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Impulsive Multiple-Bipartite Consensus Control for Networked Second-Order Multi-Agent Systems

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Abstract: In this paper, the impulsive multiple-bipartite consensus problem is discussed for networked second-order multi-agent systems (MASs) over directed network topology with acyclic partition. The definition of the multiple-bipartite consensus is introduced into second-order MASs by effectively combining the characteristics of bipartite consensus and group consensus based on the unique structure of network topology with acyclic and structural balance. By thoroughly exploring the coupling state between agents, a distributed impulsive multiple-bipartite consensus control protocol is designed for each agent by only measuring the relative information of its neighbors. Some sufficient conditions that guarantee realizing multiple-bipartite consensus are given, and the corresponding stability analysis is based on an improved Laplacian matrix associated with the network topology. Finally, some simulation examples are presented to verify the theoretical results.

Keywords: bipartite consensus; second-order MASs; impulsive control; networks; multiple-bipartite consensus



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1. Introduction

The consensus problem of multi-agent systems (MASs) enjoys valuable and potential significance in light of its physical background and engineering applications, e.g., in the deployment of distributed artificial intelligence, the smart scheduling of transportation, and the control of autonomous multiple robots and application of mobile sensor networks [1–6]. Consensus for MASs has three main categories of theoretical studies: biological community consensus theory, pure basic theory and various applications of consensus [7]. As one of the important emerging behaviors, the consensus problem requires that all the agents in the network reach an agreement on a common value or state [8–10], and consensus can be seen as the most fundamental point of many practical complex control objectives.

For the consensus problem of MASs, the mathematical model of the consensus problem formulation, the agents' dynamics and the network structure have been widely extended. Various forms of consensus have been investigated in recent decades. For instance, a quantized consensus requires that a network achieve coordinate behavior using the sampling information control method and quantized data in advance [11–13]. Bipartite consensus [14] (Altafini model) requires that all agents converge to a value of the same amplitude but a different direction [15–17]. The scaled consensus, which can be regarded as the generalized scenario of bipartite consensus, requires that the agents reach assigned proportions rather than a common value [18–20]. Cluster (group) consensus, which divides the corresponding agent set into separate groups, drives all agents in the same group to reach a complete consensus [21–23]. With regard to the agents' dynamics, the single-integrator dynamic model [24], double-integrator dynamic model [25,26] and high-order dynamic model [27] were extensively investigated. Additionally, several types of the network structure, namely fixed type [28], switching type [29] and stochastic type [30], have been involved in the

aforementioned literature. Notably, the references mentioned above mainly focused on the complete consensus or single bipartite consensus [11–18,20–30]. However, with the development of modern industrial applications, the complexity level of the control objectives is becoming increasingly higher; thus, complete consensus and single bipartite consensus cannot meet the modeling requirements in many application scenarios. Based on this, group-bipartite consensus is introduced to describe the compound tasks of the single integral MASs in [31] with a continuous control protocol.

On the other hand, the impulsive control strategy is a basic type of discontinuous control method. Owing to the advantage of simplicity, effectiveness, and robustness [32,33], the impulsive control technique [8], was deployed in the coordination control of MASs. For instance, Liu et al. [34] studied the robust impulsive synchronization of uncertain dynamical networks with bounded coupling functions. Via Lyapunov functionals and some analysis techniques, Tang et al. [35] derived some sufficient conditions for pinning the synchronization of stochastic impulsive discrete-time networks in mean square, and the developed approaches were applied to a scale-free network composed of discrete-time neural networks. In the reference [36], the synchronization of complex dynamical networks (CDNs) with system delay and multiple coupling delays was studied via impulsive distributed control. It should be noted that the aforementioned literature is all focused on designing single-order distributed impulsive control protocol algorithms for MASs in networks. To the best of our knowledge, the impulsive control strategy and second-order node dynamics which are more widely used have not been involved. The above discussion naturally prompts us to consider a problem, that is, how to combine the impulse control algorithm and the second-order dynamics characteristics to give a reasonable and feasible distributed impulsive multiple-bipartite control protocol by using only the neighbor information of agents, which forms the main motivation of this article.

In this paper, a distributed impulsive multiple-bipartite consensus protocol is designed for second-order MASs. By adopting an improved Laplacian matrix associated with the network topology, the sufficient conditions under which all second-order MASs reach multiple-bipartite consensus are presented. Compared with the relevant existing results in the literature, the innovation of this paper lies in the three following aspects. Firstly, compared with the consensus continuous protocols studied in [24–27,31], a new designed impulsive multiple-bipartite consensus control protocol corresponding to the instantaneous network connectivity only relies on the single signum information that instantaneously exchanges with agents' neighbors at discrete moments. Secondly, compared with the traditional Laplacian matrix, an improved Laplacian matrix associated with directed graph topology with acyclic partition and a structurally balanced structure is introduced to solve the stability analysis of controlled systems. Thirdly, an explicit expression of multiple-bipartite consensus states can be obtained using the proposed Laplacian matrix, which can be used to develop a unified approach yielding the desired multiple-bipartite consensus.

The rest of this paper is designed as follows. Section 2 presents preliminaries and problem formulation. Section 3 proposes the impulsive multiple-bipartite consensus control schemes for second-order MASs and carries out the stability analysis. The simulations and conclusions are presented in Section 4 and Section 5, respectively.

2. Preliminaries

Throughout this paper, $\mathbb{R}$, $\mathbb{R}^n$ and $\mathbb{R}^{m \times n}$ denote the set of real numbers, the set of the n -dimensional Euclidean space, and the set of $m \times n$ real matrices, respectively. $\mathbf{0}_n \in \mathbb{R}^n$ and $\mathbf{1}_n \in \mathbb{R}^n$ are vectors with all zeros and ones, respectively. $O_{m \times n} \in \mathbb{R}^{m \times n}$ is the zero matrix. For matrix A , A^{-1} and A^T are the inverse matrix and the transposed matrix of A , respectively. For a complex number λ , $\text{Re}(\lambda)$ and $\text{Im}(\lambda)$ represent the real and imaginary parts of λ , respectively.

2.1. Graph Theory

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a weighted directed graph of order n , where $\mathcal{V} = \{1, 2, \dots, n\}$, $\mathcal{E} \in \mathcal{V} \times \mathcal{V}$ and $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n \times n}$ are the node set, the edge set, and the weighted adjacency matrix, respectively. Here, we use node i to denote the i th agent. Moreover, $(j, i) \in \mathcal{E}$ means that there is a directed path from agent j to agent i , and $(j, i) \in \mathcal{E} \Leftrightarrow a_{ij} \neq 0$. In this paper, we assume that $a_{ii} = 0, i = 1, 2, \dots, n$. A directed path in $\mathcal{G}$ is a sequence of distinct edges of the form $(l_1, l_2), (l_2, l_3), \dots, (l_{k-1}, l_k)$ if $(l_{j-1}, l_j) \in \mathcal{E}$. The directed graph $\mathcal{G}$ has a directed spanning tree if there is at least one agent with a directed path to every other agent. Moreover, the structurally balanced graph is referred to if $\mathcal{V}$ can be divided into two subsets $\{\mathcal{P}^{(1)}, \mathcal{P}^{(2)}\}$, such that $\mathcal{P}^{(1)} \cap \mathcal{P}^{(2)} = \emptyset, \mathcal{P}^{(1)} \cup \mathcal{P}^{(2)} = \mathcal{P}$, and the weight a_{ij} in a directed graph satisfies $a_{ij} \geq 0$ if agents i and j belong to the same subgroup, and $a_{ij} \leq 0$ if agents i and j belong to different subgroups.

2.2. Problem Formulation

Assume that the i th ($i = 1, 2, \dots, n$) agent dynamics which are described by the following second-order differential equation

$$\begin{cases} \dot{r}_i(t) = v_i(t), \\ \dot{v}_i(t) = u_i(t), \end{cases} \quad (1)$$

where $r_i(t) \in \mathbb{R}^N, v_i(t) \in \mathbb{R}^N$, and $u_i(t) \in \mathbb{R}^N$ are the position, velocity and control input of the i th agent, respectively.

Let the graph $\mathcal{G}$ have a partition $\Xi = \{\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_k\}$, that is, $\cup_{w=1}^k \mathcal{P}_w = \mathcal{V}, \mathcal{P}_w \cap \mathcal{P}_z = \emptyset$; each of $\mathcal{P}_w$ is not an empty set; and $w, z = 1, 2, \dots, k$. Let $\bar{i}$ be the index of the subset associated with the aforementioned partition to which the i th agent belongs, i.e., $i \in \mathcal{P}_{\bar{i}}$. Moreover, we assume $\mathcal{P}_{\bar{i}} = \{h_{\bar{i}-1} + 1, h_{\bar{i}-1} + 2, \dots, h_{\bar{i}}\}, h_0 = 0, h_k = n$ and $h_{\bar{i}} - h_{\bar{i}-1} = n_{\bar{i}} \in \mathbb{Z}^+, \bar{i} = 1, 2, \dots, k$. That is, “ $h_{\bar{i}}$ ” denotes the node number sum of the first $\bar{i}$ subgroups, and “ $n_{\bar{i}}$ ” denotes the node number of the $\bar{i}$ th subgroup. This paper aimed to construct a multiple symmetric control task for second-order MASs (1); and accordingly, the mathematical model of this control task should be an organic unity of cluster consensus and bipartite consensus. By considering this background, we formulated the definition of a multiple bipartite consensus for second-order MASs (1).

Definition 1. Under the control input $u_i(t)$ and the partition Ξ , the second-order MASs (1) are referred to reach multiple-bipartite consensus if the following conditions hold:

- (1) Each $\mathcal{P}_{\bar{i}}$ can be divided into two subgroups as $\mathcal{P}_{\bar{i}}^{(1)}$ and $\mathcal{P}_{\bar{i}}^{(2)}$, satisfying that (i) $\mathcal{P}_{\bar{i}}^{(1)}$ and $\mathcal{P}_{\bar{i}}^{(2)}$ are all nonempty sets; (ii) $\mathcal{P}_{\bar{i}}^{(1)} \cup \mathcal{P}_{\bar{i}}^{(2)} = \mathcal{P}_{\bar{i}}$; (iii) $\mathcal{P}_{\bar{i}}^{(1)} \cap \mathcal{P}_{\bar{i}}^{(2)} = \emptyset$;
- (2) There are state vectors $\chi_{\bar{i}}(t)$, and $\kappa_{\bar{i}}(t)$, $w = 1, 2, \dots, k$, such that (i) $\lim_{t \rightarrow \infty} r_i(t) = \chi_{\bar{i}}(t)$, $\lim_{t \rightarrow \infty} v_i(t) = \kappa_{\bar{i}}(t)$, for $i \in \mathcal{P}_{\bar{i}}^{(1)}$; (ii) $\lim_{t \rightarrow \infty} r_i(t) = -\chi_{\bar{i}}(t)$, $\lim_{t \rightarrow \infty} v_i(t) = -\kappa_{\bar{i}}(t)$, for $i \in \mathcal{P}_{\bar{i}}^{(2)}, i = 1, 2, \dots, n$;
- (3) $\chi_{\bar{i}}(t)$, and $\kappa_{\bar{i}}(t), \bar{i} = 1, 2, \dots, k$, are related to the initial states $r_i(0)$ and $v_i(0), i = 1, 2, \dots, n$.

Remark 1. Compared with the group-bipartite consensus discussed in Ref. [31], Definition 1 requires not only the positions but also the velocities of the agents to converge to multiple-bipartite consensus. Moreover, Definition 1 indicates that the convergence states $\chi_{\bar{i}}(t)$, and $\kappa_{\bar{i}}(t), \bar{i}, \bar{j} = 1, 2, \dots, k$ are related to the initial states $r_i(0)$ and $v_i(0), i = 1, 2, \dots, n$, by which we can effectively select appropriate initial values to guarantee the different convergence trajectories of different groups.

Remark 2. Conversely, if each $\mathcal{G}_{\bar{i}}$ is structurally balanced, condition (1) given in Definition 1 is naturally satisfied, where $\mathcal{G}_{\bar{i}}$ is the graph associated with $\mathcal{P}_{\bar{i}}, \bar{i} = 1, 2, \dots, k$. This underlines that a structurally balanced topological structure plays an important role in realizing the multiple-

bipartite consensus of second-order MASs. Therefore, the structurally balanced topological condition should be introduced as in many existing references, such as [14,31]. If all $\mathcal{G}_{\bar{i}}$ are structurally balanced, then each $\mathcal{P}_{\bar{i}}$ can be divided into two subgroups as $\mathcal{P}_{\bar{i}}^{(1)}$ and $\mathcal{P}_{\bar{i}}^{(2)}$, satisfying that $\mathcal{P}_{\bar{i}}^{(1)}$ and $\mathcal{P}_{\bar{i}}^{(2)}$ are all nonempty sets, $\mathcal{P}_{\bar{i}}^{(1)} \cup \mathcal{P}_{\bar{i}}^{(2)} = \mathcal{P}_{\bar{i}}$ and $\mathcal{P}_{\bar{i}}^{(1)} \cap \mathcal{P}_{\bar{i}}^{(2)} = \emptyset$.

If all $\mathcal{G}_{\bar{i}}$, $\bar{i} \in \{1, 2, \dots, k\}$, are structurally balanced, then for $i \in \{1, 2, \dots, n\}$, $\exists \phi_i \in \{1, -1\}$, such that $\Phi_{\bar{i}} \mathcal{A}_{\bar{i}\bar{i}} \Phi_{\bar{i}}$ is a traditional standard Laplacian matrix, i.e., the off-diagonal elements of $\mathcal{A}_{\bar{i}\bar{i}}$ are all non-positive, where $\Phi_{\bar{i}} = \text{diag}\{\phi_{h_{\bar{i}-1}+1}, \dots, \phi_{h_{\bar{i}}}\}$ and $\mathcal{A}_{\bar{i}\bar{i}}$ is the weighted matrix associated with the agent set $\mathcal{V}_{\bar{i}}$. Now, the Laplacian matrix $\mathcal{L}$ associated with graph $\mathcal{G}$ can be defined as $\mathcal{L} = [l_{ij}]_{n \times n}$, $i, j \in \{1, 2, \dots, n\}$, where $l_{ij} = -a_{ij}$, $i \neq j$ and $l_{ii} = \sum_{j \notin \mathcal{P}_{\bar{i}}} \phi_j a_{ij} + \sum_{j \in \mathcal{P}_{\bar{i}}} |a_{ij}|$, $i, j = 1, 2, \dots, n$.

In addition to the structurally balanced topological structure, an acyclic partition [37] is an essential condition ensuring the multiple-bipartite consensus of second-order MASs (1). If partition is an acyclic partition, then the Laplacian matrix $\mathcal{L}$ has the form of [37]

$$\mathcal{L} = \begin{pmatrix} \mathcal{L}_{11} & \cdots & \mathbf{0}_{n_1 \times n_1} \\ \vdots & \ddots & \vdots \\ \mathcal{L}_{k1} & \cdots & \mathcal{L}_{kk} \end{pmatrix}, \quad (2)$$

where $\mathcal{L}_{\bar{i}\bar{i}}$ denotes the information communication between agents in $\mathcal{P}_{\bar{i}}$ and $\mathcal{L}_{\bar{i}\bar{j}}$ denotes the information communication between agents from $\mathcal{P}_{\bar{j}}$ to $\mathcal{P}_{\bar{i}}$, $\bar{i}, \bar{j} = 1, 2, \dots, k$. From the structure of the Laplacian matrix in (2), it is clear that the acyclic partition network topology allows the information to be transmitted from the front group to the back group, but the converse statement is not true.

The acyclic partition structure always demands that the effect is balanced between different groups, and therefore the following assumption should hold.

Assumption 1. $\Phi_{\bar{i}} \mathcal{L}_{\bar{i}\bar{j}} \Phi_{\bar{j}} \mathbf{1}_{n_{\bar{j}}} = \mathbf{0}_{n_{\bar{i}}}^T$, $\bar{i}, \bar{j} = 1, 2, \dots, k$.

Moreover, every group should have at least one agent that can transmit its information to any other agent, that is,

Assumption 2. Each $\mathcal{G}_{\bar{i}}$ has a spanning tree, $\bar{i} = 1, 2, \dots, k$.

Remark 3. Since the well-known work [38–40] in the early years initiated the origin of using Laplace matrix to study the consensus problem, it has been a powerful tool to study the collective behaviors of MASs. The Laplacian matrix on the graph can be imagined as the divergence of the gradient, which could better reflect the coupling interaction among MASs, and thus become our preferred method in studying this topic. The Laplacian matrix defined in our schemes is different from in a previous work, because our aim is to establish impulsive second-order cluster-to-cluster consensus structure, which involves the balance analysis among different groups. The position and the velocity states in the control input variable u_i all have two parts—one of which is a traditional feedback control input in terms of the architecture of the cooperation network topology of its own cluster, and the other is a supplementary state feedback controller applied to the agents from the other clusters. Noting that the internal coupling between the nodes of each subgroup can be positive or negative, and the impulsive control is implemented in a short interval, how to reasonably deal with the interaction between subgroups (positive or negative) in this case to achieve the multiple symmetric task control aim by combining the structural characteristics of impulsive control and multiple symmetric task convergence is the main challenge in this paper, which is very different from other existing works.

Remark 4. Assumptions 1 and 2 demonstrate two basic topological conditions in realizing the multiple-bipartite consensus of second-order MASs (1). Assumption 1 requires that the information transitions between different groups offset each other. Moreover, Assumption 2 demands the circulation of information in each group. The restrictions required in Assumptions 1 and 2 are feasible since many practical applications, such as modern industrial production lines and social networks, can satisfy these conditions.

3. Multiple-Bipartite Consensus of Second-Order MASs

To realize the multiple-bipartite consensus of second-order MASs, the distributed control protocol with local instantaneous interaction is given in the following.

$$\begin{aligned}
 u_i(t) = & \sum_{k=1}^{\infty} \left\{ \alpha \left[\sum_{j \in \bar{i}} a_{ij} (r_j(t) - \text{sgn}(a_{ij}) r_i(t)) \right. \right. \\
 & + \sum_{j \notin \bar{i}} a_{ij} (r_j(t) - \phi_j a_{ij} r_i(t)) \Big] \\
 & + \beta \left[\sum_{j \in \bar{i}} a_{ij} (v_j(t) - \text{sgn}(a_{ij}) v_i(t)) \right. \\
 & \left. \left. + \sum_{j \notin \bar{i}} a_{ij} (v_j(t) - \phi_j a_{ij} v_i(t)) \right] \right\} \delta(t - t_k),
 \end{aligned} \quad (3)$$

where $\delta(t)$ is the Dirac delta function that satisfies $\delta(t) = 0, t \neq 0$, and $\int_{-\infty}^{+\infty} f(t) \delta(t) dt = f(0)$. The time sequence $\{t_k\}_{k=0}^{\infty}$ satisfies $0 = t_0 < t_1 < \dots < t_k < \dots$, with $\lim_{t \rightarrow \infty} t_k = +\infty$ and $h = t_{k+1} - t_k$, where $k = 1, 2, 3, \dots$. Denoting $r_i(t_k^+) = \lim_{t \rightarrow t_k^+} r_i(t)$, $r_i(t_k^-) = \lim_{t \rightarrow t_k^-} r_i(t)$, $v_i(t_k^+) = \lim_{t \rightarrow t_k^+} v_i(t)$ and $v_i(t_k^-) = \lim_{t \rightarrow t_k^-} v_i(t)$; here, we assume that $v_i(t), i = 1, 2, \dots, n$ is left-continuous, that is, $v_i(t_k^-) = v_i(t_k), i = 1, 2, \dots, k$. α and $\beta > 0$ are the control gains to be designed.

By using the property of the Dirac delta function, second-order MASs (1) with the control protocol (3) have the impulsive differential equation form of:

$$\begin{cases} \dot{r}_i(t) = v_i(t), & t \neq t_k, \\ \dot{v}_i(t) = 0, & t \neq t_k, \\ \Delta r_i(t_k) = r_i(t_k^+) - r_i(t_k) = 0, \\ \Delta v_i(t_k) = v_i(t_k^+) - v_i(t_k) = \alpha \left[\sum_{j \in \bar{i}} a_{ij} (r_j(t_k) \right. \\ \quad \left. - \text{sgn}(a_{ij}) r_i(t_k)) + \sum_{j \notin \bar{i}} a_{ij} (r_j(t_k) - \phi_j a_{ij} r_i(t_k)) \right] \\ \quad + \beta \left[\sum_{j \in \bar{i}} a_{ij} (v_j(t_k) - \text{sgn}(a_{ij}) v_i(t_k)) \right. \\ \quad \left. + \sum_{j \notin \bar{i}} a_{ij} (v_j(t_k) - \phi_j a_{ij} v_i(t_k)) \right]. \end{cases} \quad (4)$$

To facilitate the analysis and design, the vector form of system (4) should be constructed. Accordingly, let $r(t) = [r_1(t)^T, r_2(t)^T, \dots, r_n(t)^T]^T$. Furthermore, $v(t) = [v_1(t)^T, v_2(t)^T, \dots, v_n(t)^T]^T$. Then, system (4) can be reformulated as

$$\begin{cases} \begin{pmatrix} \dot{r}(t) \\ \dot{v}(t) \end{pmatrix} = \begin{bmatrix} \begin{pmatrix} \mathbf{0}_{n \times n} & I_n \\ \mathbf{0}_{n \times n} & \mathbf{0}_{n \times n} \end{pmatrix} \otimes I_N \\ \begin{pmatrix} \mathbf{0}_{n \times n} & \mathbf{0}_{n \times n} \\ -\alpha \mathcal{L} & -\beta \mathcal{L} \end{pmatrix} \otimes I_N \end{bmatrix} \begin{pmatrix} r(t) \\ v(t) \end{pmatrix}, & t \neq t_k, \\ \begin{pmatrix} \Delta r(t_k) \\ \Delta v(t_k) \end{pmatrix} = \begin{bmatrix} \begin{pmatrix} \mathbf{0}_{n \times n} & \mathbf{0}_{n \times n} \\ -\alpha \mathcal{L} & -\beta \mathcal{L} \end{pmatrix} \otimes I_N \end{bmatrix} \begin{pmatrix} r(t_k) \\ v(t_k) \end{pmatrix}, & t = t_k, \end{cases} \quad (5)$$

where $\Delta r(t_k) = r(t_k^+) - r(t_k)$, $\Delta v(t_k) = v(t_k^+) - v(t_k)$, and $\otimes$ is the Kronecker product.

Furthermore, if all $\mathcal{G}_i$ are the structurally balanced directed graphs with acyclic partition, then $\Phi_i \mathcal{L}_{ii} \Phi_i$ is a standard Laplacian matrix form. Therefore, according to Ref. [31], under Assumptions 1 and 2, matrix $\Phi \mathcal{L} \Phi$ has k zero eigenvalues, and all its other eigenvalues

$$\lambda_1, \lambda_2, \dots, \lambda_{n-k} \quad (6)$$

have positive real parts, where

$$\Phi = \text{diag}\{\Phi_1, \Phi_2, \dots, \Phi_k\}. \quad (7)$$

Moreover, the k linearly independent left eigenvectors of $\Phi \mathcal{L} \Phi$ associated with a zero eigenvalue can be taken as

$$\begin{aligned} \xi_1 &= [\mu_1^T, 0_{n-h_1}^T]^T, \\ \xi_2 &= [(\sigma_1^{(2)})^T, \mu_2^T, 0_{n-h_2}^T]^T, \\ &\vdots \\ \xi_k &= [(\sigma_1^{(k)})^T, (\sigma_2^{(k)})^T, \dots, \mu_k^T]^T, \end{aligned} \quad (8)$$

satisfying $\sigma_j^{(i)} \in \mathbb{R}^{n_i}$, $(\sigma_j^{(i)})^T \mathbf{1}_{n_i} = 0$, $\mu_i = [\mu_{i,1}, \mu_{i,2}, \dots, \mu_{i,n_i}]^T \in \mathbb{R}^{n_i}$, $\mu_{i,l} \geq 0$ ($l = 1, 2, \dots, n_i$) and $(\mu_j^{(i)})^T \mathbf{1}_{n_i} = 1$, for $i = 1, 2, \dots, k$, $j = 1, 2, \dots, k-1$.

With the above preparation, we can now give our main result.

Theorem 1. Suppose that all $\mathcal{G}_i$ are the structurally balanced directed graphs with acyclic partition, and Assumptions 1 and 2 hold. If $h < \min_{1 \leq i \leq n-k} \frac{4\beta^2 \text{Re}(\lambda_i) |\lambda_i|^2 - 2\beta^3 |\lambda_i|^4}{4\alpha \text{Im}^2(\lambda_i) + \alpha \beta^2 |\lambda_i|^4}$, then by using control protocol (3), system (1) can reach multiple-bipartite consensus, i.e.,

$$\begin{aligned} \lim_{t \rightarrow \infty} r_i(t) &= \chi_i(t), \quad \lim_{t \rightarrow \infty} v_i(t) = \kappa_i(t), \quad i \in \mathcal{P}_i^{(1)}; \\ \lim_{t \rightarrow \infty} r_i(t) &= -\chi_i(t), \quad \lim_{t \rightarrow \infty} v_i(t) = -\kappa_i(t), \quad \forall i \in \mathcal{P}_i^{(2)}, \end{aligned}$$

where $i = 1, 2, \dots, k$ and $\chi(t)$, $\kappa(t)$ is explicitly expressed by

$$\chi_l(t) = [(\xi_l^T \Phi) \otimes I_N] r(0) + [(\xi_l^T \Phi) \otimes I_N] v(0) t,$$

$$\kappa_l(t) = [(\xi_l^T \Phi) \otimes I_N] v(0), l = 1, \dots, k,$$

in which λ_i , $i = 1, 2, \dots, n-k$ and ξ_l , $l = 1, 2, \dots, k$, are defined in (6) and (8), respectively.

Proof. Let

$$\chi(t) = [\mathbf{1}_{n_1}^T \otimes \chi_1^T(t), \mathbf{1}_{n_2}^T \otimes \chi_2^T(t), \dots, \mathbf{1}_{n_k}^T \otimes \chi_k^T(t)]^T$$

and

$$v(t) = [\mathbf{1}_{n_1}^T \otimes v_1^T(t), \mathbf{1}_{n_2}^T \otimes v_2^T(t), \dots, \mathbf{1}_{n_k}^T \otimes v_k^T(t)]^T.$$

The multiple-bipartite consensus error equation for system (1) is then

$$\begin{pmatrix} e_r(t) \\ e_v(t) \end{pmatrix} = \begin{pmatrix} r(t) \\ v(t) \end{pmatrix} - \begin{pmatrix} \chi(t) \\ v(t) \end{pmatrix}, \quad (9)$$

where $r(t) = [r_1^T(t), r_2^T(t), \dots, r_n^T(t)]$ and $v(t) = [v_1^T(t), v_2^T(t), \dots, v_n^T(t)]$. Note that Laplacian matrix $\mathcal{L}$ is defined by considering the effects between agents from both the same group

and the different groups. Then, by combining Equations (4) and (9), the multiple-bipartite consensus error equation in differential form can be obtained as

$$\begin{cases} \begin{pmatrix} \dot{e}_r(t) \\ \dot{e}_v(t) \end{pmatrix} = \begin{pmatrix} \mathbf{0}_{n \times n} & I_n \\ \mathbf{0}_{n \times n} & \mathbf{0}_{n \times n} \end{pmatrix} \otimes I_N \begin{pmatrix} e_r(t) \\ e_v(t) \end{pmatrix}, t \neq t_k, \\ \begin{pmatrix} e_r(t_k^+) \\ e_v(t_k^+) \end{pmatrix} = \begin{pmatrix} I_n & \mathbf{0}_{n \times n} \\ -\alpha \mathcal{L} & I_n - \beta \mathcal{L} \end{pmatrix} \otimes I_N \begin{pmatrix} e_r(t_k) \\ e_v(t_k) \end{pmatrix}, t = t_k, \end{cases} \quad (10)$$

with

$$\begin{pmatrix} e_r(0) \\ e_v(0) \end{pmatrix} = \begin{pmatrix} I_n - \Lambda & \mathbf{0}_{n \times n} \\ \mathbf{0}_{n \times n} & I_n - \Lambda \end{pmatrix} \otimes I_N \begin{pmatrix} r(0) \\ v(0) \end{pmatrix}, \quad (11)$$

where $\Lambda = [\Phi \xi_1 \mathbf{1}_{n_1}^T, \Phi \xi_2 \mathbf{1}_{n_2}^T, \dots, \Phi \xi_k \mathbf{1}_{n_k}^T]^T$. Now, two properties of matrix Λ , which play an important role in the subsequent analysis process, should be introduced in the following. (1) $\mathcal{L}\Lambda = \Lambda\mathcal{L}$: note that matrix ξ is self-similar. Then, $\mathcal{L}\Lambda = \Phi(\Phi\mathcal{L}\Phi)(\Phi\Lambda)$. From the definition of the vectors $\xi_i, i = 1, 2, \dots, k$, we have $\mathcal{L}\Lambda = \Lambda\mathcal{L}$. (2) $\Lambda^2 = \Lambda$: this property can be obtained by direct calculation based on the structure of vector $\xi_i, i = 1, 2, \dots, k$.

Furthermore, we introduce two error variables: $\bar{e}_r(t) = (\Phi \otimes I_N)e_r(t)$ and $\bar{e}_v(t) = (\Phi \otimes I_N)e_v(t)$. Then,

$$\begin{pmatrix} \bar{e}_r(t) \\ \bar{e}_v(t) \end{pmatrix} = \begin{pmatrix} \Phi & \mathbf{0}_{d \times d} \\ \mathbf{0}_{d \times d} & \Phi \end{pmatrix} \otimes I_N \begin{pmatrix} e_r(t) \\ e_v(t) \end{pmatrix}. \quad (12)$$

Then, system (10) can convert the state into two new error variables $\bar{e}_r(t)$ and $\bar{e}_v(t)$ as

$$\begin{cases} \begin{pmatrix} \dot{\bar{e}}_r(t) \\ \dot{\bar{e}}_v(t) \end{pmatrix} = \begin{pmatrix} \mathbf{0}_{n \times n} & I_n \\ \mathbf{0}_{n \times n} & \mathbf{0}_{n \times n} \end{pmatrix} \otimes I_N \begin{pmatrix} \bar{e}_r(t) \\ \bar{e}_v(t) \end{pmatrix}, t \neq t_k, \\ \begin{pmatrix} \bar{e}_r(t_k^+) \\ \bar{e}_v(t_k^+) \end{pmatrix} = \begin{pmatrix} I_n & \mathbf{0}_{n \times n} \\ -\alpha \Phi \mathcal{L} \Phi & I_n - \beta \Phi \mathcal{L} \Phi \end{pmatrix} \otimes I_N \begin{pmatrix} \bar{e}_r(t_k) \\ \bar{e}_v(t_k) \end{pmatrix}, t = t_k, \end{cases} \quad (13)$$

with

$$\begin{pmatrix} \bar{e}_r(0) \\ \bar{e}_v(0) \end{pmatrix} = \begin{pmatrix} I_n - \Phi \Lambda \Phi & \mathbf{0}_{n \times n} \\ \mathbf{0}_{n \times n} & I_n - \Phi \Lambda \Phi \end{pmatrix} \otimes I_N \times \left\{ [\text{diag}\{\Phi, \Phi\} \otimes I_N] \begin{pmatrix} r(0) \\ v(0) \end{pmatrix} \right\}. \quad (14)$$

By using the properties of matrix Λ , for $t \in [t_l, t_{l+1})$, the solution of impulsive system (13) can be given as

$$\begin{cases} \begin{pmatrix} \bar{e}_r(t) \\ \bar{e}_v(t) \end{pmatrix} = \begin{pmatrix} I_n & (t - t_l)I_n \\ \mathbf{0}_{n \times n} & I_n \end{pmatrix} \times \begin{pmatrix} I_n & hI_n \\ -\alpha \Phi \mathcal{L} \Phi & I_n - \alpha h \Phi \mathcal{L} \Phi - \beta h \Phi \mathcal{L} \Phi \end{pmatrix}^l \otimes I_N \\ \begin{pmatrix} e_r(0) \\ e_v(0) \end{pmatrix} = \left\{ \left[\begin{pmatrix} I_n & (t - t_l)I_n \\ \mathbf{0}_{n \times n} & I_n \end{pmatrix} \times Y^l \right] \otimes I_N \right\} \left\{ [\text{diag}\{\Phi, \Phi\} \otimes I_N] \begin{pmatrix} r(0) \\ v(0) \end{pmatrix} \right\}, \end{cases} \quad (15)$$

where

$$Y = \begin{pmatrix} I_n - \Lambda & h(I_n - \Phi \Lambda \Phi) \\ -\alpha \Phi \mathcal{L} \Phi & I_n - \alpha h \Phi \mathcal{L} \Phi - \beta h \Phi \mathcal{L} \Phi - \Phi \Lambda \Phi \end{pmatrix}. \quad (16)$$

Since matrix $\Phi \mathcal{L} \Phi$ has $\xi_i, i = 1, 2, \dots, k$, as its left eigenvectors associated with a zero eigenvalue, it has the Jordan decomposition as $\Phi \mathcal{L} \Phi = \Psi \Gamma \Psi^{-1}$, where

$$\Gamma = \text{diag}\{\mathbf{0}_{k \times k}, J\}, \quad (17)$$

where

$$\Psi = (p_1, p_2, \dots, p_n, \tilde{p}_{k+1}, \dots, \tilde{p}_n), \quad (18)$$

$$\Psi^{-1} = (\xi_1, \xi_2, \dots, \xi_k, \tilde{\xi}_k, \dots, \tilde{\xi}_n), \quad (19)$$

$p_1 = (\mathbf{1}_{n_1}^T, \mathbf{0}_{n-n_1}^T)^T, \dots, p_k = (\mathbf{0}_{n-n_k}^T, \mathbf{1}_{n_k}^T)^T$. Subblock J corresponds to the nonzero eigenvalues of matrix $\Phi\mathcal{L}\Phi$; therefore, $-J$ is Hurwitz stable. Considering that $\mathcal{L}\Lambda = \Lambda\mathcal{L}$ and $\Lambda^2 = \Lambda$, then $\Psi^{-1}\Lambda\Psi = \text{diag}\{I_k, \mathbf{0}_{(n-k) \times (n-k)}\}$. Therefore, we have

$$\begin{aligned}\Omega_j &= \text{diag}\{\Psi^{-1}, \Psi^{-1}\} \Upsilon \text{diag}\{\Psi, \Psi\} \\ &= \begin{pmatrix} I_n - \Psi^{-1}\Lambda\Psi & h(I_n - \Psi^{-1}\Lambda\Psi) \\ -\alpha\Gamma & I_n - \alpha h\Gamma - \beta h\Gamma - \Psi^{-1}\Lambda\Psi \end{pmatrix}.\end{aligned}\quad (20)$$

Then, the following discussion can be turned into the proof that $R_j = \det(\lambda I_{2n} - \Omega_j)$ is Schur stable if equation $h < \min_{1 \leq i \leq n-k} \frac{4\beta^2 \text{Re}(\lambda_i) |\lambda_i|^2 - 2\beta^3 |\lambda_i|^4}{4\alpha \text{Im}^2(\lambda_i) + \alpha\beta^2 |\lambda_i|^4}$ holds. Following the similar analysis of Theorem 1 in Ref. [8], we can obtain the desired result. $\square$

Remark 5. Theorem 1 gives the sufficient conditions under which the second-order MASs (1) reach multiple-bipartite consensus. Since negative weights can exist between agents from the same group and different groups, the analysis method presented in the existing work is no longer applicable here. In order to solve this problem, we construct the new multiple-bipartite consensus from error variables $\bar{e}_r(t)$ and $\bar{e}_v(t)$, based on which the new form error Equation (13) is provided. By using this transformation, the traditional stability analysis method can be applied.

Remark 6. To explicitly calculate the final multiple-bipartite consensus states, we must first determine matrix Φ . Actually, matrix Φ also determines the symbol “+” or “−”. By using matrix Φ , all second-order MASs (1) form a multiple symmetric convergence state. This symmetric state covers not only the position but also the velocity. Therefore, the multiple-bipartite consensus mathematical model defined in this paper can well describe multiple compound symmetric tasks in practical engineering applications.

Remark 7. Theorem 1 gives the sufficient conditions of realizing the multiple-bipartite consensus of second-order MASs. If

$$h \geq \min_{1 \leq i \leq n-k} \frac{4\beta^2 \text{Re}(\lambda_i) |\lambda_i|^2 - 2\beta^3 |\lambda_i|^4}{4\alpha \text{Im}^2(\lambda_i) + \alpha\beta^2 |\lambda_i|^4},$$

the second-order MASs can be in a divergent state, and this point will be illustrated in the simulation section in Section 4. Moreover, the conditions given in Theorem 1 are only composed of the eigenvalues of matrix $\Phi\mathcal{L}\Phi$ and the initial values of the system; therefore, it is easy to verify.

Remark 8. The Lyapunov stability theorem is applicable to not only linear but also nonlinear system analysis. In our paper, the impulsive second-order system considered here is linear and stochastic. It is sufficient and fit to employ the classic Schur stability theorem to carry out the analysis. Therefore, it is not pre-requisite to use the Lyapunov stability method. The final multiple-bipartite consensus convergence state is explicitly expressed by the eigenvectors of the Laplacian matrix associated with zero eigenvalue and the initial values. Then, we can select an appropriate initial value to realize the desired multiple-bipartite consensus.

4. Simulations

In this section, two simulation examples will be assessed to illustrate the aforementioned theoretical results. For convenience, we assume $N = 1$ in this section.

Example 1. Consider a network consisting of nine agents, and the information communication graph topology is shown in Figure 1. Obviously, node set $\mathcal{V} = \{1, 2, \dots, 9\}$ has an acyclic partition as $\mathcal{V}_1 = \{1, 2, 3\}$, $\mathcal{V}_2 = \{4, 5, 6\}$ and $\mathcal{V}_3 = \{7, 8, 9\}$. The corresponding graphs $\mathcal{G}_1$, $\mathcal{G}_2$, and $\mathcal{G}_3$ are all structurally balanced. Following the definition of ϕ_i , we have $\phi_1 = \phi_4 = \phi_6 = \phi_7 = \phi_8 = 1$ and $\phi_2 = \phi_3 = \phi_5 = \phi_9 = -1$. Then, $\Phi = \text{diag}\{1, -1, -1, 1, -1, 1, 1, 1, -1\}$. Furthermore, the eigenvalues of matrix $\Phi\mathcal{L}\Phi$ can be obtained by direct calculation as 0, 0, 0, 1,

1, 1, $1.5 + 0.866i$ and $1.5 - 0.866i$, where i is an imaginary unit. The corresponding vectors ξ_i , $i = 1, 2, \dots, 3$, defined in Equation (8) have the form $\xi_1 = [1, 0, 0, 0, 0, 0, 0, 0, 0]^T$, $\xi_2 = [\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0, 0]^T$ and $\xi_3 = [0, 0, 0, 0, 0, 0, 1, 0, 0]^T$. The initial values $r(0)$ and $v(0)$ are set as $r(0) = [r_1(0), r_2(0), \dots, r_9(0)]^T = [2, -1, -3, -1, 2, -1.5, -2, 3, -5]^T$ and $v(0) = [v_1(0), v_2(0), \dots, v_9(0)]^T = [2, 1, -2, 3, -7, -5, 10, 6, -3]^T$. The control gains α and β and control interval h are set as $\alpha = 1$, $\beta = 0.1$ and $h = 0.02$, respectively. Then, all the conditions of Theorem 1 are satisfied. Therefore, according to Theorem 1, through simple calculations, we can obtain the critical value of $h = \min_{1 \leq i \leq n-k} \frac{4\beta^2 \text{Re}(\lambda_i) |\lambda_i|^2 - 2\beta^3 |\lambda_i|^4}{4\alpha \text{Im}^2(\lambda_i) + \alpha \beta^2 |\lambda_i|^4} = 0.1108$. Accordingly, if $h < 0.1108$, multiple-bipartite consensus should be reached; and the final multiple-bipartite consensus states are $\lim_{t \rightarrow \infty} r_1(t) = 2 + 2t$, $\lim_{t \rightarrow \infty} r_2(t) = -2 - 2t$, $\lim_{t \rightarrow \infty} r_3(t) = -2 - 2t$, $\lim_{t \rightarrow \infty} r_4(t) = -\frac{3}{2} + \frac{8}{3}t$, $\lim_{t \rightarrow \infty} r_5(t) = \frac{3}{2} - \frac{8}{3}t$, $\lim_{t \rightarrow \infty} r_6(t) = -\frac{3}{2} + \frac{8}{3}t$, $\lim_{t \rightarrow \infty} r_7(t) = 3 + 6t$, $\lim_{t \rightarrow \infty} r_8(t) = 3 + 6t$, $\lim_{t \rightarrow \infty} r_9(t) = -3 - 6t$, $\lim_{t \rightarrow \infty} v_1(t) = 2$, $\lim_{t \rightarrow \infty} v_2(t) = -2$, $\lim_{t \rightarrow \infty} v_3(t) = -2$, $\lim_{t \rightarrow \infty} v_4(t) = \frac{8}{3}$, $\lim_{t \rightarrow \infty} v_5(t) = -\frac{8}{3}$, $\lim_{t \rightarrow \infty} v_6(t) = \frac{8}{3}$, $\lim_{t \rightarrow \infty} v_7(t) = 6$, $\lim_{t \rightarrow \infty} v_8(t) = 6$ and $\lim_{t \rightarrow \infty} v_9(t) = -6$. Figures 2 and 3 display the evolution of positions $r_i(t)$ and velocities $v_i(t)$, $i = 1, 2, \dots, 9$, respectively, as time passes. The simulation results presented in Figures 2 and 3 are accurately consistent with Theorem 1. For the system subject to the impulsive differential Equation (1), the control protocol u_i only works on the velocity states with instantaneous interval. In each interval, by the improved Laplacian matrix, we solve the one-order differential equation to obtain the velocity solution as the initial value for the next time interval. Therefore, in Figure 3, it is clear that the **velocity** states evolve with a discontinuous zigzag shape before reaching consistency. However, we can clearly see that the **position** states in Figure 2 are smooth all the time.

Furthermore, we set the control interval as $h = 0.2$. Obviously, the length of the control interval h no longer satisfies the conditions presented in Theorem 1. Figures 4 and 5 describe the trajectories of the nine agents under this control interval. Figures 4 and 5 show that the position $r_i(t)$ and velocities $v_i(t)$, $i = 1, 2, \dots, 9$, converge to an multiple-bipartite consensus state; conversely, they converge to a divergent state.

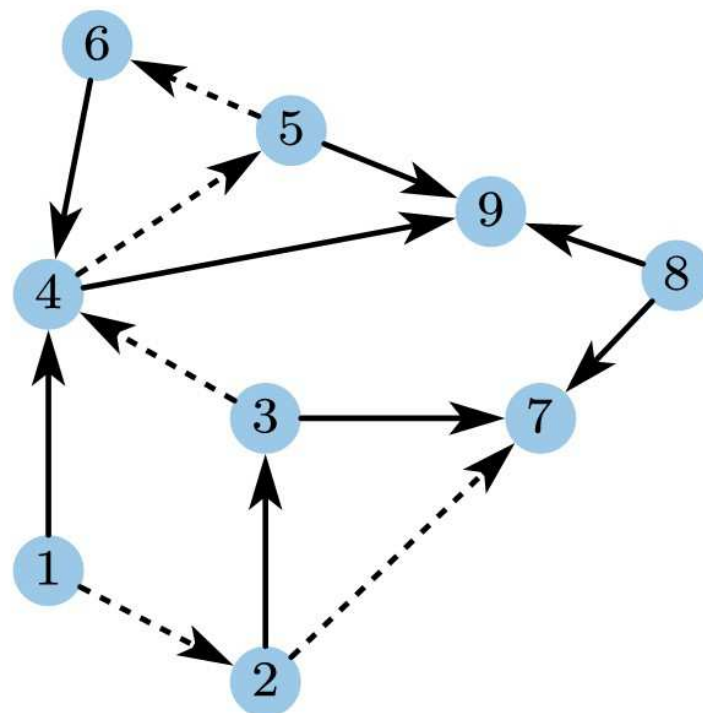


Figure 1. The topological structure of the nine agents in Example 1.

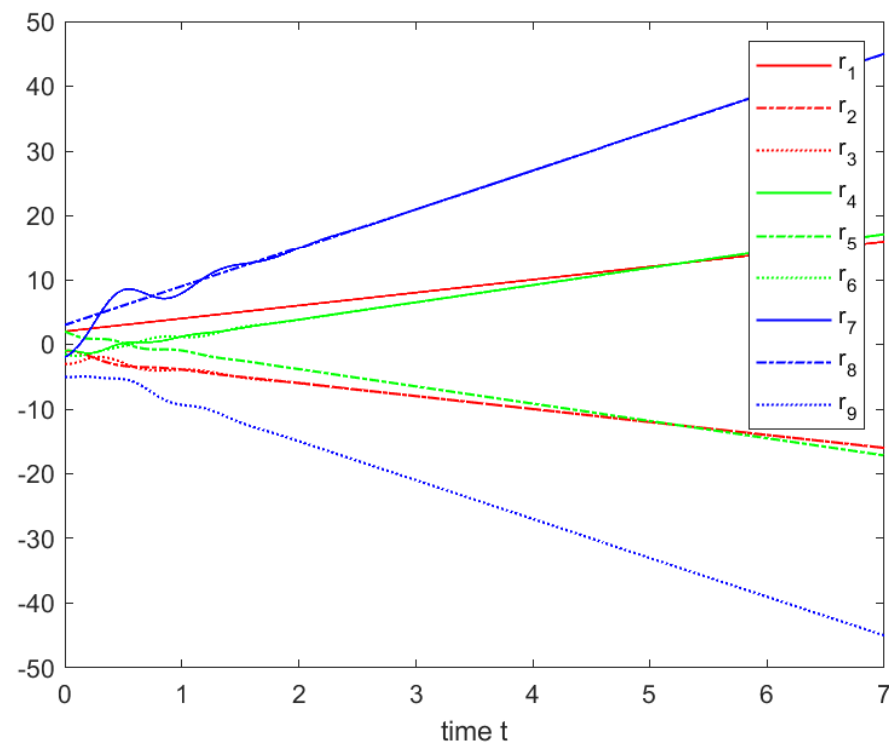


Figure 2. The time evolution trajectories of $r_i(t)$, $i = 1, 2, \dots, 9$, with $h = 0.02$ in Example 1.

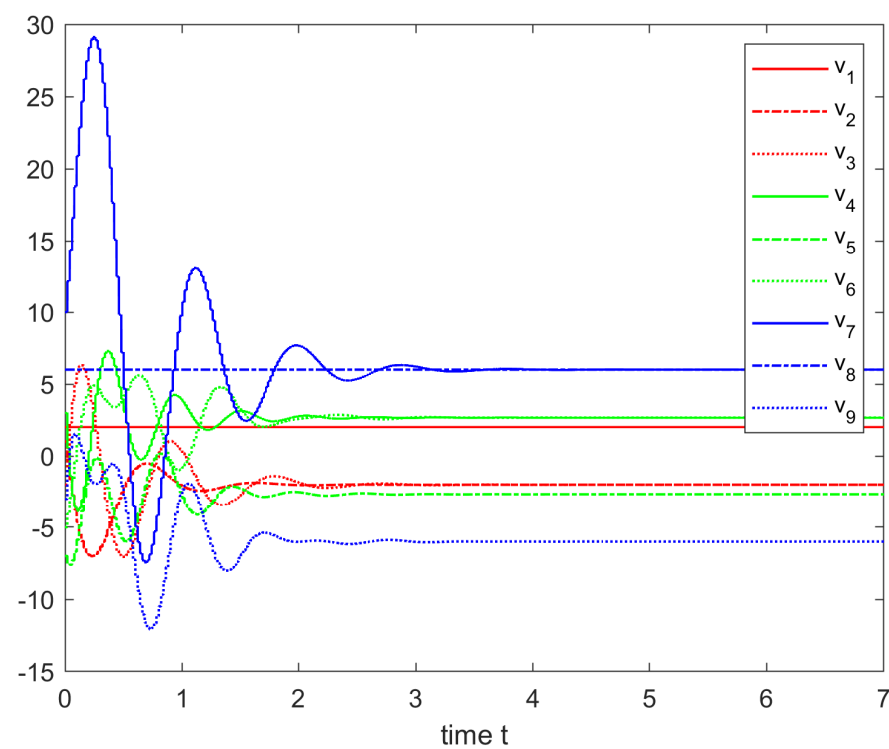


Figure 3. The time evolution trajectories of $v_i(t)$, $i = 1, 2, \dots, 9$, with $h = 0.02$ in Example 1.

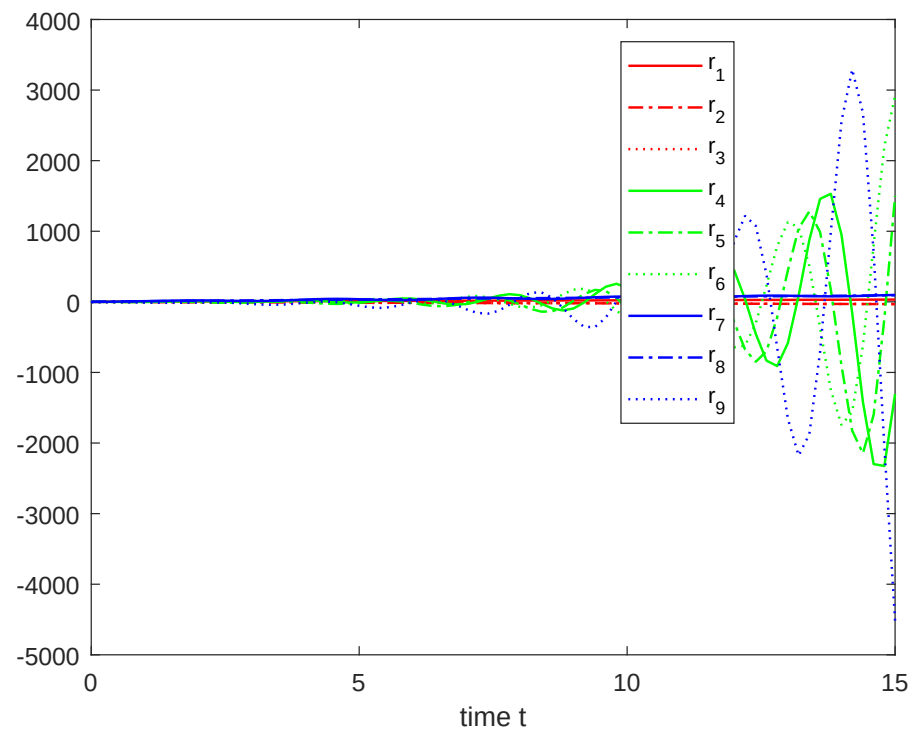


Figure 4. The time evolution trajectories of $r_i(t)$, $i = 1, 2, \dots, 9$, with $h = 0.2$ in Example 1.

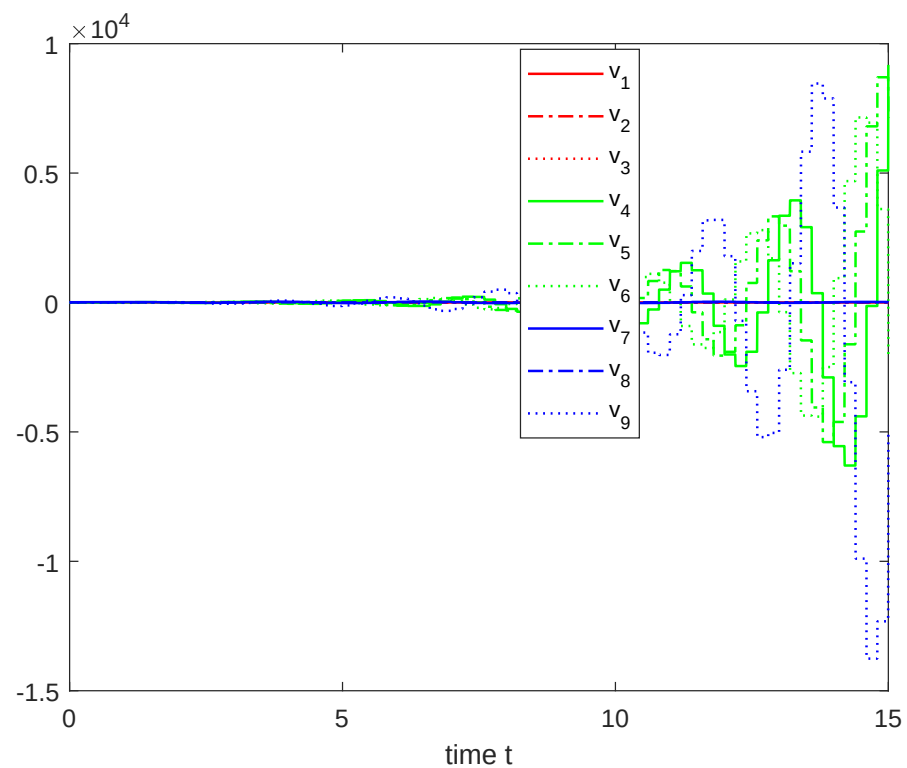


Figure 5. The time evolution trajectories of $v_i(t)$, $i = 1, 2, \dots, 9$, with $h = 0.2$ in Example 1.

Example 2. The interesting result of some subgroups being cooperative networks will be assessed through the simulation of this example. Consider the networks consisting of nine agents and the graph topology presented in Figure 6. It is easy to see that the node set $\mathcal{V} = \{1, 2, \dots, 9\}$ has an AP as $\mathcal{V}_1 = \{1, 2, 3\}$, $\mathcal{V}_2 = \{4, 5, 6\}$ and $\mathcal{V}_3 = \{7, 8, 9\}$. Since it is different from Example 1, subgraph

$\mathcal{G}_1$ corresponding to the first subgroup $\mathcal{V}_1 = \{1, 2, 3\}$ is cooperative. In addition, the second and third subgraphs $\mathcal{G}_2$ and $\mathcal{G}_3$ are structurally balanced. The control gains α and β and control interval h are set as $\alpha = 1$, $\beta = 0.1$ and $h = 0.02$, respectively. In this case, the final convergence state should be that the first group reaches a complete consensus, and the second and third groups reach bipartite consensus. Figures 7 and 8 display the trajectories of the nine agents. As can be seen from Figures 7 and 8, the first group reaches a complete consensus, and the second and third groups reach bipartite consensus, which coincides with our deduction. It can also be seen from Figure 9 that, when the second subgroup $\mathcal{V}_2 = \{4, 5, 6\}$ is cooperative, the second group reaches a complete consensus and another two subgroups reach bipartite consensus, as demonstrated in Figures 10 and 11, respectively.

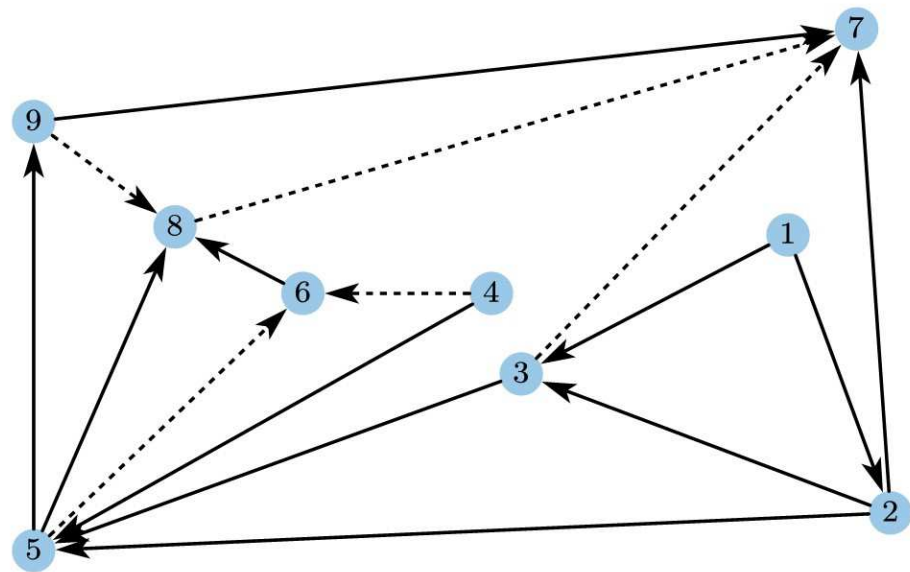


Figure 6. The topological structure of the nine agents in Example 2.

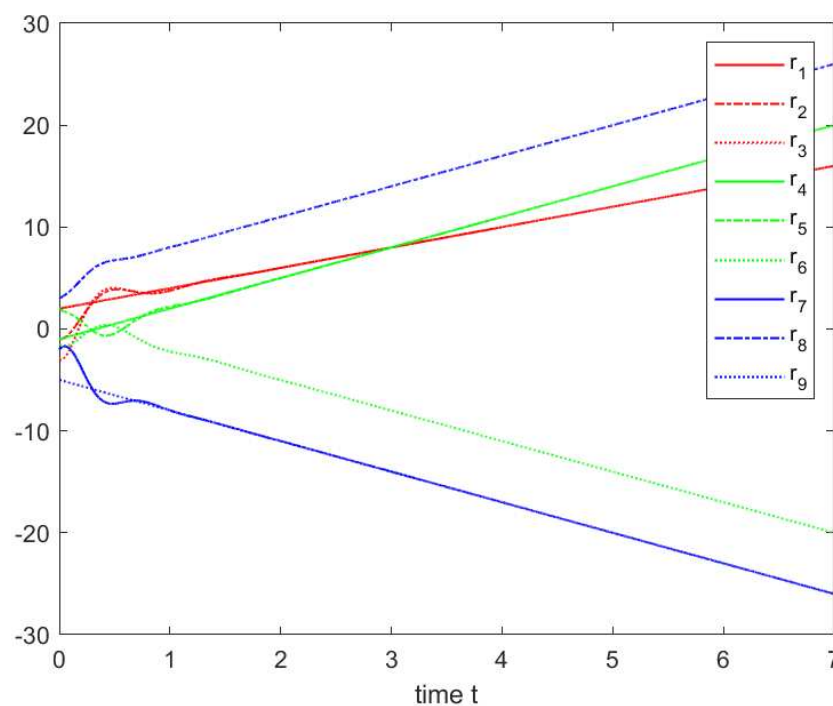


Figure 7. The time evolution trajectories of $r_i(t)$, $i = 1, 2, \dots, 9$, in Figure 6 of Example 2.

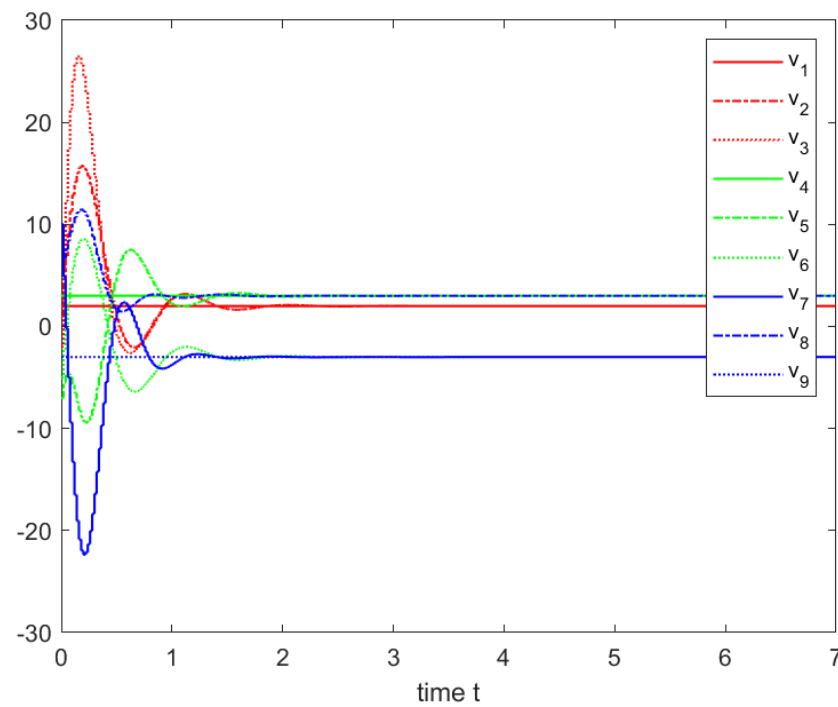


Figure 8. The time evolution trajectories of $v_i(t)$, $i = 1, 2, \dots, 9$, in Figure 6 of Example 2.

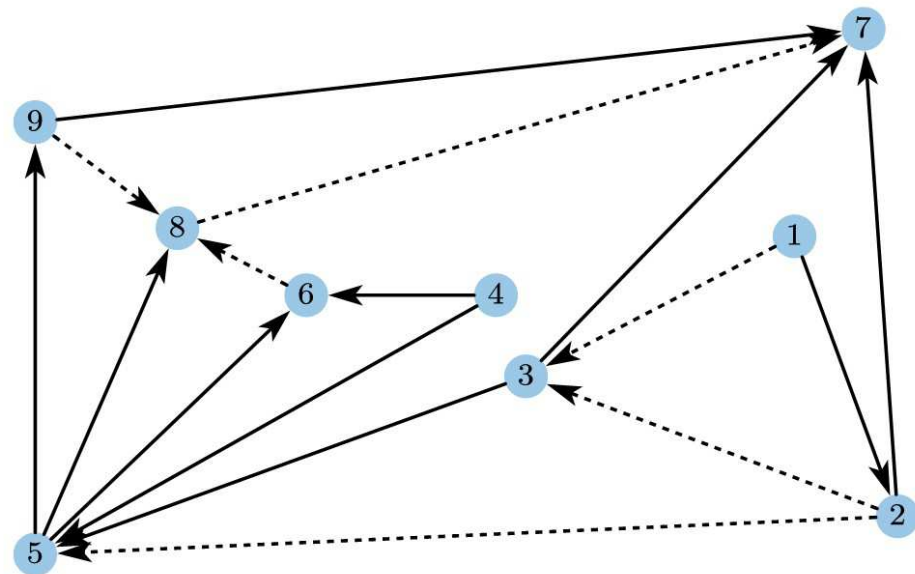


Figure 9. Another topological structure of the nine agents in Example 2.

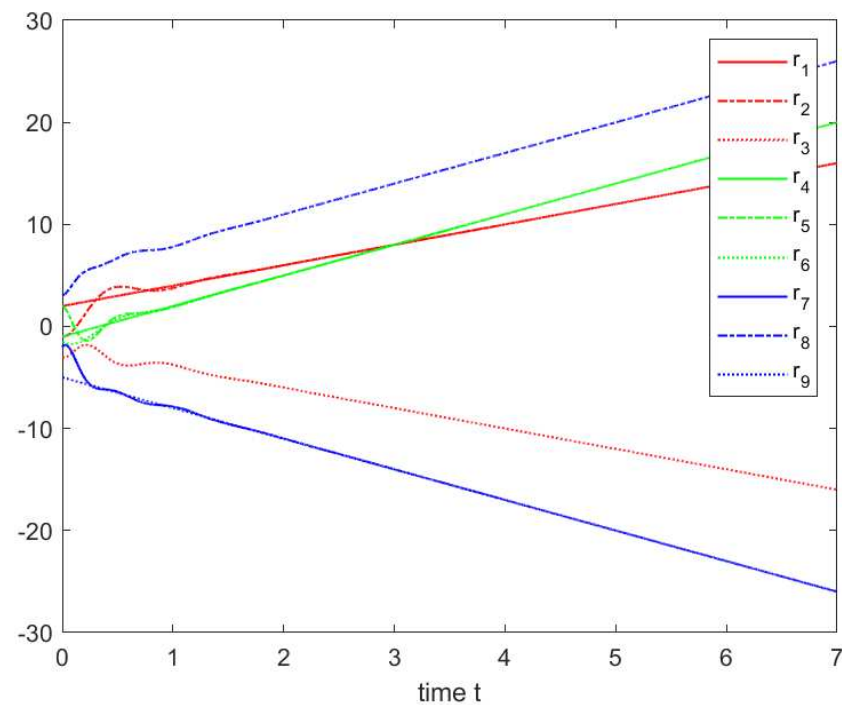


Figure 10. The time evolution trajectories of $r_i(t)$, $i = 1, 2, \dots, 9$, in Figure 9 of Example 2.

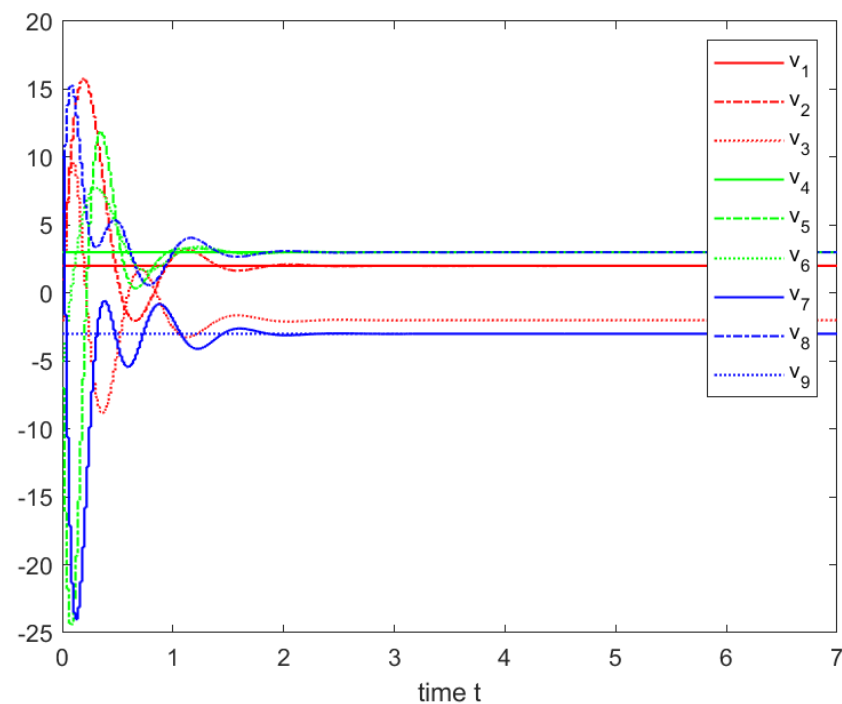


Figure 11. The time evolution trajectories of $r_i(t)$, $i = 1, 2, \dots, 9$, in Figure 9 of Example 2.

5. Conclusions

This paper studied the impulsive multiple-bipartite consensus problem of second-order MASs. First, we introduced the concept of multiple-bipartite consensus into second-order MASs, and then an improved Laplacian matrix associated with the network topology was developed according to the structure of the network. The criterion that ensures the achievement of multiple-bipartite consensus was presented. It should be noted here that the

sufficient conditions given in this paper for realizing multiple-bipartite consensus involve the eigenvalues of Laplace matrix, and this involves the connection and coupling of the entire network. That is to say, although the controller designed in this paper is distributed in form, the implementation conditions are not distributed. Based on this, our next step will focus on the research of fully distributed multiple symmetric control protocol algorithms. On the other hand, the network partition discussed in this paper is somewhat restricted; thus, another general network topology type, an almost equitable partition (AEP), which has great potential and important applications in the natural world and engineering field, such as Kuramoto oscillators and affine formation control, will be worthy of our future research [41–45].

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References

1. Li, S.; Wang, Z.; Zheng, Y.; Sun, Q.; Gao, J.; Ma, F.; Li, K. Synchronous and asynchronous parallel computation for large-scale optimal control of connected vehicles. *Transp. Res. Part C Emerg. Technol.* **2020**, *121*, 128042. [\[CrossRef\]](#)
2. Cao, Y.; Ren, W.; Meng, Z. Decentralized finite-time sliding mode estimators and their applications in decentralized finite-time formation tracking. *Syst. Cont. Lett.* **2010**, *59*, 522–529. [\[CrossRef\]](#)
3. Ferber, J. *Multi-Agent Systems: An Introduction to Distributed Artificial Intelligence*, 1st ed.; Addison-Wesley: Boston, MA, USA, 1999.
4. Russell, S.; Norvig, P. *Artificial Intelligence: A Modern Approach*; Prentice-Hall: Hoboken, NJ, USA, 1995.
5. Wooldridge, M. *An Introduction to Multiagent Systems*, 2nd ed.; John Wiley & Sons: Hoboken, NJ, USA, 2009.
6. Liu, J.; Ruan, X.; Zheng, Y. Iterative learning control for discrete-time systems with full learnability. *IEEE Trans. Neural Netw. Learn. Syst.* **2021**, *33*, 629–643. [\[CrossRef\]](#) [\[PubMed\]](#)
7. Amirkhani, A.; Barshooi, A. Consensus in multi-agent systems: A review. *Artif. Intell. Rev.* **2022**, *55*, 3897–3935. [\[CrossRef\]](#)
8. Guan, Z.; Liu, Z.; Feng, G.; Jian, M. Impulsive consensus algorithms for second-order multi-agent networks with sampled information. *Automatica* **2012**, *48*, 1397–1404. [\[CrossRef\]](#)
9. Olfati-Saber, R.; Franco, E.; Frazzoli, E.; Shamma, J. Belief consensus and distributed hypothesis testing in sensor networks. In *Networked Embedded Sensing and Control*; Antsaklis, P.J., Tabuada, P., Eds.; Lecture Notes in Control and Information Science; Springer: Berlin/Heidelberg, Germany, 2006; Volume 331.
10. Sundaram, S.; Hadjicostis, C. Finite-time distributed consensus in graphs with time-invariant topologies. In Proceedings of the 2007 American Control Conference, New York, NY, USA, 9–13 July 2007; pp. 711–716.
11. Ma, J.; Ji, H.; Sun, D.; Feng, G. An approach to quantized consensus of continuous-time linear multi-agent systems. *Automatica* **2018**, *91*, 98–104. [\[CrossRef\]](#)
12. Zhang, Z.; Zhang, L.; Hao, F.; Wang, L. Leader-following consensus for linear and Lipschitz nonlinear multiagent systems with quantized communication. *IEEE Trans. Cybern.* **2016**, *47*, 1970–1982. [\[CrossRef\]](#)
13. Chen, W.; Li, X.; Jiao, L. Quantized consensus of second-order continuous-time multi-agent systems with a directed topology via sampled data. *Automatica* **2013**, *49*, 2236–2242.
14. Altafini, C. Consensus problems on networks with antagonistic interactions. *IEEE Trans. Autom. Control* **2013**, *58*, 935–946.
15. Pan, L.; Shao, H.; Xi, Y.; Li, D. Bipartite consensus problem on matrix-valued weighted directed networks. *Sci. China Inform. Sci.* **2021**, *64*, 149204. [\[CrossRef\]](#)
16. Deng, Q.; Peng, Y.; Han, T.; Qu, D. Event-triggered bipartite consensus in networked Euler–Lagrange systems with external disturbance. *IEEE Trans. Circuits Syst. II Exp. Briefs* **2021**, *68*, 2870–2874. [\[CrossRef\]](#)

17. Ding, T.; Ge, M.; Xiong, C.; Park, J.; Li, M. Second-order bipartite consensus for networked robotic systems with quantized-data interactions and time-varying transmission delays. *ISA Trans.* **2021**, *108*, 178–187. [[CrossRef](#)] [[PubMed](#)]
18. Zhang, Z.; Chen, S.; Su, H. Scaled consensus of second-order nonlinear multiagent systems with time-varying delays via aperiodically intermittent control. *IEEE Trans. Cybern.* **2019**, *50*, 3503. [[CrossRef](#)] [[PubMed](#)]
19. Zhang, Z.; Chen, S.; Zheng, Y. Fully distributed scaled consensus tracking of high-order multiagent systems with time delays and disturbances. *IEEE Trans. Ind. Inform.* **2022**, *18*, 305–314. [[CrossRef](#)]
20. Roy, S. Scaled consensus. *Automatica* **2015**, *51*, 259–262. [[CrossRef](#)]
21. Ji, L.; Gao, T.; Liao, X. Couple-group consensus for cooperative-competitive heterogeneous multiagent systems: Hybrid adaptive and pinning methods. *IEEE Trans. Syst. Man Cybern. Syst.* **2021**, *51*, 5367–5376. [[CrossRef](#)]
22. Zhan, J.; Li, X. Cluster consensus in networks of agents with weighted cooperative-competitive interactions. *IEEE Trans. Circuits Syst. II Exp. Briefs* **2017**, *65*, 241–245. [[CrossRef](#)]
23. Liu, J.; Zhou, J. Distributed impulsive group consensus in second-order multi-agent systems under directed topology. *Int. J. Control* **2015**, *88*, 910–919. [[CrossRef](#)]
24. Adibzadeh, A.; Suratgar, A.; Menhaj, M.; Zamani, M. Constrained optimal consensus in multi-agent systems with single-and double-integrator dynamics. *Int. J. Control* **2020**, *93*, 575–587. [[CrossRef](#)]
25. Xie, G.; Wang, L. Consensus control for a class of networks of dynamic agents. *Int. J. Robust Nonlinear Control IFAC-Affil. J.* **2007**, *17*, 941–959. [[CrossRef](#)]
26. Dong, X.; Zhou, Y.; Ren, Z.; Zhong, Y. Time-varying formation control for unmanned aerial vehicles with switching interaction topologies. *Control Eng. Pract.* **2016**, *46*, 26–36. [[CrossRef](#)]
27. He, W.; Cao, J. Consensus control for high-order multi-agent systems. *IET Control Theory Appl.* **2011**, *5*, 231–238. [[CrossRef](#)]
28. Mei, J.; Ren, W.; Ma, G. Distributed containment control for Lagrangian networks with parametric uncertainties under a directed graph. *Automatica* **2012**, *48*, 653–659. [[CrossRef](#)]
29. Hu, T.; Liu, X.; He, Z.; Zhang, X.; Zhong, S. Hybrid event-triggered and impulsive control strategy for multiagent systems with switching topologies. *IEEE Trans. Cybern.* **2022**, *52*, 6283–6294. [[CrossRef](#)]
30. Fu, W.; Qin, J.; Wu, J.; Zheng, W.; Kang, Y. Interval consensus over random networks. *Automatica* **2020**, *111*, 108603. [[CrossRef](#)]
31. Liu, J.; Li, H.; Ji, J.; Luo, J. Group-bipartite consensus in the networks with cooperative-competitive Interactions. *IEEE Trans. Circuits Syst. II Exp. Briefs* **2020**, *67*, 3292–3296. [[CrossRef](#)]
32. Yang, T. *Impulsive Control Theory*; Springer: Berlin, Germany, 2001.
33. Nakadate, R.; Iwasa, T.; Onogi, S.; Arata, J.; Oguri, S.; Okamoto, Y.; Akahoshi, T.; Eto, M.; Hashizume, M. Surgical robot for intraluminal access: An ex vivo feasibility study. *Cyborg Bionic Syst.* **2020**, *2020*, 8378025. [[CrossRef](#)]
34. Liu, B.; Liu, X.; Chen, G.; Wang, H. Robust impulsive synchronization of uncertain dynamical networks. *IEEE Trans. Circuits Syst. I-Regul. Pap.* **2005**, *52*, 1431–1441.
35. Tang, Y.; Leung, S.Y.S.; Wong, W.K.; Fang, J. Impulsive pinning synchronization of stochastic discrete-time networks. *Neurocomputing* **2010**, *73*, 2132–2139. [[CrossRef](#)]
36. Guan, Z.; Liu, Z.; Feng, G.; Wang, Y. Synchronization of complex dynamical networks With time-varying delays via impulsive distributed control. *IEEE Trans. Circuits Syst. I-Regul. Pap.* **2010**, *57*, 2182–2195. [[CrossRef](#)]
37. Qin, J.; Yu, C. Cluster consensus control of generic linear multi-agent systems under directed topology with acyclic partition. *Automatica* **2013**, *49*, 2898–2905. [[CrossRef](#)]
38. Jadbabaie, A.; Lin, J.; Morse, A.S. Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Trans. Autom. Control* **2003**, *48*, 988–1001. [[CrossRef](#)]
39. Fax, J.A.; Murray, R.M. Information flow and cooperative control of vehicle formations. *IEEE Trans. Autom. Control* **2004**, *49*, 1465–1476.
40. Saber, R.O.; Murray, R.M. Consensus protocols for networks of dynamic agents. In Proceedings of the 2003 American Control Conference, Denver, CO, USA, 4–6 June 2003; pp. 951–956.
41. Monaco, S.; Celsi, L.R. On multi-consensus and almost equitable graph partitions. *Automatica* **2019**, *103*, 53–61. [[CrossRef](#)]
42. Gambuzza, L.V.; Frasca, M. Distributed control of multiconsensus. *IEEE Trans. Autom. Control* **2020**, *66*, 2032–2044. [[CrossRef](#)]
43. Mattioni, M.; Monaco, S. Cluster partitioning of heterogeneous multi-agent systems. *Automatica* **2022**, *138*, 110136. [[CrossRef](#)]
44. Cristofaro, A.; Mattioni, M. Hybrid consensus for multi-agent systems with time-driven jumps. *Nonlinear Anal.-Hybrid Syst.* **2021**, *43*, 101113. [[CrossRef](#)]
45. Cacace, F.; Mattioni, M.; Monaco, S.; Celsi, L.R. Topology-induced containment for general linear systems on weakly connected digraphs. *Automatica* **2021**, *131*, 109734.