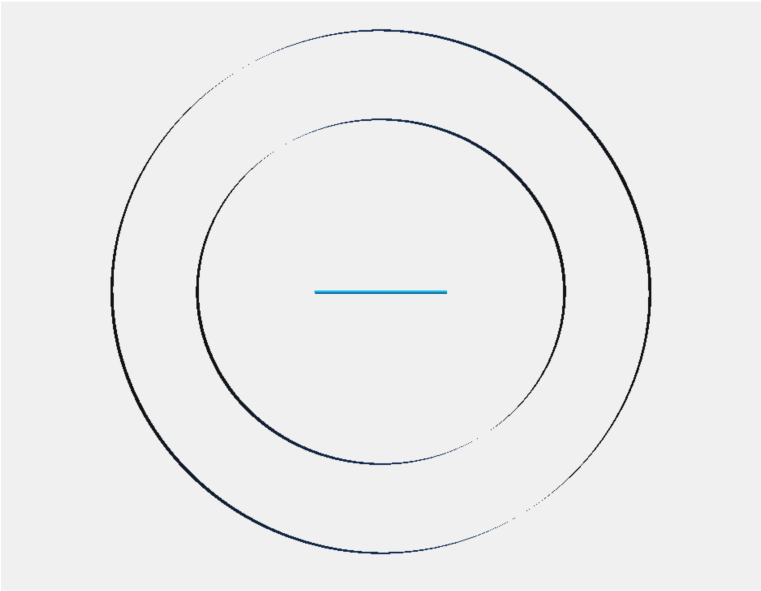


Figure 15. Pressure Iso-line in Conventional Analytical Model.

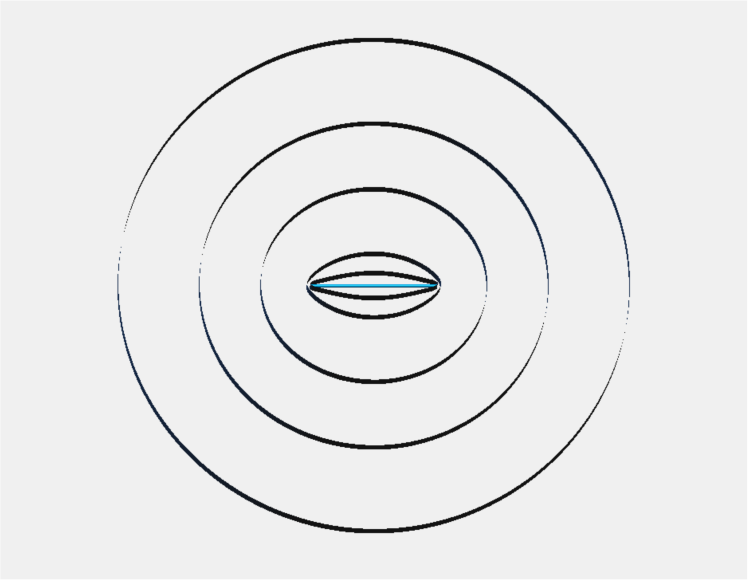


Figure 16. Pressure Iso-line Assumption in Developed Analytical Model.

4. Conclusions

In conclusion, the newly developed analytical equation provides the advantage of using a single equation to describe all of the flow periods, namely linear, transition and radial as shown in Figure 17. This analytical equation has a better advantage than the numerical solution in its ease of inversion for computation of well test derived properties such as permeability. This newly developed equation has no approximation, which provides a better accuracy in the results predicted. Thus, for any future well test analysis to be conducted in a vertically hydraulically fractured well, the use of this model will be accurate and convenient.

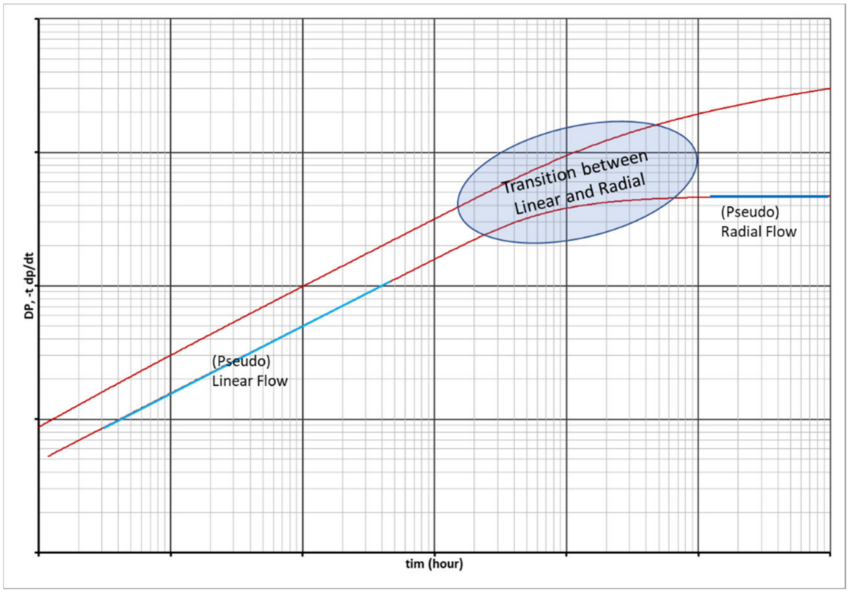


Figure 17. The New Developed Analytical Equation by Author showing All 3 Flow Periods.

Author Contributions: Conceptualization, J.H.L.; methodology, J.H.L.; software, J.H.L.; validation, J.A.B.M.S.; formal analysis, J.A.B.M.S.; investigation, J.A.B.M.S.; resources, J.A.B.M.S.; data curation, J.H.L.; writing—original draft preparation, J.A.B.M.S.; writing—review and editing, J.A.B.M.S.; visualization, J.H.L.; supervision, J.H.L.; project administration, J.H.L.; funding acquisition, J.H.L. All authors have read and agreed to the published version of the manuscript.

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Nomenclatures

$\mathbf{x_f}$	Fracture half length (ft)
2n	Number of source points
ΔL	Unit distance of the next source points (ft)
L	Distance of point of source from the center (ft).
Ei	Exponential integral function
k	Permeability (mD)
$\mathbf{p_i}$	Initial pressure (psi)
$\mathbf{c_t}$	Total Compressibility ($1/\text{psi}^{-1}$)
μ	Viscosity (cP)
ϕ	Porosity (fraction)
B	Formation Volume Factor (rb/stb)
q	Flow Rate (stb/d)
h	Formation Thickness (ft)
h	Time (hr)

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