Article

Modeling and Stability Analysis for the Vibrating Motion of Three Degrees-of-Freedom Dynamical System Near Resonance

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Abstract: The focus of this article is on the investigation of a dynamical system consisting of a linear damped transverse tuned-absorber connected with a non-linear damped-spring-pendulum, in which its hanged point moves in an elliptic path. The regulating system of motion is derived using Lagrange's equations, which is then solved analytically up to the third approximation employing the approach of multiple scales (AMS). The emerging cases of resonance are categorized according to the solvability requirements wherein the modulation equations (ME) have been found. The stability areas and the instability ones are examined utilizing the Routh–Hurwitz criteria (RHC) and analyzed in line with the solutions at the steady state. The obtained results, resonance responses, and stability regions are addressed and graphically depicted to explore the positive influence of the various inputs of the physical parameters on the rheological behavior of the inspected system. The significance of the present work stems from its numerous applications in theoretical physics and engineering.

Keywords: perturbation techniques; non-linear dynamical systems; resonance; fixed points; steady state; stability

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1. Introduction

In the last two decades, some researchers have produced numerous works trying to solve the problems of excessive vibrations of mechanical systems, including the use of absorbers to treat and absorb active and the passive vibrations, e.g., [1–6].

The motion of a pendulum vibration absorber (PVA) with a spin base is investigated in [2] to deal with vertical excitation. By altering the rotational motion, the distinctive frequency of the pendulum absorber can be modified dynamically over a large range. A longitudinal absorber is used in [3] to stabilize and regulate vibrations of a spring pendulum, with non-linear stiffness, expressing ship roll motion. To achieve a semi-closed solution for the approximation from second order, the authors used the approach of multiple scales (AMS) [7], investigating the response of the considered model near resonance cases. They applied the influence of an additional transvers absorber to generalize this problem as in [4] and [5]. It is demonstrated in [6] how to autonomously modify the rotating speed of PVA, with two degrees of freedom (DOF), by identifying the phase between the PVA and primary vibrations. The pivot's motion of a simple pendulum with rigid arm that connected with a longitudinal absorber on an elliptic trajectory is examined in [8]. All resonance cases are generally grouped, and the case of two concurrent basic external resonances is examined. The generalization of this work is found in [9] for the case of a damped elastic pendulum instead of the un-stretched one. The ME are obtained and solved numerically to check the stability and instability regions in view of RHC.

Different trajectories for a pivot point of linear and non-linear elastic pendulums with various DOF are studied in numerous works, e.g., [10–19]. The case of fixed pivot

point of 3DOF rigid body pendulum (RBP) is examined in [10], in which three resonance cases are studied simultaneously from the perspective of the AMS. In [11,12], the authors dealt with a damped pendulum motion under the influence of various forces taking into account the movement of the pivot point on a circle with a stationary angular velocity. In [13–15], Lissajous paths are considered for the motion of the spring's suspension point as constrained motion. All possible fixed points are graphed in [13] and categorized in terms of being stable or not, while in [14], the authors verified the obtained analytic solutions for the elastic RBP through comparison with the numerical results. The approximate solutions and their stability regions for a non-linear damped RBP are examined in [15] using the AMS and RHC, respectively.

Many scientific works have recently studied the motion of the pendulum's suspension point on elliptic pathways for a variety of models such as [16–19]. The linear and non-linear dampings of a spring RBP on an elliptic route are inspected in [16,17], respectively. The double pendulum motion on an elliptic pathway was studied in [18]; one of them is a simple rigid pendulum while the other one is an elastically damped. The motion was investigated when a longitudinal harmonic force and a rotating moment were applied to the pendulum end point and to the suspension point, respectively. The effect of two external perpendicular forces besides an applied moment on a spring pendulum motion movement that follows the same route is examined in [19].

Two systems were considered in [20]: the first one is a subsystem that produces a swing motion, and the other is an absorber that moves linearly with respect to it, which can be used in various fields of engineering. The harmonic balance method (HBM) [7] is applied in [21] to examine the vibrational response of a connected mass with a damped oscillator, in addition to an excited pendulum hinged to this mass. The averaging approach is employed in [22] to obtain the analytic results of the 2DOF non-linear vibrating dynamical system consisting of a non-linear spring with attached damper. The authors concluded that the oscillation and amplitude can be reduced through adjusting the parameters of the system. Therefore, it can be used to improve the design of a non-linear absorber. The vibrating motion of an auto-parametric absorber (APA) connected with a Duffing oscillator was investigated numerically and experimentally in [23]. The impacts of a magneto-rheological (MR) damper and a shape storage alloy (SMA) spring on the absorption effect and regular solution stability were investigated in [24], while the author presented in [25] an analysis of the oscillation and energy harvesting of APA vibration. The analytical, computational, and experimental studies of a dynamical system containing an electromagnetic device are presented in [26], in which the analytic solutions were achieved using the HBM. Therefore, the author constructed another dynamic model in [27] to reduce the vibration and harvesting energy simultaneously, in which it is concluded that the initial resonance's area is the ideal area for suppression. The presence of solutions for a class of multi-point boundary value problems presented on the positive half-line for a fractional differential equation at resonance is explored in [28]. The asymptotic behavior of dynamical system trajectories caused by the algorithm and generalized known results when the metric space is bounded are examined in [29].

This study addresses the behavior of a 3DOF vibrating system consisting of a non-linear damped spring pendulum linked to an absorber with linear damper, which is exposed to the impact of a force in the spring's transverse direction. It is assumed that the whole motion is in a plane in which the spring's hanging point rotates in an elliptic path with steady angular velocity. The second kind of Lagrange's equations have been used to construct the equations of motion (EOM), which are then solved using the AMS up to the third approximations. The requirements of solvability were satisfied through the elimination of secular terms, and thereby, the ME were gained. The RHC was used to examine the fixed points of the stability analysis of solutions at the steady state. As a result, the non-linear stability was examined to determine the zones of stability and instability. The system's behavior is depicted through the examination of several plots relating to various curves of the solutions' time histories, resonance, and stability zones. This model

is significant since it is used in a variety of engineering applications such as construction, bridges, and spinning machines.

2. Description of the Vibrating System

A vibrational system with 3DOF is described in Figure 1, in which a damped elastic pendulum with normal length l_0 , non-linear stiffness K_1 and K_2 , and pendulum mass m_1 is considered. A pivot point O_1 of the pendulum is limited to move in an elliptic route with stationary angular velocity Ω , while the pendulum's other end is attached to a linear absorber of mass m_2 , a normal length l_{10} , and linear stiffness K_3 . According to the sketch of Figure 1, we may write the corresponding coordinates of O_1 to the axes OX and OY in the forms $a \cos(\Omega t)$ and $b \sin(\Omega t)$, respectively. Here, the ellipse's minor and major axes are represented by a and b , respectively. On the supplementary circle b , the equivalent point of O_1 will be assigned by Q .

Figure 1. The examined model.

The system's motion is addressed to be under the influence of applied harmonic force $F(t) = F_1 \cos(\Omega_1 t)$ at the spring's radial direction, as well as a harmonic moment $M(t) = M_0 \cos(\Omega_2 t)$ at O_1 in the anticlockwise direction. Here, Ω_1, Ω_2 and F_1, M_0 are the frequencies and amplitudes of $F(t)$ and $M(t)$. The extensions of the spring and absorber are supposed to be $Z(t)$ and $\xi(t)$, respectively. Furthermore, C_1, C_2 , and C_3 are thought to indicate the coefficients of viscous damping for the spring's longitudinal, swing oscillations, and the absorber's elongation, respectively.

The kinetic and potential energies T and V of the system can be expressed in the following ways

$$\begin{aligned}
T &= \frac{1}{2}(m_1 + m_2) \left\{ \Omega^2 [a^2 \sin^2(\Omega t) + b^2 \cos^2(\Omega t)] + Z'^2 + (l + Z)^2 \Phi'^2 \right. \\
&\quad + 2\Omega Z' [b \cos(\Omega t) \sin \Phi - a \sin(\Omega t) \cos \Phi] + 2\Omega(l + Z)\Phi' \\
&\quad \times [b \cos(\Omega t) \cos \Phi + a \sin(\Omega t) \sin \Phi] \left. + \frac{1}{2}m_2 \left\{ \zeta'^2 + (l_1 + \zeta)^2 \Phi'^2 \right. \right. \\
&\quad - 2\Omega(l_1 + \zeta)\Phi' [b \cos(\Omega t) \sin \Phi + a \sin(\Omega t) \cos \Phi] + 2\Omega\zeta' \\
&\quad \times [b \cos(\Omega t) \cos \Phi - a \sin(\Omega t) \sin \Phi] + 2Z'\zeta' \sin(2\Phi) + 2(l_1 + \zeta) \\
&\quad \times Z'\Phi' \cos(2\Phi) + 2(l + Z)\Phi'\zeta' \cos(2\Phi) - 2(l + Z)(l_1 + \zeta)\Phi'^2 \sin(2\Phi) \left. \right\}, \\
V &= \frac{1}{2}K_1(Z + Z_r)^2 + \frac{1}{3}K_2(Z + Z_r)^3 + \frac{1}{2}K_3(\zeta + \zeta_r)^2 + m_1 g(l + Z)(1 - \cos \Phi) \\
&\quad + m_2 g[(l + Z)(1 - \cos \Phi) + (l_1 + \zeta) \sin \Phi]; \\
l &= l_0 + Z_r, \quad l_1 = l_{10} + \zeta_r.
\end{aligned} \tag{1}$$

Here, Z_r and ζ_r are the spring's and the absorber's static elongations, respectively; the angle formed by the vertical at O_1 and the line passing across $O_1 m_1$ are denoted by Φ ; g is the acceleration of gravity, and the primes are the differentiation in relation to time t .

According to Equation (1), Lagrange's function $L = T - V$ can be determined, and then, the controlling system of motion can be obtained using the next equations of Lagrange [30]

$$\begin{aligned}
\frac{d}{dt} \left(\frac{\partial L}{\partial Z'} \right) - \frac{\partial L}{\partial Z} &= Q_Z, \\
\frac{d}{dt} \left(\frac{\partial L}{\partial \Phi'} \right) - \frac{\partial L}{\partial \Phi} &= Q_\Phi, \\
\frac{d}{dt} \left(\frac{\partial L}{\partial \zeta'} \right) - \frac{\partial L}{\partial \zeta} &= Q_\zeta.
\end{aligned} \tag{2}$$

Here, (Q_Z, Q_Φ, Q_ζ) are the system's non-conservative generalized forces, while (Z, Φ, ζ) and (Z', Φ', ζ') are the generalized coordinates and velocities, respectively. The forms of Q_Z , Q_Φ , and Q_ζ are

$$\begin{aligned}
Q_Z &= F_1 \cos(\Omega_1 t) - C_1 Z', \\
Q_\Phi &= M_0 \cos(\Omega_2 t) - C_2 \Phi', \\
Q_\zeta &= -C_3 \zeta'.
\end{aligned} \tag{3}$$

Substituting from (1) and (3) into (2), the dimensionless form of the controlling system of EOM has the following form

$$\begin{aligned}
\ddot{z} + c_1 \dot{z} &+ z + z_r + \alpha(z + z_r)^2 + (1 - \cos \Phi)w_1^2 - (1 + z)\dot{\Phi}^2 + \beta[(\ddot{\eta} - (r + \eta)\dot{\Phi}^2) \\
&\quad \times \sin(2\Phi) + ((r + \eta)\ddot{\Phi} + 2\dot{\Phi}\dot{\eta}) \cos(2\Phi)] - \frac{\omega_1^2}{l} [b \sin(\omega\tau) \sin \Phi \\
&\quad + a \cos(\omega\tau) \cos \Phi] = f_1 \cos(p_1 \tau), \\
\left\{ (1 + z)^2 \right. &+ \beta[(r + \eta)^2 - 2(1 + z)(r + \eta) \sin(2\Phi)] \left. \right\} \ddot{\Phi} + c_2 \dot{\Phi} + 2[1 + z \\
&\quad - 2\beta(r + \eta) \sin(2\Phi)] \dot{z} \dot{\Phi} + 2[(r + \eta) - (1 + z) \sin(2\Phi)] \beta \dot{\Phi} \dot{\eta} \\
&\quad + (1 + z)w_1^2 \sin \Phi + \beta(r + \eta)w_1^2 \cos \Phi + \beta[(r + \eta)\ddot{z} + (1 + z)\ddot{\eta} \\
&\quad - 2(r + \eta)(1 + z)\dot{\Phi}^2] \cos(2\Phi) - \frac{\omega_2^2}{l} (1 + z) [b \sin(\omega\tau) \cos \Phi - a \cos(\omega\tau) \\
&\quad \times \sin \Phi] + \frac{\omega_2^2}{l} \beta(r + \eta) [b \sin(\omega\tau) \sin \Phi - a \cos(\omega\tau) \cos \Phi] = m_0 \cos(p_2 \tau), \\
\ddot{\eta} + \ddot{z} \sin(2\Phi) &+ c_3 \dot{\eta} - \frac{\omega_2^2}{l} [b \sin(\omega\tau) \cos \Phi + a \cos(\omega\tau) \sin \Phi] - [(1 + z) \sin(2\Phi) \\
&\quad + r + \eta] \dot{\Phi}^2 + [2\dot{z} \dot{\Phi} + (1 + z)\ddot{\Phi}] \cos(2\Phi) + w_1^2 \sin \Phi + w_2^2(\eta + \eta_r) = 0,
\end{aligned} \tag{4}$$

where

$$\begin{aligned}
\tau &= \omega_1 t, \quad z = \frac{Z}{l}, \quad z_r = \frac{Z_r}{l}, \quad \eta = \frac{\zeta}{l}, \quad \eta_r = \frac{\zeta_r}{l}, \quad \omega_1 = \sqrt{\frac{K_1}{m_1 + m_2}}, \quad \omega_2 = \sqrt{\frac{g}{l}}, \\
\omega_3 &= \sqrt{\frac{K_3}{m_2}}, \quad w_1 = \frac{\omega_2}{\omega_1}, \quad w_2 = \frac{\omega_3}{\omega_1}, \quad \omega = \frac{\Omega}{\omega_1}, \quad \beta = \frac{m_2}{m_1 + m_2}, \quad r = \frac{l_1}{l}, \\
\alpha &= \frac{K_2 l}{\omega_1^2 (m_1 + m_2)}, \quad c_1 = \frac{C_1}{(m_1 + m_2)\omega_1}, \quad c_2 = \frac{C_2}{(m_1 + m_2)l^2 \omega_1}, \quad c_3 = \frac{C_3}{m_2 \omega_1}, \\
f_1 &= \frac{F_1}{(m_1 + m_2)l\omega_1^2}, \quad m_0 = \frac{M_0}{(m_1 + m_2)l^2 \omega_1^2}, \quad p_1 = \frac{\Omega_1}{\omega_1}, \quad p_2 = \frac{\Omega_2}{\omega_1},
\end{aligned}$$

Denote the parameters that are dimensionless, and the dots represent the differentiation regarding to τ , in which the generalized coordinates and their corresponding first derivatives have the following initial conditions

$$\begin{aligned} z(0) &= z_0, & \Phi(0) &= \Phi_0, & \eta(0) &= \eta_0, \\ \dot{z}(0) &= 0, & \dot{\Phi}(0) &= 0, & \dot{\eta}(0) &= 0. \end{aligned}$$

3. The Desired Solutions

In this part of the present research work, we use the AMS to acquire the approximate analytic solutions of the EOM (4), categorize the different cases of resonance, and extract both the solvability criteria and the ME. Then, we look at the oscillations of the system close to the static equilibrium position [31]. To accomplish this target, we approximate the functions $\sin \Phi$ and $\cos \Phi$ up to the third-order as follows

$$\sin \Phi = \Phi - \frac{\Phi^3}{6}, \quad \cos \Phi = 1 - \frac{\Phi^2}{2}. \quad (5)$$

The damping coefficients, force's amplitudes, moment, elliptical semi-axes, and other parameters can then be represented in terms of a small parameter $0 < \varepsilon \ll 1$ as follows

$$\begin{aligned} c_1 &= \varepsilon^2 \tilde{c}_1, & c_2 &= \varepsilon^2 \tilde{c}_2, & c_3 &= \varepsilon^2 \tilde{c}_3, & \beta &= \varepsilon \tilde{\beta}, \\ f_1 &= \varepsilon^3 \tilde{f}_1, & m_0 &= \varepsilon^3 \tilde{m}_0, & a &= \varepsilon^2 \tilde{a}, & b &= \varepsilon^2 \tilde{b}, & \alpha &= \varepsilon^2 \tilde{\alpha}. \end{aligned} \quad (6)$$

As a result, we can express the functions z , Φ , and η in terms of ε and the new functions $\tilde{z}$, $\tilde{\Phi}$, and $\tilde{\eta}$ as follows

$$z = \varepsilon \tilde{z}, \quad \Phi = \varepsilon \tilde{\Phi}, \quad \eta = \varepsilon \tilde{\eta}. \quad (7)$$

According to the procedure of AMS, we can write these functions as follows

$$\begin{aligned} \tilde{z}(\tau, \varepsilon) &= \sum_{i=1}^3 \varepsilon^{i-1} \tilde{\chi}_i(\tau_0, \tau_1, \tau_2) + O(\varepsilon^3), \\ \tilde{\Phi}(\tau, \varepsilon) &= \sum_{i=1}^3 \varepsilon^{i-1} \tilde{\phi}_i(\tau_0, \tau_1, \tau_2) + O(\varepsilon^3), \\ \tilde{\eta}(\tau, \varepsilon) &= \sum_{i=1}^3 \varepsilon^{i-1} \tilde{\gamma}_i(\tau_0, \tau_1, \tau_2) + O(\varepsilon^3), \end{aligned} \quad (8)$$

where $\tau_n = \varepsilon^n \tau$ ($n = 0, 1, 2$) denote new time dependent scales on τ , in which τ_0 and τ_k ($k = 1, 2$) denote a fast and slow time scales, respectively. Because of the smallness of ε , the orders ε^3 and higher have been excluded.

In light of the supposed solutions (8), we need to transform the time derivatives regarding τ into additional ones with respect to the scales τ_0, τ_1 , and τ_2 . Therefore, we consider the following differential operators

$$\begin{aligned} \frac{d}{d\tau} &= D_0 + \varepsilon D_1 + \varepsilon^2 D_2, \\ \frac{d^2}{d\tau^2} &= D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2 (D_1^2 + 2D_0 D_2), \end{aligned} \quad (9)$$

where $D_n = \frac{\partial}{\partial \tau_n}$; $n = 1, 2$.

When (5)–(9) are substituted into (4), a family of partial differential equations (PDE) regarding ε arises. Equating the coefficients of the like powers of ε on both sides of each one of the families of PDE to gain the below groups of PDE gives as a result

Order of ε

$$\begin{aligned} D_0^2 \tilde{\chi}_1 + \tilde{\chi}_1 &= 0, \\ D_0^2 \tilde{\phi}_1 + w_1^2 \tilde{\phi}_1 &= -r \tilde{\beta} w_1^2, \\ D_0^2 \tilde{\gamma}_1 + w_2^2 \tilde{\gamma}_1 &= -(D_0^2 \tilde{\phi}_1 + w_1^2 \tilde{\phi}_1), \end{aligned} \quad (10)$$

Order of ε^2

$$\begin{aligned}
D_0^2 \tilde{\chi}_2 + \tilde{\chi}_2 &= \frac{\tilde{a}\omega^2}{\tilde{t}} \cos(\omega\tau_0) - \frac{w_1^2}{2} \tilde{\phi}_1^2 + (D_0 \tilde{\phi}_1)^2 - 2D_0 D_1 \tilde{\chi}_1 - r\tilde{\beta} D_0^2 \tilde{\phi}_1 - \tilde{a}z_r^2, \\
D_0^2 \tilde{\phi}_2 + w_1^2 \tilde{\phi}_2 &= \frac{\tilde{b}\omega^2}{\tilde{t}} \sin(\omega\tau_0) - \tilde{\chi}_1 (w_1^2 \tilde{\phi}_1 + 2D_0^2 \tilde{\phi}_1) - 2(D_0 \tilde{\chi}_1 D_0 \tilde{\phi}_1 + D_0 D_1 \tilde{\phi}_1) \\
&\quad - \tilde{\beta} [(w_1^2 \tilde{\gamma}_1 + D_0^2 \tilde{\gamma}_1) + r(D_0 \tilde{\chi}_1 + rD_0^2 \tilde{\phi}_1)], \\
D_0^2 \tilde{\gamma}_2 + w_2^2 \tilde{\gamma}_2 &= \frac{\tilde{b}\omega^2}{\tilde{t}} \sin(\omega\tau_0) - (D_0^2 \tilde{\phi}_2 + w_1^2 \tilde{\phi}_2 + \tilde{\chi}_1 D_0^2 \tilde{\phi}_1) - D_0 \tilde{\phi}_1 (2D_0 \tilde{\chi}_1 - rD_0 \tilde{\phi}_1) \\
&\quad - 2(D_0 D_1 \tilde{\gamma}_1 + D_0 D_1 \tilde{\phi}_1 + \tilde{\phi}_1 D_0^2 \tilde{\chi}_1),
\end{aligned} \tag{11}$$

Order of ε^3

$$\begin{aligned}
D_0^2 \tilde{\chi}_3 + \tilde{\chi}_3 &= \tilde{f}_1 \cos(p_1 \tau_0) - \tilde{c}_1 D_0 \tilde{\chi}_1 + \tilde{\chi}_1 [(D_0^2 \tilde{\phi}_1)^2 - 2\tilde{a}z_r] + \tilde{\phi}_1 [\frac{\tilde{b}}{\tilde{t}} \omega^2 \sin(\omega\tau_0) \\
&\quad - w_1^2 \tilde{\phi}_2 - 2\tilde{\beta} D_0^2 \tilde{\gamma}_1] - D_1^2 \tilde{\chi}_1 + 2D_0 \tilde{\phi}_1 (D_0 \tilde{\phi}_2 + D_1 \tilde{\phi}_1 - \tilde{\beta} D_0 \tilde{\gamma}_1) \\
&\quad - 2(D_0 D_2 \tilde{\chi}_1 + D_0 D_1 \tilde{\chi}_2 + r\tilde{\beta} D_0 D_1 \tilde{\phi}_1) - \tilde{\beta} (\tilde{\gamma}_1 D_0^2 \tilde{\phi}_1 + rD_0^2 \tilde{\phi}_2), \\
D_0^2 \frac{\partial^2 \tilde{\phi}_3}{\partial \tau_0^2} + w_1^2 \tilde{\phi}_3 &= \tilde{m}_0 \cos(p_2 \tau_0) + \frac{\tilde{a}}{\tilde{t}} \omega^2 \cos(\omega\tau_0) (r\tilde{\beta} - \tilde{\phi}_1) - \tilde{c}_2 D_0 \tilde{\phi}_1 + \tilde{\chi}_1 [\frac{\tilde{b}}{\tilde{t}} \omega^2 \\
&\quad \times \sin(\omega\tau_0) - w_1^2 \tilde{\phi}_2 - 2D_0 \tilde{\chi}_1 D_0 \tilde{\phi}_1 - 4D_0 D_1 \tilde{\phi}_1 - \tilde{\beta} D_0^2 \tilde{\gamma}_1 - \tilde{\chi}_1 D_0^2 \tilde{\phi}_1 \\
&\quad - 2D_0^2 \tilde{\phi}_2] - \tilde{\phi}_1 (w_1^2 \tilde{\chi}_2 - \frac{1}{2} r\tilde{\beta} w_1^2 \tilde{\phi}_1 - \frac{1}{6} w_1^2 \tilde{\phi}_1^2 - 4r\tilde{\beta} D_0^2 \tilde{\phi}_1) - \tilde{\gamma}_1 (\tilde{\beta} D_0^2 \tilde{\chi}_1 \\
&\quad + 2r\tilde{\beta} D_0^2 \tilde{\phi}_1) - \tilde{\beta} w_1^2 \tilde{\gamma}_2 - D_1^2 \tilde{\phi}_1 - 2D_0 \tilde{\chi}_1 (D_1 \tilde{\phi}_1 + D_0 \tilde{\phi}_2) - 2D_0 \tilde{\phi}_1 (D_1 \tilde{\chi}_1 \\
&\quad + r\tilde{\beta} D_0 \tilde{\gamma}_1 + D_0 \tilde{\chi}_2 - r\tilde{\beta} D_0 \tilde{\phi}_1) - 2(D_0 D_2 \tilde{\phi}_1 + \tilde{\beta} D_0 D_1 \tilde{\gamma}_1 + r\tilde{\beta} D_0 D_1 \tilde{\chi}_1 \\
&\quad + r^2 \tilde{\beta} D_0 D_1 \tilde{\phi}_1 + D_0 D_1 \tilde{\phi}_2 + \tilde{\chi}_2 D_0^2 \tilde{\phi}_1) - \tilde{\beta} (D_0^2 \tilde{\gamma}_2 + rD_0^2 \tilde{\chi}_2 + r^2 D_0^2 \tilde{\phi}_2), \\
D_0^2 \tilde{\gamma}_3 + w_2^2 \tilde{\gamma}_3 &= \tilde{\phi}_1 [\frac{\tilde{a}}{\tilde{t}} \omega^2 \cos(\omega\tau_0) + \frac{1}{6} w_1^2 \tilde{\phi}_1^2 + 2(D_0 \tilde{\phi}_1)^2 - 4D_0 D_1 \tilde{\chi}_1 - 2D_0^2 \tilde{\chi}_2 \\
&\quad + 2\tilde{\phi}_1 D_0^2 \tilde{\phi}_1] - \tilde{\chi}_1 (2D_0 D_1 \tilde{\phi}_1 + D_0^2 \tilde{\phi}_2) - \tilde{c}_3 D_0 \tilde{\gamma}_1 - (D_0^2 \tilde{\phi}_3 + w_1^2 \tilde{\phi}_3) \\
&\quad - 2D_0 \tilde{\phi}_1 (D_1 \tilde{\chi}_1 - rD_1 \tilde{\phi}_1 + D_0 \tilde{\chi}_2 - \frac{1}{2} \tilde{\gamma}_1 D_0 \tilde{\phi}_1 - rD_0 \tilde{\phi}_2) - 2D_0 \tilde{\chi}_1 (D_1 \tilde{\phi}_1 \\
&\quad + D_0 \tilde{\phi}_2) - [D_0^2 \tilde{\gamma}_1 + (1 + \tilde{\chi}_2) D_0^2 \tilde{\phi}_1] - 2(D_0 D_2 \tilde{\gamma}_1 + D_0 D_2 \tilde{\phi}_1 + D_0 D_1 \tilde{\gamma}_2 \\
&\quad + D_0 D_1 \tilde{\phi}_2 + \tilde{\phi}_2 D_0^2 \tilde{\chi}_1).
\end{aligned} \tag{12}$$

Based on the preceding groups of Equations (10)–(12), we can solve them sequentially. Accordingly, we will start with the general solutions of the system of Equation (10) which take the following forms

$$\begin{aligned}
\tilde{\chi}_1 &= B_1 e^{i\tau_0} + \bar{B}_1 e^{-i\tau_0}, \\
\tilde{\phi}_1 &= B_2 e^{iw_1 \tau_0} + \bar{B}_2 e^{-iw_1 \tau_0} - r\tilde{\beta}, \\
\tilde{\gamma}_1 &= B_3 e^{iw_2 \tau_0} + \bar{B}_3 e^{-iw_2 \tau_0} + \frac{w_2^2}{w_1^2} r\tilde{\beta}; \quad i = \sqrt{-1}.
\end{aligned} \tag{13}$$

Here, B_j ($j = 1, 2, 3$) represent functions of τ_k ($k = 1, 2$) and $\bar{B}_j$ denote their complex conjugates.

Making use of the above solutions (13) into the second group of PDE (11) yields secular terms, the removal of which demands that

$$\begin{aligned}
D_1 B_1 &= 0, \\
D_1 B_2 &= -\frac{i\tilde{\beta}}{2} r^2 w_1 B_2, \\
D_1 B_3 &= \frac{i\tilde{\beta}}{2w_2} (w_2^2 - w_1^2) B_3.
\end{aligned} \tag{14}$$

As a result, the second-order solutions are as follows:

$$\begin{aligned}
\tilde{\chi}_2 &= \frac{\omega^2 \tilde{a}}{2l(1-\omega^2)} e^{i\omega\tau_0} - \frac{3w_1^2 B_2^2}{2(1-4w_1^2)} e^{2iw_1 \tau_0} + \frac{2r\tilde{\beta}}{1-w_1^2} w_1^2 B_2 e^{iw_1 \tau_0} - (\frac{1}{2} r^2 \tilde{\beta}^2 w_1^2 + \tilde{a}z_r^2 \\
&\quad - w_1^2 B_2 \bar{B}_2) + c.c., \\
\tilde{\phi}_2 &= \frac{2-w_1}{1-2w_1} w_1 B_1 \bar{B}_2 e^{i(1-w_1)\tau_0} - \frac{i\omega^2 \tilde{b}}{2l(w_1^2 - \omega^2)} e^{i\omega\tau_0} - \frac{1+w_1^2}{1-w_1^2} r\tilde{\beta} B_1 e^{i\tau_0} \\
&\quad - \frac{2+w_1}{1+2w_1} w_1 B_1 B_2 e^{i(1+w_1)\tau_0} - \tilde{\beta} B_3 e^{iw_2 \tau_0} - \frac{w_1^2}{w_2^2} r\tilde{\beta}^2 + c.c., \\
\tilde{\gamma}_2 &= \frac{i\omega^2 \tilde{b}}{2l(w_2^2 - \omega^2)} e^{i\omega\tau_0} - \frac{rw_1^2 B_2^2}{(w_2^2 - 4w_1^2)} e^{2iw_1 \tau_0} + \frac{2B_1 \bar{B}_2}{w_2^2 - (1-w_1)^2} e^{i(1-w_1)\tau_0} \\
&\quad + \frac{2B_1 B_2}{w_2^2 - (1+w_1)^2} e^{i(1+w_1)\tau_0} + \frac{3+w_1^2}{1-w_2^2} r\tilde{\beta} B_1 e^{i\tau_0} - \frac{r^2 \tilde{\beta} w_1^2}{w_2^2 - w_1^2} B_2 e^{iw_1 \tau_0} \\
&\quad + \frac{w_1^4}{w_2^4} r\tilde{\beta}^2 + \frac{2}{w_2^2} rw_1^2 B_2 \bar{B}_2 + c.c.
\end{aligned} \tag{15}$$

Here, *c.c.* signifies the complex conjugate of the prior terms.

Substituting (13)–(15) into the third group of PDE (12) and removing terms that produce the secular one to gain the requirements of solvability of the third order of approximation results as follows

$$\begin{aligned}
 2iD_2B_1 & - \left[\frac{(1+w_1^2)^2}{(w_1^2-1)} r^2 \tilde{\beta}^2 - i\tilde{c}_1 - 2\tilde{\alpha}z_r - \frac{6(1-w_1^2)}{(1-4w_1^2)} w_1^2 B_2 \bar{B}_2 \right] B_1 = 0, \\
 2iw_1 D_2 B_2 & + w_1 \left\{ i\tilde{c}_2 + \tilde{\alpha}z_r^2 w_1 - \frac{(1-w_1^2)w_1}{2(1-4w_1^2)} [(1+8w_1^2)B_2 \bar{B}_2 - 12B_1 \bar{B}_1] \right. \\
 & \quad \left. - \frac{r^2 \tilde{\beta}^2 w_1}{4(1-w_1^2)w_2^2} [(14-3r^2)w_2^2 - 2(4-w_2^2)w_1^4 + (8+3r^2w_2^2)w_1^2] \right\} B_2 = 0, \\
 2iw_2 D_2 B_3 & - \left\{ \tilde{\beta}^2 (w_1^2 - w_2^2) \left[1 + \frac{1}{4w_2^2} (w_1^2 - w_2^2) \right] - i\tilde{c}_3 w_2 + r^2 \tilde{\beta}^2 w_2^2 + 2w_1^2 B_2 \bar{B}_2 \right\} B_3 = 0.
 \end{aligned} \tag{16}$$

Based on the foregoing, we can phrase the third-order solutions as in the forms

$$\begin{aligned}
 \tilde{\chi}_3 & = \frac{\tilde{f}_1}{2(1-p_1^2)} e^{ip_1 \tau_0} + \frac{ir\tilde{\beta}\tilde{\omega}^4}{2l(\omega^2-w_1^2)(1-\omega^2)} e^{i\omega \tau_0} - q_1 B_2 e^{i(w_1+\omega)\tau_0} + q_2 B_2 e^{i(w_1-\omega)\tau_0} \\
 & - q_3 B_2^2 e^{2iw_1 \tau_0} + q_4 B_1 B_2 e^{i(1+w_1)\tau_0} - \frac{3(1+w_1)}{4(1+2w_1)} w_1 B_1 B_2^2 e^{i(1+2w_1)\tau_0} - q_5 B_1 \bar{B}_2 e^{i(1-w_1)\tau_0} \\
 & - r^2 \tilde{\beta} w_1^2 \left(\frac{w_1^2}{w_2^2} \tilde{\beta}^2 + 2B_2 \bar{B}_2 \right) + \frac{3(1-w_1)}{4(1-2w_1)} w_1 B_1 \bar{B}_2^2 e^{i(1-2w_1)\tau_0} - \frac{r\tilde{\beta}^2}{(1-w_2^2)} (w_1^2 + 3w_2^2) \\
 & \times B_3 e^{iw_2 \tau_0} + q_6 B_2 B_3 e^{i(w_1+w_2)\tau_0} + q_7 B_2 \bar{B}_3 e^{i(w_1-w_2)\tau_0} - q_8 B_2 e^{iw_1 \tau_0} + c.c., \\
 \tilde{\phi}_3 & = \frac{\tilde{m}_0}{2(\omega_2^2-p_2^2)} e^{ip_2 \tau_0} - q_9 e^{i\omega \tau_0} + q_{10} B_1 e^{i(1+\omega)\tau_0} - q_{11} B_1 e^{i(1-\omega)\tau_0} + q_{12} B_1 e^{i\tau_0} \\
 & + \frac{1+w_1^2}{1-w_1^2} r\tilde{\beta} B_1^2 e^{2i\tau_0} + q_{13} B_2 e^{i(w_1+\omega)\tau_0} + q_{14} B_2 e^{i(w_1-\omega)\tau_0} + q_{15} B_2^2 e^{2iw_1 \tau_0} \\
 & + \frac{w_1}{4(1+2w_1)} (6+5w_1+w_1^2) B_1^2 B_2 e^{i(2+w_1)\tau_0} - \frac{(1-49w_1^2)}{48(1-4w_1^2)} B_2^3 e^{3iw_1 \tau_0} \\
 & + q_{16} B_1 B_3 e^{i(1+w_2)\tau_0} - q_{17} B_2 B_3 e^{i(w_1+w_2)\tau_0} - \frac{r^2 \tilde{\beta}^2 w_2^2}{w_1^2 - w_2^2} B_3 e^{iw_2 \tau_0} + q_{18} B_1 \bar{B}_3 e^{i(1-w_2)\tau_0} \\
 & - \frac{w_1 B_1^2 \bar{B}_2}{4(1-2w_1)} (6-5w_1+w_1^2) e^{i(2-w_1)\tau_0} + \frac{2(w_1-w_2)w_1}{(2w_1-w_2)w_2} r\tilde{\beta} \bar{B}_2 \bar{B}_3 e^{i(w_1-w_2)\tau_0} \\
 & - q_{19} + q_{20} B_1 B_2 e^{i(1+w_1)\tau_0} - q_{21} B_1 \bar{B}_2 e^{i(1-w_1)\tau_0} + c.c., \\
 \tilde{\gamma}_3 & = \frac{i\tilde{\omega}^2 B_1}{l(\omega^2-w_1^2)[w_2^2-(1+\omega)^2]} e^{i(1+\omega)\tau_0} - \frac{\tilde{m}_0}{2(\omega_2^2-p_2^2)} e^{ip_2 \tau_0} - q_{22} e^{i\omega \tau_0} \\
 & - \frac{i\tilde{\omega}^2 B_1}{l(\omega^2-w_1^2)[w_2^2-(1-\omega)^2]} e^{i(1-\omega)\tau_0} + \frac{r\tilde{\beta}(1+w_1^2)^2 A_1^2}{(w_1^2-1)(w_2^2-4)} e^{2i\tau_0} - q_{23} \bar{B}_2 e^{i(w_1+\omega)\tau_0} \\
 & - q_{24} B_2 e^{i(w_1-\omega)\tau_0} - \frac{2(2+w_1)w_1 B_1^2 B_2}{(1+2w_1)[w_2^2-(2+w_1)^2]} e^{i(2+w_1)\tau_0} + q_{25} B_1 B_2^2 e^{i(1+2w_1)\tau_0} \\
 & - q_{26} B_2^3 e^{3iw_1 \tau_0} + \frac{3+w_1^2}{1+2w_2} \tilde{\beta} B_1 B_3 e^{i(1+w_2)\tau_0} + \frac{2r\tilde{\beta} w_1}{w_1+2w_2} B_2 B_3 e^{i(w_1+w_2)\tau_0} \\
 & - q_{27} \bar{B}_1 B_2^2 e^{i(2w_1-1)\tau_0} - \frac{3+w_1^2}{2w_2-1} \tilde{\beta} \bar{B}_1 B_3 e^{i(w_2-1)\tau_0} + \frac{w_1 B_2^2 B_3}{4(w_1+w_2)} e^{i(2w_1+w_2)\tau_0} \\
 & + \frac{2(2-w_1)w_1 \bar{B}_1^2 B_2}{(1-2w_1)[w_2^2-(2-w_1)^2]} e^{i(-2+w_1)\tau_0} + \frac{2r\tilde{\beta} w_1}{w_1+2w_2} \bar{B}_2 B_3 e^{i(w_2-w_1)\tau_0} \\
 & + \frac{w_1 \bar{B}_2^2 B_3}{4(w_1-w_2)} e^{i(w_2-2w_1)\tau_0} + q_{28} B_1 e^{i\tau_0} + q_{29} B_1 B_2 e^{i(1+w_1)\tau_0} + q_{30} B_2^2 e^{2iw_1 \tau_0} \\
 & + q_{31} B_2 e^{iw_1 \tau_0} + q_{32} \bar{B}_1 B_2 e^{i(-1+w_1)\tau_0} + q_{33} + c.c.,
 \end{aligned} \tag{17}$$

where q_s ($s = 1, 2, 3, \dots, 33$) are given in Appendix A.

In light of the removal conditions (14) and (16) of secular terms, we can estimate the functions B_j ($j = 1, 2, 3$). We can easily acquire the desired approximate analytical expressions of z , Φ , and η up to the third approximation in view of the uses of the hypothesis (7), solutions (8), and the attained solutions (13), (15), and (17).

4. Resonance Categorizes and Modulation Equations (ME)

In this section, we look at how to categorize the various cases of resonance based on the aforementioned solutions, which are legitimate as long as their dominators are not zero [8]. As these dominators go closer to zero, resonance cases emerge. As a result, these cases can be categorized into the fundamental external case of resonance which is met at $p_1 = 1$, $p_2 = w_1$, $p_2 = w_2$; the internal case of resonance which is found at $w_1 = 1$, $w_2 = 1$, $\omega = 1$, $w_1 = 2$, $2w_1 = 1$, $w_2 = 2$, $\omega = w_1$, $\omega = w_2$, $w_1 = w_2$, $3w_1 = w_2$, $2w_1 = w_2$, $w_1 = 0$, $w_2 = 0$, $\omega = 0$; and the combined resonance case which is encountered

at $w_1 \pm \omega = 1$, $w_1 \pm w_2 = 1$, $w_2 \pm \omega = 1$, $w_2 \pm \omega = w_1$, $\pm(w_2 - w_1) = 1$, $\pm(w_2 - w_1) = 2$, $\pm(w_2 - 2w_1) = 1$.

It should be emphasized that if any of the prior resonance cases is achieved, the behavior of the examined system would be difficult. As a result, the methods employed would have to be modified.

We will look at two fundamental external resonances and one internal resonance that are carried out simultaneously to handle this situation. As a consequence, we take into account the occurrence of all three of the following cases at the same time:

$$p_1 \approx 1, \quad p_2 \approx w_1, \quad 3w_1 \approx w_2. \quad (18)$$

These relations (18) indicate how close p_1 , p_2 , and $3w_1$ are to 1, w_1 , and w_2 , respectively. To do this, the dimensionless values defined by the parameters σ_j ($j = 1, 2, 3$) of detuning (which characterize the distance between the oscillations and the rigorous resonance) can be inserted as follows:

$$\begin{aligned} p_1 &= 1 + \sigma_1, \\ p_2 &= w_1 + \sigma_2, \\ 3w_1 &= w_2 + \sigma_3. \end{aligned} \quad (19)$$

Therefore, the order of σ_j can thus be inferred as follows:

$$\sigma_1 = \varepsilon \tilde{\sigma}_1, \quad \sigma_2 = \varepsilon \tilde{\sigma}_2, \quad \sigma_3 = \varepsilon \tilde{\sigma}_3. \quad (20)$$

Substituting (19) and (20) into (11) and (12), and then removing the generated secular terms, we get the relevant solvability requirements that are based on the approximated equations as follows:

$$\begin{aligned} D_1 B_1 &= 0, \\ D_1 B_2 &= -\frac{i}{2} r^2 \tilde{\beta} w_1 B_2, \\ D_1 B_3 &= -\frac{i \tilde{\beta}}{2 w_2} (w_1^2 - w_2^2) B_3, \\ 2i D_2 B_1 &= \frac{\tilde{f}_1}{2} e^{i \tilde{\sigma}_1 \tau_1} - \left[\frac{(1+w_1^2)^2}{(1-w_1^2)} r^2 \tilde{\beta}^2 + i \tilde{c}_1 + 2 \tilde{\alpha} z_r + \frac{6(1-w_1^2)}{(1-4w_1^2)} w_1^2 B_2 \bar{B}_2 \right] B_1, \\ 2i D_2 B_2 &= \left\{ \frac{r^2 \tilde{\beta}^2 w_1}{4(1-w_1^2)w_2^2} [(14-3r^2)w_2^2 - 2(4-w_2^2)w_1^4 + (8+3r^2w_2^2)w_1^2] \right\} B_2 \\ &\quad - i \tilde{c}_2 - \tilde{\alpha} z_r^2 w_1 - \frac{(1-w_1^2)w_1}{2(1-4w_1^2)} [12B_1 \bar{B}_1 - (1+8w_1^2)B_2 \bar{B}_2] + \frac{\tilde{m}_0}{2} e^{i \tilde{\sigma}_2 \tau_1}, \\ 2i w_2 D_2 B_3 &= \left\{ \tilde{\beta}^2 (w_1^2 - w_2^2) \left[1 + \frac{1}{4w_2^2} (w_1^2 - w_2^2) \right] - i \tilde{c}_3 w_2 + r^2 \tilde{\beta}^2 w_2^2 + 2w_1^2 B_2 \bar{B}_2 \right\} B_3 \\ &\quad - \frac{4w_1^2}{1-4w_1^2} (1-w_1^2) B_2^3 e^{i \tilde{\sigma}_3 \tau_1}. \end{aligned} \quad (21)$$

According to a careful inspection of the foregoing solvability conditions, we have a system of six non-linear PDE in terms of the functions B_j ($j = 1, 2, 3$) that are dependent on the slow scales τ_k ($k = 1, 2$). Then, we can introduce the following polar form of these functions:

$$B_j = \frac{\tilde{a}_j(\tau_1, \tau_2)}{2} e^{i \psi_j(\tau_1, \tau_2)}, \quad a_j = \varepsilon \tilde{a}_j \quad (j = 1, 2, 3), \quad (22)$$

where ψ_j and $\tilde{a}_j$ denote real functions of the phases and their amplitudes of $\tilde{z}$, $\tilde{\Phi}$, and $\tilde{\eta}$.

Based on the above analysis, the first-order derivative of operators B_j ($j = 1, 2, 3$) can be stated as follows:

$$\frac{dB_j}{d\tau} = \varepsilon D_1 B_j + \varepsilon^2 D_2 B_j. \quad (23)$$

Therefore, we can convert the PDE (21) into ordinary differential equations (ODE) according to the uses of (22), (23), and the next modified phases,

$$\begin{aligned}\theta_1(\tau_1, \tau_2) &= \tilde{\sigma}_1 \tau_1 - \psi_1(\tau_1, \tau_2), \\ \theta_2(\tau_1, \tau_2) &= \tilde{\sigma}_2 \tau_1 - \psi_2(\tau_1, \tau_2), \\ \theta_3(\tau_1, \tau_2) &= \tilde{\sigma}_3 \tau_1 + 3\psi_2(\tau_1, \tau_2) - \psi_3(\tau_1, \tau_2),\end{aligned}\quad (24)$$

into the requirement of solvability (21). Partitioning the real parts and the imaginary ones yields the next system of six first-order ODE in terms of a_j and θ_j ($j = 1, 2, 3$)

$$\begin{aligned}\frac{da_1}{d\tau} &= \frac{f_1}{2} \sin \theta_1 - \frac{1}{2} c_1 a_1, \\ a_1 \frac{d\theta_1}{d\tau} &= a_1 \left[\sigma_1 - \alpha z_r - \frac{(1+w_1^2)^2}{2(1-w_1^2)} r^2 \beta^2 - \frac{3(1-w_1^2)}{4(1-4w_1^2)} w_1^2 a_2^2 \right] + \frac{f_1}{2} \cos \theta_1, \\ \frac{da_2}{d\tau} &= \frac{m_0}{2w_1} \sin \theta_2 - \frac{1}{2} c_2 a_2, \\ a_2 \frac{d\theta_2}{d\tau} &= a_2 \left\{ \sigma_2 - w_1 [\alpha z_r^2 - r^2 \beta + \frac{(w_1^2-1)}{8(4w_1^2-1)} [12a_1^2 - (1+8w_1^2)a_2^2] \right. \\ &\quad \left. - \frac{r^2 \beta^2}{4(1-w_1^2)w_2^2} (w_2^2(14-3r^2) - 2(4-w_2^2)w_1^4 + (8+3r^2w_2^2)w_1^2) \right\} + \frac{m_0}{2w_1} \cos \theta_2, \\ \frac{da_3}{d\tau} &= -\frac{1}{2} \left[\frac{(1-w_1^2)w_1^2}{(1-4w_1^2)w_2} a_2^3 \sin \theta_3 + c_3 a_3 \right], \\ a_3 \frac{d\theta_3}{d\tau} &= a_3 \left\{ \sigma_3 + 3w_1 \left[\alpha z_r^2 - r^2 \beta + \frac{(1-w_1^2)}{8(1-4w_1^2)} [12a_1^2 - (1+8w_1^2)a_2^2] \right. \right. \\ &\quad \left. \left. - \frac{r^2 \beta^2}{4(1-w_1^2)w_2^2} [w_2^2(14-3r^2) - 2(4-w_2^2)w_1^4 + (8+3r^2w_2^2)w_1^2] \right] \right. \\ &\quad \left. - \frac{3m_0}{2w_1 a_2} \cos \theta_2 + \frac{1}{2w_2} [\beta(w_1^2 - w_2^2) + \frac{1}{2} w_1^2 a_2^2 + \frac{\beta^2}{4w_2^2} (4r^2 w_2^4 + 4w_2^2(w_1^2 - w_2^2) \right. \right. \\ &\quad \left. \left. + (w_1^2 - w_2^2)^2) \right] \right\} - \frac{(1-w_1^2)w_1^2}{2(1-4w_1^2)w_2} a_2^3 \cos \theta_3.\end{aligned}\quad (25)$$

This system reveals the ME for both a_j and θ_j ($j = 1, 2, 3$) of the studied three cases of resonances. For the following selected values of the physical parameters of the considered model, the solutions of the system are graphically displayed in distinct plots as drawn in Figures 2–6

$$\begin{aligned}a &= 0.3 \text{ m}, \quad b = 0.5 \text{ m}, \quad g = 9.8 \text{ m/sec}^2, \quad \beta = 0.25, \quad f_1 = 0.0004, \\ m_0 &= 0.00007, \quad \sigma_1 = -0.001, \quad \sigma_2 = 0.01, \quad \sigma_3 = 0.001.\end{aligned}$$

A more in-depth study of Figures 2–4 shows that they are calculated when $c_1(=0.01, 0.015, 0.02)$, $c_2(=0.018, 0.021, 0.024)$, and $c_3(=0.19, 0.2, 0.21)$, respectively. These figures are drawn at $\omega_1 = 3.354$ and $\omega_2 = 3.131$ when $(c_2 = 0.02, c_3 = 0.2)$, $(c_1 = 0.01, c_3 = 0.2)$, and $(c_1 = 0.01, c_2 = 0.02)$. Moreover, Figures 5 and 6 are drawn at $c_1 = 0.01$, $c_2 = 0.02$, and $c_3 = 0.2$ when $\omega_1(=3.316, 3.354, 3.391)$ and $\omega_2(=3.212, 3.131, 3.084)$, respectively, for stationary values of $\omega_2 = 3.131$ as in Figure 5 and $\omega_1 = 3.354$ as in Figure 6.

When the amplitudes a_j and the adjusted phases θ_j are varied with time τ for distinct values of the damping coefficients c_j ($j = 1, 2, 3$) and the frequencies ω_k ($k = 1, 2$), we can predict that the above system (25) has a good influence according to these values.

According to the plotted curves in Figure 2, we observe that when c_1 has various values, the time histories of the amplitude a_1 and the adjusted phase θ_1 behave as decaying waves until reaching a stationary behavior at the end of the investigated period of time, as seen in Figure 2a,b. The fluctuations of a_2 and θ_2 waves with time τ become clear in the first quarter of the period of time and become stationary after that as explored in Figure 2c,d. On the other hand, a sharp descent of the curves describing the waves of a_3 and θ_3 is observed in the curves of Figure 2e,f, which is due to the last two equations of the system (25). There is no variation of the a_2, a_3 and θ_2, θ_3 curves with the change of c_1 values due to the formulations of the equations of these curves.

Figure 2. Relations of amplitudes $a_j(\tau)$ ($j = 1, 2, 3$) and the modified phases $\theta_j(\tau)$ at $c_2 = 0.02$, $c_3 = 0.2$, $\omega_1 = 3.354$, and $\omega_2 = 3.131$: (a–f) when $c_1 (=0.01, 0.015, 0.02)$.

Figure 3. Relations of $a_j(\tau)$ ($j = 1, 2, 3$) and $\theta_j(\tau)$ at $c_1 = 0.01$, $c_3 = 0.2$, $\omega_1 = 3.354$ and $\omega_2 = 3.131$: (a–f) when $c_2 (=0.018, 0.021, 0.024)$.

Figure 4. Characterization of the time histories of $a_j(\tau)$ ($j = 1, 2, 3$) and $\theta_j(\tau)$ at $c_1 = 0.01$, $c_2 = 0.02$, $\omega_1 = 3.354$, and $\omega_2 = 3.131$: (a–f) when $c_3 (=0.19, 0.2, 0.21)$.

Figure 5. Changes of a_j ($j = 1, 2, 3$) and θ_j with τ at $\omega_2 = 3.131$, $c_1 = 0.01$, $c_2 = 0.02$, and $c_3 = 0.2$: (a–f) when $\omega_1 (=3.316, 3.354, 3.391)$.

Figure 6. Changes of a_j ($j = 1, 2, 3$) and θ_j with τ at $\omega_1 = 3.354$, $c_1 = 0.01$, $c_2 = 0.02$, and $c_3 = 0.2$: (a–f) when $\omega_2 = (3.212, 3.131, 3.084)$.

The change of the various values of c_2 is evident in the curves describing the time histories of the amplitude a_2 and the modified phase θ_2 because the third and fourth equations of system (25) are dependent on c_2 , as seen in Figure 3c,d, while the other equations do not depend explicitly on c_2 . Therefore, there is no variation, to some extent, of the curves describing a_1, a_3 and θ_1, θ_3 as drawn in the other parts of Figure 3.

Since the last two equations of system (25) depend on c_3 , an observed variation of the curves describes the modified phase θ_3 , as seen in Figure 4f. There is no observed variation in the curves of the other variables because the first four equations of system (25) do not depend on c_3 explicitly as indicated in the other parts of Figure 4.

An examination of the system of equations (25) shows that these equations are dependent on ω_1 and ω_2 . Therefore, we expect a good impact of these parameters on the time histories of a_j ($j = 1, 2, 3$) and θ_j which met with the plotted curves of Figures 5 and 6. These curves describing the waves of these variables oscillate in a decaying manner as drawn in parts (a)–(d) of these figures or monotonically decrease with time as seen in parts (e) and (f) of the same figures. Based on this analysis, we come to the conclusion that the behavior of the system of equations (25) is stable and free of chaos.

Figures 7–11 present the phase plane diagrams of a_j ($j = 1, 2, 3$) and θ_j when c_j and ω_k ($k = 1, 2$) have various values. An inspection of the curves of these figures shows that we have spirals curves that are directed to one point, which gives an impression of the steady motion of these amplitudes and phases.

Figure 7. Phase planes $a_j\theta_j$ ($j = 1, 2, 3$) at $c_2 = 0.02$, $c_3 = 0.2$, $\omega_1 = 3.354$, and $\omega_2 = 3.131$: (a–c) when $c_1 (=0.01, 0.015, 0.02)$.

Figure 8. Phase planes $a_j\theta_j$ ($j = 1, 2, 3$) at $c_1 = 0.01$, $c_3 = 0.2$, $\omega_1 = 3.354$ and $\omega_2 = 3.131$: (a–c) when $c_2 (=0.018, 0.021, 0.024)$.

Figure 9. Phase planes $a_j \theta_j$ ($j = 1, 2, 3$) when $c_1 = 0.01$, $c_2 = 0.02$, $\omega_1 = 3.354$ and $\omega_2 = 3.131$: (a,b) when $c_3 (=0.19, 0.2, 0.21)$, (c) when $c_3 = 0.19$, (d) when $c_3 = 0.2$, and (e) when $c_3 = 0.21$.

Figure 10. Phase planes $a_j\theta_j$ ($j = 1, 2, 3$) at $\omega_2 = 3.131$, $c_1 = 0.01$, $c_2 = 0.02$, and $c_3 = 0.2$: (a,b) when $\omega_1 (= 3.316, 3.354, 3.391)$, (c) when $\omega_1 = 3.316$, (d) when $\omega_1 = 3.354$, and (e) when $\omega_1 = 3.391$.

Figure 11. Phase planes $a_j \theta_j$ ($j = 1, 2, 3$) at $\omega_1 = 3.354$, $c_1 = 0.01$, $c_2 = 0.02$, and $c_3 = 0.2$: (a,b) when $\omega_2 (=3.212, 3.131, 3.084)$, (c) when $\omega_2 = 3.212$, (d) when $\omega_2 = 3.131$, and (e) when $\omega_2 = 3.084$.

Parts of Figures 7–9 are drawn when c_1 , c_2 , and c_3 have different values, respectively, to reveal the variation of curves of the phase planes $a_j \theta_j$ ($j = 1, 2, 3$) with these values, while Figures 10 and 11 describe the change of these planes that happened at different values of ω_1 and ω_2 , respectively.

According to the curves of these figures and the system of equations (25), we observe that the plane $a_1 \theta_1$ is impacted by the various values of c_1 , as seen in Figure 7a, while there is no variation of the curves of planes $a_2 \theta_2$ and $a_3 \theta_3$ as indicated in Figure 7b,c. The curves of the phase planes $a_2 \theta_2$ and $a_3 \theta_3$ have been impacted with the variation of c_2 values of as seen in Figure 8b,c. On the other hand, there is no observed variation of the curves drawn in plane $a_1 \theta_1$ when c_2 changes, as noticed in Figure 8a. According to the plotted curves in Figure 9, we can see that the curves shown in parts (a) and (b) that describe the phase planes $a_1 \theta_1$ and $a_2 \theta_2$ respectively, have no variation with various values of c_3 . The good impact of the values of c_3 is observed in parts (c), (d), and (e) of Figure 9 for the phase plane $a_3 \theta_3$.

The influence of the frequencies $\omega_1 (=3.316, 3.354, 3.391)$ and $\omega_2 (=3.212, 3.131, 3.084)$ on the phase plane diagrams $a_j\theta_j$ ($j = 1, 2, 3$) is observed from the curves of Figures 10 and 11, respectively. Therefore, we can say that these curves have a spiral form from the outside to the inside, and it is directed toward a single point for each curve, which means that all values of ω_k ($k = 1, 2$) have a significant impact on the curves of these planes. The reason goes back to the equations of system (25) that depend explicitly on ω_k ($k = 1, 2$).

It is important to remember that the obtained approximate solutions z , Φ , and η describe the spring's elongation, the rotation angle at the point O_1 , and the elongation of the transverse absorber, respectively.

Figures 12–16 depict the graphical representations of these solutions over time τ , revealing their behaviors across the analyzed time interval as presented in parts (a), (b), and (c) when $\tau \in [0, 100]$ and in parts (d), (e), and (f) when $\tau \in [0, 1000]$ of these figures. Figures 12–14 are performed at $\omega_1 = 3.354$ and $\omega_2 = 3.131$ at different values of $c_1 (= 0.01, 0.015, 0.02)$, $c_2 (= 0.018, 0.021, 0.024)$, and $c_3 (= 0.19, 0.2, 0.21)$ respectively. When these figures are compared with each other, we can see that the variations with different values of c_j ($j = 1, 2, 3$) seem to be slight. This is because the approximate solutions of the variables z , Φ , and η do not depend on c_j explicitly. In addition, Figures 15 and 16 are portrayed at $c_1 = 0.01$, $c_2 = 0.02$, and $c_3 = 0.2$ when $\omega_1 (= 3.316, 3.354, 3.391)$ and $\omega_2 (= 3.212, 3.131, 3.084)$, respectively. It is noted that parts of Figure 15 have a phase shift, while parts of Figure 16 do not have a phase shift. The reason is that the curves of Figures 15 and 16 are graphed when ω_1 and ω_2 have different values, respectively.

Figure 12. Solutions' time histories when $c_1 (= 0.01, 0.015, 0.02)$, $c_2 = 0.02$, $c_3 = 0.2$, $\omega_1 = 3.354$, and $\omega_2 = 3.131$: (a–c) at $\tau \in [0, 100]$ and (d–f) at $\tau \in [0, 1000]$.

Figure 13. Change of z , Φ , and η via τ when $c_1 = 0.01$, $c_2(= 0.018, 0.021, 0.024)$, $c_3 = 0.2$, $\omega_1 = 3.354$ and $\omega_2 = 3.131$: (a–c) at $\tau \in [0, 100]$ and (d–f) at $\tau \in [0, 1000]$.

Figure 14. Evolution of solutions z , Φ , and η throughout time τ at $c_3(=0.19, 0.2, 0.21)$, $c_1 = 0.01$, $c_2 = 0.02$, $\omega_1 = 3.354$, and $\omega_2 = 3.131$: (a–c) at $\tau \in [0, 100]$ and (d–f) at $\tau \in [0, 1000]$.

Figure 15. Progression of solutions z , Φ , and η over time τ when $\omega_1 (= 3.316, 3.354, 3.391)$, $\omega_2 = 3.131$, $c_1 = 0.01$, $c_2 = 0.02$, and $c_3 = 0.2$: (a–c) at $\tau \in [0, 100]$ and (d–f) at $\tau \in [0, 1000]$.

Figure 16. Solutions' temporal histories when $\omega_2(=3.212, 3.131, 3.084)$, $\omega_1 = 3.354$, $c_1 = 0.01$, $c_2 = 0.02$, and $c_3 = 0.2$: (a–c) at $\tau \in [0, 100]$ and (d–f) at $\tau \in [0, 1000]$.

Based on the sketched curves of the solutions z , Φ , and η , we observe that the waves describing these solutions have a periodic manner in which the number of oscillations and their wavelengths remain stationary to some extent with the variation of c_j ($j = 1, 2, 3$) values. Parts (a) of these figures have an explicitly periodic form for the wave of the solution z . It is notable from parts (b) of these figures that each period of the wave contains a constant number of vibrations that are repeated for each period. This is due to the analytical form of the rotation angle Φ , in which its behavior has a spinning form. On the other hand, the wave describing spring's elongation η experiences rapid oscillations at the beginning of the motion due to the absorber's effect and damping impact on the investigated dynamical system, in which it settles down after that and vanishes at the end of time interval, as seen in parts (c) of these figures.

According to the calculations of Figures 15 and 16, we get to the conclusion that the change of the ω_k ($k = 1, 2$) values has a considerable impact on the attitude of the describing waves of the attained solutions. Regardless of the fact that the wave's behavior of the solutions is periodic, we observe that the amplitudes of these waves increase and decrease with the increasing of ω_k as seen in Figures 15a–f and 16a–f, respectively.

5. Steady State Solutions

The major objective of the present section is to study the oscillations of the examined system in the case of steady state. From the equations of system (25), we can obtain both of the modified phases θ_j ($j = 1, 2, 3$) and amplitudes a_j in the steady state case. Alternatively, the zero values of the left-hand sides of the equations of this system are considered. Therefore, we consider $\frac{da_j}{d\tau} = 0$ and $\frac{d\theta_j}{d\tau} = 0$ [32], to obtain the next algebraic system of six equations of the functions θ_j and a_j .

$$\begin{aligned}
 f_1 \sin \theta_1 - c_1 a_1 &= 0, \\
 f_1 \cos \theta_1 + 2a_1 [\sigma_1 - \alpha z_r - \frac{(1+w_1^2)^2}{2(1-w_1^2)} r^2 \beta^2 - \frac{3(1-w_1^2)}{4(1-4w_1^2)} w_1^2 a_2^2] &= 0, \\
 m_0 \sin \theta_2 - w_1 c_2 a_2 &= 0, \\
 m_0 \cos \theta_2 + 2w_1 a_2 \left\{ \sigma_2 - w_1 (\alpha z_r^2 - r^2 \beta + \frac{(1-w_1^2)}{8(1-4w_1^2)} [12a_1^2 - (1+8w_1^2) a_2^2] \right. \\
 &\quad \left. - \frac{r^2 \beta^2}{4(1-w_1^2) w_2^2} [(14-3r^2) w_2^2 - 2(4-w_2^2) w_1^4 + (8+3r^2 w_2^2) w_1^2] \right\} = 0, \\
 \frac{(1-w_1^2)}{(1-4w_1^2)} w_1^2 a_2^3 \sin \theta_3 + w_2 c_3 a_3 &= 0, \\
 \frac{(1-w_1^2)}{(1-4w_1^2)} w_1^2 a_2^3 \cos \theta_3 - 2w_2 a_3 \left\{ \sigma_3 + 3w_1 \left\{ \alpha z_r^2 - r^2 \beta + \frac{(1-w_1^2)}{8(1-4w_1^2)} [12a_1^2 \right. \right. \\
 &\quad \left. \left. - (1+8w_1^2) a_2^2] - \frac{r^2 \beta^2}{4(1-w_1^2) w_2^2} [(14-3r^2) w_2^2 - 2(4-w_2^2) w_1^4 + (8+3r^2 w_2^2) w_1^2] \right\} \right. \\
 &\quad \left. - \frac{3m_0}{2w_1 a_2} \cos \theta_2 + \frac{1}{2w_2} [\beta (w_1^2 - w_2^2) + \frac{1}{2} w_1^2 a_2^2 + \frac{\beta^2}{4w_2^2} (4r^2 w_2^4 + 4w_2^2 (w_1^2 - w_2^2) \right. \\
 &\quad \left. + (w_1^2 - w_2^2)^2)] \right\} = 0,
 \end{aligned} \tag{26}$$

Now, we can remove the adjusted phases θ_j from the preceding system to produce the following three non-linear algebraic equations between longitudinal amplitude a_1 , the swing oscillations a_2 , and the frequency represented by the detuning parameters σ_j and the absorber's amplitude a_3 .

$$\begin{aligned}
 f_1^2 &= a_1^2 \left\{ c_1^2 + 4 \left[\sigma_1 - \alpha z_r - \frac{(1+w_1^2)^2}{2(1-w_1^2)} r^2 \beta^2 - \frac{3(1-w_1^2)}{4(1-4w_1^2)} w_1^2 a_2^2 \right]^2 \right\}, \\
 m_0^2 &= w_1^2 a_2^2 \left\{ c_2^2 + 4 \left[\sigma_2 - w_1 (\alpha z_r^2 - r^2 \beta + \frac{(1-w_1^2)}{8(1-4w_1^2)} [12a_1^2 - (1+8w_1^2) a_2^2] \right. \right. \\
 &\quad \left. \left. - \frac{r^2 \beta^2}{4(1-w_1^2) w_2^2} [(14-3r^2) w_2^2 - 2(4-w_2^2) w_1^4 + (8+3r^2 w_2^2) w_1^2] \right]^2 \right\}, \\
 \frac{(1-w_1^2)^2}{(1-4w_1^2)^2} w_1^4 &= w_2^2 a_3^2 \left\{ c_3^2 + 4 \left[\sigma_3 + 3w_1 \left\{ \beta^2 w_1^4 + 2(2\beta + \beta^2 + a_2^2) w_1^2 w_2^2 \right. \right. \right. \\
 &\quad \left. \left. - \beta [4 + (3-4r^2) \beta] w_2^4 \right\} \right]^2 \right\}.
 \end{aligned} \tag{27}$$

Stability testing is considered a crucial aspect of the vibrations in the steady state case. To explore such a circumstance, the behavior of the system in a domain relatively near to fixed points is investigated. Therefore, the substitutions listed below are employed in (25) to achieve this purpose.

$$\begin{aligned}
 a_1 &= a_{10} + a_{11}, & \theta_1 &= \theta_{10} + \theta_{11}, \\
 a_2 &= a_{20} + a_{21}, & \theta_2 &= \theta_{20} + \theta_{21}, \\
 a_3 &= a_{30} + a_{31}, & \theta_3 &= \theta_{30} + \theta_{31}.
 \end{aligned} \tag{28}$$

Here, a_{j0} and θ_{j0} ($j = 1, 2, 3$) denote the steady state solutions, whereas a_{j1} and θ_{j1} represent relatively small disturbances in relation to a_{j0} and θ_{j0} . As a result of linearization and the reality of the fixed points of (25), we get

$$\begin{aligned}
\frac{da_{11}}{d\tau} &= \frac{1}{2}(f_1\theta_{11}\cos\theta_{10} - c_1a_{11}), \\
a_{10}\frac{d\theta_{11}}{d\tau} &= a_{11}[\sigma_1 - \alpha z_r - \frac{(1+w_1^2)^2}{2(1-w_1^2)}r^2\beta^2] - \frac{3(1-w_1^2)}{4(1-4w_1^2)}w_1^2a_{20}(a_{20}a_{11} + 2a_{10}a_{21}) \\
&\quad + \frac{1}{2}f_1\theta_{11}\sin\theta_{10}, \\
\frac{da_{21}}{d\tau} &= \frac{1}{2}(\frac{m_0}{w_1}\theta_{21}\cos\theta_{20} - c_2a_{21}), \\
a_{20}\frac{d\theta_{21}}{d\tau} &= a_{21}\left\{\sigma_2 - w_1(\alpha z_r^2 - r^2\beta - \frac{r^2\beta^2}{4(1-w_1^2)w_2^2}[(14-3r^2)w_2^2 - 2(4-w_2^2)w_1^4] \right. \\
&\quad \left. + (8+3r^2w_2^2)w_1^2)\right\} - \frac{(1-w_1^2)w_1}{8(1-4w_1^2)}[12a_{10}(a_{10}a_{21} + 2a_{20}a_{11}) \\
&\quad - 3(1+8w_1^2)a_{20}a_{21}] - \frac{m_0}{2w_1}\theta_{21}\sin\theta_{20}, \\
\frac{da_{31}}{d\tau} &= -\frac{1}{2}[\frac{(1-w_1^2)w_1^2}{(1-4w_1^2)w_2}a_{20}^2(a_{20}\theta_{31}\cos\theta_{30} + 3a_{21}\sin\theta_{30}) + c_3a_{31}], \\
a_{20}a_{30}\frac{d\theta_{31}}{d\tau} &= (a_{20}a_{31} + a_{21}a_{30})\left\{\sigma_3 + 3w_1(\alpha z_r^2 - r^2\beta - \frac{r^2\beta^2}{4(1-w_1^2)w_2^2}[(14-3r^2)w_2^2 \right. \\
&\quad \left. - 2(4-w_2^2)w_1^4 + (8+3r^2w_2^2)w_1^2]) + \frac{1}{2w_2}[\beta(w_1^2 - w_2^2) + \frac{\beta^2}{4w_2^2}(4r^2w_2^4 \right. \\
&\quad \left. + 4w_2^2(w_1^2 - w_2^2) + (w_1^2 - w_2^2)^2)]\right\} + \frac{3(1-w_1^2)}{8(1-4w_1^2)}w_1[12a_{10}(a_{10}a_{20}a_{31} \\
&\quad + a_{10}a_{30}a_{21} + 2a_{20}a_{30}a_{11}) - (1+8w_1^2)(a_{20}^3a_{31} + 3a_{20}^2a_{30}a_{21})] + \frac{w_1^2}{4w_2} \\
&\quad \times (a_{20}^3a_{31} + 3a_{20}^2a_{30}a_{21}) - \frac{3m_0}{2w_1}(a_{31}\cos\theta_{20} - a_{30}\theta_{21}\sin\theta_{20}) \\
&\quad - \frac{(1-w_1^2)w_1^2}{2(1-4w_1^2)w_2}a_{20}^3(4a_{21}\cos\theta_{30} - a_{20}\theta_{31}\sin\theta_{30}).
\end{aligned} \tag{29}$$

Since a_{j1} and θ_{j1} ($j = 1, 2, 3$) are perturbed functions of amplitudes and phases of the aforementioned linear system. Then, the linear function $k_s e^{\lambda\tau}$ ($s = 1, 2, 3, 4, 5, 6$) of the exponential form can be used to express about their solutions, where k_s and λ are constants and the perturbation's eigenvalue, respectively. The real parts of the roots of the next characteristic equation of (29) should be negative if the steady state solutions are asymptotically stable [33,34].

$$\lambda^6 + \Gamma_1\lambda^5 + \Gamma_2\lambda^4 + \Gamma_3\lambda^3 + \Gamma_4\lambda^2 + \Gamma_5\lambda + \Gamma_6 = 0, \tag{30}$$

where Γ_s ($s = 1, 2, \dots, 6$) are functions of a_{j0} , θ_{j0} , and c_j (see Appendix B).

The required and sufficient requirements of stability for certain solutions at steady state can be expressed as follows

$$\begin{aligned}
\Gamma_1 &> 0, \quad \Gamma_3(\Gamma_1\Gamma_2 - \Gamma_3) - \Gamma_4\Gamma_1^2 + \Gamma_1\Gamma_5 > 0, \quad \Gamma_1\Gamma_2 - \Gamma_3 > 0, \\
\Gamma_3\Gamma_4(\Gamma_1\Gamma_2 - \Gamma_3) - \Gamma_1^2\Gamma_4^2 - \Gamma_1\Gamma_2^2\Gamma_5 + \Gamma_2\Gamma_3\Gamma_5 + 2\Gamma_1\Gamma_4\Gamma_5 - \Gamma_5^2 + \Gamma_1^2\Gamma_2\Gamma_6 - \Gamma_1\Gamma_3\Gamma_6 &> 0, \\
\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5 - \Gamma_3^2\Gamma_4\Gamma_5 - \Gamma_1^2\Gamma_4^2\Gamma_5 - \Gamma_1\Gamma_2^2\Gamma_5^2 + \Gamma_2\Gamma_3\Gamma_5^2 + 2\Gamma_1\Gamma_4\Gamma_5^2 - \Gamma_5^3 - \Gamma_1\Gamma_2\Gamma_3^2\Gamma_6 \\
+ \Gamma_3^3\Gamma_6 + \Gamma_1^2\Gamma_3\Gamma_4\Gamma_6 + 2\Gamma_1^2\Gamma_2\Gamma_5\Gamma_6 - 3\Gamma_1\Gamma_3\Gamma_5\Gamma_6 - \Gamma_1^3\Gamma_6^2 &> 0, \\
\Gamma_6(\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5 - \Gamma_3^2\Gamma_4\Gamma_5 - \Gamma_1^2\Gamma_4^2\Gamma_5 - \Gamma_1\Gamma_2^2\Gamma_5^2 + \Gamma_2\Gamma_3\Gamma_5^2 + 2\Gamma_1\Gamma_4\Gamma_5^2 - \Gamma_5^3 - \Gamma_1\Gamma_2\Gamma_3^2\Gamma_6 \\
+ \Gamma_3^3\Gamma_6 + \Gamma_1^2\Gamma_3\Gamma_4\Gamma_6 + 2\Gamma_1^2\Gamma_2\Gamma_5\Gamma_6 - 3\Gamma_1\Gamma_3\Gamma_5\Gamma_6 - \Gamma_1^3\Gamma_6^2) &> 0.
\end{aligned} \tag{31}$$

6. The Stability Analysis

In this section, we investigate the model's stability as well as its non-linear evolution using the Routh–Hurwitz non-linear stability approach. It must be remembered that a damped spring connected with a transverse absorber under the action of $F(t)$ and $M(t)$. Some factors have been revealed to play a substantial impact in the stability processes such as damping's constants c_j ($j = 1, 2, 3$), frequencies ω_k ($k = 1, 2$), and parameters of detuning σ_j . To gain the stability plots of system (25), a specific process with various parameters of the system has been used. The adjusted amplitudes a_j ($j = 1, 2, 3$) plotted via time are for various parametrical regions, in addition to the graphical representations

of their characteristics through the phase plane paths. Curves of frequency responses of a_j versus σ_2 and the system's fixed points have been portrayed in Figures 17–31, in which the flowing data have been taken into account besides the previous ones.

$$\sigma_1 = (-0.001, 0, 0.001), \quad \sigma_3 = 0.001, \quad p_1 = 1 + \sigma_1, \\ p_2 = w_1 + \sigma_2, \quad 3w_1 = w_2 + \sigma_3, \quad f_1 = 0.0004, \quad m_0 = 0.00007.$$

Figure 17. Frequency response for a_j ($j = 1, 2, 3$) via σ_2 at when $c_2 = 0.02$, $c_3 = 0.2$, $\omega_1 = 3.354$, $\omega_2 = 3.131$, and $\sigma_1 = -0.001$: (a) when $c_1 = 0.01$, (b) when $c_1 = 0.015$, (c) when $c_1 = 0.02$, and (d,e) when $c_1 (=0.01, 0.015, 0.02)$.

Figure 18. Frequency response of a_j ($j = 1, 2, 3$) via σ_2 at $c_2 = 0.02$, $c_3 = 0.2$, $\omega_1 = 3.354$, $\omega_2 = 3.131$, and $\sigma_1 = 0$: (a) when $c_1 = 0.01$, (b) when $c_1 = 0.015$, (c) when $c_1 = 0.02$, and (d,e) when $c_1 (=0.01, 0.015, 0.02)$.

Figure 19. Frequency response of a_j ($j = 1, 2, 3$) via σ_2 at $c_2 = 0.02$, $c_3 = 0.2$, $\omega_1 = 3.354$, $\omega_2 = 3.131$, and $\sigma_1 = 0.001$: (a) when $c_1 = 0.01$, (b) when $c_1 = 0.015$, (c) when $c_1 = 0.02$, and (d,e) when $c_1 = (0.01, 0.015, 0.02)$.

Figure 20. Curves of the planes $\sigma_2 a_j$ ($j = 1, 2, 3$) at when $c_1 = 0.01$, $c_3 = 0.2$, $\omega_1 = 3.354$, $\omega_2 = 3.131$, and $\sigma_1 = -0.001$: (a-c) when $c_2 = (0.018, 0.021, 0.024)$.

The significance impact of the damping coefficient c_1 has been explored in the curves of Figures 17–19 when $\sigma_1 = -0.001$, $\sigma_1 = 0$, and $\sigma_1 = 0.001$, respectively. The impact of c_2 and c_3 , for the same values of σ_1 , has been indicated in Figures 20–25. The inspection of parts of Figures 17–19 shows the effect of the selected values of $c_1 = (0.01, 0.015, 0.02)$ on the frequency response curves when $c_2 = 0.02$, $c_3 = 0.2$, $\omega_1 = 3.354$ and $\omega_2 = 3.131$.

Figure 21. Curves of the planes $\sigma_2 a_j$ ($j = 1, 2, 3$) at $c_1 = 0.01$, $c_3 = 0.2$, $\omega_1 = 3.354$, $\omega_2 = 3.131$, and $\sigma_1 = 0$: (a–c) when $c_2 (=0.018, 0.021, 0.024)$.

Figure 22. Curves of the planes $\sigma_2 a_j$ ($j = 1, 2, 3$) at $c_1 = 0.01$, $c_3 = 0.2$, $\omega_1 = 3.354$, $\omega_2 = 3.131$, and $\sigma_1 = 0.001$: (a–c) when $c_2 (=0.018, 0.021, 0.024)$.

Figure 23. Curves of the planes $\sigma_2 a_j$ ($j = 1, 2, 3$) at $c_1 = 0.01$, $c_2 = 0.02$, $\omega_1 = 3.354$, $\omega_2 = 3.131$, and $\sigma_1 = -0.001$: (a–c) when $c_3 (=0.19, 0.2, 0.21)$.

Figure 24. Curves of the planes $\sigma_2 a_j$ ($j = 1, 2, 3$) at $c_1 = 0.01$, $c_2 = 0.02$, $\omega_1 = 3.354$, $\omega_2 = 3.131$, and $\sigma_1 = 0$: (a–c) when $c_3 (=0.19, 0.2, 0.21)$.

Figure 25. Curves of the planes $\sigma_2 a_j$ ($j = 1, 2, 3$) at $c_1 = 0.01$, $c_2 = 0.02$, $\omega_1 = 3.354$, $\omega_2 = 3.131$, and $\sigma_1 = 0.001$: (a–c) when $c_3 (=0.19, 0.2, 0.21)$.

Figure 26. Curves of the planes $\sigma_2 a_j$ ($j = 1, 2, 3$) at $\omega_2 = 3.131$, $c_1 = 0.01$, $c_2 = 0.02$, $c_3 = 0.2$, and $\sigma_1 = -0.001$: (a) when $\omega_1 = 3.316$, (b) when $\omega_1 = 3.354$, (c) when $\omega_1 = 3.391$, and (d,e) when $\omega_1 (=3.316, 3.354, 3.391)$.

Figure 27. Curves of the planes $\sigma_2 a_j$ ($j = 1, 2, 3$) at $\omega_2 = 3.131$, $c_1 = 0.01$, $c_2 = 0.02$, $c_3 = 0.2$, and $\sigma_1 = 0$: (a) when $\omega_1 = 3.316$, (b) when $\omega_1 = 3.354$, (c) when $\omega_1 = 3.391$, and (d,e) when $\omega_1 (=3.316, 3.354, 3.391)$.

Figure 28. Curves of the planes $\sigma_2 a_j$ ($j = 1, 2, 3$) at $\omega_2 = 3.131$, $c_1 = 0.01$, $c_2 = 0.02$, $c_3 = 0.2$, and $\sigma_1 = 0.001$: (a) when $\omega_1 = 3.316$, (b) when $\omega_1 = 3.354$, (c) when $\omega_1 = 3.391$, and (d,e) when $\omega_1 (=3.316, 3.354, 3.391)$.

It is obvious that c_1 has a bigger role on the curves of plane $a_1 \sigma_2$ than on the frequency response curves of planes $a_2 \sigma_2$ and $a_3 \sigma_2$, which is due to the formulation of system (25). It is noted that all parts of these figures have only one critical fixed point, and this means that we have one only area for both stability and instability. Stable fixed points have been detected in the area $-0.5 \leq \sigma_2 \leq -0.04$ while the unstable fixed points have been found in the area $-0.04 < \sigma_2 \leq 0.5$, in which the stable points and the unstable ones have been presented by solid and dashed curves, respectively.

Figure 29. Curves of the planes $\sigma_2 a_j$ ($j = 1, 2, 3$) at when $\omega_1 = 3.354$, $c_1 = 0.01$, $c_2 = 0.02$, $c_3 = 0.2$, and $\sigma_1 = -0.001$: (a) when $\omega_2 = 3.212$, (b) when $\omega_2 = 3.131$, (c) when $\omega_2 = 3.084$, and (d,e) when $\omega_2 = (3.212, 3.131, 3.084)$.

Figure 30. Curves of the planes $\sigma_2 a_j$ ($j = 1, 2, 3$) at $\omega_1 = 3.354$, $c_1 = 0.01$, $c_2 = 0.02$, $c_3 = 0.2$, and $\sigma_1 = 0$: (a) when $\omega_2 = 3.212$, (b) when $\omega_2 = 3.131$, (c) when $\omega_2 = 3.084$, and (d,e) when $\omega_2 = (3.212, 3.131, 3.084)$.

Alternatively, Figures 20–22 explore the impact of the $c_2 (=0.018, 0.021, 0.024)$ values on a_j when $c_1 = 0.01$, $c_3 = 0.2$, $\omega_1 = 3.354$, and $\omega_2 = 3.131$. One critical fixed point has been found with the stability and instability areas $-0.5 \leq \sigma_2 \leq -0.04$ and $-0.04 < \sigma_2 \leq 0.5$, respectively. The location of the peak points has been shifted with the increasing of c_2 values as seen in parts (a) and (b) of Figures 20–22, while it will be stationary as in part (c) of these figures. In addition, Figures 23–25 reveal the influence of $c_3 (=0.19, 0.2, 0.21)$ on the amplitudes a_j when $c_1 = 0.01$, $c_2 = 0.02$, $\omega_1 = 3.354$, and $\omega_2 = 3.131$, in which one peak and one critical point have been explored for each curve. The stable and unstable fixed points have been found in the ranges $-0.5 \leq \sigma_2 \leq -0.04$ and $-0.04 < \sigma_2 \leq 0.5$, respectively.

Figure 31. Curves of the planes $\sigma_2 a_j$ ($j = 1, 2, 3$) when $\omega_1 = 3.354$, $c_1 = 0.01$, $c_2 = 0.02$, $c_3 = 0.2$, and $\sigma_1 = -0.001$: (a) at $\omega_2 = 3.212$, (b) at $\omega_2 = 3.131$, (c) at $\omega_2 = 3.084$, and (d,e) at $\omega_2 (=3.212, 3.131, 3.084)$.

The influence of the frequencies $\omega_1 (=3.316, 3.354, 3.391)$ at $\omega_2 = 3.131$ and $\omega_2 (=3.212, 3.131, 3.084)$ at $\omega_1 = 3.354$ has been examined through the curves of Figures 26–28, respectively. These figures are calculated when $c_1 = 0.01$, $c_2 = 0.02$, and $c_3 = 0.2$ at $\sigma_1 = -0.001$, $\sigma_1 = 0$, and $\sigma_1 = 0.001$. Based on the included curves in Figures 26–28, we conclude that there exists one peak point with different locations, and each curve has just one essential fixed point. It is clear that ω_1 has a significant impact on the frequency response curves because the equations of system (25) depend directly on the frequencies parameters. Moreover, the stable and unstable regions of the fixed points are calculated as in Table 1.

Table 1. Stability and instability area when $\omega_1 (=3.316, 3.354, 3.391)$ at $\omega_2 = 3.131$.

Figure	ω_1	ω_2	Stable Area	Unstable Area
Figures 26–28	3.316	3.131	$-0.5 < \sigma_2 < -0.06$	$-0.06 < \sigma_2 < 0.5$
Figures 26–28	3.354	3.131	$-0.5 < \sigma_2 < -0.04$	$-0.04 < \sigma_2 < 0.5$
Figures 26–28	3.391	3.131	$-0.5 < \sigma_2 < -0.03$	$-0.03 < \sigma_2 < 0.5$

The above remarks can be applied to the curves of Figures 29–31 when the frequency values $\omega_2 (=3.212, 3.131, 3.084)$ are considered. The ranges of stable fixed points and unstable ones have been given in Table 2.

Table 2. Stability and instability area when $\omega_2 (=3.212, 3.131, 3.084)$ at $\omega_1 = 3.354$.

Figure	ω_1	ω_2	Stable Area	Unstable Area
Figures 29–31	3.354	3.212	$-0.5 < \sigma_2 < -0.17$	$-0.17 < \sigma_2 < 0.5$
Figures 29–31	3.354	3.131	$-0.5 < \sigma_2 < -0.04$	$-0.04 < \sigma_2 < 0.5$
Figures 29–31	3.354	3.084	$-0.5 < \sigma_2 < 0$	$0 < \sigma_2 < 0.5$

7. Non-Linear Interpretations

This section focuses on elucidating the non-linear amplitude's characteristics of system (25) as well as evaluating their stability. As a result, the following transformations are taken into account [31,35]

$$\begin{aligned} B_1 &= [U_1(\tau_1, \tau_2) + iV_1(\tau_1, \tau_2)] e^{i\tilde{\sigma}_1 \tau_1}, \\ B_2 &= [U_2(\tau_1, \tau_2) + iV_2(\tau_1, \tau_2)] e^{i\tilde{\sigma}_2 \tau_1}, \\ B_3 &= [U_3(\tau_1, \tau_2) + iV_3(\tau_1, \tau_2)] e^{i(\tilde{\sigma}_3 + 3\tilde{\sigma}_2)\tau_1}. \end{aligned} \quad (32)$$

First, (32) were substituted into (25), and then, the real and imaginary parts were separated to produce

$$\begin{aligned} \dot{u}_1 &= \frac{v_1}{2} [2\sigma_1 - 2\alpha z_r - \frac{r^2 \beta^2}{(1-w_1^2)} (1+w_1^2)^2 - \frac{6w_1^2}{(1-4w_1^2)} (1-w_1^2)(u_2^2 + v_2^2)] - \frac{c_1}{2} u_1, \\ \dot{v}_1 &= \frac{u_1}{2} [2\alpha z_r - 2\sigma_1 + \frac{r^2 \beta^2}{(1-w_1^2)} (1+w_1^2)^2 - \frac{6w_1^2}{(1-4w_1^2)} (1-w_1^2)(u_2^2 + v_2^2)] - \frac{c_1}{2} v_1 - \frac{f_1}{4}, \\ \dot{u}_2 &= \frac{v_2}{2} \left\{ 2\sigma_2 + r^2 \beta w_1 - w_1 (\alpha z_r^2 + \frac{(1-w_1^2)}{2(1-4w_1^2)}) [12(u_1^2 + v_1^2) - (1+8w_1^2)(u_2^2 + v_2^2)] \right. \\ &\quad \left. - \frac{r^2 \beta^2}{4(1-w_1^2)w_2^2} [(14-3r^2)w_2^2 - 2(4-w_2^2)w_1^4 + (8+3r^2w_2^2)w_1^2] \right\} - \frac{c_2}{2} u_2, \\ \dot{v}_2 &= \frac{u_2}{2} \left\{ w_1 (\alpha z_r^2 + \frac{(1-w_1^2)}{2(1-4w_1^2)}) [12(u_1^2 + v_1^2) - (1+8w_1^2)(u_2^2 + v_2^2)] - \frac{r^2 \beta^2}{4(1-w_1^2)w_2^2} \right. \\ &\quad \left. \times [(14-3r^2)w_2^2 - 2(4-w_2^2)w_1^4 + (8+3r^2w_2^2)w_1^2] - 2\sigma_2 - r^2 \beta w_1 \right\} - \frac{c_2}{2} v_2 - \frac{m_0}{4w_1}, \\ \dot{u}_3 &= \frac{v_3}{2w_2} [2w_2(\sigma_3 + 3\sigma_2) + \beta(w_1^2 - w_2^2) + 2w_1^2(u_2^2 + v_2^2) + \frac{\beta^2}{4w_2^2} (4r^2w_2^4 + 4w_2^2 \\ &\quad \times (w_1^2 - w_2^2) + (w_1^2 - w_2^2)^2)] - \frac{c_3}{2} u_3 + \frac{2(1-w_1^2)}{(1-4w_1^2)w_2} w_1^2 v_2 (v_2^2 - 3u_2^2), \\ \dot{v}_3 &= \frac{u_3}{-2w_2} [2w_2(\sigma_3 + 3\sigma_2) + \beta(w_1^2 - w_2^2) + 2w_1^2(u_2^2 + v_2^2) + \frac{\beta^2}{4w_2^2} (4r^2w_2^4 + 4w_2^2 \\ &\quad \times (w_1^2 - w_2^2) + (w_1^2 - w_2^2)^2)] - \frac{c_3}{2} v_3 + \frac{2(1-w_1^2)}{(1-4w_1^2)w_2} w_1^2 u_2 (u_2^2 - 3v_2^2). \end{aligned} \quad (33)$$

where $U_j = \varepsilon u_j$, $V_j = \varepsilon v_j$; ($j = 1, 2, 3$).

The adjusted amplitudes were then modified throughout time in various parametric zones and the amplitude's properties were depicted in phase-plane curves. Then, the parameters prior values are taken into account to plot Figures 32–36.

The time histories of the new modified phases u_j and v_j ($j = 1, 2, 3$) are portrayed in parts (a), (d), and (g) and (b), (e), and (h) of Figures 32–36, while parts (c), (f), and (i) show the projections of the trajectories of modulation equations on the phase planes $u_j v_j$. Figures 32–34 are drawn at $\omega_1 = 3.354$ and $\omega_2 = 3.131$ when $c_1 (=0.01, 0.015, 0.02)$, $c_2 (=0.018, 0.021, 0.024)$, and $c_3 (=0.19, 0.2, 0.21)$, respectively. Moreover, Figures 35 and 36 are calculated at $c_1 = 0.01$, $c_2 = 0.02$, and $c_3 = 0.2$ when $\omega_1 (=3.316, 3.354, 3.391)$ and $\omega_2 (=3.212, 3.131, 3.084)$, respectively.

According to these figures, we can see a decaying manner in the trajectories of the waves characterizing u_1, v_1, u_2, v_2 and u_3, v_3 as drawn in parts (a), (b), (d), (e), (g), and (h) of Figures 32–34 when c_j ($j = 1, 2, 3$) take various values, while the same conclusion for the mentioned parts can be found in Figures 35 and 36 when ω_k ($k = 1, 2$) has distinct values. Moreover, the curves in planes $u_j v_j$, represented in parts (c), (f), and (i), behave as spiral curves and move toward a single point to confirm that the motion is free of chaos.

A closer look to Figures 32–34 reveals that the new parameters u_1 and v_1 besides the phase plane curves $u_1 v_1$ have been impacted with the change of c_1 more than the others values of c_2 and c_3 . The time histories of u_2, v_2 and u_3, v_3 in addition to the curves of the planes $u_2 v_2$ and $u_3 v_3$ have been impacted with the change of the damping parameters c_2 and c_3 , respectively. The principal reason goes back to the nature formulation of the system of equations (33). The spiral curves are directed from the outside to the inside, describing the stability of the studied system.

Figure 32. Time histories of u_j ($j = 1, 2, 3$), v_j , and the projection of phase planes tracks $u_j v_j$ when $c_2 = 0.02$, $c_3 = 0.2$, $\omega_1 = 3.354$, and $\omega_2 = 3.131$: (a–i) at $c_1 (=0.01, 0.015, 0.02)$.

Figure 33. Time histories of u_j ($j = 1, 2, 3$), v_j , and the projection of phase planes tracks $u_j v_j$ when $c_1 = 0.01$, $c_3 = 0.2$, $\omega_1 = 3.354$, and $\omega_2 = 3.131$: (a–i) at $c_2 = (0.018, 0.021, 0.024)$.

Figure 34. Time histories of u_j ($j = 1, 2, 3$), v_j , and the projection of phase planes tracks $u_j v_j$ when $c_1 = 0.01$, $c_2 = 0.02$, $\omega_1 = 3.354$, and $\omega_2 = 3.131$: (a–i) at $c_3 = (0.19, 0.2, 0.21)$.

Figure 35. Time histories of u_j ($j = 1, 2, 3$), v_j , and the projection of phase planes tracks $u_j v_j$ when $\omega_2 = 3.131$, $c_1 = 0.01$, $c_2 = 0.02$, and $c_3 = 0.2$: (a–i) at $\omega_1 (=3.316, 3.354, 3.391)$.

Figure 36. Time histories of u_j ($j = 1, 2, 3$), v_j , and the projection of phase planes tracks $u_j v_j$ when $\omega_1 = 3.354$, $c_1 = 0.01$, $c_2 = 0.02$, and $c_3 = 0.2$: (a–i) at $\omega_2 (=3.212, 3.131, 3.084)$.

The good effectiveness of the change of the frequency parameters ω_1 and ω_2 on the dynamical behavior of considered system (33) has been shown in parts of Figures 35 and 36. Time histories curves of the new parameters u_j ($j = 1, 2, 3$) and v_j are plotted in parts (a), (d), and (g) and (b), (e), and (h), respectively. Whereas the plane curves $u_j v_j$ have been drawn in parts (c), (f), and (i) of Figures 35 and 36. It is noted that these waves have been impacted with the various values of the frequency parameters, in which decay curves of the time histories are obtained and spiral ones of the phase plane toward one single point are plotted, indicating that the motion is smooth, steady and without disorder.

8. Conclusions

The non-linear motion of a damped spring pendulum with an attached linear damped transverse absorber in the direction of the spring has been investigated. Under the impact of a harmonic force and moment, the motion of the pendulum's hanging point has been constrained to an elliptic path. The EOM have been derived applying Lagrange's equations from the second kind. The AMS has been used to achieve the approximate solutions up to the third order. Based on the solvability requirements, the ME have been recognized. Three resonance cases of primary external and internal resonance were investigated simultaneously. The RHC was used to investigate and evaluate the stability of fixed points' locations. The time histories of the achieved solutions, resonance responses, and the stability and instability zones at the steady state case were drawn and analyzed. The impact of various inputs of physical parameters on the performance of the system under investigation was examined. This system carries a lot of weight due to its use in engineering vibrational control applications.

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Appendix A

$$\begin{aligned}
q_1 &= \frac{i\tilde{b}\omega^3(\omega+2w_1)}{2l(\omega^2-w_1^2)[1-(w_1+\omega)^2]}, \\
q_2 &= \frac{i\tilde{b}\omega^3(\omega-2w_1)}{2l(\omega^2-w_1^2)[1-(w_1-\omega)^2]}, \\
q_3 &= \frac{(1+2w_1^2)}{(1-4w_1^2)^2} r^2 \tilde{\beta} w_1^2, \\
q_4 &= \frac{2r\tilde{\beta}(2+w_1)w_1^3}{(1+2w_1)(w_1^2-1)}, \\
q_5 &= \frac{2r\tilde{\beta}(2-w_1)w_1^3}{(1-2w_1)(w_1^2-1)}, \\
q_6 &= \frac{2\tilde{\beta}(w_1+w_2)^2}{[1-(w_1+w_2)^2]}, \\
q_7 &= \frac{2\tilde{\beta}(w_1-w_2)^2}{[1-(w_1-w_2)^2]}, \\
q_8 &= \frac{r\tilde{\beta}^2 w_1^2}{(w_1^2-1)^2 w_2^2} [2w_1^4 + r^2 w_2^2 - w_1^2(2-r^2 w_2^2)], \\
q_9 &= \frac{\tilde{\beta}\omega^2}{2l(w_1^2-\omega^2)} \left\{ \frac{r\tilde{a}}{\omega^2-1} (2-\omega^2+w_1^2) \right. \\
&\quad \left. + i\tilde{b} \left[\frac{2w_1^2}{w_2^2-\omega^2} + \omega^2 \left(\frac{r^2}{w_1^2-\omega^2} - \frac{2}{w_2^2-\omega^2} \right) \right] \right\}, \\
q_{10} &= \frac{i\omega^3(\omega+2)\tilde{b}}{2l(\omega^2-w_1^2)[w_1^2-(1+\omega)^2]}, \\
q_{11} &= \frac{i\omega^3(\omega-2)\tilde{b}}{2l(\omega^2-w_1^2)[w_1^2-(1-\omega)^2]}, \\
q_{12} &= \frac{r\tilde{\beta}^2}{(w_1^2-1)^2 w_2^2 (w_2^2-1)} \{ w_1^4 w_2^2 - w_1^6 (1-2w_2^2) \\
&\quad + w_2^2 (3-r^2+r^2 w_2^2) + w_1^2 [1-(6+r^2)w_2^2 + r^2 w_2^4] \}, \\
q_{13} &= \frac{\tilde{a}\omega}{2l(1-\omega^2)(\omega+2w_1)} (1-\omega^2-2\omega w_1-w_1^2), \\
q_{14} &= \frac{\tilde{a}\omega}{2l(1-\omega^2)(\omega-2w_1)} (1-\omega^2+2\omega w_1-w_1^2), \\
q_{15} &= \frac{r\tilde{\beta}}{2(1-5w_1^2+4w_1^4)(4w_2^2-w_2^2)} [100w_1^6-4w_2^2-3w_1^4 \\
&\quad \times (22+9w_2^2)+w_1^2(14+19w_2^2)], \\
q_{16} &= \frac{\tilde{\beta}}{w_2^2-(1+w_2)^2} [1+w_1^2-w_2(2+w_2)], \\
q_{17} &= \frac{2(w_1+w_2)}{w_2(2w_1+w_2)} r\tilde{\beta} w_1, \\
q_{18} &= \frac{\tilde{\beta}}{w_2^2-(1-w_2)^2} [1+w_1^2+w_2(2-w_2)], \\
q_{19} &= \frac{r\tilde{\beta}}{6w_2^4} \left\{ \tilde{\beta}^2 (6w_1^4-2r^2w_2^4+3r^2w_1^2w_2^4) + 6\tilde{\alpha}z_r^2w_2^4 + 12w_1^2w_2^2B_2\bar{B}_2 \right. \\
&\quad \left. + \frac{6w_2^4}{w_1^2-1} [2(1+w_1^2)B_1\bar{B}_1 - (4-w_1^2+w_1^4)B_2\bar{B}_2] \right\}, \\
q_{20} &= \tilde{\beta} \{ 2+r^2w_1^3[17w_1+5w_1^2+(22-5w_2^2)] \\
&\quad + w_1^2(8+13r^2-7r^2w_2^2) + w_1(8+3r^2-3r^2w_2^2) \}, \\
q_{21} &= \tilde{\beta} \{ r^2w_1^3[5w_1^2-17w_1+(22-5w_2^2)] - 2 \\
&\quad - w_1^2(8+13r^2-7r^2w_2^2) + w_1(8+3r^2-3r^2w_2^2) \}, \\
q_{22} &= \frac{\tilde{\beta}\omega^2}{2l(w_2^2-\omega^2)} \left\{ \frac{r\tilde{a}}{1-\omega^2} (3+w_1^2) + \frac{i\tilde{b}}{(\omega^2-w_1^2)(\omega^2-w_2^2)} \right. \\
&\quad \left. \times [\omega^2(4w_1^2-r^2w_2^2) - (2-r^2)\omega^4 - 2w_1^4] \right\}, \\
q_{23} &= \frac{\omega^2[i\tilde{r}\tilde{b}\omega(\omega^2-1)w_1+\tilde{a}(\omega^2-w_1^2)]}{l(\omega^2-1)(\omega^2-w_1^2)[(w_2^2-(\omega+w_1)^2)]}, \\
q_{24} &= \frac{\omega^2[i\tilde{r}\tilde{b}\omega(\omega^2-1)w_1+\tilde{a}(\omega^2-w_1^2)]}{l(\omega^2-1)(\omega^2-w_1^2)[(w_2^2-(w_1-\omega)^2)]}, \\
q_{25} &= \frac{2(2+3w_1+w_1^2)rw_1^2}{(1+2w_1)[w_2^2-(1+2w_1)^2]}, \\
q_{26} &= \frac{4(1-w_1^2)w_1^2}{(1-4w_1^2)(w_2^2-9w_1^2)}, \\
q_{27} &= \frac{2(2-3w_1+w_1^2)rw_1^2}{(2w_1-1)[w_2^2-(-1+2w_1)^2]},
\end{aligned}$$

$$\begin{aligned}
q_{28} &= \frac{r}{w_2^2-1} \left\{ \frac{\tilde{\beta}^2}{w_2^2(1-w_1^2)} [w_1^2(3-2w_1^2-w_1^4) + (3-r^2-(8+r^2)w_1^2 \right. \\
&\quad \left. + 3w_1^4 + 2w_1^6)w_2^2 + r^2(1+w_1^2)w_2^4] - \frac{4w_1^2(2-5w_1^2)(1-w_2^2)B_2\bar{B}_2}{1-4w_1^2} \right\}, \\
q_{29} &= \frac{\tilde{\beta}}{(1+2w_1)(w_1^2-1)[w_2^2-(1+w_1)^2]} \{ (1+w_1) \{ 2+w_1[6(1+r^2) \\
&\quad + w_1(18r^2-w_1[8-14r^2-r^2w_1(1-7w_1-7w_1^2+w_1^3)])] \} \\
&\quad + r^2w_1[4+w_1(10+w_1(6-w_1-6w_1^2-w_1^3))] \} w_2^2 \}, \\
q_{30} &= \frac{rw_1^2}{2(1-5w_1^2+4w_1^4)w_2^2(w_2^2-4w_1^2)^2} \{ 2r^2(1-5w_1^2+4w_1^4)w_2^4 \\
&\quad + \tilde{\beta}[8(w_1^4-5w_1^6+4w_1^8)-4w_1^2(22-63w_1^2+17w_1^4)w_2^2 + (23-68w_1^2+21w_1^4)w_2^4] \}, \\
q_{31} &= \frac{w_1}{4(1-4w_1^2)(w_1^2-w_2^2)^2w_2^2} \{ r^2\tilde{\beta}^2w_1(1-4w_1^2)[w_1^2(5r^2+4w_1^2)w_2^2-8w_1^4 \\
&\quad + 4(2-r^2-w_1^2)w_2^4] - 4w_2^2(w_1^2-w_2^2)[i\tilde{c}_2(1-4w_1^2)+4w_1(3B_1\bar{B}_1-(1-w_1^2)B_2\bar{B}_2)] \}, \\
q_{32} &= \frac{\tilde{\beta}}{(1-2w_1)(w_1^2-1)[w_2^2-(1-w_1)^2]} \{ (1-w_1) \{ 2+w_1[w_1(18r^2+w_1 \\
&\quad \times [8-14r^2+r^2w_1(1+7w_1-7w_1^2+w_1^3)]) - 6(1+r^2)] \} \\
&\quad + r^2w_1[4-w_1(10-w_1(6+w_1-6w_1^2+w_1^3))] \} w_2^2 \}, \\
q_{33} &= \frac{r\tilde{\beta}}{2w_2^2} \left\{ \tilde{\beta}^2[r^2w_1^2(w_1^2-1) + \frac{2}{w_2^2}w_1^6] + \frac{1}{(w_1^2-1)w_2^2} [2w_2^2[\tilde{\alpha}w_1^2(w_1^2-1)z_r^2 \right. \\
&\quad \left. + 2(1+w_1^2)^2B_1\bar{B}_1] - 2w_1^2[4w_1^2(1-w_1^2) + (7-2r^2+2r^2w_1^2+w_1^4)w_2^2]B_2\bar{B}_2] \right\},
\end{aligned}$$

Appendix B

$$\begin{aligned}
\Gamma_1 &= \frac{1}{2}[c_1+c_2+c_3+\frac{2}{a_{10}}d_3+\frac{2}{a_{20}a_{30}}(a_{30}d_7-d_{12})], \\
\Gamma_2 &= \frac{1}{4a_{10}a_{20}^2a_{30}} \{ a_{20}[a_{30}(2a_{20}[(c_1+c_2+c_3)d_3-2d_1d_2]+4d_3d_7+a_{10}[a_{20}(c_1c_2 \\
&\quad + (c_1+c_2)c_3)-4d_5d_6+2(c_1+c_2+c_3)d_7]) + 4a_{10}d_9d_{11}] - 2[2a_{20}d_3+a_{10} \\
&\quad \times (a_{20}(c_1+c_2+c_3)+2d_7)]d_{12} \}, \\
\Gamma_3 &= \frac{1}{8a_{10}a_{20}^2a_{30}} \{ 2a_{20}^2a_{30}[c_3(c_1d_3-2d_1d_2)-c_2(2d_1d_2-(c_1+c_3)d_3)] \\
&\quad - 8d_3d_7d_{12} - 4a_{20}[a_{30}(2d_1d_2d_7+d_3(2d_5d_6-(c_1+c_2+c_3)d_7)) \\
&\quad - 2d_1d_2d_{12} - d_3(2d_9d_{11}-(c_1+c_2+c_3)d_{12})] + a_{10}[a_{20}^2a_{30}c_1c_2c_3 \\
&\quad + 8d_5d_6d_{12} + 4d_7(2d_9d_{11}-(c_1+c_2+c_3)d_{12}) - 2a_{20}(a_{30}(c_3(2d_5d_6 \\
&\quad - c_2d_7) + c_1(2d_5d_6-(c_2+c_3)d_7)) - c_2(2d_9d_{11}-c_3d_{12}) - c_1(2d_9d_{11} \\
&\quad - (c_2+c_3)d_{12}))] \}, \\
\Gamma_4 &= \frac{1}{-8a_{10}a_{20}^2a_{30}} \{ a_{20}^2a_{30}c_2c_3(c_1d_3-2d_1d_2) + 4[2d_3d_7-a_{10}(2d_5d_6-(c_1+c_2)d_7)]d_9d_{11} \\
&\quad + 2[2(a_{10}(c_1+c_3)+2d_3)d_5d_6-(a_{10}(c_1c_2+(c_1+c_2)c_3)-4d_1d_2 \\
&\quad + 2(c_1+c_2+c_3)d_3)d_7]d_{12} - a_{20}[a_{10}c_1[a_{30}c_3(c_2d_7-2d_5d_6)+c_2(2d_9d_{11} \\
&\quad - c_3d_{12})] + 2[2(c_1+c_2)d_3-2d_1d_2]d_9d_{11} + [2(c_2+c_3)d_1d_2-(c_2c_3 \\
&\quad + c_1(c_2+c_3))d_3]d_{12} + a_{30}(d_3[(c_2c_3+c_1(c_2+c_3))d_7-2(c_1+c_3)d_5d_6] \\
&\quad - 2d_1[d_2(-2d_5d_6+(c_2+c_3)d_7)d_4^2d_5w_1])] \}, \\
\Gamma_5 &= \frac{1}{8a_{10}a_{20}^2a_{30}} \{ 4c_1d_3d_7d_9d_{11} + 4c_2d_3d_7d_9d_{11} - 8d_3d_5d_6d_9d_{11} - 8d_1d_2d_7d_9d_{11} \\
&\quad - 8d_1d_2d_5d_6d_{12} + 4c_1d_3d_5d_6d_{12} + 4c_3d_3d_5d_6d_{12} + 4c_2d_1d_2d_7d_{12} + 4c_3d_1d_2d_7d_{12} \\
&\quad - 2c_1c_2d_3d_7d_{12} - 2c_1c_3d_3d_7d_{12} - 2c_2c_3d_3d_7d_{12} - a_{10}c_1(2d_5d_6-c_2d_7)(2d_9d_{11}-c_3d_{12}) \\
&\quad + a_{20}(2d_1d_2-c_1d_3)[a_{30}c_3(2d_5d_6-c_2d_7)-c_2(2d_9d_{11}-c_3d_{12})] \\
&\quad - 2d_1d_4^2d_5(a_{20}a_{30}c_3-2d_{12})w_1 \}, \\
\Gamma_6 &= \frac{1}{8a_{10}a_{20}^2a_{30}} (2d_9d_{11}-c_3d_{12}) \{ c_1d_3(c_2d_7-2d_5d_6) \\
&\quad - 2d_1[d_2(c_2d_7-2d_5d_6)+d_4^2d_5w_1] \},
\end{aligned}$$

where,

$$\begin{aligned}
 d_1 &= \frac{1}{2} f_1 \cos \theta_{10}, \\
 d_2 &= \sigma_1 - \alpha z_r - \frac{r^2 \beta^2}{2(1-w_1^2)} (1+w_1^2)^2 - \frac{3(1-w_1^2)}{4(1-4w_1^2)} w_1^2 a_{20}^2, \\
 d_3 &= \frac{1}{2} f_1 \sin \theta_{10}, \\
 d_4 &= \frac{3(1-w_1^2)}{(1-4w_1^2)} w_1 a_{10} a_{20}, \\
 d_5 &= \frac{m_0}{2w_1} \cos \theta_{20}, \\
 d_6 &= \sigma_2 - w_1 \left\{ \alpha z_r^2 - r^2 \beta + \frac{(1-w_1^2)}{8(1-4w_1^2)} [12a_{10}^2 - 3(1+8w_1^2)a_{20}^2] \right. \\
 &\quad \left. - \frac{r^2 \beta^2}{4(1-w_1^2)w_2^2} [(14-3r^2)w_2^2 - 2(4-w_2^2)w_1^4 + (8+3r^2w_2^2)w_1^2] \right\}, \\
 d_7 &= \frac{m_0}{2w_1} \sin \theta_{20}, \\
 d_8 &= \frac{3(1-w_1^2)}{2(1-4w_1^2)w_2} w_1^2 a_{20}^2 \sin \theta_{30}, \\
 d_9 &= \frac{(1-w_1^2)}{2(1-4w_1^2)w_2} w_1^2 a_{30}^2 \cos \theta_{30}, \\
 d_{10} &= a_{30} \left\{ \sigma_3 + 3w_1 [\alpha z_r^2 - r^2 \beta - \frac{r^2 \beta^2}{4(1-w_1^2)w_2^2} [(14-3r^2)w_2^2 - 2(4-w_2^2)w_1^4 \right. \\
 &\quad \left. + (8+3r^2w_2^2)w_1^2]] + \frac{1}{2w_2} [\beta(w_1^2 - w_2^2) + \frac{\beta^2}{4w_2^2} (4r^2w_2^4 + 4w_2^2(w_1^2 - w_2^2) \right. \\
 &\quad \left. + (w_1^2 - w_2^2)^2)] \right\} + \frac{3w_1}{8(1-4w_1^2)} (1-w_1^2) [12a_{10}^2 a_{30} - 3(1+8w_1^2)a_{20}^2 a_{30}] \\
 &\quad + \frac{3}{2w_2} w_1^2 a_{20}^2 a_{30} - \frac{2(1-w_1^2)}{(1-4w_1^2)w_2} w_1^2 a_{20}^3 \cos \theta_{30}, \\
 d_{11} &= a_{20} \left\{ \sigma_3 + 3w_1 [\alpha z_r^2 - r^2 \beta - \frac{r^2 \beta^2}{4(1-w_1^2)w_2^2} [(14-3r^2)w_2^2 - 2(4-w_2^2)w_1^4 \right. \\
 &\quad \left. + (8+3r^2w_2^2)w_1^2]] + \frac{\beta}{2w_2} [(w_1^2 - w_2^2) + \frac{\beta}{4w_2^2} (4r^2w_2^4 + 4w_2^2(w_1^2 - w_2^2) \right. \\
 &\quad \left. + (w_1^2 - w_2^2)^2)] \right\} + \frac{3w_1(1-w_1^2)}{8(1-4w_1^2)} [12a_{10}^2 a_{30} - 3(1+8w_1^2)a_{20}^3] \\
 &\quad + \frac{w_1^2}{4w_2} a_{20}^3 - \frac{3}{2w_1} m_0 \cos \theta_{20}, \\
 d_{12} &= \frac{(1-w_1^2)}{2(1-4w_1^2)w_2} w_1^2 a_{20}^4 \sin \theta_{30}.
 \end{aligned}$$

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