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Analysis of the Vibration Behaviors of Rotating Composite Nano-Annular Plates Based on Nonlocal Theory and Different Plate Theories

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Abstract: Rotating machinery has significant applications in the fields of micro and nano meters, such as nano-turbines, nano-motors, and biomolecular motors, etc. This paper takes rotating nano-annular plates as the research object to analyze their free vibration behaviors. Firstly, based on Kirchhoff plate theory, Mindlin plate theory, and Reddy plate theory, combined with nonlocal constitutive relations, the differential motion equations of rotating functionally graded nano-annular plates in a thermal environment are derived. Subsequently, the numerical method is used to discretize and solve the motion equations. The effects of nonlocal parameter, temperature change, inner and outer radius ratio, and rotational velocity on the vibration frequencies of the nano-annular plates are analyzed through numerical examples. Finally, the relationship between the fundamental frequencies and the thickness-to-radius ratio of the nano-annular plates of clamped inner and outer rings is discussed, and the differences in the calculation results among the three plate theories are compared.

Keywords: nonlocal elasticity theory; rotating nano-annular plates; Kirchhoff plate theory; Mindlin plate theory; Reddy plate theory

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1. Introduction

At present, nanotechnology is important in the fields of material preparation, computer technology, aerospace, environmental energy, and biotechnology. Equipment made of nanomaterials are lighter in weight, stronger in hardness, longer in life, lower in maintenance costs, and more convenient in design. At the same time, nanotechnology can also be used to produce materials with specific properties, such as biological materials and bionic materials. In recent years, the applications of nanotechnology to the field of rotating machinery have become a development trend, and corresponding results have been achieved, such as biomolecular motors, nanoturbines, Micro-Electro-Mechanical System (MEMS) gyroscopes, rotating micromirrors, nanowheels, etc. [1–5]. When these micro-rotating components work, due to the uneven distribution of their own materials, mechanical vibration caused by external excitation and other reasons will cause work failure, structural fatigue, and other problems, so dynamic analysis of rotating components is of great significance. Based on the above backgrounds, many scholars simplified the micro-rotating devices into rods, beams, plates, and other models for mechanical analysis and obtained corresponding academic results. Alireza [6] et al. used rotating nanorods as the research object to explore the effects of rotational velocity on natural frequencies. Azimi [7] et al. analyzed the vibration behaviors of a rotating functionally graded nano-Timoshenko beam in a thermal environment and explored the influence of the change in rotational velocity on the vibration characteristics. M. Baghani [8] studied the effects of small-scale parameters, rotational velocity, magnetic field, and surface energy on the vibration characteristics and stability of rotating beams based on nonlocal elastic theory. T. Murmu [9] et al. simulated

the free vibration behaviors of a nano-turbine based on the Euler–Bernoulli beam model and nonlocal elastic theory and explored the effects of temperature, rotational velocity, and gradient index on natural frequencies. Based on the Kirchhoff plate theory, Majid Ghadiri [10] et al. explored the influence of small-scale parameters, rotational velocity, and other physical parameters on the vibration frequencies of rotating nanoplates. Based on the modified couple stress theory and the Mindlin disc model, Mohammad Mahinzare [11] et al. explored the influence of the material gradient index, small-scale parameters, rotational velocity, and Winkler coefficient of elasticity on natural frequencies and summarized the critical rotational velocity range. Mohammad Hosseini [12] et al. explored the influence of small-scale parameters, rotational velocity, and the elastic damping coefficient on the vibration characteristics of rotating laminated circular plates based on the third-order shear deformation theory and nonlocal strain gradient theory.

For nano-scale matter, since quantum effects begin to affect the structure and properties of matter, the classical elastic theory at the macro level appears powerless when dealing with small-scale issues. In this case, a variety of new theoretical systems are also applied, such as nonlocal theory used in this paper. The nonlocal theory replaces the related variables in traditional continuum mechanics by introducing nonlocal parameters and writing internal forces as nonlocal expressions. This theory is a modified classical elastic theory. In the classical local continuum mechanics, it is assumed that the stress state at a point depends only on the strain at that point. However, in 1972, Eringen and Edelen [13] defined the nonlocal continuum mechanics theory and pointed out that the stress of a certain point in the medium is related to the strain of all points and then deduced the constitutive relationship under the nonlocal theory. Based on the theory of nonlocal elasticity to study the mechanical behaviors of nanostructures such as deformation, buckling, and vibration, some scholars have done a lot of research [14–19]. Li et al. [20–23] reported a lot of research work in the fields of nonlocal theory and nanomechanics and obtained many meaningful results.

The rotating nano annular plates are of great application value in micro-nano structures (such as the disk on the upper shaft of the nanomotor). At the same time, the nano annular plates are affected by the surrounding physical environment and vibrate freely during the working process. Therefore, the main purpose of the paper is to study the influence of the changes of various physical parameters on the natural frequencies of the nano annular plates based on three different plate theories. The actual working conditions of the nano annular plates are affected by temperature. At the same time, compared with the inner and outer diameter surfaces, the upper and lower surfaces of the nano annular plates have more contact with the working environment and are more affected by external factors. Therefore, functionally graded materials are used to manufacture the equipment. The physical properties of the materials are designed according to the law of power function changes along the thickness of the annular plates. Judging from the current research results, many scholars often choose one of the Kirchhoff plates, Mindlin plates or Reddy plates for analysis when studying the mechanical behaviors of annular plates or circular plates. In this paper, the free vibration behaviors of rotating annular plates are analyzed, and the influence of rotational velocity, temperature, and other parameters on the vibration frequencies is studied. The numerical results under the three theories are analyzed and compared, and the applicable conditions of the three plate theories are summarized.

2. Materials and Methods

As shown in Figure 1, the functionally graded nano-annular plate with inner radius r_1 , outer radius r_2 , and thickness h rotates around the z -axis at an angular velocity Ω . Establish a (r, θ, z) polar coordinate system, where r , θ , and z are the radial coordinate, hoop coordinate, and normal coordinate, respectively. The (r, θ) coordinate plane coincides with the geometric midplane.

Figure 1. Schematic diagram of the cross-section rotating functionally graded nano-annular plate.

2.1. The Distribution of Material Properties

The physical properties of the functionally graded nano-annular plate (such as elastic modulus E , mass density ρ , thermal expansion coefficient α) change according to the power function law along the thickness direction. Since each physical parameter is affected by the temperature T , it can be expressed as

$$P(z, T) = [P_c(T) - P_m(T)] \left(\frac{z}{h} + \frac{1}{2} \right)^k + P_m(T), -\frac{h}{2} \leq z \leq \frac{h}{2} \quad (1)$$

where k is a gradient index of the material, the subscript c represents the properties of the material on the upper surface ($z = -h/2$), and the subscript m represents the properties of the material on the lower surface ($z = h/2$). Considering the change in the physical properties of the material with temperature, $P(T)$ can be expressed as [24]

$$P(T) = P_0 (P_{-1} T^{-1} + 1 + P_1 T + P_2 T^2 + P_3 T^3) \quad (2)$$

among them, P_0 , P_{-1} , P_1 , P_2 , and P_3 are coefficients related to temperature, which are generally directly given by experiments. Since the Poisson's ratio of the functionally graded annular plates do not change much with the plate thickness and temperature, Poisson's ratio μ is regarded as a constant.

2.2. Nonlocal Elasticity Theory

According to Eringen's nonlocal elastic theory, the stress at any point in the continuum depends on the strain at all points in the continuum. Regarding the axisymmetric annular plate structure, the constitutive equation of the linear elastic nonlocal structure is [25]

$$\left[1 - (e_0 a)^2 \nabla^2 \right] \sigma = \sigma' \quad (3)$$

where, $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r}$ is the Laplacian equation; $\sigma = (\sigma_r \ \sigma_\theta \ \tau_{rz})^T$ and $\sigma' = (\sigma'_r \ \sigma'_\theta \ \tau'_{rz})^T$ represent nonlocal stress and classical stress, respectively; $e_0 a$ represents nonlocal parameter, e_0 is the nonlocal constant of material, and a is the internal characteristic scale (such as the lattice constant, bond length of a single carbon-carbon bond).

2.3. Axisymmetric Annular Plate Governing Equations

There are different theories for the strain in the thickness direction of the plates; the commonly used ones are the Kirchhoff plate, Mindlin plate, and Reddy plate strain theory. Kirchhoff plate theory and Mindlin plate theory assume that the cross-section of the plate remains flat after bending deformation, while the Kirchhoff plate theory ignores the influence of shear deformation and cross-section rotation around the neutral axis. The Reddy plate theory assumes that the cross-section of the plate is cubic after bending

and deformation. The displacement modes of the three plate theories can be uniformly expressed as

$$\begin{cases} u_r(r, z, t) = u(r, t) + z[(\lambda - 1) - \beta z^2] \frac{\partial w(r, t)}{\partial r} + z(\lambda - \beta z^2) \varphi(r, t) \\ u_z(r, z, t) = w(r, t) \end{cases} \quad (4)$$

where $u_r(r, z, t)$ and $u_z(r, z, t)$ represent the respective radial displacement and lateral displacement of any point in the annular plate; $u(r, t)$ and $w(r, t)$ represent the respective radial displacement and lateral displacement of any point in the midplane of the annular plate; $\varphi(r, t)$ is the rotation angle of the surface normal of the annular plate, and t is the time variable. When $\lambda = \beta = 0$, expression (4) is the displacement expression of Kirchhoff plate theory; when $\lambda = 1, \beta = 0$, expression (4) is the displacement expression of Mindlin plate theory; when $\lambda = 1, \beta = 4/(3h^2)$, expression (4) represents the displacement expression for the Reddy plate theory.

From the geometric relationship, the relationship between the strain and displacement of the annular plate is

$$\begin{pmatrix} \varepsilon_r \\ \varepsilon_\theta \\ \gamma_{rz} \end{pmatrix} = \begin{bmatrix} \frac{\partial}{\partial r} & \frac{1}{r} & \frac{\partial}{\partial z} \\ 0 & 0 & \frac{\partial}{\partial r} \end{bmatrix}^T \begin{pmatrix} u_r \\ u_z \end{pmatrix} \quad (5)$$

where ε_r and ε_θ represent the respective radial strain and hoop strain of the annular plate; γ_{rz} is the shear strain. Since the Kirchhoff plate theory ignores the shear strain, the corresponding shear stress and shear force are no longer considered.

According to the physical equations, the expressions of classical stress can be derived as

$$\boldsymbol{\sigma}' = \frac{E(z, T)}{1 - \mu^2} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & 1/2\kappa \end{bmatrix} \begin{pmatrix} \varepsilon_r \\ \varepsilon_\theta \\ \gamma_{rz} \end{pmatrix} \quad (6)$$

where κ is the shear correction factor; take $\kappa = 1$ for Reddy plate theory; take $\kappa = 12/\pi^2$ for Mindlin plate theory; for Kirchhoff plate theory, $\kappa \rightarrow \infty$, then $\tau'_{rz} = 0$.

From the expressions (6), the internal force of the annular plate under the classical theory can be derived

$$\mathbf{s}' = \mathbf{A}\mathbf{B}\mathbf{f} \quad (7)$$

where $\mathbf{s}'$, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{f}$ are the internal force vector, coefficient matrix, operator matrix, displacement vector, respectively. The specific representations are

$$\mathbf{s}' = (N'_r N'_\theta M'_r M'_\theta P'_r P'_\theta Q'_r R'_r)^T \quad (8)$$

$$\begin{cases} (N'_r M'_r P'_r) = \int_{-h/2}^{h/2} \sigma'_r(1, z, z^3) dz \\ (N'_\theta M'_\theta P'_\theta) = \int_{-h/2}^{h/2} \sigma'_\theta(1, z, z^3) dz \\ (Q'_r R'_r) = \int_{-h/2}^{h/2} \tau'_{rz}(1, z^2) dz \end{cases} \quad (9)$$

$$\mathbf{A} = \begin{bmatrix} A_{11} & 0 & (\lambda B_{11} - \beta E_{11}) & 0 & [(\lambda - 1)B_{11} - \beta E_{11}] & 0 & 0 \\ 0 & A_{11} & 0 & (\lambda B_{11} - \beta E_{11}) & 0 & [(\lambda - 1)B_{11} - \beta E_{11}] & 0 \\ B_{11} & 0 & (\lambda D_{11} - \beta F_{11}) & 0 & [(\lambda - 1)D_{11} - \beta F_{11}] & 0 & 0 \\ 0 & B_{11} & 0 & (\lambda D_{11} - \beta F_{11}) & 0 & [(\lambda - 1)D_{11} - \beta F_{11}] & 0 \\ E_{11} & 0 & (\lambda F_{11} - \beta H_{11}) & 0 & [(\lambda - 1)F_{11} - \beta H_{11}] & 0 & 0 \\ 0 & E_{11} & 0 & (\lambda F_{11} - \beta H_{11}) & 0 & [(\lambda - 1)F_{11} - \beta H_{11}] & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \left(\lambda \frac{A_{44}}{\kappa} - 3\beta \frac{D_{44}}{\kappa}\right) \\ 0 & 0 & 0 & 0 & 0 & 0 & \left(\lambda \frac{D_{44}}{\kappa} - 3\beta \frac{F_{44}}{\kappa}\right) \end{bmatrix} \quad (10)$$

$$\mathbf{B} = \begin{pmatrix} \frac{\partial}{\partial r} + \frac{u}{r} & \mu \frac{\partial}{\partial r} + \frac{1}{r} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\partial}{\partial r} + \frac{u}{r} & \mu \frac{\partial}{\partial r} + \frac{1}{r} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \frac{\partial^2}{\partial r^2} + \frac{u}{r} \frac{\partial}{\partial r} & \mu \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} & \frac{\partial}{\partial r} \end{pmatrix}^T \quad (11)$$

$$\mathbf{f} = (u \ \varphi \ w)^T \quad (12)$$

The coefficients of each component in the coefficient matrix $\mathbf{A}$ are

$$(A_{11}, B_{11}, D_{11}, E_{11}, F_{11}, H_{11}) = \int_{-h/2}^{h/2} \frac{E(z, T)}{1 - \mu^2} (1, z, z^2, z^3, z^4, z^6) dz \quad (13)$$

$$(A_{44}, D_{44}, F_{44}) = \int_{-h/2}^{h/2} \frac{E(z, T)}{2(1 + \mu)} (1, z^2, z^4) dz \quad (14)$$

Substituting expression (7) into the nonlocal theory expression (3), the internal force relationship between the classical theory and the nonlocal theory can be obtained

$$\left[1 - (e_0 a)^2 \nabla^2\right] \mathbf{s} = \mathbf{s}' = \mathbf{A} \mathbf{B} \mathbf{f} \quad (15)$$

$$\mathbf{s} = (N_r N_\theta M_r M_\theta P_r P_\theta Q_r R_r)^T \quad (16)$$

$$\begin{cases} (N_r M_r P_r) = \int_{-h/2}^{h/2} \sigma_r(1, z, z^3) dz \\ (N_\theta M_\theta P_\theta) = \int_{-h/2}^{h/2} \sigma_\theta(1, z, z^3) dz \\ (Q_r R_r) = \int_{-h/2}^{h/2} \tau_{rz}(1, z^2) dz \end{cases} \quad (17)$$

where N_r, N_θ represent nonlocal axial force; M_r, M_θ represent nonlocal bending moment; Q_r is the nonlocal shear force; P_r, P_θ are higher order nonlocal bending moments; R_r is a higher order nonlocal shear force;

From expressions (4) to (7), the expressions of strain energy, external force potential energy, and kinetic energy of the annular plates under the classical theory can be derived:

Strain energy

$$U = \int_V (\sigma'_r \varepsilon_r + \sigma'_\theta \varepsilon_\theta + \tau'_{rz} \gamma_{rz}) dV \quad (18)$$

Kinetic energy

$$T = \int_V \frac{1}{2} \rho(z, T) \left[\left(\frac{\partial u_r}{\partial t} \right)^2 + \left(\frac{\partial u_z}{\partial t} \right)^2 \right] dV \quad (19)$$

External force potential energy

$$H = -\frac{1}{2} \int_{r_1}^{r_2} \int_0^{2\pi} r N^{RT} \left(\frac{\partial w}{\partial r} \right)^2 d\theta dr \quad (20)$$

where $N^{RT} = N^R + N^T$, N^R and N^T are the radial internal force generated due to the rotation and the radial internal force caused by temperature changes, respectively.

The internal force produced by the temperature change is

$$N^T = - \int_{-h/2}^{h/2} \frac{E(z, T)}{1 - \mu} \alpha(z, T) \Delta T dz \quad (21)$$

N^R can be solved by the plane stress problem. When the boundary conditions of the inner and outer rings of the annular plates are hinged or clamped, the boundary conditions can be expressed as $u_r|_{r=r_1, r_2} = 0$, then we can obtain

$$N^R = I_0 \frac{\Omega^2}{8} \left[(1 + \mu)(r_1^2 + r_2^2) + (1 - \mu) \frac{r_1^2 r_2^2}{r^2} - (3 + \mu)r^2 \right] \quad (22)$$

When the boundary conditions of the inner rings of the annular plates are clamped and the boundary conditions of the outer rings are free, the boundary conditions can be expressed as $u_r|_{r=r_1} = 0, \sigma_r|_{r=r_2} = 0$, then we can get

$$N^R = I_0 \frac{\Omega^2}{8} \left\{ (3 + \mu) + \frac{(1 - \mu)[(3 + \mu)r_2^2 - (1 + \mu)r_1^2]}{(1 - \mu)r_1^2 + (1 + \mu)r_2^2} \frac{r_1^2}{r^2} \right\} (r_2^2 - r^2) \quad (23)$$

According to the Hamilton principle, the stationary value of energy is zero, that is

$$\delta \int_{t_0}^{t_1} (-U + T + H) dt = 0 \quad (24)$$

substituting expressions (18)–(20) into expression (24), the unified governing equations are

$$\frac{1}{r} \frac{\partial(rN'_r)}{\partial r} - \frac{N'_\theta}{r} = I_0 \frac{\partial^2 u}{\partial t^2} + v_1 \frac{\partial^2 \varphi}{\partial t^2} + v_2 \frac{\partial^3 w}{\partial r \partial t^2} \quad (25)$$

$$\frac{\lambda}{r} \frac{\partial(rM'_r)}{\partial r} - \frac{\beta}{r} \frac{\partial(rP'_r)}{\partial r} + \beta \frac{P'_\theta}{r} - \lambda \frac{M'_\theta}{r} + 3\beta R'_r - Q'_r = v_1 \frac{\partial^2 u}{\partial t^2} + v_3 \frac{\partial^2 \varphi}{\partial t^2} + v_4 \frac{\partial^3 w}{\partial r \partial t^2} \quad (26)$$

$$\begin{aligned} & \frac{(1-\lambda)}{r} \frac{\partial^2(rM'_r)}{\partial r^2} + \frac{\beta}{r} \frac{\partial^2(rP'_r)}{\partial r^2} + \frac{(\lambda-1)}{r} \frac{\partial M'_\theta}{\partial r} - \frac{\beta}{r} \frac{\partial P'_\theta}{\partial r} + \frac{\lambda}{r} \frac{\partial(rQ'_r)}{\partial r} - \frac{3\beta}{r} \frac{\partial(rP'_r)}{\partial r} + \frac{1}{r} \frac{\partial}{\partial r} \left(rN^{RT} \frac{\partial w}{\partial r} \right) \\ & = -v_2 \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial^2 u}{\partial t^2} \right) - v_4 \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial^2 \varphi}{\partial t^2} \right) + I_0 \frac{\partial^2 w}{\partial t^2} - v_5 \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial^3 w}{\partial r \partial t^2} \right) \end{aligned} \quad (27)$$

The expressions of each coefficient are

$$(I_0, I_1, I_2, I_3, I_4, I_6) = \int_{-h/2}^{h/2} \rho(z, T) (1, z, z^2, z^3, z^4, z^6) dz \quad (28)$$

$$\begin{aligned} v_1 &= \lambda I_1 - \beta I_3, v_2 = (\lambda - 1) I_1 - \beta I_3, v_3 = \lambda^2 I_2 - 2\lambda \beta I_4 + \beta^2 I_6 \\ v_4 &= \lambda(\lambda - 1) I_2 + \beta(1 - 2\lambda) I_4 + \beta^2 I_6, v_5 = (\lambda - 1)^2 I_2 + \beta(2\lambda - 1) I_4 + \beta^2 I_6 \end{aligned} \quad (29)$$

When the nano-annular plates are in free linear vibration, the displacement functions $u(r, t)$, $w(r, t)$, $\varphi(r, t)$ can be expressed in the form of space–time separation

$$u(r, t) = \tilde{u}(r)e^{i\omega t}, w(r, t) = \tilde{w}(r)e^{i\omega t}, \varphi(r, t) = \tilde{\varphi}(r)e^{i\omega t} \quad (30)$$

where $\tilde{u}(r)$, $\tilde{w}(r)$, $\tilde{\varphi}(r)$ are the mode functions; ω represents the vibration frequencies of the annular plates.

Combining Equations (15), (25)–(27) and (30), the mechanical variables in the governing Equations (25)–(27) can be replaced by displacement variables, and the nonlocal governing equations can be obtained after sorting.

$$A_{11} \frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial r} (r\tilde{u}) + \bar{B}_1 \frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial r} (r\tilde{\varphi}) + \bar{B}_2 \frac{\partial}{\partial r} (\nabla^2 \tilde{w}) = -\omega^2 [1 - (e_0 a)^2 \nabla^2] \left(I_0 \tilde{u} + v_1 \tilde{\varphi} + v_2 \frac{\partial \tilde{w}}{\partial r} \right) \quad (31)$$

$$\bar{B}_1 \frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial r} (r\tilde{u}) + \bar{D}_1 \frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial r} (r\tilde{\varphi}) + \bar{D}_2 \frac{\partial}{\partial r} (\nabla^2 \tilde{w}) - \frac{\bar{A}_4}{\kappa} \left(\tilde{\varphi} + \frac{\partial \tilde{w}}{\partial r} \right) = -\omega^2 [1 - (e_0 a)^2 \nabla^2] \left(v_1 \tilde{u} + v_3 \tilde{\varphi} + v_4 \frac{\partial \tilde{w}}{\partial r} \right) \quad (32)$$

$$\begin{aligned} & -\bar{B}_2 \nabla^2 \left(\frac{1}{r} \frac{\partial}{\partial r} (r\tilde{u}) \right) - \bar{D}_2 \nabla^2 \left(\frac{1}{r} \frac{\partial}{\partial r} (r\tilde{\varphi}) \right) - 2\bar{D}_3 \nabla^4 \tilde{w} + \frac{\bar{A}_4}{\kappa} \left(\nabla^2 \tilde{w} + \frac{\partial \tilde{\varphi}}{\partial r} + \frac{\tilde{\varphi}}{r} \right) \\ & = -\omega^2 [1 - (e_0 a)^2 \nabla^2] \left[-v_2 \frac{1}{r} \frac{\partial}{\partial r} (r\tilde{u}) - v_4 \frac{1}{r} \frac{\partial}{\partial r} (r\tilde{\varphi}) + I_0 \tilde{w} - v_5 \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \tilde{w}}{\partial r} \right) - \frac{1}{r} \frac{\partial}{\partial r} \left(rN^{RT} \frac{\partial \tilde{w}}{\partial r} \right) \right] \end{aligned} \quad (33)$$

In Equations (31)–(33), the related coefficients are expressed as

$$\begin{aligned} \bar{B}_1 &= \lambda B_{11} - \beta E_{11}, \bar{B}_2 = (\lambda - 1) B_{11} - \beta E_{11}, \bar{D}_1 = \lambda^2 D_{11} - 2\lambda \beta F_{11} + \beta^2 H_{11}, \\ \bar{D}_2 &= \lambda(\lambda - 1) D_{11} - 2\lambda \beta F_{11} + \beta^2 H_{11}, \bar{A}_4 = \lambda^2 A_{44} - 6\lambda \beta D_{44} + 9\beta^2 F_{44} \end{aligned} \quad (34)$$

For the derived governing Equations (31)–(33), it is difficult to solve high-order variable coefficient differential equations due to the existence of the nonlocal parameter, rotational velocity, and other parameters, so in this paper, the differential quadrature method (DQM) was used to discretize the governing equations. This method first selects a finite number of nodes in the solution domain and then approximates the derivative value of each point of the function in the solution domain with the weighted linear sum of the selected node's function values. By the DQM method, the partial differential equations can be discretized into algebraic equations, and the determinant of the coefficient matrix is zero according to the necessary and sufficient conditions for the matrix equations to be solved, and finally the eigenvalues are calculated and the results are analyzed. The specific discrete form of Equations (31)–(33) has been given in Appendix A. The specific discrete method refers to the literature [26].

3. Calculation Results and Analysis

3.1. Validation

In order to verify the validity of the model in this paper and the accuracy of the calculation, we first calculated the free vibration numerical results of the non-rotating circular plates and compared this with the results of [27]. The governing equations derived from the Reddy plate theory are complicated, and the omission of related parameters can simplify the Reddy plate governing equations to the Kirchhoff plate governing equations and Mindlin plate governing equations. Therefore, this paper selects the numerical results under the Reddy plate theory for comparison. According to the data in Table 1, it can be found that the numerical results in this paper are in good agreement with those in the literature. Therefore, the validity of the model and the accuracy of the calculation can be verified.

Table 1. Comparison of natural frequencies for Reddy circular plates ($\bar{\Omega} = 0$, $k \rightarrow \infty$, $s = 0$).

Mode		1	2	3	4	5	6
C-C	This paper	10.215	39.771	89.102	158.18	246.99	355.54
	Literature [27]	10.215	39.771	89.102	158.18	246.99	355.54
C-F	This paper	9.0026	38.443	87.749	156.81	245.62	354.17
	Literature [27]	9.0026	38.443	87.749	156.81	245.62	354.17

Next, based on three different plate theories, the free vibration behaviors of the rotating functionally graded nano-annular plates under two boundary conditions are analyzed, and the effects of various parameters such as the inner and outer radius ratio, rotational velocity, nonlocal parameter, functional gradient index, and temperature change on the vibration frequencies under two boundary conditions are examined. The two boundary conditions of the nano-annular plates are inner and outer rings clamped (C-C), inner rings clamped and outer rings free (C-F). Take the thickness h of the annular plates as 1 nm and the outer radius r_2 as 50 nm. The upper surface of the annular plates is made of ceramic (Si_3N_4) with density of $\rho_c = 2707 \text{ kg/m}^3$, and the lower surface material is made of metal (SUS304) with density of $\rho_m = 3800 \text{ kg/m}^3$. Assume the initial ambient temperature $T_0 = 300 \text{ K}$ at which the annular plates do not undergo initial stress and deformation, and the modulus of elasticity and the coefficient of thermal expansion change with the change in the ambient temperature. The specific temperature-related coefficients are shown in Table 2. Since Poisson's ratio does not change much, the constant is taken as 0.3.

Table 2. Material coefficients of ceramics and metals.

	Material	P_{-1}	P_0	P_1	P_2	P_3
E_c/Pa	Si_3N_4	0	348.43×10^9	-3.070×10^{-4}	2.160×10^{-7}	-8.946×10^{-11}
E_m/Pa	SUS304	0	201.04×10^9	3.079×10^{-4}	-6.534×10^{-7}	0
α_c/K^{-1}	Si_3N_4	0	5.8723×10^{-6}	9.095×10^{-4}	0	0
α_m/K^{-1}	SUS304	0	12.330×10^{-6}	8.086×10^{-4}	0	0

In order to ensure the numerical results have universal applications, the physical quantities are made dimensionless

$$\begin{aligned}\bar{u} &= \frac{\tilde{u}}{r_2}, \bar{w} = \frac{\tilde{w}}{r_2}, \bar{r} = \frac{r}{r_2}, s = \frac{r_1}{r_2}, \delta = \frac{h}{r_2}, \tau = \frac{e_0 a}{r_2}, \bar{t} = \frac{t}{r_2^2} \sqrt{\frac{E_m h^3}{12 \rho_m h (1 - \mu^2)}} \\ \bar{\Omega}^2 &= \frac{12 \rho_m h r_2^2 (1 - \mu^2)}{E_m h^3} \Omega^2, \bar{\omega}^2 = \frac{12 \rho_m h r_2^2 (1 - \mu^2)}{E_m h^3} \omega^2, \bar{N}^{RT} = \frac{12 N^{RT} r_2^2 (1 - \mu^2)}{E_m h^3}\end{aligned}\quad (35)$$

The research contents of the Sections 3.1–3.6 are the influence of various physical parameters on the natural frequencies of the rotating nano-annular plates. The results obtained through Kirchhoff's theory are the same as the conclusions of the literature [28].

3.2. The Influence of the Ratio of the Inner and Outer Radius on Frequencies

Figure 2 studies the change law of the first three order dimensionless natural frequencies of the nano annular plates under the influence of the ratio of the inner and outer radius. $\tau = 0.05$, $\bar{\Omega} = 5$, $\Delta T = 10$ K. It can be seen from the figure that the dimensionless natural frequencies gradually increase with the increase in the ratio of inner and outer radius, and the larger the ratio of the inner and outer radius, the greater the influence on the dimensionless natural frequencies. The boundary conditions play an important role in the natural frequencies of the nano annular plates. Compared with the C-C boundary condition, the equivalent stiffness of the nano-annular plates under the C-F boundary condition is weakened, so the natural frequencies decrease. Comparing the data in the figures, it can be found that the differences in the data results under the three plate theories are small.

Figure 2. Cont.

Figure 2. The relationship between dimensionless natural frequencies and the ratio of the inner and outer radius under two different boundary conditions. (a) C-C Kirchhoff plate; (b) C-C Mindlin plate; (c) C-C Reddy plate; (d) C-F Kirchhoff plate; (e) C-F Mindlin plate; (f) C-F Reddy plate.

3.3. The Influence of Rotational Velocity on Frequencies

Figure 3 studies the influence of the dimensionless rotational velocity on the first three order dimensionless natural frequencies of the nano-annular plates under two boundary conditions. $\tau = 0.05$, $s = 0.1$, $\Delta T = 0$ K. It can be seen from Figure 3 that the influence of rotational velocity on natural frequencies is related to boundary conditions. During the rotation of the nano-annular plates under the C-C boundary condition, the increase in the rotational velocity weakens the equivalent stiffness of the nano-annular plates and reduces their natural frequencies. When the rotational velocity reaches a certain value, the natural frequencies drop to zero, which leads to the instability of the nano-plates. For the nano-annular plates with the C-F boundary condition, the natural frequencies gradually increase with the increase in the rotational velocity, which means that the increase in the rotational velocity enhances the equivalent stiffness of the nano-annular plates. Analyzing the essential reason through these two phenomena, the centrifugal forces linearly distributed along the radial direction are generated when the nano-annular plates rotate. From expression 16b, it can be analyzed that the internal forces N^R in the local unit width along the radial direction in the annular plates is negative, and local pressures are generated. As the rotational velocity increases, the local pressure values gradually increase. When the rotational velocity increases to a certain extent, the radial equivalent pressure values of the nano-annular plates exceed the critical load, resulting in instability, and the free vibration frequencies become zero. Therefore, the rotational velocity of the nano-annular plates with the C-C boundary condition should not be too high when working, otherwise they will be unstable and cannot work normally. It can be analyzed from expression 16a that the linearly distributed centrifugal forces N^R generated in the radial direction are positive when the annular plates rotate, so that the radial cross-sections are stretched, and as the rotational velocity increases, the tensile deformation increases. Thereby, the equivalent stiffness of the nano-annular plates is enhanced, and their natural frequencies increase.

3.4. The Influence of Functionally Gradient Index on Frequencies

Figure 4 studies the influence of the functionally gradient index k on the first three order dimensionless natural frequencies of the nano-annular plates. $\tau = 0.05$, $\bar{\Omega} = 5$, $\Delta T = 0$ K. It can be seen from Figure 4 that under the two boundary conditions, the first three order dimensionless natural frequencies decrease with the increase in the functionally gradient parameter k , and the rate of change changes from fast to slow and finally converges to a fixed value. The increase in the functionally gradient parameter k weakens the equivalent stiffness of the annular plates and reduces their natural frequencies. For the nano-annular plates made of Si_3N_4 and SUS304, the size of the functional gradient parameter k reflects the ratio of the two material components. When k increases from 0, the change in k changes the values of the physical parameters and the position of the physical midplane. This change

has significant impacts on the stiffness of the annular plates, and as the value of k gradually increases, the change in the physical parameters and the position of the physical midplane have little effect on the stiffness of the nano-annular plates. It can also be observed from the figure that the data under the three plate theories have little differences and are basically the same.

Figure 3. The relationship between dimensionless natural frequencies and the rotational velocity under two different boundary conditions. (a) C-C Kirchhoff plate; (b) C-C Mindlin plate; (c) C-C Reddy plate; (d) C-F Kirchhoff plate; (e) C-F Mindlin plate; (f) C-F Reddy plate.

3.5. The Influence of Nonlocal Parameter on Frequencies

Figures 5–7 show the influence of the nonlocal parameter on the natural frequencies during the rotation of the nano-annular plates. $\bar{\Omega} = 5$, $s = 0.1$, $\Delta T = 0$ K. It can be seen from the figure that the increase in the nonlocal parameter reduces the equivalent stiffness of the nano-annular plates and decreases the first three order natural frequencies. Compared with the second and third order natural frequencies, the fundamental frequencies change less obviously. Comparing the law of the influence of nonlocal parameter on frequencies with the numerical results of literature [29], it can be found that the law of the influence of the nonlocal parameter on frequencies is not affected by the state of motion and rotational velocity and is consistent with the law of change without rotation. In the figures, the natural frequencies under the two boundary conditions are compared. When the values of the dimensionless rotational velocity, gradient index, and other parameters are constant, it can be found that the natural frequencies of the nano-annular plates under the C-C boundary condition are higher than the natural frequencies of nano-annular plates under the C-F boundary. When the inner and outer rings of the annular plates are clamped, the mechanical properties are enhanced, and there is good stability during the working process.

Figure 4. The relationship between dimensionless natural frequencies and the functionally gradient index under two different boundary conditions. (a) C-C Kirchhoff plate; (b) C-C Mindlin plate; (c) C-C Reddy plate; (d) C-F Kirchhoff plate; (e) C-F Mindlin plate; (f) C-F Reddy plate.

Figure 5. The relationship between dimensionless natural frequencies and the nonlocal parameter under two different boundary conditions (Kirchhoff plate). (a) The first-order frequencies; (b) The second-order frequencies; (c) The third-order frequencies.

Figure 6. The relationship between dimensionless natural frequencies and the nonlocal parameter under two different boundary conditions (Mindlin plate). (a) The first-order frequencies; (b) The second-order frequencies; (c) The third-order frequencies.

Figure 7. The relationship between dimensionless natural frequencies and the nonlocal parameter under two different boundary conditions (Reddy plate). (a) The first-order frequencies; (b) The second-order frequencies; (c) The third-order frequencies.

3.6. The Influence of Temperature Change on Frequencies

Figures 8–10 show the influence of temperature change ΔT on the natural frequencies of the nano-annular plates. $\bar{\Omega} = 5$, $s = 0.1$, $\tau = 0.1$. It can be seen from the figure that when the boundary condition is C-C, with the positive increase in ΔT , that is, the temperature gradually increases, the first three order dimensionless natural frequencies of the annular plates gradually decrease. When the boundary condition is C-F, the fundamental frequencies of the annular plates increase with the increase in temperature, and the second and third order dimensionless natural frequencies decrease with the increase in temperature. When the value of rotational velocity is fixed, the change in temperature changes the internal forces in the plane, so that the natural frequencies of the nano-annular plates change. When the value of ΔT increases in the positive direction, the absolute values of the radial forces N^T generated by the temperature change can be analyzed through expression (28), which gradually compresses the radial cross sections of the annular plates. This change increases the equivalent stiffness under the first-order vibration of the nano annular plates with the C-F boundary condition and increases the fundamental frequencies. For the nano-annular plates, with inner and outer rings clamped, the change trend is just the opposite. Therefore, the influence of temperature change on the fundamental frequencies of the nano-annular plates during the rotation is related to the boundary conditions.

Figure 8. The relationship between dimensionless natural frequencies and the temperature change under two different boundary conditions (Kirchhoff plate). (a) The first-order frequencies; (b) The second-order frequencies; (c) The third-order frequencies.

Figure 9. The relationship between dimensionless natural frequencies and the temperature change under two different boundary conditions (Mindlin plate). (a) The first-order frequencies; (b) The second-order frequencies; (c) The third-order frequencies.

Figure 10. The relationship between dimensionless natural frequencies and the temperature change under two different boundary conditions (Reddy plate). (a) The first-order frequencies; (b) The second-order frequencies; (c) The third-order frequencies.

3.7. The Influence of Different Plate Theories on Frequencies

Figures 11 and 12 are based on different deformation theories to study the influence of the thickness-to-radius ratio of the inner and outer ring clamped annular plates on the fundamental frequencies and to compare the numerical results under different theories to compare the differences. Take the outer radius $r_2 = 50$ nm; it can be seen from Figure 11 that when the thickness-to-radius ratio is greater than 0.05 and less than 0.15 and the value of rotational velocity is greater than 10 and less than 15, the numerical results based on the Kirchhoff plate theory are significantly greater than the numerical results obtained by the Mindlin plate theory and the Reddy plate theory. In this case, Kirchhoff plate theory is not suitable for the analysis of the mechanical behaviors of the annular plates; it is more reasonable to apply Mindlin plate theory or Reddy plate theory. When the thickness-to-radius ratio is greater than 0.15 and the value of rotational velocity is greater than 15, it can be seen that the numerical results under the Mindlin plate theory and the Reddy plate theory are very different, and the Reddy plate theory gradually shows its superiority. It can be seen from Figure 12 that when the thickness-to-radius ratio is less than 0.05, the results obtained based on the three deformation theories are relatively close, and they are all suitable for analyzing the vibration behaviors of the nano-annular plates. When the thickness-to-radius ratio is greater than 0.05, the numerical results based on the Kirchhoff plate theory are quite different from the numerical results obtained from the Mindlin plate theory and the Reddy plate theory. In this case, the Kirchhoff plate theory is not suitable for the vibration analysis of the nano-annular plates, the Mindlin plate theory and Reddy plate

theory can be used at this time. It can be seen that when the ratio of thickness to radius is greater than 0.15, the differences between the results obtained based on the Mindlin plate theory and the Reddy plate theory are smaller, but the results obtained by the Reddy plate theory are more accurate.

Figure 11. The influence of the thickness-to-radius ratio and rotational velocity on the first-order natural frequencies under different plate theories ($k = 5$, $s = 0.1$, $\tau = 0.05$, $\Delta T = 0$ K).

Figure 12. The influence of the thickness-to-radius ratio and nonlocal parameter on the first-order natural frequencies under different plate theories ($k = 5$, $s = 0.1$, $\bar{\Omega} = 5$, $\Delta T = 0$ K).

4. Discussion

Based on the three plate theories of Kirchhoff, Mindlin, and Reddy and the nonlocal elastic theory, this paper studies the free vibration behaviors of the rotating nano-annular plates in a thermal environment. The governing equations are derived by Hamilton principle, and the differential quadrature method is used to discretize the governing equations and the natural frequencies of the functionally graded nano-annular plates under the two boundary conditions are obtained. The research results show that the temperature change, functionally gradient index parameter, nonlocal parameter, rotational velocity, and other parameters have an important influence on the free vibration behaviors of the nano-annular plates.

The three plate theories and nonlocal constitutive relations used in this paper to study the vibration behaviors of the nano plates have been applied by a large number of scholars in previous studies. In recent years, the new plate theories and shape functions have been proved and applied by many scholars. Good numerical results have been obtained, such as [30,31]. At the same time, when studying the mechanical properties of micro-nano structures and choosing physical constitutive relations, in addition to the nonlocal theory used in this paper, there are strain gradient theory and nonlocal strain gradient theory, etc. However, the nonlocal theory and the strain gradient theory have different or even completely opposite conclusions in practical applications. The nonlocal strain gradient theory considers both nonlocal effects and strain gradient effects and resolves the confusion and controversy caused by the two theories [32]. Therefore, the authors' next work is to study the mechanical characteristics of micro-nano structures based on the nonlocal strain gradient theory and the new shear plate theories.

5. Conclusions

The contribution of this paper was to unify the vibration equations of rotating nano annular plates based on the three plate theories and establish a unified mathematical model, which is a feature of this paper. Then, the differences of the calculation results about the three plate theories were compared. It is concluded that (i) When studying the vibration problems of the nano-annular plates based on the three plate theories, the differences in the numerical results calculated by the three theories become obvious as the thickness-to-radius ratio increases. Kirchhoff plate theory is suitable for vibration analysis of thin plates; Mindlin plate theory and Reddy plate theory are more suitable for mechanical studies of thick plates, and the numerical results of Reddy plate theory are more accurate for thick plates analysis. (ii) The effects of rotational velocity on the natural frequencies of the nano-annular plates are related to the boundary conditions. During the rotation of the annular-plates with inner and outer rings clamped, local pressures are generated in plane. When the rotational velocity reaches a certain value, instability occurs. For the nano-annular plates with the inner rings clamped and the outer ring free, during the rotation process, the micro segments in the plane are stretched under the action of the rotational velocity to increase the stiffness, and the natural frequencies gradually increase as the rotational velocity increases. (iii) The influence of the nonlocal parameter on the natural frequencies of the nano-annular plates is not affected by the rotational velocity. The increase in the nonlocal parameter weakens the equivalent stiffness of the nano-annular plates and reduce their natural frequencies. At the same time, in the case of considering the nonlocal parameter, the numerical results calculated based on Reddy plate theory and Mindlin plate theory are more accurate, while the results calculated based on Reddy plate theory have better accuracy. (iv) During the rotation of the nano-annular plates, the influence of temperature changes on the fundamental frequencies is related to the boundary conditions. For the nano-annular plates with the inner rings clamped and the outer rings free, the increase in temperature increases the fundamental frequencies, but for the nano-annular plates with inner and outer rings clamped, its dynamics are the opposite.

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Appendix A

According to the differential quadrature method, the coefficients of the interpolation points are calculated, so that the governing Equations (31)–(33) are transformed into

$$A_{11} \sum_{j=1}^n c_{ij}^2 \tilde{u}_j + \frac{A_{11}}{r_i} \sum_{j=1}^n c_{ij}^1 \tilde{u}_j - \frac{A_{11}}{r_i^2} \tilde{u}_i + \bar{B}_1 \sum_{j=1}^n c_{ij}^2 \tilde{\varphi}_j + \frac{\bar{B}_1}{r_i} \sum_{j=1}^n c_{ij}^1 \tilde{\varphi}_j - \frac{\bar{B}_1}{r_i^2} \tilde{\varphi}_i + \bar{B}_2 \sum_{j=1}^n c_{ij}^3 \tilde{w}_j + \frac{\bar{B}_2}{r_i} \sum_{j=1}^n c_{ij}^2 \tilde{w}_j - \frac{\bar{B}_2}{r_i^2} \sum_{j=1}^n c_{ij}^1 \tilde{w}_j$$

$$= -\omega^2 \left\{ \begin{aligned} & I_0 \tilde{u}_i - (e_0 a)^2 I_0 \sum_{j=1}^n c_{ij}^2 \tilde{u}_j - \frac{(e_0 a)^2 I_0}{r_i} \sum_{j=1}^n c_{ij}^1 \tilde{u}_j + v_1 \tilde{\varphi}_i - (e_0 a)^2 v_1 \sum_{j=1}^n c_{ij}^2 \tilde{\varphi}_j - \frac{(e_0 a)^2 v_1}{r_i} \sum_{j=1}^n c_{ij}^1 \tilde{\varphi}_j + v_2 \sum_{j=1}^n c_{ij}^1 \tilde{w}_j \\ & - (e_0 a)^2 v_2 \sum_{j=1}^n c_{ij}^3 \tilde{w}_j - \frac{(e_0 a)^2 v_2}{r_i} \sum_{j=1}^n c_{ij}^2 \tilde{w}_j \end{aligned} \right\} \quad (A1)$$

$$\bar{B}_1 \sum_{j=1}^n c_{ij}^2 \tilde{u}_j + \frac{\bar{B}_1}{r_i} \sum_{j=1}^n c_{ij}^1 \tilde{u}_j - \frac{\bar{B}_1}{r_i^2} \tilde{u}_i + \bar{D}_1 \sum_{j=1}^n c_{ij}^2 \tilde{\varphi}_j + \frac{\bar{D}_1}{r_i} \sum_{j=1}^n c_{ij}^1 \tilde{\varphi}_j - \frac{\bar{D}_1}{r_i^2} \tilde{\varphi}_i$$

$$- \frac{\bar{A}_4}{\kappa} \tilde{\varphi}_i + \bar{D}_2 \sum_{j=1}^n c_{ij}^3 \tilde{w}_j + \frac{\bar{D}_2}{r_i} \sum_{j=1}^n c_{ij}^2 \tilde{w}_j - \frac{\bar{D}_2}{r_i^2} \sum_{j=1}^n c_{ij}^1 \tilde{w}_j - \frac{\bar{A}_4}{\kappa} \sum_{j=1}^n c_{ij}^1 \tilde{w}_j$$

$$= -\omega^2 \left\{ \begin{aligned} & v_1 \tilde{u}_i - (e_0 a)^2 v_1 \sum_{j=1}^n c_{ij}^2 \tilde{u}_j - \frac{(e_0 a)^2 v_1}{r_i} \sum_{j=1}^n c_{ij}^1 \tilde{u}_j + v_3 \tilde{\varphi}_i - (e_0 a)^2 v_3 \sum_{j=1}^n c_{ij}^2 \tilde{\varphi}_j \\ & - \frac{(e_0 a)^2 v_3}{r_i} \sum_{j=1}^n c_{ij}^1 \tilde{\varphi}_j + v_4 \sum_{j=1}^n c_{ij}^1 \tilde{w}_j - (e_0 a)^2 v_4 \sum_{j=1}^n c_{ij}^3 \tilde{w}_j - \frac{(e_0 a)^2 v_4}{r_i} \sum_{j=1}^n c_{ij}^2 \tilde{w}_j \end{aligned} \right\} \quad (A2)$$

$$- \frac{\bar{B}_2}{r_i^3} \tilde{u}_i + \frac{\bar{B}_2}{r_i^2} \sum_{j=1}^n c_{ij}^1 \tilde{u}_j - \frac{2\bar{B}_2}{r_i} \sum_{j=1}^n c_{ij}^2 \tilde{u}_j - \bar{B}_2 \sum_{j=1}^n c_{ij}^3 \tilde{u}_j + \left(\frac{\bar{A}_4}{\kappa} \frac{1}{r_i} - \frac{\bar{D}_2}{r_i^3} \right) \tilde{\varphi}_i + \left(\frac{\bar{D}_2}{r_i^2} + \frac{\bar{A}_4}{\kappa} \right) \sum_{j=1}^n c_{ij}^1 \tilde{\varphi}_j - \frac{2\bar{D}_2}{r_i} \sum_{j=1}^n c_{ij}^2 \tilde{\varphi}_j$$

$$- \bar{D}_2 \sum_{j=1}^n c_{ij}^3 \tilde{\varphi}_j + \left(\frac{\bar{A}_4}{\kappa} \frac{1}{r_i} - \frac{\bar{D}_3}{r_i^3} \right) \sum_{j=1}^n c_{ij}^1 \tilde{w}_j + \left(\frac{2\bar{D}_3}{r_i^2} + \frac{\bar{A}_4}{\kappa} \right) \sum_{j=1}^n c_{ij}^2 \tilde{w}_j - \frac{4\bar{D}_3}{r_i} \sum_{j=1}^n c_{ij}^3 \tilde{w}_j - 2\bar{D}_3 \sum_{j=1}^n c_{ij}^4 \tilde{w}_j$$

$$= -\omega^2 \left\{ \begin{aligned} & - \frac{v_2}{r_i} \tilde{u}_i - v_2 \sum_{j=1}^n c_{ij}^1 \tilde{u}_j + (e_0 a)^2 v_2 \sum_{j=1}^n c_{ij}^3 \tilde{u}_j + \frac{(e_0 a)^2 v_2}{r_i^3} \tilde{u}_i - \frac{(e_0 a)^2 v_2}{r_i^2} \sum_{j=1}^n c_{ij}^1 \tilde{u}_j + \frac{2(e_0 a)^2 v_2}{r_i} \sum_{j=1}^n c_{ij}^2 \tilde{u}_j - v_4 \sum_{j=1}^n c_{ij}^1 \tilde{\varphi}_j \\ & - \frac{v_4}{r_i} \tilde{\varphi}_i + (e_0 a)^2 v_4 \sum_{j=1}^n c_{ij}^3 \tilde{\varphi}_j + \frac{2(e_0 a)^2 v_4}{r_i} \sum_{j=1}^n c_{ij}^2 \tilde{\varphi}_j - \frac{(e_0 a)^2 v_4}{r_i^2} \sum_{j=1}^n c_{ij}^1 \tilde{\varphi}_j + \frac{(e_0 a)^2 v_4}{r_i^3} \tilde{\varphi}_i + I_0 \tilde{w}_i - I_0 (e_0 a)^2 \sum_{j=1}^n c_{ij}^2 \tilde{w}_j \\ & - \frac{I_0 (e_0 a)^2}{r_i} \sum_{j=1}^n c_{ij}^1 \tilde{w}_j - v_5 \sum_{j=1}^n c_{ij}^2 \tilde{w}_j - \frac{v_5}{r_i} \sum_{j=1}^n c_{ij}^1 \tilde{w}_j + v_5 (e_0 a)^2 \sum_{j=1}^n c_{ij}^4 \tilde{w}_j + N^{RT} (e_0 a)^2 \sum_{j=1}^n c_{ij}^4 \tilde{w}_j + \frac{2(e_0 a)^2 v_5}{r_i} \sum_{j=1}^n c_{ij}^3 \tilde{w}_j \\ & - \frac{(e_0 a)^2 v_5}{r_i^2} \sum_{j=1}^n c_{ij}^2 \tilde{w}_j + \frac{(e_0 a)^2 v_5}{r_i^3} \sum_{j=1}^n c_{ij}^1 \tilde{w}_j - N^{RT} \sum_{j=1}^n c_{ij}^2 \tilde{w}_j - \frac{\partial N^{RT}}{\partial r_i} \sum_{j=1}^n c_{ij}^1 \tilde{w}_j - \frac{N^{RT}}{r_i} \sum_{j=1}^n c_{ij}^1 \tilde{w}_j + 3(e_0 a)^2 \frac{\partial N^{RT}}{\partial r_i} \sum_{j=1}^n c_{ij}^3 \tilde{w}_j \\ & + \frac{2(e_0 a)^2 N^{RT}}{r_i} \sum_{j=1}^n c_{ij}^3 \tilde{w}_j + 3(e_0 a)^2 \frac{\partial^2 N^{RT}}{\partial r_i^2} \sum_{j=1}^n c_{ij}^2 \tilde{w}_j + \frac{4(e_0 a)^2}{r_i} \frac{\partial N^{RT}}{\partial r_i} \sum_{j=1}^n c_{ij}^2 \tilde{w}_j + \frac{N^{RT} (e_0 a)^2}{r_i^2} \sum_{j=1}^n c_{ij}^2 \tilde{w}_j \\ & + (e_0 a)^2 \frac{\partial^3 N^{RT}}{\partial r_i^3} \sum_{j=1}^n c_{ij}^1 \tilde{w}_j + \frac{2(e_0 a)^2}{r_i} \frac{\partial^2 N^{RT}}{\partial r_i^2} \sum_{j=1}^n c_{ij}^1 \tilde{w}_j + \frac{(e_0 a)^2}{r_i} \frac{\partial N^{RT}}{\partial r_i} \sum_{j=1}^n c_{ij}^1 \tilde{w}_j - \frac{3N^{RT} (e_0 a)^2}{r_i^3} \sum_{j=1}^n c_{ij}^1 \tilde{w}_j \end{aligned} \right\} \quad (A3)$$

where $i = 2, \dots, N-1, N$ is the total number of sampling points on the radical of the annular plate, and c_{ij}^k is the j -th weighting coefficient of the k -th differential in the i equation, and the expressions of the correlation coefficients are as follows

$$c_{ij}^k = k \left[c_{ii}^{k-1} c_{ij}^1 - \frac{c_{ij}^{k-1}}{(r_i - r_j)} \right] \quad (i, j = 1, 2, \dots, N, i \neq j \text{ and } 2 \leq k \leq N-1) \quad (A4)$$

$$\left\{ \begin{array}{l} c_{ij}^1 = \frac{M(r_i)}{(r_i - r_j)M(r_j)} (i, j = 1, 2, \dots, N, i \neq j) \\ c_{ii}^k = - \sum_{j=1, i \neq j}^n c_{ij}^k (i = 1, 2, \dots, n, k = 1, 2, \dots, N-1) \\ M(r_i) = \prod_{j=1, i \neq j}^n (r_i - r_j) \quad (i = 1, 2, \dots, N) \\ x_i = r_1 + \frac{(r_2 - r_1)}{2} \left\{ 1 - \cos \left[\frac{\pi(i-1)}{N-1} \right] \right\} \quad (i = 1, 2, \dots, N) \end{array} \right. \quad (A5)$$

For example, the boundary conditions of the annular plate with inner and outer rings clamped are expressed as

$$\tilde{u}_1 = \tilde{\varphi}_1 = \tilde{w}_1 = 0, r = r_1 \quad (A6)$$

$$\tilde{u}_N = \tilde{\varphi}_N = \tilde{w}_N = 0, r = r_2 \quad (A7)$$

Define the modal vector as

$$\mathbf{d}^* = \{ \tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_N, \tilde{\varphi}_1, \tilde{\varphi}_2, \dots, \tilde{\varphi}_N, \tilde{w}_1, \tilde{w}_2, \dots, \tilde{w}_N \}^T \quad (A8)$$

The governing equations and boundary conditions can be expressed as a matrix

$$(\mathbf{K} - \Omega^2 \mathbf{M}) \mathbf{d}^* = 0 \quad (A9)$$

where $\mathbf{M}$ and $\mathbf{K}$ are the equivalent mass matrix and equivalent stiffness matrix composed of coefficients, respectively; Ω represents the vibration frequencies of the annular plate.

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