

Article

Einstein Aggregation Operators under Bipolar Neutrosophic Environment with Applications in Multi-Criteria Decision-Making

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Abstract: In this article, we introduce bipolar neutrosophic (BN) aggregation operators (AOs) as a revolutionary notion in aggregation operators (AOs) by applying Einstein operations to bipolar neutrosophic aggregation operators (AOs), with its application related to a real-life problem. The neutrosophic set is able to draw out the incomplete, inconsistent and indeterminate information pretty efficiently. Initially, we present essential definitions along with operations correlated to the neutrosophic set (NS) and its generalization, the bipolar neutrosophic set (BNS). The Einstein aggregation operators are our primary targets, such as the BN Einstein weighted average (BNEWA), BN Einstein ordered weighted average (BNEOWA), BN Einstein hybrid average (BNEHA), BN Einstein weighted geometric (BNEWG), BN Einstein ordered weighted geometric (BNEOWG) and BN Einstein hybrid geometric (BNEHG), as well as their required properties. The most important benefit of using the suggested approaches is that they provide decision-makers with complete sight of the issue. These techniques, when compared to other methods, provide complete, progressive and precise findings. Lastly, by means of diverse types of newly introduced aggregation operators and a numerical illustration by an example, we suggest an innovative method to be used for multi-criteria community decision-making (DM). This illustrates the utility and applicability of this new strategy when facing real-world problems.

Keywords: aggregation operator; decision-making; BNEWA; BNEOWA; BNEHA; BNEWG; BNEOWG; BNEHG



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1. Introduction

In the modern age of managerial decision-making (DM), knowledge frequently remains incomplete, undetermined and incompatible. L. A. Zadeh was the first to propose the fuzzy set theory [1], which deals with uncertainty and has applications in a wide range of modern fields of present and future society. Yet, fuzzy set has a problem in that it can simply convey the value of membership but not non-membership. To address this, Atanassov [2] developed the intuitionistic fuzzy set (IFS), as well as associated theory, in order to summarize the concept of fuzzy set. A pair of membership values is used to represent each IFS element, truth-membership $\mathfrak{S}(\chi)$ and a non-membership value (falsity-membership) $f(\chi)$, as well as satisfy the condition $\mathfrak{S}(\chi), f(\chi) \in [0, 1]$ with $0 \leq \mathfrak{S}(\chi) + f(\chi) \leq 1$. IFS can only handle incomplete data, meaning indeterminate data are not an option.

Florentin Smarandache [3] developed the novel concept of NS, which adds an indeterminacy membership value $I(\chi)$ to IFS. NS has a strong ability to deal with knowledge that is imperfect, uncertain and contradicting.

When $\mathfrak{S}(\chi) + I(\chi) + f(\chi) < 1$, the information acquired is indeterminate.

When $\mathfrak{S}(\chi) + I(\chi) + f(\chi) > 1$, this represents inconsistency in an NS.

Wang and others devised the single-valued neutrosophic set (SVNS) to deal with real-world situations [4], with the conditions $\mathfrak{S}(\chi), I(\chi), f(\chi) \in [0, 1]$ and $0 \leq \mathfrak{S}(\chi) + I(\chi) + f(\chi) \leq 3$. Ye [5] proposed a method for comparing SVNS, and defined the correlation coefficient. Wang et al. formulated an interval-valued neutrosophic set [6] and broadened the values of truth, indeterminacy and false memberships, which are all between 0 and 1.

For researchers, aggregation operators (AOs) are extremely important. Since its introduction, several scientists [7–14] have contributed significantly to the theory development of IFS. Xu et al. [15] introduced the concept of distinct IF aggregation operators based on IFS (AOs). Wang et al. and Zhao et al. [16,17] created Einstein aggregation operators (AOs). The Einstein t-norm typically gives the same smooth approximations as the product and sum of algebra.

The algebraic t-norm (product $\otimes$) and t-conorm (sum $\oplus$), respectively, are as below:

$$\begin{aligned} T(a, b) &= a \otimes b = ab \\ T^*(a, b) &= a \oplus b = a + b - ab \end{aligned}$$

The Einstein operations t-norm (product $\otimes$) and t-conorm (sum $\oplus$), respectively, are as below:

$$\begin{aligned} T(a, b) &= a \otimes b = \frac{ab}{1+(1-a)(1-b)} \\ T^*(a, b) &= a \oplus b = \frac{a+b}{1+ab} \end{aligned}$$

The bipolar fuzzy set (BFS) [18–20] has emerged as a new technique to deal with vagueness in DM situations. The bipolar fuzzy set's membership degree varies between -1 and 1 . Positive as well as negative membership degrees are available in BFS. BFSs are incredibly beneficial for several fields of study, as well as DM [21,22]. Gul [23] defined bipolar averaging and geometric aggregations operators (AOs). The bipolar neutrosophic set was created by Irfan and others [24,25] by using fundamental operations and a comparison mechanism [26,27]. Jamil and others created aggregation operators (AOs) based on BN values and applied them to DM problems. Heronian mean aggregation operators were developed by Fan and others [28]. Abdullah et al. introduced the idea of a bipolar soft set and its application to decision-making [29]. Jafar and other developed a bipolar neutrosophic soft set and applied it to decision-making [30]. Ali et al. introduced complex fuzzy set and its properties [31]. Broumi et al. introduced bipolar complex fuzzy set and its aggregation operators [32]. Jamil and others applied a multi-criteria decision-making approach to the bipolar neutrosophic set [33].

Regardless of the available information, there is a lot of literature related to the subject. The following features of the bipolar neutrosophic set inspired the present research team to perform a systematic and in-depth inquiry into decision analysis. Our findings are given below:

SVNSs deal with ambiguous details easily. This set combines the generalization of prior sets such as the classical set, the FS set and the IFS set. BFSs are highly effective for dealing with unpredictable, unexpected real-world circumstances because these can handle both positive as well as negative membership values.

The current study's main and most important goals were as follows:

- To suggest different bipolar neutrosophic Einstein AOs as well as desired properties to study;
- Based on BNN, establish a multi-criteria DM approach in the direction of real life problem-solving;
- Give a numerical description of a multi-criteria DM example.

The remainder of the paper is organized as follows. The second segment presents a basic definition as well as its associated properties. The BNEWA and BNEWG aggregation operators are introduced in the Section 3. These innovative AOs are applied to multi-criteria

decision-making in Section 4, and a numerical example is also presented. Finally, we offer comparative research, as well as some concluding thoughts, in Section 5.

2. Preliminaries

In this section, we provide fundamental definitions related to neutrosophic set theory. We define different fuzzy sets, BNS, score functions, accuracy functions, and certainty functions, as well as the Einstein operations.

Definition 1 [3]. Consider R to represent a fixed set. Then, the neutrosophic set N is defined as:

$$N = \{ (\chi, \mathfrak{S}(\chi), I(\chi), f(\chi)) \mid \chi \in R \}$$

Now, the mapping for membership functions of truth, indeterminacy and falsity are $\mathfrak{S} : N \rightarrow Q$, $I : N \rightarrow Q$ and $f : N \rightarrow Q$, respectively; here, $Q =]0^-, 1^+[$ and $0^- \leq \mathfrak{S}(\chi) + I(\chi) + f(\chi) \leq 3^+$.

Definition 2 [4]. Consider P to represent a fixed set; the single-valued neutrosophic set (SVNS) of A is stated as:

$$A_{NS} = \{ (\chi, \mathfrak{S}(\chi), I(\chi), f(\chi)) \mid \chi \in P \}$$

Now, here, the mapping for membership functions of truth, indeterminacy and falsity are $\mathfrak{S} : A_{NS} \rightarrow L$, $I : A_{NS} \rightarrow L$ and $f : A_{NS} \rightarrow L$, respectively; here, $L = [0, 1]$ and there is the condition $0 \leq \mathfrak{S}(\chi) + I(\chi) + f(\chi) \leq 3$.

Some fundamental operations related to single-valued neutrosophic sets (SVNS) are listed below:

$$A_{NS} = \{ (\chi, \mathfrak{S}_{A_{NS}}(\chi), I_{A_{NS}}(\chi), f_{A_{NS}}(\chi)) \mid \chi \in P \}, B_{NS} = \{ (\chi, \mathfrak{S}_{B_{NS}}(\chi), I_{B_{NS}}(\chi), f_{B_{NS}}(\chi)) \mid \chi \in P \},$$

The subset $A_{NS} \subseteq B_{NS}$ is represented as

$$\mathfrak{S}_{A_{NS}}(\chi) \leq \mathfrak{S}_{B_{NS}}(\chi), I_{A_{NS}}(\chi) \geq I_{B_{NS}}(\chi), f_{A_{NS}}(\chi) \geq f_{B_{NS}}(\chi).$$

$A_{NS} = B_{NS}$ is represented as

$$\mathfrak{S}_{A_{NS}}(\chi) = \mathfrak{S}_{B_{NS}}(\chi), I_{A_{NS}}(\chi) = I_{B_{NS}}(\chi), f_{A_{NS}}(\chi) = f_{B_{NS}}(\chi).$$

The complement A'_{NS} is

$$A'_{NS} = \{ (\chi, f_{A_{NS}}(\chi), 1 - I_{A_{NS}}(\chi), \mathfrak{S}_{A_{NS}}(\chi)) \mid \chi \in P \}$$

$A_{NS} \cap B_{NS}$ is represented as

$$A_{NS} \cap B_{NS} = \{ (\chi, \min\{\mathfrak{S}_{A_{NS}}(\chi), \mathfrak{S}_{B_{NS}}(\chi)\}, \max\{I_{A_{NS}}(\chi), I_{B_{NS}}(\chi)\}, \max\{f_{A_{NS}}(\chi), f_{B_{NS}}(\chi)\}) \mid \chi \in P \}$$

The union is defined by

$$A_{NS} \cup B_{NS} = \{ (\chi, \max\{\mathfrak{S}_{A_{NS}}(\chi), \mathfrak{S}_{B_{NS}}(\chi)\}, \min\{I_{A_{NS}}(\chi), I_{B_{NS}}(\chi)\}, \min\{f_{A_{NS}}(\chi), f_{B_{NS}}(\chi)\}) \mid \chi \in P \}$$

Definition 3 [5]. Consider two single-valued neutrosophic numbers (SVNNs) $u_1 = (\mathfrak{S}_1, I_1, f_1)$ and $u_2 = (\mathfrak{S}_2, I_2, f_2)$. The following are the distinct basic operations for SVNNs:

$$\begin{aligned} u_1 + u_2 &= (\mathfrak{S}_1 + \mathfrak{S}_2 - \mathfrak{S}_1 \mathfrak{S}_2, I_1 I_2, f_1 f_2); \\ u_1 \cdot u_2 &= (\mathfrak{S}_1 + \mathfrak{S}_2, I_1 + I_2 - I_1 I_2, f_1 + f_2 - f_1 f_2); \\ \lambda(u_1) &= (1 - (1 - \mathfrak{S}_1)^\lambda, (I_1)^\lambda, (f_1)^\lambda); \\ (u_1)^\lambda &= ((\mathfrak{S}_1)^\lambda, 1 - (1 - I_1)^\lambda, 1 - (1 - f_1)^\lambda); \end{aligned}$$

where $\lambda > 0$.

Definition 4 [5]. Let $u_1 = (\mathfrak{S}_1, I_1, f_1)$ be a single-valued neutrosophic number (SVNN). Then, the score function $s(u_1)$ is defined below:

$$s(u_1) = \frac{(\mathfrak{S}_1 + 1 - I_1 + 1 - f_1)}{3};$$

Definition 5 [5]. The accuracy function for an SVNN $u_1 = (\mathfrak{S}_1, I_1, f_1)$ is denoted by (u_1) , defined below:

$$(u_1) = (\mathfrak{S}_1 - f_1);$$

Definition 6 [5]. The certainty function for an SVNN $u_1 = (\mathfrak{S}_1, I_1, f_1)$ is denoted by $c(u_1)$, defined below:

$$c(u_1) = \mathfrak{S}_1$$

Definition 7 [5]. Consider two SVNNs $u_1 = (\mathfrak{S}_1, I_1, f_1)$ and $u_2 = (\mathfrak{S}_2, I_2, f_2)$. The following is the relationship between them:

- i. In condition $s(u_1) > s(u_2)$, then u_1 is greater than u_2 , denoted by $u_1 > u_2$;
- ii. In condition $s(u_1) = s(u_2)$ and $a(u_1) > a(u_2)$, then u_1 is greater than u_2 , denoted by $u_1 > u_2$;
- iii. In condition $s(u_1) = s(u_2)$, $a(u_1) = a(u_2)$ and $c(u_1) > c(u_2)$, in that case, u_1 is superior to u_2 , denoted by $u_1 > u_2$;
- iv. In condition $s(u_1) = s(u_2)$, $a(u_1) = a(u_2)$ and $c(u_1) = c(u_2)$, in that case, u_1 is equal to u_2 , denoted by $u_1 = u_2$;

Definition 8 [21]. Consider P as a universal set. Then, the bipolar fuzzy set (BFS) is stated below:

$$F = \{ \langle \chi, \mathfrak{S}_F^+(\chi), f_F^-(\chi) \rangle | \chi \in P \}$$

where $\mathfrak{S}_F^+(\chi) : F \rightarrow L^+$ represents a positive membership function and the negative membership function is $f_F^-(\chi) : F \rightarrow K^-$, where $L^+ = [0, 1]$ and $K^- = [-1, 0]$.

Definition 9 [25]. Suppose A is a bipolar neutrosophic set contained by universal set P , stated as:

$$A = \{ (\chi, \mathfrak{S}^+(\chi), I^+(\chi), f^+(\chi), \mathfrak{S}^-(\chi), I^-(\chi), f^-(\chi)) | \chi \in P \}$$

Let $\mathfrak{S}^+(\chi), I^+(\chi), f^+(\chi) = BN^+$ as well as $\mathfrak{S}^-(\chi), I^-(\chi), f^-(\chi) = BN^-$.

Here, $\mathfrak{S}^+(\chi), I^+(\chi), f^+(\chi)$ are positive membership functions representing true, indeterminate and false for $\chi \in P$, and $\mathfrak{S}^-(\chi), I^-(\chi), f^-(\chi)$ represent true, indeterminate and false for negative membership functions. Now, when mapping $BN^+ : A \rightarrow L^+$ and $BN^- : A \rightarrow L^-$, here, $L^+ = [0, 1]$ and $L^- = [-1, 0]$. This applies with the condition $0 \leq \mathfrak{S}^+(\chi) + I^+(\chi) + f^+(\chi) + \mathfrak{S}^-(\chi) + I^-(\chi) + f^-(\chi) \leq 6$.

Example 1. Let $P = \{\chi_1, \chi_2, \chi_3\}$, then

$$A = \left\{ \begin{array}{l} (\chi_1, 0.1, 0.5, 0.3, -0.4, -0.5, -0.6), \\ (\chi_2, 0.2, 0.7, 0.4, -0.3, -0.6, -0.2), \\ (\chi_3, 0.4, 0.6, 0.7, -0.2, -0.3, -0.1) \end{array} \right\}$$

is a subset of universal set P that is BNS.

Some fundamental operations [12] used for BNSs are stated below:

Let $A_1 = \{(\chi, \mathfrak{S}_1^+(\chi), I_1^+(\chi), f_1^+(\chi), \mathfrak{S}_1^-(\chi), I_1^-(\chi), f_1^-(\chi)) | \chi \in P\}$ and $A_2 = \{(\chi, \mathfrak{S}_2^+(\chi), I_2^+(\chi), f_2^+(\chi), \mathfrak{S}_2^-(\chi), I_2^-(\chi), f_2^-(\chi)) | \chi \in P\}$ be two BNSs.

i. Then, $A_1 \subseteq A_2$ if and only if

$$\mathfrak{S}_1^+(\chi) \leq \mathfrak{S}_2^+(\chi), I_1^+(\chi) \leq I_2^+(\chi), f_1^+(\chi) \geq f_2^+(\chi),$$

and

$$\mathfrak{S}_1^-(\chi) \geq \mathfrak{S}_2^-(\chi), I_1^-(\chi) \geq I_2^-(\chi), f_1^-(\chi) \leq f_2^-(\chi)$$

ii. $A_1 = A_2$ if and only if

$$\mathfrak{S}_1^+(\chi) = \mathfrak{S}_2^+(\chi), I_1^+(\chi) = I_2^+(\chi), f_1^+(\chi) = f_2^+(\chi),$$

and

$$\mathfrak{S}_1^-(\chi) = \mathfrak{S}_2^-(\chi), I_1^-(\chi) = I_2^-(\chi), f_1^-(\chi) = f_2^-(\chi)$$

iii. The union is defined as below:

$$(A_1 \cup A_2) = \left\{ \left(\max(\mathfrak{S}_1^+(\chi), \mathfrak{S}_2^+(\chi)), \frac{I_1^+(\chi) + I_2^+(\chi)}{2}, \min(f_1^+(\chi), f_2^+(\chi)), \min(\mathfrak{S}_1^-(\chi), \mathfrak{S}_2^-(\chi)), \frac{I_1^-(\chi) + I_2^-(\chi)}{2}, \max(f_1^-(\chi), f_2^-(\chi)) \right) \right\},$$

iv. The intersection is defined as:

$$(A_1 \cap A_2) = \left\{ \left(\min(\mathfrak{S}_1^+(\chi), \mathfrak{S}_2^+(\chi)), \frac{I_1^+(\chi) + I_2^+(\chi)}{2}, \max(f_1^+(\chi), f_2^+(\chi)), \max(\mathfrak{S}_1^-(\chi), \mathfrak{S}_2^-(\chi)), \frac{I_1^-(\chi) + I_2^-(\chi)}{2}, \min(f_1^-(\chi), f_2^-(\chi)) \right) \right\},$$

v. Let $A = \{(\chi, \mathfrak{S}^+(\chi), I^+(\chi), f^+(\chi), \mathfrak{S}^-(\chi), I^-(\chi), f^-(\chi)) | \chi \in P\}$ be BNSs.

Then, the complement A' is defined as:

$$\mathfrak{S}_{A'}^+(\chi) = \{1^+\} - \mathfrak{S}_A^+(\chi), I_{A'}^+(\chi) = \{1^+\} - I_A^+(\chi), f_{A'}^+(\chi) = \{1^+\} - f_A^+(\chi),$$

and

$$\mathfrak{S}_{A'}^-(\chi) = \{1^-\} - \mathfrak{S}_A^-(\chi), I_{A'}^-(\chi) = \{1^-\} - I_A^-(\chi), f_{A'}^-(\chi) = \{1^-\} - f_A^-(\chi),$$

Definition 10 [25]. Consider two bipolar neutrosophic numbers (BNNs) $u_1 = (\mathfrak{S}_1^+, I_1^+, f_1^+, \mathfrak{S}_1^-, I_1^-, f_1^-)$ and $u_2 = (\mathfrak{S}_2^+, I_2^+, f_2^+, \mathfrak{S}_2^-, I_2^-, f_2^-)$. The following are the basic operations between two BNNs:

- i. $u_1 + u_2 = (\mathfrak{S}_1^+ + \mathfrak{S}_2^+ - \mathfrak{S}_1^+ \mathfrak{S}_2^+, I_1^+ I_2^+, f_1^+ f_2^+, -\mathfrak{S}_1^- \mathfrak{S}_2^-, -(-I_1^- - I_2^- - I_1^- I_2^-), -(-f_1^- - f_2^- - f_1^- f_2^-));$
- ii. $u_1 \cdot u_2 = (\mathfrak{S}_1^+ \mathfrak{S}_2^+, I_1^+ + I_2^+ - I_1^+ I_2^+, f_1^+ + f_2^+ - f_1^+ f_2^+, -(-\mathfrak{S}_1^- - \mathfrak{S}_2^- - \mathfrak{S}_1^- \mathfrak{S}_2^-), -I_1^- I_2^-, -f_1^- f_2^-);$
- iii. $\lambda(u_1) = \left(1 - (1 - \mathfrak{S}_1^+)^{\lambda}, (I_1^+)^{\lambda}, (f_1^+)^{\lambda}, -(-\mathfrak{S}_1^-)^{\lambda}, -(-I_1^-)^{\lambda}, -\left(1 - (1 - (-f_1^-))^{\lambda} \right) \right);$
- iv. $(u_1)^{\lambda} = \left((\mathfrak{S}_1^+)^{\lambda}, 1 - (1 - I_1^+)^{\lambda}, 1 - (1 - f_1^+)^{\lambda}, -\left(1 - (1 - (-\mathfrak{S}_1^-))^{\lambda} \right), -(-I_1^-)^{\lambda}, -(-f_1^-)^{\lambda} \right);$

where $\lambda > 0$.

Definition 11 [25]. The score function for a bipolar neutrosophic number $u = (\mathfrak{S}^+, I^+, f^+, \mathfrak{S}^-, I^-, f^-)$ denoted by $S(u)$ is:

$$S(u) = \frac{1}{6} (\mathfrak{S}^+ + 1 - I^+ + 1 - f^+ + 1 + \mathfrak{S}^- - I^- - f^-)$$

Definition 12 [25]. The accuracy function defined for a BNN $u = (\mathfrak{S}^+, I^+, f^+, \mathfrak{S}^-, I^-, f^-)$ is $a(u)$, as below:

$$a(u) = \Im^+ - f^+ + \Im^- - f^-$$

Definition 13 [25]. Consider $u = (\Im^+, I^+, f^+, \Im^-, I^-, f^-)$ to represent the neutrosophic number. Then, the certainty function is:

$$c(u) = \Im^+ - f^-$$

Definition 14 [25]. Consider $u_1 = (\Im_1^+, I_1^+, f_1^+, \Im_1^-, I_1^-, f_1^-)$ and $u_2 = (\Im_2^+, I_2^+, f_2^+, \Im_2^-, I_2^-, f_2^-)$ to represent two BNNs. Then, a comparison method among BNNs is:

- In condition $S(u_1) \succ S(u_2)$ then u_1 is greater than u_2 , denoted by $u_1 \succ u_2$
- In condition $S(u_1) = S(u_2)$ and $a(u_1) \succ a(u_2)$ then u_1 is greater than u_2 , denoted by $u_1 \succ u_2$;
- In condition $S(u_1) = S(u_2)$, $a(u_1) = a(u_2)$ and $c(u_1) \succ c(u_2)$ in that case, u_1 is superior to u_2 , denote by $u_1 \succ u_2$;
- In condition $S(u_1) = S(u_2)$, $a(u_1) = a(u_2)$ and $c(u_1) = c(u_2)$ in that case, u_1 is equal to u_2 , denoted by $u_1 = u_2$.

Definition 15. Consider $u = (\Im^+, I^+, f^+, \Im^-, I^-, f^-)$, $u_1 = (\Im_1^+, I_1^+, f_1^+, \Im_1^-, I_1^-, f_1^-)$ and $u_2 = (\Im_2^+, I_2^+, f_2^+, \Im_2^-, I_2^-, f_2^-)$ to be BNNs, and $\lambda \succ 0$ any real value. Then, the fundamental Einstein operations of BNNs are:

$$\begin{aligned} u_1 \oplus u_2 &= \left(\frac{\Im_1^+ + \Im_2^+}{1 + \Im_1^+ \Im_2^+}, \frac{I_1^+ I_2^+}{1 + (1 - I_1^+)(1 - I_2^+)}, \frac{f_1^+ f_2^+}{1 + (1 - f_1^+)(1 - f_2^+)}, \frac{-\Im_1^- \Im_2^-}{1 + (\Im_1^- + \Im_2^-)}, \frac{-(-I_1^- - I_2^-)}{1 + I_1^- I_2^-}, \frac{-(-f_1^- - f_2^-)}{1 + f_1^- f_2^-} \right) \\ u_1 \otimes u_2 &= \left(\frac{\Im_1^+ \Im_2^+}{1 + (1 - \Im_1^+)(1 - \Im_2^+)}, \frac{I_1^+ + I_2^+}{1 + I_1^+ I_2^+}, \frac{f_1^+ + f_2^+}{1 + f_1^+ f_2^+}, \frac{-(-\Im_1^- - \Im_2^-)}{1 + \Im_1^- \Im_2^-}, \frac{-I_1^- I_2^-}{1 + (1 + I_1^-)(1 + I_2^-)}, \frac{-f_1^- f_2^-}{1 + (1 + f_1^-)(1 + f_2^-)} \right) \\ \lambda(u) &= \left(\frac{(1 + \Im^+)^{\lambda} - (1 - \Im^+)^{\lambda}}{(1 + \Im^+)^{\lambda} + (1 - \Im^+)^{\lambda}}, \frac{2(I^+)^{\lambda}}{(2 - I^+)^{\lambda} + (I^+)^{\lambda}}, \frac{2(f^+)^{\lambda}}{(2 - f^+)^{\lambda} + (f^+)^{\lambda}}, \frac{-2|\Im^-|^{\lambda}}{(2 + \Im^-)^{\lambda} + |\Im^-|^{\lambda}}, \frac{-2|I^-|^{\lambda}}{(2 + I^-)^{\lambda} + |I^-|^{\lambda}}, \frac{-((1 + |f^-|)^{\lambda} - (1 + f^-)^{\lambda})}{(1 + |f^-|)^{\lambda} + (1 + f^-)^{\lambda}} \right) \\ (u)^{\lambda} &= \left(\frac{2(\Im^+)^{\lambda}}{(2 - \Im^+)^{\lambda} + (\Im^+)^{\lambda}}, \frac{(1 + I^+)^{\lambda} - (1 - I^+)^{\lambda}}{(1 + I^+)^{\lambda} + (1 - I^+)^{\lambda}}, \frac{(1 + f^+)^{\lambda} - (1 - f^+)^{\lambda}}{(1 + f^+)^{\lambda} + (1 - f^+)^{\lambda}}, \frac{-((1 + |\Im^-|)^{\lambda} - (1 + \Im^-)^{\lambda})}{(1 + |\Im^-|)^{\lambda} + (1 + \Im^-)^{\lambda}}, \frac{-2|I^-|^{\lambda}}{(2 + I^-)^{\lambda} + |I^-|^{\lambda}}, \frac{-2|f^-|^{\lambda}}{(2 + f^-)^{\lambda} + |f^-|^{\lambda}} \right) \end{aligned}$$

3. Bipolar Neutrosophic Einstein Average AOs

Here, we suggest a number of fundamental properties for bipolar Einstein average AOs in the current section of the article, such as BNEWA, BNEOWA and BNEHA.

3.1. Bipolar Neutrosophic Einstein Weighted-Average Aggregation Operators

Consider $u_l = (\Im_l^+, I_l^+, f_l^+, \Im_l^-, I_l^-, f_l^-)$ to represent a family of BNNs. Here, $l \in \{1, 2, 3, \dots, n\}$

Definition 16. The (BNEWA) aggregation operator is defined as follows:

$$BNEWA_v(u_1, u_2, \dots, u_n) = \bigoplus_{l=1}^n (v_l u_l) = v_1 u_1 \oplus v_2 u_2 \oplus \dots \oplus v_n u_n \quad (1)$$

where $v = (v_1, v_2, \dots, v_n)^T$ represents weighted vectors of u_l , that is, $v_l > 0$ and $\sum_{l=1}^n v_l = 1$.

Theorem 1. The (BNEWA) operator gives in return a (BNV) by

$$BNEWA_v(u_1, u_2, \dots, u_n) = \left(\frac{1 - 2}{1 + 2}, \frac{2\zeta_1}{\zeta_2 + \zeta_1}, \frac{2f_1}{f_2 + f_1}, \frac{-2\zeta_3}{4 + 3}, -\frac{\zeta_3 - \zeta_4}{\zeta_3 + \zeta_4}, -\frac{f_3 - f_4}{f_3 + f_4} \right) \quad (2)$$

where

$$\begin{aligned} 1 &= \prod_{l=1}^n (1 + \mathfrak{S}_l^+)^{v_l}, 2 = \prod_{l=1}^n (1 - \mathfrak{S}_l^+)^{v_l}, 3 = \prod_{l=1}^n |\mathfrak{S}_l^-|^{v_l}, 4 = \prod_{l=1}^n (2 + \mathfrak{S}_l^-)^{v_l}, \\ \zeta_1 &= \prod_{l=1}^n (I_l^+)^{v_l}, \zeta_2 = \prod_{l=1}^n (2 - I_l^+)^{v_l}, \zeta_3 = \prod_{l=1}^n (1 + |I_l^-|)^{v_l}, \zeta_4 = \prod_{l=1}^n (1 + I_l^-)^{v_l}, \\ f_1 &= \prod_{l=1}^n (f_l^+)^{v_l}, f_2 = \prod_{l=1}^n (2 - f_l^+)^{v_l}, f_3 = \prod_{l=1}^n (1 + |f_l^-|)^{v_l}, f_4 = \prod_{l=1}^n (1 + f_l^-)^{v_l}. \end{aligned}$$

where $v = (v_1, v_2, \dots, v_n)^T$ is the weighted vector of u_l , such that $v_l > 0$ and $\sum_{l=1}^n v_l = 1$.

Proof. Now, by mathematical induction:

For $n = 2$,

$$v_1 u_1 = \left(\begin{array}{c} \frac{(1+\mathfrak{S}_1^+)^{v_1} - (1-\mathfrak{S}_1^+)^{v_1}}{(1+\mathfrak{S}_1^+)^{v_1} + (1-\mathfrak{S}_1^+)^{v_1}}, \frac{2(I_1^+)^{v_1}}{(2-I_1^+)^{v_1} + (I_1^+)^{v_1}}, \frac{2(f_1^+)^{v_1}}{(2-f_1^+)^{v_1} + (f_1^+)^{v_1}}, \\ \frac{-2|\mathfrak{S}_1^-|^{v_1}}{(2+\mathfrak{S}_1^-)^{v_1} + |\mathfrak{S}_1^-|^{v_1}}, \frac{-2|I_1^-|^{v_1}}{(2+I_1^-)^{v_1} + |I_1^-|^{v_1}}, \frac{-((1+|f_1^-|)^{v_1} - (1+f_1^-)^{v_1})}{(1+|f_1^-|)^{v_1} + (1+f_1^-)^{v_1}} \end{array} \right)$$

and

$$v_2 u_2 = \left(\begin{array}{c} \frac{(1+\mathfrak{S}_2^+)^{v_2} - (1-\mathfrak{S}_2^+)^{v_2}}{(1+\mathfrak{S}_2^+)^{v_2} + (1-\mathfrak{S}_2^+)^{v_2}}, \frac{2(I_2^+)^{v_2}}{(2-I_2^+)^{v_2} + (I_2^+)^{v_2}}, \frac{2(f_2^+)^{v_2}}{(2-f_2^+)^{v_2} + (f_2^+)^{v_2}}, \\ \frac{-2|\mathfrak{S}_2^-|^{v_2}}{(2+\mathfrak{S}_2^-)^{v_2} + |\mathfrak{S}_2^-|^{v_2}}, \frac{-2|I_2^-|^{v_2}}{(2+I_2^-)^{v_2} + |I_2^-|^{v_2}}, \frac{-((1+|f_2^-|)^{v_2} - (1+f_2^-)^{v_2})}{(1+|f_2^-|)^{v_2} + (1+f_2^-)^{v_2}} \end{array} \right)$$

and for

$$v_1 u_1 = \left(\begin{array}{c} \frac{(1+\mathfrak{S}_1^+) - (1-\mathfrak{S}_1^+)}{(1+\mathfrak{S}_1^+) + (1-\mathfrak{S}_1^+)}, \frac{2(I_1^+)}{(2-I_1^+) + (I_1^+)}, \frac{2(f_1^+)}{(2-f_1^+) + (f_1^+)}, \\ \frac{-2|\mathfrak{S}_1^-|}{(2+\mathfrak{S}_1^-) + |\mathfrak{S}_1^-|}, \frac{-(1+|I_1^-|) - (1+I_1^-)}{(1+|I_1^-|) + (1+I_1^-)}, \frac{-(1+|f_1^-|) - (1+f_1^-)}{(1+|f_1^-|) + (1+f_1^-)} \end{array} \right), \quad (3)$$

thus satisfying $n = 1$.

We put $n = r$ into Equation (3)

$$BNEWA_v(u_1, u_2, \dots, u_r) = \left(\begin{array}{c} \frac{\prod_{l=1}^r (1+\mathfrak{S}_l^+)^{v_l} - \prod_{l=1}^r (1-\mathfrak{S}_l^+)^{v_l}}{\prod_{l=1}^r (1+\mathfrak{S}_l^+)^{v_l} + \prod_{l=1}^r (1-\mathfrak{S}_l^+)^{v_l}}, \frac{2 \prod_{l=1}^r (I_l^+)^{v_l}}{\prod_{l=1}^r (2-I_l^+)^{v_l} + \prod_{l=1}^r (I_l^+)^{v_l}}, \frac{2 \prod_{l=1}^r (f_l^+)^{v_l}}{\prod_{l=1}^r (2-f_l^+)^{v_l} + \prod_{l=1}^r (f_l^+)^{v_l}}, \\ \frac{-2 \prod_{l=1}^r |\mathfrak{S}_l^-|^{v_l}}{\prod_{l=1}^r (2+\mathfrak{S}_l^-)^{v_l} + \prod_{l=1}^r |\mathfrak{S}_l^-|^{v_l}}, \frac{-2 \prod_{l=1}^r |I_l^-|^{v_l}}{\prod_{l=1}^r (2+I_l^-)^{v_l} + \prod_{l=1}^r |I_l^-|^{v_l}}, \frac{-\prod_{l=1}^r (1+|f_l^-|)^{v_l} - \prod_{l=1}^r (1+f_l^-)^{v_l}}{\prod_{l=1}^r (1+|f_l^-|)^{v_l} + \prod_{l=1}^r (1+f_l^-)^{v_l}} \end{array} \right),$$

Assume it is also satisfied for $n = r$.

Now, for $n = r + 1$,

$$\begin{aligned}
& BNEWA_v(u_1, u_2, \dots, u_r) = \\
& \left(\frac{\prod_{l=1}^r (1+\mathfrak{S}_l^+)^{v_l} - \prod_{l=1}^r (1-\mathfrak{S}_l^+)^{v_l}}{\prod_{l=1}^r (1+\mathfrak{S}_l^+)^{v_l} + \prod_{l=1}^r (1-\mathfrak{S}_l^+)^{v_l}}', \frac{2 \prod_{l=1}^r (I_l^+)^{v_l}}{\prod_{l=1}^r (2-I_l^+)^{v_l} + \prod_{l=1}^r (I_l^+)^{v_l}}', \frac{2 \prod_{l=1}^r (f_l^+)^{v_l}}{\prod_{l=1}^r (2-f_l^+)^{v_l} + \prod_{l=1}^r (f_l^+)^{v_l}}', \right. \\
& \left. \frac{-2 \prod_{l=1}^r |\mathfrak{S}_l^-|^{v_l}}{\prod_{l=1}^r (2+\mathfrak{S}_l^-)^{v_l} + \prod_{l=1}^r |\mathfrak{S}_l^-|^{v_l}}', \frac{-2 \prod_{l=1}^r |I_l^-|^{v_l}}{\prod_{l=1}^r (2+I_l^-)^{v_l} + \prod_{l=1}^r |I_l^-|^{v_l}}', -\frac{\prod_{l=1}^r (1+|f_l^-|)^{v_l} - \prod_{l=1}^r (1+f_l^-)^{v_l}}{\prod_{l=1}^r (1+|f_l^-|)^{v_l} + \prod_{l=1}^r (1+f_l^-)^{v_l}} \right) \\
& \oplus \left(\frac{(1+\mathfrak{S}_{r+1}^+)^{v_{r+1}} - (1-\mathfrak{S}_{r+1}^+)^{v_{r+1}}}{(1+\mathfrak{S}_{r+1}^+)^{v_{r+1}} + (1-\mathfrak{S}_{r+1}^+)^{v_{r+1}}}, \frac{2(I_{r+1}^+)^{v_{r+1}}}{(2-I_{r+1}^+)^{v_{r+1}} + (I_{r+1}^+)^{v_{r+1}}}, \frac{2(f_{r+1}^+)^{v_{r+1}}}{(2-f_{r+1}^+)^{v_{r+1}} + (f_{r+1}^+)^{v_{r+1}}}, \right. \\
& \left. \frac{-2|\mathfrak{S}_{r+1}^-|^{v_{r+1}}}{(2+\mathfrak{S}_{r+1}^-)^{v_{r+1}} + |\mathfrak{S}_{r+1}^-|^{v_{r+1}}}, \frac{-2|I_{r+1}^-|^{v_{r+1}}}{(2+I_{r+1}^-)^{v_{r+1}} + |I_{r+1}^-|^{v_{r+1}}}, -\frac{(1+|f_{r+1}^-|)^{v_{r+1}} - (1+f_{r+1}^-)^{v_{r+1}}}{(1+|f_{r+1}^-|)^{v_{r+1}} + (1+f_{r+1}^-)^{v_{r+1}}} \right) \\
& = \left(\frac{\prod_{l=1}^{r+1} (1+\mathfrak{S}_l^+)^{v_l} - \prod_{l=1}^{r+1} (1-\mathfrak{S}_l^+)^{v_l}}{\prod_{l=1}^{r+1} (1+\mathfrak{S}_l^+)^{v_l} + \prod_{l=1}^{r+1} (1-\mathfrak{S}_l^+)^{v_l}}', \frac{2 \prod_{l=1}^{r+1} (I_l^+)^{v_l}}{\prod_{l=1}^{r+1} (2-I_l^+)^{v_l} + \prod_{l=1}^{r+1} (I_l^+)^{v_l}}', \frac{2 \prod_{l=1}^{r+1} (f_l^+)^{v_l}}{\prod_{l=1}^{r+1} (2-f_l^+)^{v_l} + \prod_{l=1}^{r+1} (f_l^+)^{v_l}}', \right. \\
& \left. \frac{-2 \prod_{l=1}^{r+1} |\mathfrak{S}_l^-|^{v_l}}{\prod_{l=1}^{r+1} (2+\mathfrak{S}_l^-)^{v_l} + \prod_{l=1}^{r+1} |\mathfrak{S}_l^-|^{v_l}}', \frac{-2 \prod_{l=1}^{r+1} |I_l^-|^{v_l}}{\prod_{l=1}^{r+1} (2+I_l^-)^{v_l} + \prod_{l=1}^{r+1} |I_l^-|^{v_l}}', -\frac{\prod_{l=1}^{r+1} (1+|f_l^-|)^{v_l} - \prod_{l=1}^{r+1} (1+f_l^-)^{v_l}}{\prod_{l=1}^{r+1} (1+|f_l^-|)^{v_l} + \prod_{l=1}^{r+1} (1+f_l^-)^{v_l}} \right)
\end{aligned}$$

Hence, Equation (3) is satisfied for all values of $n = r + 1$.

Hence, the theorem is proven. $\square$

Theorem 2. (Idempotency) Let $u_l = (\mathfrak{S}_l^+, I_l^+, f_l^+, \mathfrak{S}_l^-, I_l^-, f_l^-)$, where $l \in \{1, 2, \dots, n-1, n\}$ represents a collection of equal BNNs, that is, $u_l = u$. Then

$$BNEWA_v(u_1, u_2, \dots, u_n) = u$$

Proof. Since

$$\begin{aligned}
& BNEWA_v(u_1, u_2, \dots, u_n) \\
& = \bigoplus_{l=1}^n (v_l u_l) = \bigoplus_{l=1}^n (v_l) u = (u) \sum_{l=1}^n v_l = u
\end{aligned}$$

the proof is complete. $\square$

Theorem 3. (Bounded) Suppose that $u^+ = \max_l u_l$ and $u^- = \min_l u_l$ represent the minimum and maximum BNN, respectively

$$u^- \leq BNEWA_v(u_1, u_2, \dots, u_n) \leq u^+$$

Proof. Let

$$BNEWA_v(u_1, u_2, \dots, u_n) = u(\mathfrak{S}^+, I^+, f^+, \mathfrak{S}^-, I^-, f^-)$$

Then,

$$\begin{aligned}
s(u^-) & \leq s(\text{BNEWA operator}) \\
s(u^-) & \leq s(\text{BNEWA operator}) \\
s(u^+) & \geq s(\text{BNEWA operator})
\end{aligned}$$

Combining both equations, we have

$$u^- \leq BNEWA_v(u_1, u_2, \dots, u_n) \leq u^+$$

Thus, the proof is complete. $\square$

Theorem 4. (Monotonicity) Suppose $u_l = (\mathfrak{S}_l^+, I_l^+, f_l^+, \mathfrak{S}_l^-, I_l^-, f_l^-)$ and $u'_l = (\mathfrak{S}_l'^+, I_l'^+, f_l'^+, \mathfrak{S}_l'^-, I_l'^-, f_l'^-)$, where $l \in \{1, 2, \dots, n-1, n\}$ represents BNNs. If $u_l \leq u'_l$, then

$$BNEWA_v(u_1, u_2, \dots, u_n) \leq BNEWA_v(u'_1, u'_2, \dots, u'_n)$$

Proof. Assume that

$$BNEWA_v(u_1, u_2, \dots, u_n) = \bigoplus_{l=1}^n (v_l u_l)$$

and

$$BNEWA_v(u'_1, u'_2, \dots, u'_n) = \bigoplus_{l=1}^n (v_l u'_l)$$

Since $u_l \leq u'_l$, then

$$\bigoplus_{l=1}^n (v_l u_l) \leq \bigoplus_{l=1}^n (v_l u'_l)$$

Thus, the proof is complete. $\square$

3.2. BN Einstein Ordered Weighted Average Aggregation Operators

Definition 17. The (BNEOWA) BN Einstein ordered weighted-average AO is stated as:

$$BNEOWA_v(u_1, u_2, \dots, u_n) = \bigoplus_{l=1}^n (v_l u_{\rho(l)}) = v_1 u_{\rho(1)} \oplus v_2 u_{\rho(2)} \oplus v_3 u_{\rho(3)} \oplus \dots \oplus v_n u_{\rho(n)}, \quad (4)$$

where $(\rho(1), \rho(2), \rho(3), \dots, \rho(n))$ represents a permutation with $u_{\rho(l-1)} \geq u_{\rho(l)}$, $\forall l \in Z$, $Z = \{1, 2, \dots, n-1, n\}$ and $v = (v_1, v_2, \dots, v_n)^T$ representing associated weighting vectors for u_l with $v_l > 0$, $\sum_{l=1}^n v_l = 1$.

Theorem 5. The (BNEOWA) operator gives, in return, a BNV, by

$$BNEOWA_v(u_1, u_2, \dots, u_n) = \left(\frac{\partial_1 - \partial_2}{\partial_1 + \partial_2}, \frac{2\zeta_{\partial 1}}{\zeta_{\partial 2} + \zeta_{\partial 1}}, \frac{2f_{\partial 1}}{f_{\partial 2} + f_{\partial 1}}, \frac{-2\partial_3}{\partial_4 + \partial_3}, -\frac{\zeta_{\partial 3} - \zeta_{\partial 4}}{\zeta_{\partial 3} + \zeta_{\partial 4}}, -\frac{f_{\partial 3} - f_{\partial 4}}{f_{\partial 3} + f_{\partial 4}} \right) \quad (5)$$

where,

$$\begin{aligned} \partial_1 &= \prod_{l=1}^n (1 + \mathfrak{S}_{\rho(l)}^+)^{v_l}, \partial_2 = \prod_{l=1}^n (1 - \mathfrak{S}_{\rho(l)}^+)^{v_l}, \partial_3 = \prod_{l=1}^n |\mathfrak{S}_{\rho(l)}^-|^{v_l}, \partial_4 = \prod_{l=1}^n (2 + \mathfrak{S}_{\rho(l)}^-)^{v_l}, \\ \zeta_{\partial 1} &= \prod_{l=1}^n (I_{\rho(l)}^+)^{v_l}, \zeta_{\partial 2} = \prod_{l=1}^n (2 - I_{\rho(l)}^+)^{v_l}, \zeta_{\partial 3} = \prod_{l=1}^n (1 + |I_{\rho(l)}^-|)^{v_l}, \zeta_{\partial 4} = \prod_{l=1}^n (1 + I_{\rho(l)}^-)^{v_l}, \\ f_{\partial 1} &= \prod_{l=1}^n (f_{\rho(l)}^+)^{v_l}, f_{\partial 2} = \prod_{l=1}^n (2 - f_{\rho(l)}^+)^{v_l}, f_{\partial 3} = \prod_{l=1}^n (1 + |f_{\rho(l)}^-|)^{v_l}, f_{\partial 4} = \prod_{l=1}^n (1 + f_{\rho(l)}^-)^{v_l}. \end{aligned}$$

where $(\rho(1), \rho(2), \dots, \rho(n))$ represents a permutation with $u_{\rho(l-1)} \geq u_{\rho(l)}$, $\forall l \in Z$, $Z = \{1, 2, \dots, n-1, n\}$ and $v = (v_1, v_2, \dots, v_n)^T$ representing associated weighting vectors for u_l with $v_l > 0$ and $\sum_{l=1}^n v_l = 1$.

Proof. The proof follows from Theorem 1. $\square$

Theorem 6. (Commutativity) Let $u_l = (\mathfrak{S}_l^+, I_l^+, f_l^+, \mathfrak{S}_l^-, I_l^-, f_l^-)$, as well as $u'_l = (\mathfrak{S}_l'^+, I_l'^+, f_l'^+, \mathfrak{S}_l'^-, I_l'^-, f_l'^-)$, where $l \in Z$ represents two BNNs.

$$\text{BNEOWA}_\omega(u_1, u_2, \dots, u_n) = \text{BNEOWA}_\omega(u'_1, u'_2, \dots, u'_n)$$

where $(u_1, u_2, \dots, u_n)$ is any permutation of $(u'_1, u'_2, \dots, u'_n)$.

Proof. Assume that

$$\text{BNEOWA}_v(u_1, u_2, \dots, u_n) = \bigoplus_{l=1}^n (v_l u_{\rho(l)})$$

and

$$\text{BNEOWA}_v(u'_1, u'_2, \dots, u'_n) = \bigoplus_{l=1}^n (v_l u'_{\rho(l)})$$

Since $(u_1, u_2, \dots, u_n)$ is any permutation of $(u'_1, u'_2, \dots, u'_n)$, then

$$\bigoplus_{l=1}^n (v_l u_{\rho(l)}) = \bigoplus_{l=1}^n (v_l u'_{\rho(l)})$$

The proof is complete. $\square$

Theorem 7. (Idempotency) Let $u_l = (\mathfrak{S}_l^+, I_l^+, f_l^+, \mathfrak{S}_l^-, I_l^-, f_l^-)$, where $l \in Z$. $Z = \{1, 2, \dots, n-1, n\}$ is a collection of all equal BNNs, i.e., $u_l = u$:

$$\text{BNEOWA}_v(u_1, u_2, u_3, \dots, u_n) = u$$

Proof. The proof follows from Theorem 2. $\square$

Theorem 8. (Bounded) Consider $u^- = \min_l u_l$, $u^+ = \max_l u_l$ to represent the minimum and maximum BNN, respectively

$$u^- \leq \text{BNEOWA}_v(u_1, u_2, \dots, u_n) \leq u^+$$

Proof. The proof follows from Theorem 3. $\square$

Theorem 9. (Monotonicity) Let $u_l = (\mathfrak{S}_l^+, I_l^+, f_l^+, \mathfrak{S}_l^-, I_l^-, f_l^-)$, and $u'_l = (\mathfrak{S}_l'^+, I_l'^+, f_l'^+, \mathfrak{S}_l'^-, I_l'^-, f_l'^-)$, where $l \in Z$ represents BNNs. If $u_l \leq u'_l$,

$$\text{BNEOWA}_\omega(u_1, u_2, \dots, u_n) \leq \text{BNEOWA}_\omega(u'_1, u'_2, \dots, u'_n)$$

Proof. The proof follows from Theorem 4. $\square$

3.3. BN Einstein Hybrid Average Aggregation Operators

Definition 18. The (BNEHA) BN Einstein hybrid average AO is stated as:

$$\text{BNEHA}_{w,v}(u_1, u_2, \dots, u_n) = \bigoplus_{l=1}^n (v_l \dot{u}_{\rho(l)}) = v_1 \dot{u}_{\rho(1)} \oplus v_2 \dot{u}_{\rho(2)} \oplus v_3 \dot{u}_{\rho(3)} \oplus \dots \oplus v_n \dot{u}_{\rho(n)}, \quad (6)$$

where $w = (w_1, w_2, w_3, \dots, w_{n-1}, w_n)$ is a weighting vector of u_l , such that $w_l \in [0, 1]$, $\sum_{l=1}^n w_l = 1$ and $\dot{u}_{\rho(l)}$ are the l -th biggest component of the BN argument $\dot{u}_l$ ($\dot{u}_l = (nv_l)u_l, l = 1, 2, \dots, n$), and

$v = (v_1, v_2, \dots, v_n)$ is the weighting vector of BN argument u_l , along with $v_l \in [0, 1]$, $\sum_{l=1}^n v_l = 1$. Here, n represents a balancing coefficient.

Note that BNEHA reduces to BNEWA if $w = \left(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}\right)^T$ or the BNEOWA operator if $v = \left(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}\right)$.

Theorem 10. The (BNEHA) operator precedes a BN number with

$$BNEHA_{w,v}(u_1, u_2, \dots, u_n) = \left(\frac{\dot{1}-\dot{2}}{\dot{1}+\dot{2}}, \frac{2\dot{\zeta}_1}{\dot{\zeta}_2+\dot{\zeta}_1}, \frac{2\dot{f}_1}{\dot{f}_2+\dot{f}_1}, \frac{-2\dot{3}}{\dot{4}+\dot{3}}, -\frac{\dot{\zeta}_3-\dot{\zeta}_4}{\dot{\zeta}_3+\dot{\zeta}_4}, -\frac{\dot{f}_3-\dot{f}_4}{\dot{f}_3+\dot{f}_4} \right) \quad (7)$$

where,

$$\begin{aligned} \dot{1} &= \prod_{l=1}^n \left(1 + \dot{\mathfrak{S}}_{\rho(l)}^+\right)^{v_l}, \dot{2} = \prod_{l=1}^n \left(1 - \dot{\mathfrak{S}}_{\rho(l)}^+\right)^{v_l}, \dot{3} = \prod_{l=1}^n \left|\dot{\mathfrak{S}}_{\rho(l)}^-\right|^{v_l}, \dot{4} = \prod_{l=1}^n \left(2 + \dot{\mathfrak{S}}_{\rho(l)}^-\right)^{v_l}, \\ \dot{\zeta}_1 &= \prod_{l=1}^n \left(\dot{I}_{\rho(l)}^+\right)^{v_l}, \dot{\zeta}_2 = \prod_{l=1}^n \left(2 - \dot{I}_{\rho(l)}^+\right)^{v_l}, \dot{\zeta}_3 = \prod_{l=1}^n \left(1 + \left|\dot{I}_{\rho(l)}^-\right|\right)^{v_l}, \dot{\zeta}_4 = \prod_{l=1}^n \left(1 + \dot{I}_{\rho(l)}^-\right)^{v_l}, \\ \dot{f}_1 &= \prod_{l=1}^n \left(\dot{f}_{\rho(l)}^+\right)^{v_l}, \dot{f}_2 = \prod_{l=1}^n \left(2 - \dot{f}_{\rho(l)}^+\right)^{v_l}, \dot{f}_3 = \prod_{l=1}^n \left(1 + \left|\dot{f}_{\rho(l)}^-\right|\right)^{v_l}, \dot{f}_4 = \prod_{l=1}^n \left(1 + \dot{f}_{\rho(l)}^-\right)^{v_l}. \end{aligned}$$

where $w = (w_1, w_2, w_3, \dots, w_{n-1}, w_n)$ is a weighting vector for u_l , by means of $w_l \in [0, 1]$, $\sum_{l=1}^n w_l = 1$, and $\dot{u}_{\rho(l)}$ is the l -th biggest component of BN argument $\dot{u}_l$ ($\dot{u}_l = (nv_l)u_l, l = 1, 2, \dots, n$).

Moreover, $v = (v_1, v_2, \dots, v_n)$ is the weighting vector of BN argument u_l , with $v_l \in [0, 1]$, $\sum_{l=1}^n v_l = 1$.

Proof. The theorem is straightforward. $\square$

4. Bipolar Neutrosophic Einstein Geometric AOs

In the current section, we extend our study to bipolar neutrosophic Einstein geometric AOs, such as BNEWG, BNEOWG and BNHA, as well as their required properties.

4.1. Bipolar Neutrosophic EinsteinWeighted Geometric AO

Let $u_l = (\mathfrak{S}_l^+, I_l^+, f_l^+, \mathfrak{S}_l^-, I_l^-, f_l^-)$ ($l = 1, 2, \dots, n$) represent a family of bipolar neutrosophic values.

Definition 19. The (BNEWG) bipolar neutrosophic Einstein weighted geometric operator is defined as:

$$BNEWG_v(u_1, u_2, u_3, \dots, u_n) = \bigotimes_{l=1}^n (v_l u_l) = v_1 u_1 \otimes v_2 u_2 \otimes \dots \otimes v_n u_n \quad (8)$$

where $v = (v_1, v_2, v_3, \dots, v_n)^T$ is a weighting vector of u_l , with $v_l > 0$ and $\sum_{l=1}^n v_l = 1$.

Theorem 11. The (BNEWG) operators return a BN value with

$$BNEWG_v(u_1, u_2, u_3, \dots, u_n) = \left(\frac{2\dot{1}}{2+\dot{1}}, \frac{\dot{\zeta}_1-\dot{\zeta}_2}{\dot{\zeta}_1+\dot{\zeta}_2}, \frac{\dot{f}_1-\dot{f}_2}{\dot{f}_1+\dot{f}_2}, -\frac{\dot{3}-\dot{4}}{\dot{3}+\dot{4}}, \frac{-2\dot{\zeta}_3}{\dot{\zeta}_4+\dot{\zeta}_3}, \frac{-2\dot{f}_3}{\dot{f}_4+\dot{f}_3} \right) \quad (9)$$

where

$$\begin{aligned}\tilde{1} &= \prod_{l=1}^n (\mathfrak{S}_l^+)^{v_l}, \tilde{2} = \prod_{l=1}^n (2 - \mathfrak{S}_l^+)^{v_l}, \tilde{3} = \prod_{l=1}^n (1 + |\mathfrak{S}_l^-|)^{v_l}, \tilde{4} = \prod_{l=1}^n (1 + \mathfrak{S}_l^-)^{v_l}, \\ \zeta_{\tilde{1}} &= \prod_{l=1}^n (1 + I_l^+)^{v_l}, \zeta_{\tilde{2}} = \prod_{l=1}^n (1 - I_l^+)^{v_l}, \zeta_{\tilde{3}} = \prod_{l=1}^n |I_l^-|^{v_l}, \zeta_{\tilde{4}} = \prod_{l=1}^n (2 + I_l^-)^{v_l}, \\ f_{\tilde{1}} &= \prod_{l=1}^n (1 + f_l^+)^{v_l}, f_{\tilde{2}} = \prod_{l=1}^n (1 - f_l^+)^{v_l}, f_{\tilde{3}} = \prod_{l=1}^n |f_l^-|^{v_l}, f_{\tilde{4}} = \prod_{l=1}^n (2 + f_l^-)^{v_l}.\end{aligned}$$

where $v = (v_1, v_2, v_3, \dots, v_n)^T$ is a weighting vector of u_l , with $v_l > 0$ as well as $\sum_{l=1}^n v_l = 1$.

Proof. The proof follows from Theorem 1. $\square$

Theorem 12. (Idempotency) Let $u_l = (\mathfrak{S}_l^+, I_l^+, f_l^+, \mathfrak{S}_l^-, I_l^-, f_l^-)$ ($l = 1, 2, 3, \dots, n$) be a set of equal BNVs, that is, $u_l = u$

$$BNEWG_v(u_1, u_2, u_3, \dots, u_n) = u$$

Proof. Since

$$\begin{aligned}BNEWG_v(u_1, u_2, \dots, u_n) \\ = \bigotimes_{l=1}^n (v_l u_l) = \bigotimes_{l=1}^n (v_l) u = (u) \sum_{l=1}^n v_l = u\end{aligned}$$

The proof is complete. $\square$

Theorem 13. (Bounded) Suppose that $u^- = \min_l u_l$, $u^+ = \max_l u_l$, then

$$u^- \leq BNEWG_v(u_1, u_2, \dots, u_n) \leq u^+$$

Proof. Let

$$BNEWG_v(u_1, u_2, \dots, u_n) = u(\mathfrak{S}^+, I^+, f^+, \mathfrak{S}^-, I^-, f^-)$$

Then

$$s(u^-) \leq s(BNEWG \text{ operator}) s(u^+) \geq s(BNEWG \text{ operator})$$

Combining both equations, we have

$$u^- \leq BNEWG_v(u_1, u_2, \dots, u_n) \leq u^+$$

Thus, the proof is complete. $\square$

Theorem 14. (Monotonicity) Let $u_l = (\mathfrak{S}_l^+, I_l^+, f_l^+, \mathfrak{S}_l^-, I_l^-, f_l^-)$ ($l = 1, 2, \dots, n$) and $u'_l = (\mathfrak{S}'_l^+, I'_l^+, f'_l^+, \mathfrak{S}'_l^-, I'_l^-, f'_l^-)$ ($l = 1, 2, \dots, n$) be a collection of BNVs. If $u_l \leq u'_l$, then

$$BNEWG_v(u_1, u_2, \dots, u_n) \leq BNEWG_v(u'_1, u'_2, \dots, u'_n)$$

Proof. Assume that

$$BNEWG_v(u_1, u_2, \dots, u_n) = \bigotimes_{l=1}^n (v_l u_l)$$

and

$$BNEWG_v(u'_1, u'_2, \dots, u'_n) = \bigotimes_{l=1}^n (v_l u'_l)$$

Since $u_l \leq u'_l$, then

$$\bigotimes_{l=1}^n (v_l u_l) \leq \bigotimes_{l=1}^n (v_l u'_l)$$

Thus, the proof is complete. $\square$

4.2. BN Einstein OrderedWeighted Geometric Aggregation Operators

Definition 20. The (BNEOWG) BN Einstein ordered weighted geometric AO is stated as:

$$\text{BNEOWG}_v(u_1, u_2, \dots, u_{n-1}, u_n) = \bigotimes_{l=1}^n (v_l u_{\sigma(l)}) = v_1 u_{\sigma(1)} \otimes v_2 u_{\sigma(2)} \otimes v_3 u_{\sigma(3)} \otimes \dots \otimes v_l u_{\sigma(l)}, \quad (10)$$

where $(\sigma(1), \dots, \sigma(n-1), \sigma(n))$ represents permutation $(1, 2, 3, \dots, n-1, n)$, that is, $u_{\sigma(l-1)} \geq u_{\sigma(l)}$ and $(j = 2, \dots, n-1, n)$, and $v = (v_1, v_2, v_3, \dots, v_n)^T$ is a weighting vector of u_l , with $v_l > 0$ and $\sum_{l=1}^n v_l = 1$.

Theorem 15. The (BNEOWG) operators return a BN value with

$$\text{BNEOWG}_v(u_1, u_2, \dots, u_n) = \left(\frac{2\tilde{\partial}_1}{\tilde{\partial}_2 + \tilde{\partial}_1}, \frac{\tilde{\zeta}_{\tilde{\partial}_1} - \tilde{\zeta}_{\tilde{\partial}_2}}{\tilde{\zeta}_{\tilde{\partial}_1} + \tilde{\zeta}_{\tilde{\partial}_2}}, \frac{f_{\tilde{\partial}_1} - f_{\tilde{\partial}_2}}{f_{\tilde{\partial}_1} + f_{\tilde{\partial}_2}}, -\frac{\tilde{\partial}_3 - \tilde{\partial}_4}{\tilde{\partial}_3 + \tilde{\partial}_4}, \frac{-2\tilde{\zeta}_{\tilde{\partial}_3}}{\tilde{\zeta}_{\tilde{\partial}_4} + \tilde{\zeta}_{\tilde{\partial}_3}}, \frac{-2f_{\tilde{\partial}_3}}{f_{\tilde{\partial}_4} + f_{\tilde{\partial}_3}} \right) \quad (11)$$

where,

$$\begin{aligned} \tilde{\partial}_1 &= \prod_{j=1}^n (\mathfrak{S}_{\sigma(j)}^+)^{\omega_j}, \tilde{\partial}_2 = \prod_{j=1}^n (2 - \mathfrak{S}_{\sigma(j)}^+)^{\omega_j}, \tilde{\partial}_3 = \prod_{j=1}^n (1 + |\mathfrak{S}_{\sigma(j)}^-|)^{\omega_j}, \tilde{\partial}_4 = \prod_{j=1}^n (1 + \mathfrak{S}_{\sigma(j)}^-)^{\omega_j}, \\ \tilde{\zeta}_{\tilde{\partial}_1} &= \prod_{j=1}^n (1 + I_{\sigma(j)}^+)^{\omega_j}, \tilde{\zeta}_{\tilde{\partial}_2} = \prod_{j=1}^n (1 - I_{\sigma(j)}^+)^{\omega_j}, \tilde{\zeta}_{\tilde{\partial}_3} = \prod_{j=1}^n |I_{\sigma(j)}^-|^{\omega_j}, \tilde{\zeta}_{\tilde{\partial}_4} = \prod_{j=1}^n (2 + I_{\sigma(j)}^-)^{\omega_j}, \\ f_{\tilde{\partial}_1} &= \prod_{j=1}^n (1 + f_{\sigma(j)}^+)^{\omega_j}, f_{\tilde{\partial}_2} = \prod_{j=1}^n (1 - f_{\sigma(j)}^+)^{\omega_j}, f_{\tilde{\partial}_3} = \prod_{j=1}^n |f_{\sigma(j)}^-|^{\omega_j}, f_{\tilde{\partial}_4} = \prod_{j=1}^n (2 + f_{\sigma(j)}^-)^{\omega_j}. \end{aligned}$$

where $(\sigma(1), \sigma(2), \sigma(3), \dots, \sigma(n))$ represents a permutation of $(1, 2, \dots, n)$ with $u_{\sigma(l-1)} \geq u_{\sigma(l)}$ and $(l = 2, 3, \dots, n)$, and $v = (v_1, v_2, v_3, \dots, v_n)^T$ is a weighting vector of u_l , with $v_l > 0$ and $\sum_{l=1}^n v_l = 1$.

Proof. The theorem is simple to understand. $\square$

Theorem 16. (Commutativity) Let $u_l = (\mathfrak{S}_l^+, I_l^+, f_l^+, \mathfrak{S}_l^-, I_l^-, f_l^-)$, as well as $u'_l = (\mathfrak{S}_l'^+, I_l'^+, f_l'^+, \mathfrak{S}_l'^-, I_l'^-, f_l'^-)$, where $l \in Z$ represents two BNNs.

$$\text{BNEOWG}_\omega(u_1, u_2, \dots, u_n) = \text{BNEOWG}_\omega(u'_1, u'_2, \dots, u'_n)$$

where $(u_1, u_2, \dots, u_n)$ is any permutation of $(u'_1, u'_2, \dots, u'_n)$.

Proof. Assume that

$$\text{BNEOWG}_v(u_1, u_2, \dots, u_n) = \bigotimes_{l=1}^n (v_l u_{\rho(l)})$$

and

$$\text{BNEOWG}_v(u'_1, u'_2, \dots, u'_n) = \bigotimes_{l=1}^n (v_l u'_{\rho(l)})$$

Since $(u_1, u_2, \dots, u_n)$ is any permutation of $(u'_1, u'_2, \dots, u'_n)$, then

$$\bigotimes_{l=1}^n (v_l u_{\rho(l)}) = \bigotimes_{l=1}^n (v_l u'_{\rho(l)})$$

The proof is complete. $\square$

Theorem 17. (Idempotency) Let $u_l = (\mathfrak{S}_l^+, I_l^+, f_l^+, \mathfrak{S}_l^-, I_l^-, f_l^-)$ ($l = 1, 2, \dots, n$) represent a set of all equal BNVs, that is, $u_l = u$

$$\text{BNEOWG}_v(u_1, u_2, u_3, \dots, u_{n-1}, u_n) = u$$

Theorem 18. (Bounded) Consider $u^- = \min_l u_l, u^+ = \max_l u_l$; then,

$$u^- \leq \text{BNEOWG}_v(u_1, u_2, \dots, u_n) \leq u^+$$

Theorem 19. (Monotonicity) Let $u_l = (\mathfrak{S}_l^+, I_l^+, f_l^+, \mathfrak{S}_l^-, I_l^-, f_l^-)$ ($l = 1, 2, \dots, n$) and $u'_l = (\mathfrak{S}'_l^+, I'_l^+, f'_l^+, \mathfrak{S}'_l^-, I'_l^-, f'_l^-)$ ($l = 1, 2, \dots, n$) be a collection of two BNVs. If $u_l \leq u'_l$,

$$\text{BNEOWG}_v(u_1, u_2, u_3, \dots, u_n) \leq \text{BNEOWG}_v(u'_1, u'_2, u'_3, \dots, u'_n)$$

4.3. BN Einstein Hybrid Geometric Aggregation Operators

Definition 21. The BN Einstein hybrid geometric (BNEHG) aggregation operator is stated as follows:

$$\text{BNEHG}_{w,v}(u_1, u_2, u_3, \dots, u_{n-1}, u_n) = \bigotimes_{l=1}^n (v_l \dot{u}_{\sigma(l)}) = v_1 \dot{u}_{\sigma(1)} \otimes v_2 \dot{u}_{\sigma(2)} \otimes v_3 \dot{u}_{\sigma(3)} \otimes \dots \otimes v_l \dot{u}_{\sigma(l)}, \quad (12)$$

where $w = (w_1, w_2, w_3, \dots, w_{n-1}, w_n)$ is a weighting vector of u_l , such that $w_l \in [0, 1]$, $\sum_{l=1}^n w_l = 1$ and $\dot{u}_{\rho(l)}$ represent the l -th biggest component of the BN argument $\dot{u}_l$ ($\dot{u}_l = (nv_l)u_l, l = 1, 2, \dots, n$), and $v = (v_1, v_2, \dots, v_n)$ is the weighting vector of BN arguments u_l , along with $v_l \in [0, 1]$, $\sum_{l=1}^n v_l = 1$.

Note that BNEHG reduces to BNEWG if $w = \left(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}\right)^T$ or the BNEOWG operator if $v = \left(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}\right)$.

Theorem 20. The (BNEHG) operator gives in return a BNV with

$$\text{BNEHG}_{w,v}(u_1, u_2, u_3, \dots, u_n) = \left(\frac{2\dot{\gamma}_1}{2+\dot{\gamma}_1}, \frac{\dot{\zeta}_1 - \dot{\zeta}_2}{\dot{\zeta}_1 + \dot{\zeta}_2}, \frac{\dot{f}_1 - \dot{f}_2}{\dot{f}_1 + \dot{f}_2}, -\frac{\dot{\gamma}_3 - \dot{\gamma}_4}{\dot{\gamma}_3 + \dot{\gamma}_4}, \frac{-2\dot{\zeta}_3}{\dot{\zeta}_4 + \dot{\zeta}_3}, \frac{-2\dot{f}_3}{\dot{f}_4 + \dot{f}_3} \right) \quad (13)$$

where

$$\begin{aligned} \dot{\gamma}_1 &= \prod_{j=1}^n \left(\dot{\mathfrak{S}}_{\sigma(j)}^+ \right)^{\omega_j}, \dot{\gamma}_2 = \prod_{j=1}^n \left(2 - \dot{\mathfrak{S}}_{\sigma(j)}^+ \right)^{\omega_j}, \dot{\gamma}_3 = \prod_{j=1}^n \left(1 + \left| \dot{\mathfrak{S}}_{\sigma(j)}^- \right| \right)^{\omega_j}, \dot{\gamma}_4 = \prod_{j=1}^n \left(1 + \dot{\mathfrak{S}}_{\sigma(j)}^- \right)^{\omega_j}, \\ \dot{\zeta}_1 &= \prod_{j=1}^n \left(1 + \dot{I}_{\sigma(j)}^+ \right)^{\omega_j}, \dot{\zeta}_2 = \prod_{j=1}^n \left(1 - \dot{I}_{\sigma(j)}^+ \right)^{\omega_j}, \dot{\zeta}_3 = \prod_{j=1}^n \left| \dot{I}_{\sigma(j)}^- \right|^{\omega_j}, \dot{\zeta}_4 = \prod_{j=1}^n \left(2 + \dot{I}_{\sigma(j)}^- \right)^{\omega_j}, \\ \dot{f}_1 &= \prod_{j=1}^n \left(1 + \dot{f}_{\sigma(j)}^+ \right)^{\omega_j}, \dot{f}_2 = \prod_{j=1}^n \left(1 - \dot{f}_{\sigma(j)}^+ \right)^{\omega_j}, \dot{f}_3 = \prod_{j=1}^n \left| \dot{f}_{\sigma(j)}^- \right|^{\omega_j}, \dot{f}_4 = \prod_{j=1}^n \left(2 + \dot{f}_{\sigma(j)}^- \right)^{\omega_j}. \end{aligned}$$

where $w = (w_1, w_2, w_3, \dots, w_{n-1}, w_n)$ is a weighting vector of u_l , such that $w_l \in [0, 1]$, $\sum_{l=1}^n w_l = 1$ and $\dot{u}_{\rho(l)}$ represent the l -th biggest component of the BN argument $\dot{u}_l$ ($\dot{u}_l = (nv_l)u_l, l = 1, 2, \dots, n$), and $v = (v_1, v_2, \dots, v_n)$ is the weighting vector of BN arguments u_l , along with $v_l \in [0, 1]$, $\sum_{l=1}^n v_l = 1$. Here, n represents a balancing coefficient.

Proof. The theorem is straightforward. $\square$

5. Multi-Criteria Group DM Problem Based on BN Einstein Aggregation Operators

This section includes a multi-criteria application based on BN Einstein aggregation operators (AOs), along with crisp numbers serving as attributes' weights and BN numbers serving as attributes' values.

5.1. Algorithm

Consider $G = \{G_1, G_2, \dots, G_m\}$ to represent a collection of finite m alternatives, $L = \{L_1, L_2, L_3, \dots, L_n\}$ to represent a collection of finite n attributes and $D = \{D_1, D_2, D_3, \dots, D_k\}$ to be a finite number of k decision-makers. Suppose $v = (v_1, v_2, \dots, v_n)^T$ is the weighted vector for decision-makers $\bar{D}^s$ ($s = 1, 2, 3, \dots, k-1, k$), with $v_l \in [0, 1]$ and $\sum_{l=1}^n v_l = 1$. Suppose $w = (w_1, w_2, w_3, \dots, w_n)^T$ is the weighted vector representing the set of attributes $L = \{L_1, L_2, L_3, \dots, L_n\}$, with $w_l \in [0, 1]$ and $\sum_{l=1}^n w_l = 1$. The decision-maker assesses an option based on a set of criteria, the values of which are defined by BNVs. Let $u_{ij}^{(s)} = \left[\left(\mathfrak{S}_{ij}^+, I_{ij}^+, f_{ij}^+, \mathfrak{S}_{ij}^-, I_{ij}^-, f_{ij}^- \right) \right]_{m \times n}$ be the decision matrices provided by an expert decision-maker, and $u_{ij}^{(s)}$ be a BNN for $L_{\hat{i}}$ attribute associated with alternatives. We have $\mathfrak{S}_{ij}^+, I_{ij}^+, f_{ij}^+, \mathfrak{S}_{ij}^-, I_{ij}^-, f_{ij}^- \in [0, 1]$ with condition $0 \leq \mathfrak{S}_{ij}^+ + I_{ij}^+ + f_{ij}^+ + \mathfrak{S}_{ij}^- + I_{ij}^- + f_{ij}^- \leq 6$, where $\hat{i} = 1, 2, 3, \dots, m-1, m$ and $\hat{j} = 1, 2, 3, \dots, n$. (Figure 1)

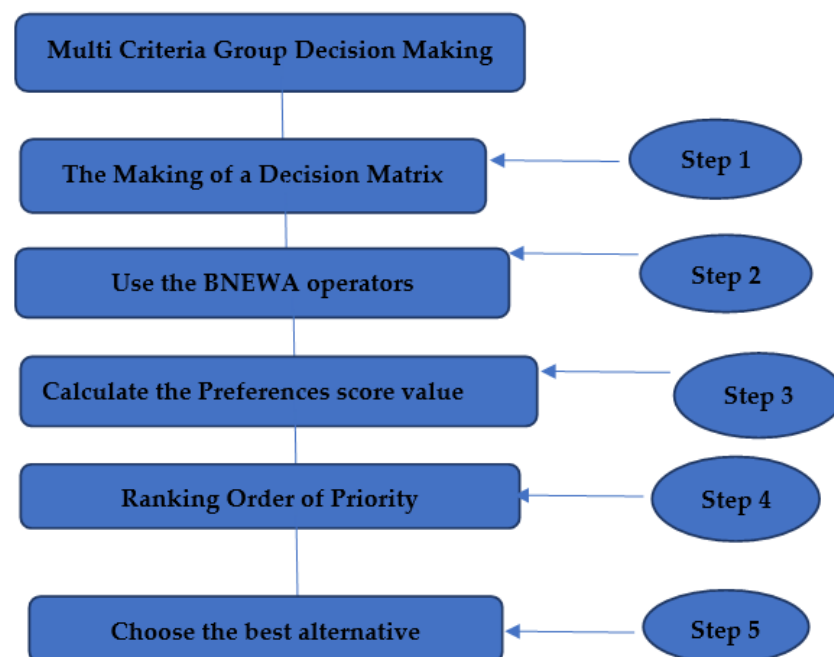


Figure 1. Flow Chart for MCGDM.

Step 1: Matrix construction (Table 1 Table 2 Table 3) for decision-making

$$\overline{D}^s = \left[\widehat{u}_{ij}^{(s)} \right]_{m \times n} \quad (s = 1, 2, 3, \dots, k-1, k).$$

Table 1. BN decision matrix by decision-maker D₁.

	L_1	L_2	L_3	L_4
G ₁	(0.2,0.6,0.5,−0.3,−0.9,−0.6)	(0.4,0.6,0.7,−0.5,−0.4,−0.3)	(0.8,0.3,0.5,−0.6,−0.1,−0.9)	(0.5,0.4,0.6,−0.8,−0.5,−0.4)
G ₂	(0.5,0.7,0.3,−0.8,−0.5,−0.7)	(0.3,0.4,0.6,−0.8,−0.7,−0.6)	(0.4,0.6,0.9,−0.5,−0.4,−0.5)	(0.3,0.8,0.9,−0.1,−0.4,−0.3)
G ₃	(0.5,0.7,0.8,−0.4,−0.7,−0.4)	(0.3,0.7,0.7,−0.5,−0.3,−0.2)	(0.5,0.3,0.4,−0.6,−0.7,−0.9)	(0.4,0.6,0.5,−0.3,−0.5,−0.8)
G ₄	(0.8,0.5,0.4,−0.7,−0.6,−0.5)	(0.2,0.4,0.5,−0.8,−0.6,−0.3)	(0.3,0.7,0.4,−0.5,−0.7,−0.5)	(0.9,0.4,0.6,−0.5,−0.4,−0.7)

Table 2. BN decision matrix by decision-maker D₂.

	L_1	L_2	L_3	L_4
G ₁	(0.4,0.6,0.5,−0.7,−0.4,−0.8)	(0.5,0.4,0.6,−0.8,−0.5,−0.7)	(0.4,0.6,0.5,−0.4,−0.8,−0.5)	(0.5,0.6,0.3,−0.4,−0.6,−0.8)
G ₂	(0.4,0.7,0.5,−0.6,−0.3,−0.9)	(0.4,0.7,0.8,−0.3,−0.5,−0.4)	(0.2,0.5,0.7,−0.6,−0.5,−0.4)	(0.8,0.4,0.2,−0.8,−0.1,−0.4)
G ₃	(0.6,0.3,0.6,−0.3,−0.7,−0.8)	(0.6,0.4,0.6,−0.7,−0.5,−0.8)	(0.6,0.3,0.2,−0.1,−0.4,−0.7)	(0.5,0.6,0.7,−0.3,−0.5,−0.6)
G ₄	(0.2,0.3,0.4,−0.7,−0.6,−0.8)	(0.8,0.5,0.4,−0.7,−0.4,−0.6)	(0.8,0.4,0.5,−0.7,−0.5,−0.1)	(0.6,0.5,0.8,−0.7,−0.6,−0.4)

Table 3. BN decision matrix by decision-maker D₃.

	L_1	L_2	L_3	L_4
G ₁	(0.5,0.6,0.4,−0.7,−0.4,−0.3)	(0.2,0.5,0.6,−0.4,−0.7,−0.5)	(0.5,0.7,0.2,−0.9,−0.5,−0.3)	(0.5,0.6,0.3,−0.7,−0.5,−0.2)
G ₂	(0.9,0.2,0.4,−0.5,−0.4,−0.8)	(0.5,0.1,0.2,−0.9,−0.6,−0.4)	(0.1,0.4,0.8,−0.6,−0.5,−0.3)	(0.6,0.5,0.3,−0.9,−0.3,−0.5)
G ₃	(0.4,0.5,0.6,−0.1,−0.6,−0.5)	(0.3,0.4,0.8,−0.5,−0.4,−0.3)	(0.4,0.6,0.3,−0.4,−0.5,−0.3)	(0.5,0.4,0.9,−0.5,−0.4,−0.7)
G ₄	(0.1,0.4,0.5,−0.4,−0.8,−0.7)	(0.4,0.3,0.6,−0.2,−0.7,−0.5)	(0.7,0.5,0.6,−0.4,−0.3,−0.9)	(0.1,0.5,0.7,−0.5,−0.8,−0.3)

Step 2: Computing $BNEWA_v(r_{i1}, r_{i2}, \dots, r_{in})$ (Table 4) for every $i = 1, 2, 3, \dots, m$.

$$r_{\widehat{i}} = \left(\mathfrak{S}_{\widehat{i}}^+, I_{\widehat{i}}^+, f_{\widehat{i}}^+, \mathfrak{S}_{\widehat{i}}^-, I_{\widehat{i}}^-, f_{\widehat{i}}^- \right) =$$

$$BNEWA_v(r_{i1}, r_{i2}, \dots, r_{in}) = \bigoplus_{l=1}^n (v_l r_{il}) =$$

$$\left(\frac{1-2}{1+2}, \frac{2\zeta_1}{\zeta_2+\zeta_1}, \frac{2f_1}{f_2+f_1}, \frac{-23}{4+3}, -\frac{\zeta_3-\zeta_4}{\zeta_3+\zeta_4}, -\frac{f_3-f_4}{f_3+f_4} \right)$$

Table 4. A collective BN decision matrix R.

	L_1	L_2
G ₁	(0.4234,0.6000,0.4581,−0.6501,−0.4842,−0.6306)	(0.3783,0.4568,0.6096,−0.5917,−0.5810,−0.5943)
G ₂	(0.6940,0.4453,0.4360,−0.5766,−0.3620,−0.8517)	(0.4219,0.3288,0.4750,−0.5426,−0.5640,−0.4224)
G ₃	(0.5161,0.4057,0.6187,−0.2024,−0.6627,−0.6704)	(0.4632,0.4251,0.6870,−0.5952,−0.4423,−0.6001)
G ₄	(0.2462,0.3555,0.4381,−0.5675,−0.6938,−0.7403)	(0.6286,0.4014,0.4839,−0.4527,−0.5567,−0.5351)
	L_3	L_4
G ₁	(0.4940,0.6011,0.3536,−0.5954,−0.6522,−0.4973)	(0.5000,0.5776,0.3233,−0.5453,−0.5520,−0.5868)
G ₂	(0.1818,0.4670,0.7589,−0.5895,−0.4905,−0.3718)	(0.6950,0.4726,0.2807,−0.7190,−0.2130,−0.4321)
G ₃	(0.5161,0.4021,0.2533,−0.2160,−0.4764,−0.6073)	(0.4905,0.5135,0.7552,−0.3708,−0.4614,−0.6659)
G ₄	(0.7293,0.4649,0.5274,−0.5482,−0.4504,−0.6005)	(0.4884,0.4893,0.7387,−0.5952,−0.6796,−0.3989)

Step 3: Computing the scores values $S(r_i^-)$.

Step 4: Various software systems are ranked for $BNEWA_v(u_{i1}^-, u_{i2}^-, \dots, u_{in}^-)$ in accordance with their scores.

Step 5: Picking the best option.

5.2. Illustrative Example

We believe that a medicine business company should hire a professional manager. For this reason, the organization forms a working group with three decision-makers in weighting vectors $v = (0.1, 0.5, 0.4)^T$. When choosing the most knowledgeable management approach, there are various factors to evaluate, but in this situation, the committee just analyses the four criteria mentioned below, along with associated weighted vector $v = (0.1, 0.4, 0.2, 0.3)^T$. Four managers $G_j (j = 1, 2, 3, 4)$ will advance to the next round of the procedure after passing the first screening exam.

L_1 : Salary;

L_2 : Experience;

L_3 : Working skill;

L_4 : Dealing with public.

Step 1. Matrices construction for decisions.

Step 2: Compute $BNEWA_v(r_{i1}^-, r_{i2}^-, \dots, r_{in}^-)$

$$r_i^- = (\Im_i^+, I_i^+, f_i^+, \Im_i^-, I_i^-, f_i^-) = BNEWA_v(r_{i1}^-, r_{i2}^-, \dots, r_{in}^-) = \bigoplus_{l=1}^n (v_l r_{il}) = \left(\frac{1-2}{1+2}, \frac{2\zeta_1}{\zeta_2+\zeta_1}, \frac{2f_1}{f_2+f_1}, \frac{-23}{4+3}, -\frac{\zeta_3-\zeta_4}{\zeta_3+\zeta_4}, -\frac{f_3-f_4}{f_3+f_4} \right)$$

Step 3: Now, compute the scoring function of

$$\begin{aligned} r_1 &= (0.3029, 0.5336, 0.4445, -0.5839, -0.4416, -0.4277), \\ r_2 &= (0.2934, 0.4068, 0.4501, -0.6059, -0.3806, -0.3667), \\ r_3 &= (0.3556, 0.4435, 0.5882, -0.3867, -0.3570, -0.4616), \\ r_4 &= (0.4670, 0.4341, 0.5578, -0.5236, -0.4085, -0.4405), \\ S(r_i^-) &= \frac{1}{6}(\Im^+ + 1 - I^+ + 1 - f^+ + 1 + \Im^- - I^- - f^-), \\ S(r_1) &= .4350, S(r_2) = 0.4296, S(r_3) = .4593, S(r_4) = 0.4668 \end{aligned}$$

Step 4: We have arrived at a conclusion by calculating the scores:

$$G_4 \succ G_3 \succ G_1 \succ G_2$$

Step 5: The best option is G_4 .

6. Comparison

Different researchers have used a wide range of DM approaches so far. Chen et al. [28] used FSs, Atanassov [2] used IFSs, Zavadskas et al. [21] used NSs, Dubois et al. [34] used BFSs and Irfan et al. [25] used BNSs, among other study approaches. We employed Einstein operators to apply bipolarity to neutrosophic sets in this paper.

The aggregation operators in this study were broader and more versatile, representing an advantage of our proposed method, which led us to determine that G_4 was the best manager for the job.

7. Conclusions

The goal of this research was to look at different BN aggregation operators developed with the help of Einstein t-norms/t-conorms for multi-criteria community DM using BNVs

as the criteria. The Einstein t-norm typically gives the same smooth approximations as the product and sum of an algebraic t-norm. Motivated by Einstein operations, we suggested bipolar neutrosophic Einstein aggregation operators for decision-making problems. To begin with, we looked at BN Einstein aggregation operators and the properties that they must have. Along with their attributes, these AOs were (BNEWA), (BNEOWA), (BNEHWA), (BNEWG), (BNEOWG) and (BNEHWG). Finally, we showed how to make multi-criteria decisions using such a framework. A descriptive case of manager selection was studied. The outcomes of the present paper show that the proposed approaches are accurate and practical when put into practice. We plan to extend the proposed approach to other domains and employ it in future research projects, such as pattern identification and risk analysis.

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