applied sciences

MDPI

Article

Comparison of Traffic Flow Models with Real Traffic Data Based on a Quantitative Assessment

Aleksandra Romanowska * and Kazimierz Jamroz

Faculty of Civil and Environmental Engineering, Gdansk University of Technology, 11/12 Narutowicza Str., 80-233 Gdansk, Poland; kjamroz@pg.edu.pl

* Correspondence: aleksandra.romanowska@pg.edu.pl

Abstract: The fundamental relationship of traffic flow and bivariate relations between speed and flow, speed and density, and flow and density are of great importance in transportation engineering. Fundamental relationship models may be applied to assess and forecast traffic conditions at uninterrupted traffic flow facilities. The objective of the article was to analyze and compare existing models of the fundamental relationship. To that end, we proposed a universal and quantitative method for assessing models of the fundamental relationship based on real traffic data from a Polish expressway. The proposed methodology seeks to address the problem of finding the best deterministic model to describe the empirical relationship between fundamental traffic flow parameters: average speed, flow, and density based on simple and transparent criteria. Both single and multi-regime models were considered: a total of 17 models. For the given data, the results helped to identify the best performing models that meet the boundary conditions and ensure simplicity, empirical accuracy, and good estimation of traffic flow parameters.

Keywords: traffic flow models; fundamental relationship of traffic flow; uninterrupted traffic flow; traffic conditions analysis; single-regime models; multi-regime models

check for updates

Citation: Romanowska, A.; Jamroz, K. Comparison of Traffic Flow Models with Real Traffic Data Based on a Quantitative Assessment. *Appl. Sci.* **2021**, *11*, 9914. https://doi.org/10.3390/app11219914

Academic Editor: Sehyun Tak

Received: 23 September 2021
Accepted: 20 October 2021
Published: 23 October 2021

Publisher's Note: MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Copyright: © 2021 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https://creativecommons.org/licenses/by/4.0/).

1. Introduction

Traffic flow models describe vehicle flows using three basic parameters: average speed (v), density (k), and flow (q). When traffic flow is considered a stationary phenomenon, the parameters are linked with one another using a relationship (1), which is known as the fundamental relationship.

$$q = kv \quad (1)$$

The relationship (1) and bivariate relations between speed and flow, speed and density, and flow and density are of great importance in transportation engineering. They are used in planning, design and redesign, operation, and control of transportation facilities; they help to determine the actual capacity of an existing road, assess its traffic conditions or select the cross-section for a new road depending on forecasted traffic volumes.

The relations between speed, flow, and density have been proved empirically in many studies over the last nine decades, starting from Greenshields in the 1930s [1]. Since then, researchers have been exploring ways to offer the best possible mathematical description of the relations. Historically, these were established either empirically by looking for mathematical models that fit the observed data or derived from microscopic traffic flow characteristics or from the analogy to the movement of fluid. The models proposed in the literature are predominantly deterministic and describe the average behavior of a traffic stream. The output of the model is fully determined by the initial conditions and the parameters. Since they are simple to use, interpret, and determine traffic characteristics, the models have a wide variety of applications, including methods for forecasting and assessing traffic conditions.

In 1995, Castillo and Benitez [2] concluded that while the literature includes a number of proposed solutions, a mathematical deterministic model still has not been found to

give a sufficiently good description of the fundamental relationship for all facility types and ranges of traffic density. Later literature studies also suggest that the problem is still open [3,4]. Similarly, it is not clear from the literature, which of the existing models can be perceived as the best and what the criteria would be. The latter are not commonly agreed, and even if used, the assessment or benchmarking of the models is often a subjective matter.

The problem of selecting traffic flow model has practical implications. The models representing the relationship between speed, flow, and density serve as a basis for traffic conditions forecasts and assessments such as in the case of United States' HCM [5] or Germany's HBS [6]. Thus, it was also one of the problems to be solved in the context of the recent development of a new Polish method for highway capacity analyses. How to choose the right model for this purpose? What expectations should be given? How to assess whether the model performs well or not? These questions motivated us to the work presented in the paper.

The main objective of the article is to review and compare existing models of the fundamental relationship using empirical data from Polish roads. In particular, the article aims to: (1) Identify the criteria to be met by a good model of the fundamental relationship; (2) Develop a transparent and quantitative method for assessing fundamental relationship models based on identified criteria; (3) Assess, compare, and rank the existing models of the fundamental relationship.

The paper is organized as follows: Section 2 gives an overview of deterministic mathematical models of the fundamental relationship and criteria and methods for assessing them. Section 3 discusses the proposed methodology for assessing models of the fundamental relationship. Section 4 presents an application of the methodology for field data. The results are discussed in Section 5.

2. Theoretical Background

2.1. Modeling Equilibrium Traffic Flow Relationships

Fundamental relationship models are divided into single-regime and multi-regime models, depending on how traffic conditions are described. Single-regime models describe the relationship using a single mathematical function for an entire range of traffic conditions. Multi-regime models use two or more mathematical functions for this purpose, each representing different conditions of traffic.

An important feature of speed–flow–density relationships is the presence of characteristic, boundary values of the particular relations (boundary points, extremes, and points of inflection):

- Boundary points: free flow speed (v_{sw}) occurs theoretically when volume and density approach zero ($k \to 0$, $q \to 0$); maximal density (k_{max}) occurs when flow and speed approach zero ($q \to 0$, $v \to 0$);
- Extremes and points of inflection of the function: maximal flow (q_{max}) and the corresponding optimal speed (v_{opt}) and optimal density (k_{opt}).

The values of these parameters define the particular traffic regimes: free-flow conditions when density increases, flow rises from zero until the maximal value is reached ($q \to q_{max}$), and speed decreases from the maximal value (v_{sw}) until the optimal value is reached ($v \to v_{opt}$); congested conditions when density keeps increasing until it reaches the maximal value ($k \to k_{max}$), causing speed and flow to decrease to zero ($v \to 0$, $q \to 0$). The characteristics helps to identify the boundary values of the fundamental relationship (further referred to as BC1 and BC2): (a) at zero density, the speed is equal to the free flow speed, (b) at maximum density (jam density), the speed is zero. These conditions are referred to as static properties of traffic flow [2].

2.1.1. Single-Regime Traffic Flow Models

Conducted nearly ninety years ago, Greenshields' work [1] was followed by a number of models of the fundamental relationship, which differed on functional form, the number of parameters, and the relationship they represented. Table 1 lists selected models.

Table 1. List of selected single-regime models.

Author (Year of Publication)	Basic Function	Parameters
Greenshields (1935)	$v = v_{sw}\left(1 - \frac{k}{k_{max}}\right)$	v_{sw}, k_{max}
Greenberg (1959)	$v = v_{opt} \ln\left(\frac{k_{max}}{k}\right)$	v_{opt}, k_{max}
Underwood (1960)	$v = v_{sw} \exp\left(-\frac{k}{k_{opt}}\right)$	v_{sw}, k_{opt}
Newell (1961)	$v = v_{sw}\left(1 - \exp\left(-\frac{a}{v_{sw}}\left(\frac{1}{k} - \frac{1}{k_{max}}\right)\right)\right)$	v_{sw}, k_{max}, a
Pipes-Munjal (1967)	$v = v_{sw}\left(1 - \left(\frac{k}{k_{max}}\right)^n\right)$	v_{sw}, k_{max}, n
Northwestern(1967)	$v = v_{sw} \exp\left(-\frac{1}{2}\left(\frac{k}{k_{opt}}\right)^2\right)$	v_{sw}, k_{opt}
Drew (1965)	$v = v_{sw}\left(1 - \left(\frac{k}{k_{max}}\right)^{n+\frac{1}{2}}\right)$	v_{sw}, k_{max}, n
Krystek (1980)	$v = v_{sw}\left(1 - \frac{k}{k_{max}}\right)^4$	v_{sw}, k_{max}
Kerner and Konhäuser (1995)	$v = v_{sw}\left(\frac{1}{1+\exp\left(\frac{\frac{k}{k_{max}} - 0.25}{0.06}\right)} - 3.72 \times 10^{-6}\right)$	v_{sw}, k_{max}
Del Castillo and Benitez (1995)	$v = v_{sw}\left(1 - \exp\left(\frac{\lvert C_j \rvert}{v_{sw}}\left(1 - \frac{k_{max}}{k}\right)\right)\right)$	v_{sw}, k_{max}, C_j
Van Aerde (1995)	$k = \frac{1}{c_1 + \frac{c_2}{v_{sw} - v} + c_3 v}$	v_{sw}, c_1, c_2, c_3
MacNicholas (2008)	$v = v_{sw}\left(\frac{k_{max}^n - k^n}{k_{max}^n + mk^n}\right)$	$v_{sw}, k_{max}, n,$ m
Wang (2011)	$v = v_{min} + \left(\frac{v_{sw} - v_{min}}{\left(1+\exp\left(\frac{k - k_{opt}}{a}\right)\right)^b}\right)$	$v_{sw}, v_{min}, k_{opt}, a, b$
Kucharski and Drabicki (2017)	$v = \frac{v_{sw}}{1 + a\left(\frac{k}{k_{opt}}\right)^b}$	v_{sw}, k_{opt}, a, b

While Greenshields' linear model [1] is very simple to use, it comes with a limitation that is evident in almost every study, namely its inconsistency with the field data for any facility types and ranges of traffic density. Greenberg [7] applied empirical data from Lincoln's tunnel and matched them with a logarithmic function derived from hydrodynamic analogy [8]. The model's downside is that it does not meet BC1—when density tends to zero, speed tends to infinity. In response to the limitation in Greenberg's model, Underwood [9] proposed an exponential model. The issue here is that the model does not have a finite density value and as a result fails to satisfy BC2. The literature offers a modification to both models, i.e., Greenberg's and Underwood's, using Taylor's series [10]. Pipes [11] suggested a modification of Greenshields' model by adding a new parameter deduced from the car-following theory. Depending on its value, the relationship v(k) takes a concave or convex form or when n = 1, and it is transformed to Greenshields' model. A similar approach was proposed by Krystek [12] and Drew [13]. Developed from urban arterial data, Krystek's model modification of Greenshields' model involves the use of an exponent of 4. Drew's model [13] is similar to Pipes–Munjal's model where the n parameter, which is the exponent, is replaced with the expression $n + \frac{1}{2}$.

There are other more complex models where the function v(k) is reverse S-shaped. The outcome is a better representation of the nature of the phenomenon of how vehicle flows move. Northwestern's was the first such model [14]; it was a modification of Underwood's model that was built by adding constants that change the shape of the relation v(k) to that of a bell. The model comes with a limitation similar to that of Underwood's model (failure to satisfy BC2) and just as previously can only be solved by using Taylor's series. Similarly to Underwood or Drake, the exponential function was also applied by Newell, Del Castillo

and Benitez [2], and Kerner and Konhäuser [15]. Newell's exponential model, similarly to Pipes' model, was deduced from the car-following theory. The model satisfies both BC1 and BC2. Speed in the model falls rapidly as density increases, which is considered a limitation [4]. Del Castillo and Benitez [2] assumed that vehicle flow is strongly affected by the parameter of kinematic wave speed at jam density C_j—the parameter was included in the model. Kerner and Konhäuser [15] proposed a model based on two parameters extended by three constant numerical values that cannot be interpreted in terms of physical parameters and may be seen as a downside. Van Aerde's model [16] was derived from a microscopic model of following the leader and combines Pipes' and Greenshields' model into a single-regime model [17]. The advantage of the model is that it ensures a good match to data for a wide variety of facility types and for entire ranges of densities. Its disadvantage is that it is computationally complex (c_1, c_2, c_3 are parameters that must be determined), and variable k is used as a dependent variable, which makes the model inconvenient to use [18]. MacNicholas [18] offered a simpler alternative to Van Aerde's model. His own model reported comparable empirical accuracy to Van Aerde's model. Wang [4] proposed using the logistic function to describe the relationship $v = f(k)$. He assumed that even when traffic is very dense, vehicles move at a finite minimal speed v_{min}. In the model, a, b are shape parameters. The outcome of modeling is a sigmoid curve v(k) with a very good match to empirical data [4]. On the downside, speed never reaches zero in the model and density never reaches a finite value, which is inconsistent with BC2. One of the recent models is the four-parameter v = f(k) model proposed by Kucharski and Drabicki [19] built based on a BPR speed–flow equation [20] and overcoming its limitation to represent only a free-flow traffic regime. The downside is that the model fails to satisfy BC2.

Figure 1 graphically shows how the relationship $v = f(k)$ is represented by selected models. The evolution from the simplest linear form to the relationship being represented by an S-shaped curve can be seen from the figure.

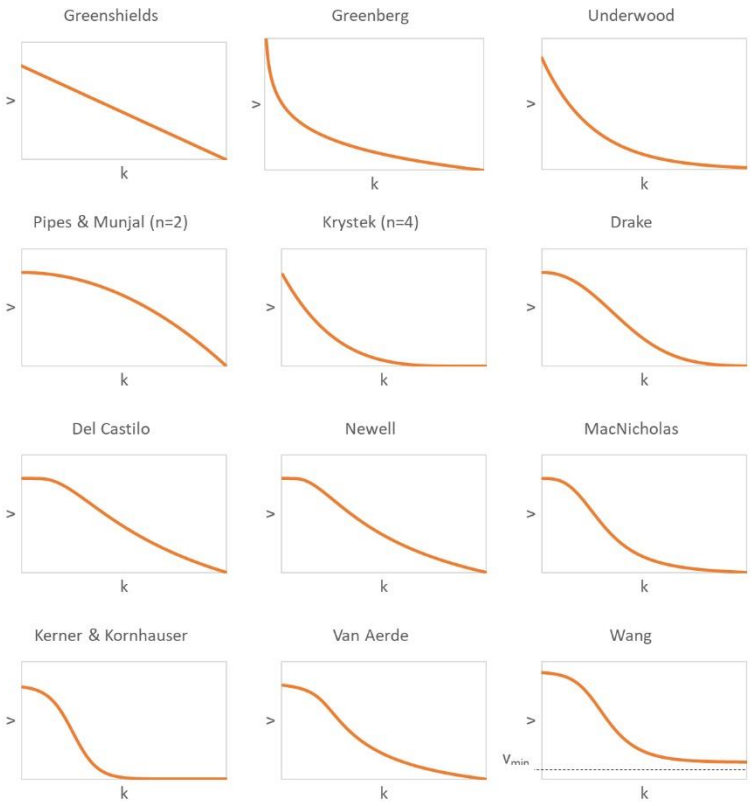


Figure 1. Representation of the speed–density relationship in selected traffic flow models.

There is also a group of models (with v = f(q) BPR function [20] as an example) that represent the speed–flow–density relationship for a free-flow regime only and do not consider traffic flow characteristics after the maximum flow and corresponding optimum density are reached. It is not a limitation if the purpose is to analyze traffic under free-flow conditions; however, to estimate values of traffic flow parameters at the maximum flow (q_{max}, k_{opt}, v_{opt}), some initial assumptions regarding the boundary of the free-flow regime need to be made. As a result of these reasons, these kinds of models are not included in Table 1.

2.1.2. Multi-Regime Traffic Flow Models

Researchers proposed multi-regime models primarily for their empirical accuracy. However, this can only be obtained at the cost of a more complicated functional form and much more complicated calibration with field data. The main problem with multi-regime models is determining the boundaries of the particular traffic conditions described by separate functions and determining the points of transition from one state to another. Table 2 lists selected two-regime models.

Table 2. List of selected multi-regime models.

Author (Year of Publication)	Basic Function	Parameters
Edie (1961)	$v = \begin{cases} v_{sw} \exp\left(-\frac{k}{k_{opt}}\right) & k < k_1 \\ v_{opt} \ln\left(\frac{k_{max}}{k}\right) & k \geq k_1 \end{cases}$	v_{sw}, k_{opt}, v_{opt}, k_{max}, k_1
Smulders (1989)	$v = \begin{cases} v_{sw} - \alpha k & k \leq k_1 \\ d\left(\frac{1}{k} - \frac{1}{k_{max}}\right) & k > k_1 \end{cases}$, where $d = \frac{v_{sw} - \alpha k_{opt}}{\frac{1}{k_{opt}} - \frac{1}{k_{max}}}$	v_{sw}, α, k_{opt}, k_{max}, k_1
Triangular	$q = \begin{cases} v_{sw} k & k < k_1 \\ q_{max} - \frac{k - k_{opt}}{k_{max} - k_{opt}} q_{max} & k \geq k_1 \end{cases}$	v_{sw}, q_{max}, k_{opt}, k_{max}, k_1
Daganzo (1997)	$q = \begin{cases} v_{sw} k & k < k_1 \\ v_{sw} k_1 & k_1 < k < k_2 \\ q_{max} - \frac{k - k_2}{k_{max} - k_2} q_{max} & k \geq k_2 \end{cases}$	
Wu (2002)	$q = \begin{cases} k\left(1 - \left(\frac{k}{k_1}\right)^{l-1} v_{sw} + \left(\frac{k}{k_1}\right)^{l-1} v_{opt} & v > v_{opt} \\ v_{opt} k_1 - \frac{k - k_1}{k_{max} - k_1} v_{sw} k_1 & k \geq k_1 \end{cases}$	v_{sw}, v_{opt}, k_{max}, l, k_1

where: k_1, k_2—boundary values of density at which there is a transition between traffic flow regimes.

Edie was one of the first researchers to suggest a description of the relationship between volume and density using a non-continuous curve. In 1961, he presented a two-regime traffic model making the distinction between free-flow and congested operation. The model is a combination of Greenberg's and Underwood's models [7,9]. Smulders [21] proposed a two-regime model where speed falls linearly in free-flow traffic as density increases. In congested traffic, volume falls linearly as density increases. The triangular model is very popular in urban traffic control. It represents the relationship q(k) by two straight lines. Following a modification by Daganzo, the model is a truncated triangle where the same relationship is represented by three straight lines [22,23]. Wu proposed a two-regime model of volume variability q relative to density k, where the shape of the curve in free-flow will depend on the number of lanes l [23].

Figure 2 presents how the models summarized above represent the relationships.

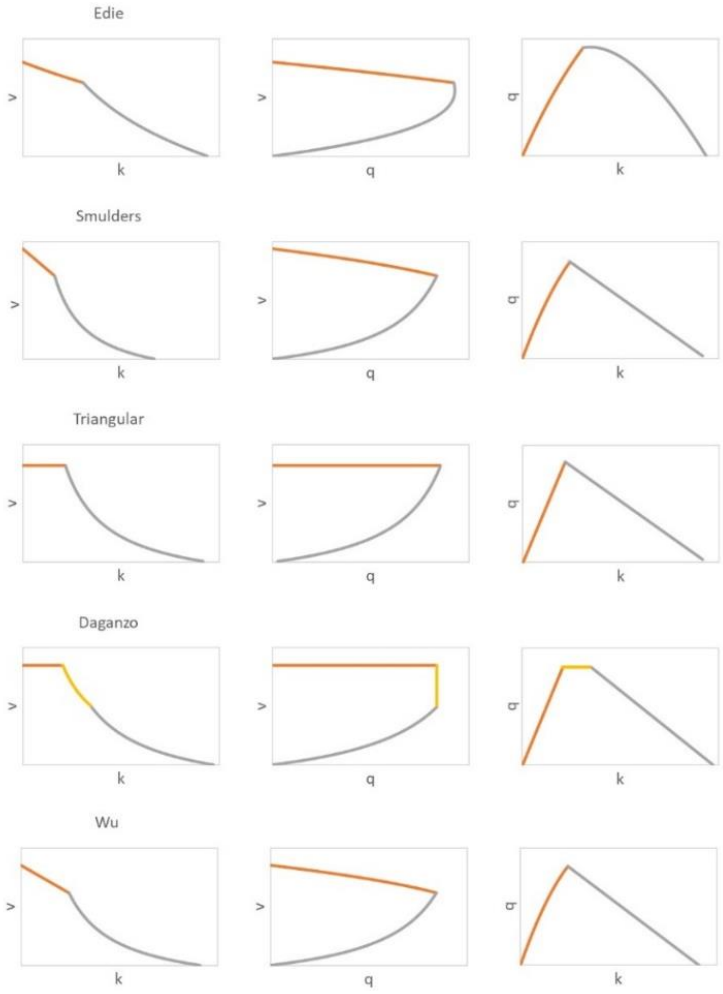


Figure 2. Relationships v(k), v(q), and q(k) in selected multi-regime models (assuming model continuity).

2.2. Criteria and Methods for the Comparison of Traffic Flow Models

The need to compare models becomes evident when scientists or practitioners must decide which model will work best for their data and select the best from several models. There are a number of criteria that may be used for assessment and comparison of computational models [24]: goodness of fit (which is the empirical accuracy of the model), simplicity or complexity (regarding models' functional form and number of parameters), generalizability (ability to use the model for forecasting future observations), faithfulness (whether the model represents the regularities that are the basis of the phenomenon being modeled), interpretability (whether the model's parameters are clear and linked to the phenomena being modeled), explanatory adequacy (whether the model's theoretical representation is sufficient to understand the data being observed), and falsifiability (whether potential observations exist that are inconsistent with the model). As pointed out by Myung and Pitt [24], among these, the most important and the most widely used are criteria lending to quantification (i.e., goodness of fit, simplicity). However, qualitative criteria may provide a relevant and valuable complement to model assessment. It is the researcher's decision to select criteria to match those features of the model that they consider most important.

Most of the above criteria can also apply to models of the fundamental relationship. What makes a good traffic flow model? According to Wang et al. [4], it should have a simple functional form (be "mathematically elegant"), fit the data well, have interpretable parameters, and allow the entire range of traffic conditions to be modeled. The latter suggests that a single-regime model is preferred over a multi-regime model, which is confirmed in the works of MacNicholas and Underwood [9,18]. In principle, the criteria given by

MacNicholas and Underwood are consistent with those identified by Wang. MacNicholas mentions an additional aspect of having to meet boundary conditions. Underwood, on the other hand, points out that the model should easily lend itself to mathematical analysis and that its limitations should be known in advance.

A review of the above works suggests that researchers are in agreement as to what criteria should be met by the fundamental relationship model. As a result, the criteria they identified may be used to assess and compare models of the fundamental relationship. This type of comparison in the literature usually has an expert basis rather than concrete criteria and measures to assess the criteria. The criteria researchers refer to most often include goodness of fit, which is a quantitative measure. In addition, faithfulness of the model is often assessed by checking how the model fits in with the empirical data and how the phenomena are represented on the fundamental diagram. An example is the work of Gaddam and Rao [25], who assess models matching data from two sections of an urban arterial in Delhi with goodness of fit, using root mean square error, average relative error, and cumulative residual plots for the assessment. To complement the criterion assessment, expert judgement on the properties of the models on the speed–density diagrams is applied. To assess existing models for a 55 mph motorway section, May [26] used two quantitative measures: goodness of fit, measured by mean deviation, and parameter value, which is assessed by checking if they are consistent with the ranges established from empirical data. It is not clear from the publication how these ranges were identified. The same criterion was used by Rakha [27] for the same data and the same ranges of parameters as May [26], comparing known single and multi-regime models with Van Aerde's model. Cheng et al. [28] compared 10 single-regime models using goodness of fit and tested the models' stability by comparing how the same model parameters change on different sections of the same road. In many other studies, models are compared with a qualitative assessment only, e.g., a visual check is made of the match between the models and empirical data, limitations are identified, and model properties are assessed [4,17,29].

The conclusion from the review of the literature is that while researchers are in agreement about the criteria of a good model of the fundamental relationship, there is no agreement as to how traffic flow models should be assessed or compared. There are no commonly agreed criteria that would apply to the assessment, comparison, or benchmarking of traffic flow models. Similarly, ways to assess particular criteria are not agreed, and it is usually up to the individual interpretation of the researcher.

3. Materials and Methods

3.1. Data

To develop and apply the methodology, the data were sourced from a section of the S6 express road located in Gdansk, Poland (54°25′ N, 18°29′ E). The section is part of a dual carriageway with four lanes running within the conurbation. The speed limit for passenger cars is 120 km/h and 80 km/h for heavy goods vehicles. The share of trucks in overall traffic is 9% on average. Annual average daily traffic is 74,000 vehicles per day. During peaks of traffic (summer holidays), volumes are observed to exceed 100,000 vehicles per day.

The data [30] come from a continuous traffic measurement station which is operated by a double induction loop. Vehicles crossing the station are automatically detected by devices installed on each lane, which register the time a vehicle is detected, ascertain its spot speed, and identify the type of vehicle and lane it is using.

The data cover a period of 36 months between 2014 and 2017 on the southbound carriageway. A total of 37.5 million vehicles were recorded over that period.

Structured Query Language (SQL) was used for data processing and initial analysis. The data processing was divided into the following stages:

1. Raw traffic data that were provided by the national road authority (General Director for National Roads and Motorways, GDDKiA) in the text file format were imported to the database on the installed SQL server.

2. The data were verified in terms of empty rows, zero values, vehicle speeds beyond the expected range, and unusual vehicle lengths. The problem of zeros or unusual values concerned approximately 2% of registered vehicles and had marginal impact on the number of registered vehicles—the records were excluded from further processing.
3. Individual vehicle headways were calculated for each record.
4. The data were aggregated into 5 min intervals generating information about traffic volume, space-mean speed, share of heavy goods vehicles, or average headways. The traffic volume was calculated into flow rate using passenger car equivalents [5]. Traffic density was determined from the relation (1).

The database was extended with weather information (condition and intensity of precipitation, condition of road surface, horizontal visibility), conditions of natural lighting (dusk, day, dawn, night), and presence of road events (road works, collisions, accidents, stationary vehicles, etc.). Precipitation, wet or snow-covered/icy road surface, lack of natural lighting, and visibility below 200 m were proved to have a significant impact on space-mean speed [31]. Thus, periods of road events and adverse weather and lighting conditions were excluded from further analyses.

3.2. Method

Empirical data are shown in the fundamental diagram (Figure 3). They are represented by the scattered cloud of empirical data for the entire range of traffic conditions. The phenomena are reported by many studies and explained by e.g., heterogeneity of drivers, non-stationary dynamical features of traffic flow, randomness in individual driving behaviors, and errors in measurement methods or data processing [32–35]. The existence of the scatter raises some questions: What should be the functional form of the deterministic model to ensure the highest empirical accuracy? Should it be represented by a single or multi-regime model? How to estimate boundary parameters and what should be their values?

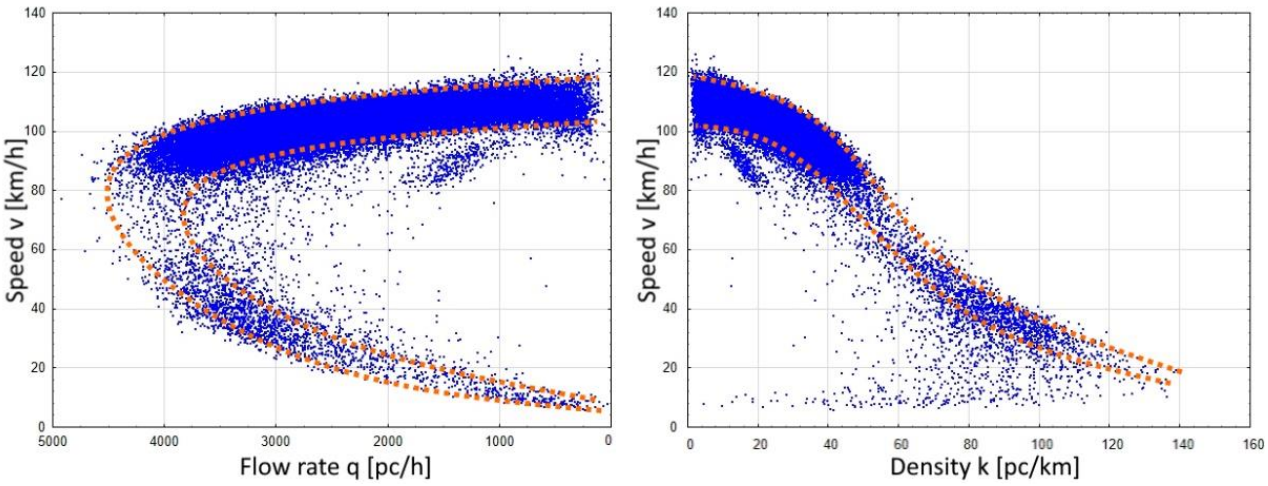


Figure 3. Empirical relations between speed, density, and flow.

To answer the questions, work was divided into the following stages: (stage I) data preparation and determination of expected values of boundary parameters, (stage II) selection of criteria for model assessment and adoption of detailed principles for criteria assessment and criteria acceptance levels, (stage III) model calibration, (stage IV) model assessment and comparison.

3.2.1. Preparation of the Data and Determination of Expected Parameter Values

The scatter on the empirical fundamental diagram may make model assessment and the identification of real boundary traffic parameters more difficult. The classical definition of the fundamental relationship assumes that traffic is stationary and homogeneous, which

means that vehicles behave the same way in similar traffic circumstances [22]. This suggests that for the same traffic density, vehicles in a stream will move with the same speed. Building on this, an averaged representation of traffic conditions was determined from the data. This helps to significantly reduce the number of points in the diagram, making a visual assessment easier. It will be possible to state whether the actual traffic conditions are correctly represented in the model. In the proposed approach, observed densities are divided into narrow ranges e.g., 0.1 pc/km wide. The empirical mean of the other parameters, i.e., speed and flow rate, are computed. A similar approach can be found in the works of Rakha and Arafeh and Zheng et al. [36,37]. Figure 4 presents the data averaged against density. As we can see, the scatter has been significantly reduced with only the congested traffic still represented by a cloud of points. This may be partly due to a much smaller sample of observed congested traffic—from nearly 110,000 of analyzed time intervals, only 3000 represented densities above 20 pc/h/lane (as a result for high traffic densities, the parameters are averaged on the basis of just a few time intervals).

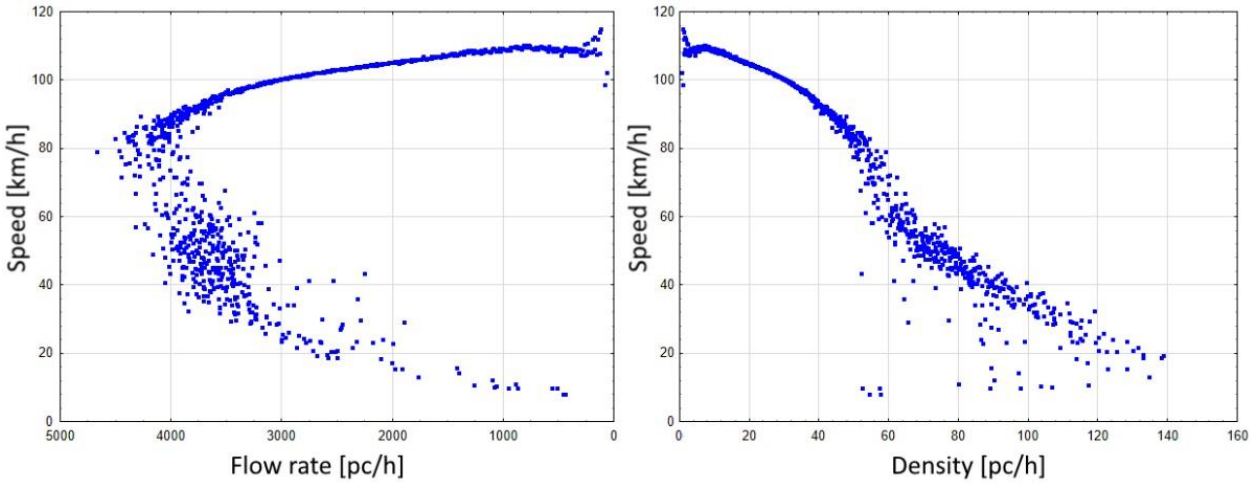


Figure 4. Curves v(q) and v(k) that determine the averaged representation of traffic conditions.

Once aggregated, the data may also be used to determine expected ranges of boundary parameters. However, to determine parameters within the maximal flow area (q_{max}, k_{opt}, v_{opt}), the data must represent both free-flow and congested traffic. The expected ranges of boundary parameters are determined as follows (Table 3):

- v_{sw}, by determining the 5th and 95th percentiles of speed (v_5, v_{95}) under conditions of low traffic volume, i.e., for corresponding volumes of $q \leq 500$ pc/h/lane;
- q_{max}, by determining the 95th and 99th percentiles of traffic volume (q_{95}, q_{99});
- v_{opt}, by determining the 5th and 95th percentiles of speed (v_5, v_{95}) under conditions of high traffic volume, where $q_{max_1} \leq q \leq q_{max_2}$ and in free-flow traffic, i.e., assuming that $v \geq 60$ km/h (to exclude the effect of vehicle stream speed in congested traffic);
- k_{opt}, by determining the 5th and 95th percentiles of density (k_5, k_{95}) occurring in the expected range of optimal speeds ($v_{opt_1} \leq v \leq v_{opt_2}$).

With the 5th and 95th percentile as the boundary values of the range v_{sw}, v_{opt}, k_{opt} it is possible to determine areas that include the value of the given parameter with a 90% certainty. This approach helps to exclude the effects of extreme values and outliers on the results (meaning parameter ranges). In the case of q_{max}, using the 95th percentile helps to identify the lower boundary of the expected range, which determines 5% of the highest traffic volumes; the 99th percentile in the upper boundary of the range helps to exclude the effects of 1% of the highest values on the expected parameter ranges.

In the case of maximal density, which is difficult to spot in empirical data, results from a Polish study [38] were used to identify the expected range k_{max}. Aerial observation of traffic flows (an aeromobile platform about 400 m high covering a section of about 1.5 km)

helped to register jams on a selected dual carriageway and four lane sections of Polish motorways. The closest to a maximal traffic flow density was when a platoon consisting exclusively of passenger cars was moving at an average speed of 3.1 km/h, when the average traffic density per lane was 96.6 veh/km/lane. Building on the study, we can assume that the lower boundary of the range k_{max} is 100 pc/km/lane. According to this definition and the research quoted before, when density is in the order of 100 pc/km/lane, speed should be equal to zero or be close to zero. The upper boundary may be determined based on average vehicle length and minimal headways; for an average vehicle length of 5 m and minimal headway of 5.5 m, maximal density will be about 150 pc/km/lane.

Table 3. Determining the expected range of boundary parameter values from an averaged representation of traffic conditions.

Par.	The Lower Range Limit	The Upper Range Limit	Condition	The Expected Range (Field Data)
v_{sw}	$v_{sw_1} = v_5$	$v_{sw_2} = v_{95}$	$q \leq 500$ pc/h/lane	107 ÷ 113 km/h
q_{max}	$q_{max_1} = q_{95}$	$q_{max_2} = q_{99}$		4175 ÷ 4386 pc/h
v_{opt}	$v_{opt_1} = v_5$	$v_{opt_2} = v_{95}$	$q_{max_1} \leq q \leq q_{max_2}$ $v \geq 60$ km/h	72 ÷ 86 km/h
k_{opt}	$k_{opt_1} = k_5$	$k_{opt_2} = k_{95}$	$v_{opt_1} \leq v \leq v_{opt_2}$	47 ÷ 58 pc/km
k_{max}	based on aerial measurements	calculated assuming minimal headways of 5.5 m	n.a.	200 ÷ 300 pc/km

3.2.2. Selection of Criteria and Adoption of Rules for Model Assessment

Based on the literature review, the proposed solution is to adopt the following criteria for model assessment:

1. Boundary conditions are met. A model representing the full range of traffic conditions should meet the following conditions: BC1: $v \rightarrow v_{sw}$ when $k \rightarrow 0$ and $q \rightarrow 0$ BC2: $v \rightarrow 0$ when $k \rightarrow k_{max}$ and exclusively in the case of two-regime models: BC3: $v_{sw_1} = v_{sw2}$, $v_{opt_1} = v_{opt_2}$, $k_{opt_1} = k_{opt_2}$,$q_{max_1} = q_{max_2}$, $k_{max_1} = k_{max_2}$ The third boundary condition applies to identical values of boundary parameters in free-flow and congested traffic (input boundary parameters from the free-flow model can be input into the congested traffic model as constant values). Please note that BC3 excludes non-continuous models (with BC3, it is possible to compare single and two-regime models).
2. Simplicity, which is measured with the number of parameters and equations in the model. It was assumed that the fewer the equations and parameters, the simpler, more practical, and more adaptable the model will be for the given road and traffic conditions.
3. Empirical accuracy, which is measured with absolute and relative measures of goodness of fit, the RMSE, and MAPE, which are classical accuracy measures. The RMSE is expressed with units of a dependent variable, which means that it will be possible to establish the average difference in km/h between real and estimated values of speed. The MAPE expresses the average percentage difference between a real and estimated value.
4. Values of boundary parameters are assessed mainly by comparing estimated boundary parameters with expected ranges as determined. If the value of an estimated parameter fits in within the expected range, we can say that the model is a good estimator of the particular parameter.

3.2.3. Model Calibration

The models' boundary parameters and others were determined using the non-linear method of the least squares with Levenberg–Marquardt algorithm for non-linear optimization [39]. Starting from the given initial values of the parameters, the algorithm iteratively reduces the sum of squares of errors between real and modeled values of the explanatory value by sequentially updating model parameters. All model parameters were estimated.

The initial values of boundary parameters were treated as the centers of the expected ranges of values that were determined during data preparation. The boundary density for traffic conditions in two-regime models was set as the middle value from the range of expected k_{opt} values.

3.2.4. Model Assessment and Comparison

Given the criteria that were used, we must ensure that the models under comparison are related to the same dependent variable. The assessment was conducted for general models (2) or models transformed to a general form (2), where speed is a function of density, of parameters related to the traffic flow (boundary parameters), and of other non-dimensional parameters.

$$v = f(k, \text{boundary parameters}, \text{non-dimensional parameters}) \quad (2)$$

From the models listed in Tables 1 and 2, all but Van Aerde's model meet the requirement. Unless the condition is met, the model cannot be transformed directly to (2). As a result, the model cannot be analyzed further.

A two-stage assessment procedure was adopted as presented in Figure 5. In the first stage, the model is assessed for whether it meets the boundary conditions (criterion 1). If it does, the assessment continues with criteria 2, 3 and 4. If it does not, the model's analysis is discontinued. The procedure allows for a conditional acceptance, if the model does not meet the boundary conditions. This may be the case when, given the objective of the analysis, representing a specific condition of traffic is not relevant (in case of failure to meet BC1 or BC2) or when we agree that the model will meet the requirements if within the expected range of k_{max}, speeds are close to zero, even though density does not take a finite value (in case of failure to meet BC2).

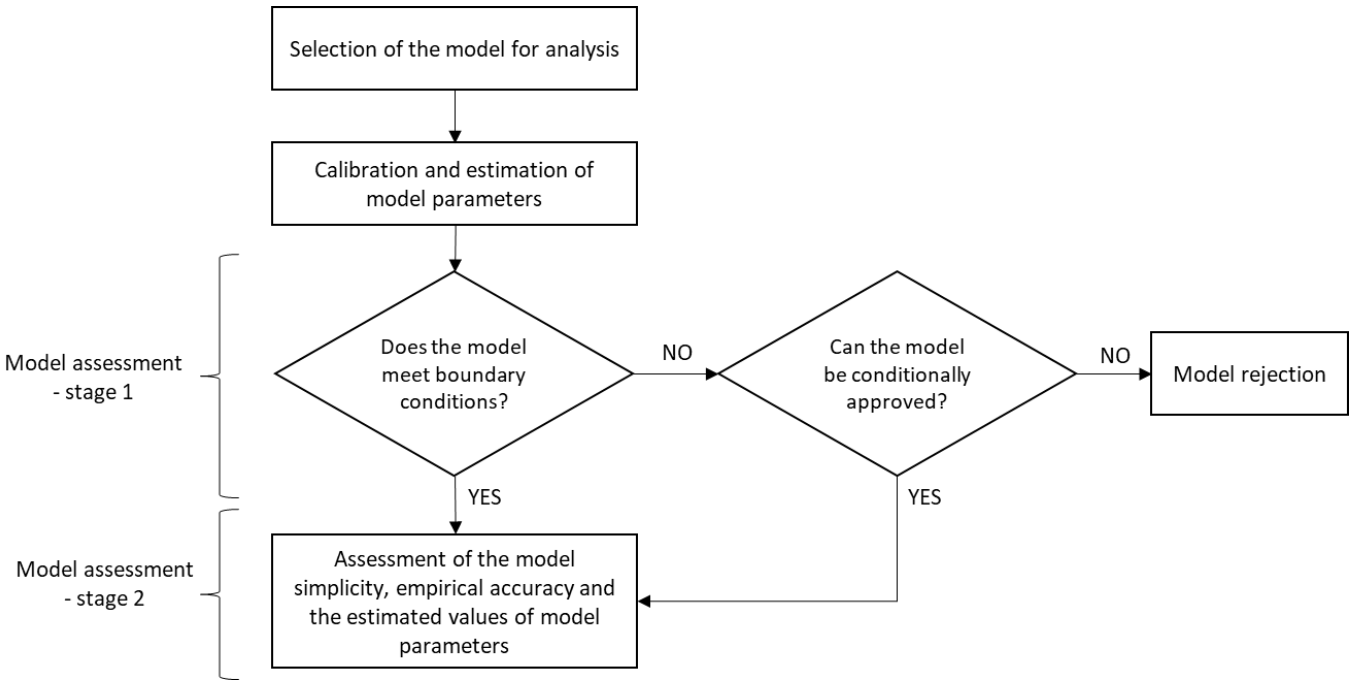


Figure 5. Procedure for model assessment.

Criteria 2, 3 and 4 are assessed using indicators determined in accordance with Table 4. The model's final assessment is a weighted average of scores for the particular criteria. The weights help to address the essence of the criteria, i.e., the objective of the analysis.

Table 4. Formulas for model assessment and comparison—criteria 2, 3 and 4.

Criterion	Equation	Explanation
2 Simplicity	$ca_2 = w_{21}\left(\frac{\max(np)-np_i}{\max(np)-\min(np)}\right) + w_{22}ne_i^{-1}$	np—number of parameters ne—number of equations i—the analyzed model w_{21}, w_{22}—weights (sum up to 1)
3 Empirical accuracy	$ca_3 = \frac{\max(MAPE)-MAPE_i}{\max(MAPE)-\min(MAPE)}$ or $ca_3 = \frac{\max(RMSE)-RMSE_i}{\max(RMSE)-\min(RMSE)}$	i—the analyzed model
4 Traffic flow parameters	$ca_4 = \frac{nve_i}{5}$	nve_i—number of the parameters of model i, which are estimated correctly
Final assessment	$ca = w_2ca_2 + w_3ca_3 + w_4ca_4$	w_1, w_2, w_3—weights (sum up to 1)

4. Results

To facilitate comparisons between the models, single-regime models were divided into two groups: group 1 includes the simplest models with two to three parameters, group 2 contains models that represent the relation v(k) with S-shaped curves. Finally, a group of two-regime models was distinguished. All models that meet criterion 1 unconditionally were cumulatively assessed for criteria 2, 3 and 4 (Table 5).

Group 1, single-regime models. The models do not give a correct representation of any of the traffic conditions. In free-flow traffic relative to the empirical fall, speed falls rapidly as density and flow increase. Where the flow is maximal, the estimated speed is significantly underrepresented (Figure 6). From the analyzed models, those not meeting the boundary conditions are Greenberg's (BC1, BC2), Krystek's (BC2), and Underwood's (BC2) models. The other models in the group show major errors in estimations and incorrectly estimate the values of nearly all boundary parameters (Table 5).

Group 2, single-regime models. Group 2 models offer a much better representation of the empirical relation with the S-shaped curve performing significantly better for characteristics of traffic flow both for free flow and congested traffic (Figure 7). The models by Northwestern, Wang, Kerner and Konhäuser, and Kucharski and Drabicki do not meet BC2; hence, as set out in the assessment procedure (Figure 5), they are not included in further analysis. From those left, the two models by Newell and Del Castillo in the chart are practically the same, and the estimated boundary parameters are almost identical (Table 5). The models that are the best at meeting the criteria and have the highest scores are Newell's and MacNicholas' models. However, both do not seem to give a good enough representation of the area of maximal traffic (Figure 7).

Two-regime models. Edie's model is non-continuous, and the boundary parameters in free-flow and congested traffic are not consistent (Figure 8). As a result, BC3 is not satisfied. The triangular model does not represent the actual traffic patterns at all. In visual terms, Smulders's and Wu's models perform the best with a relatively small error and an assessment of criterion 4 comparable with that of group 2 models. In the case where model simplicity is not a priority, both models can compete with the best single-regime models that have been analyzed.

The results of the analyses show that group 2 single-regime models offer the best characteristics and give the best representation of real traffic conditions. However, none of them are able to correctly represent the area of maximal traffic, and the values they produce are too low for q_{max} and v_{opt} (Table 5).

Given the situation, models that do not satisfy BC2 could be included in the analysis conditionally. They may be applied e.g., in methods for traffic condition assessment where our main focus is on free-flowing traffic before the road reaches its capacity.

Table 5. Assessment and comparison of traffic flow models for the field data.

Assessment Criteria		Single-Regime Models—Group 1			Single-Regime Models—Group 2			Two-Regime Models		
		Greenshields	Drew	Pipes-Munjal	Newell	Del Castillo	MacNicholas	Smulders	Triangular	Wu
1	*Satisfied boundary conditions*									
	BC1	+	+	+	+	+	+	+	+	+
	BC2	+	+	+	+	+	+	+	+	+
	BC3	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.	+	+	+
2	*Simplicity*									
	No. of parameters	2	3	3	3	3	4	3	3	7
	No. of equations	1	1	1	1	1	1	2	2	2
	ca_2	1.00	0.90	0.90	0.90	0.90	0.80	0.65	0.65	0.25
3	*Empirical accuracy*									
	RMSE (km/h)	7.95	7.86	5.64	6.29	6.29	6.04	6.00	22.58	6.85
	MAPE (%)	13.91	14.12	14.12	10.56	10.56	9.93	10.06	35.27	11.72
	ca_3	0.84	0.83	0.83	0.98	0.98	1.00	0.99	0.00	0.93
4	*Parameter value (expected value)*									
	v_{sw} (km/h)	121	109	118	109	109	110	114	100	120
	v_{opt} (km/h)	60	69	66	67	65	69	90	99	86
	k_{max} (pc/km)	138	-	134	160	160	370	221	383	182
	k_{opt} (pc/km)	69	49	67	56	62	58	45	47	49
	q_{max} (pc/h)	4153	4010	4198	4001	4005	4010	4080	3936	4214
	ca_4	0.00	0.20	0.20	0.40	0.20	0.40	0.20	0.20	0.40
Final assessment		0.61	0.64	0.64	0.76	0.69	0.73	0.61	0.28	0.53

* The highlighted cells (criterion 4) mean that the estimated values of the parameters are consistent with the expected value ranges (Table 3). ** Weights for the criteria assessment were adopted as follows: $w_{21} = w_{22} = 0.5$, $w_2 = w_3 = w_4 = 0.33$.

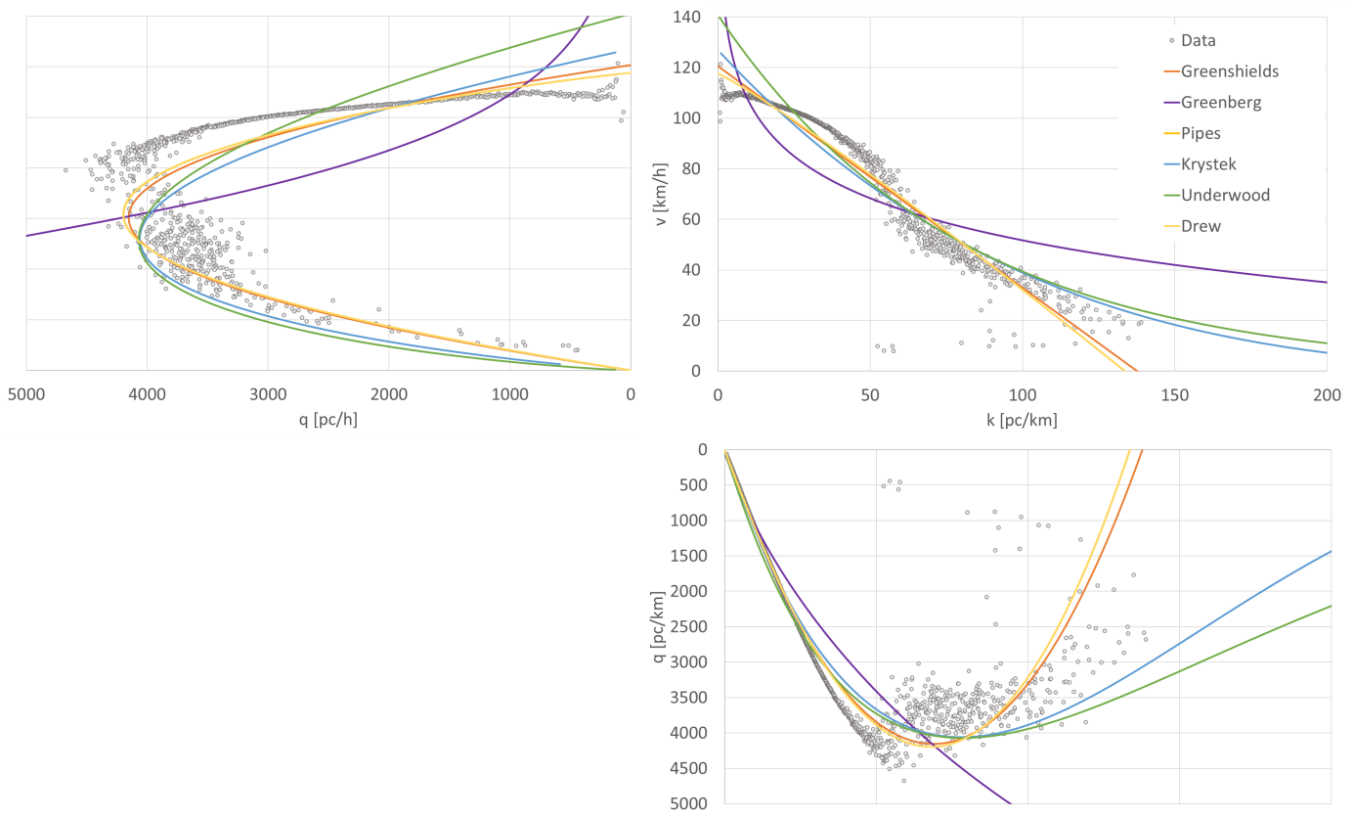


Figure 6. Fundamental diagram represented by single-regime models—group 1.

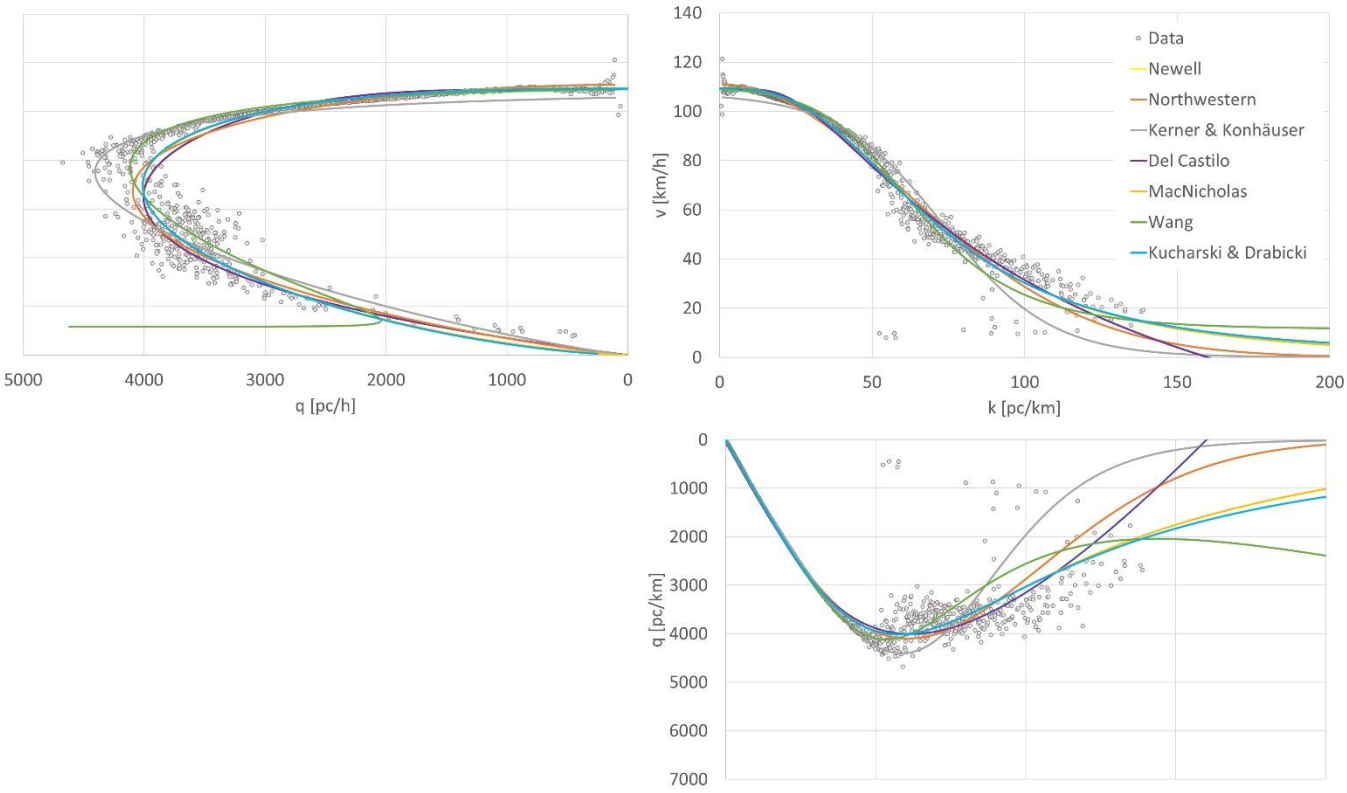


Figure 7. Fundamental diagram represented by single-regime models—group 2.

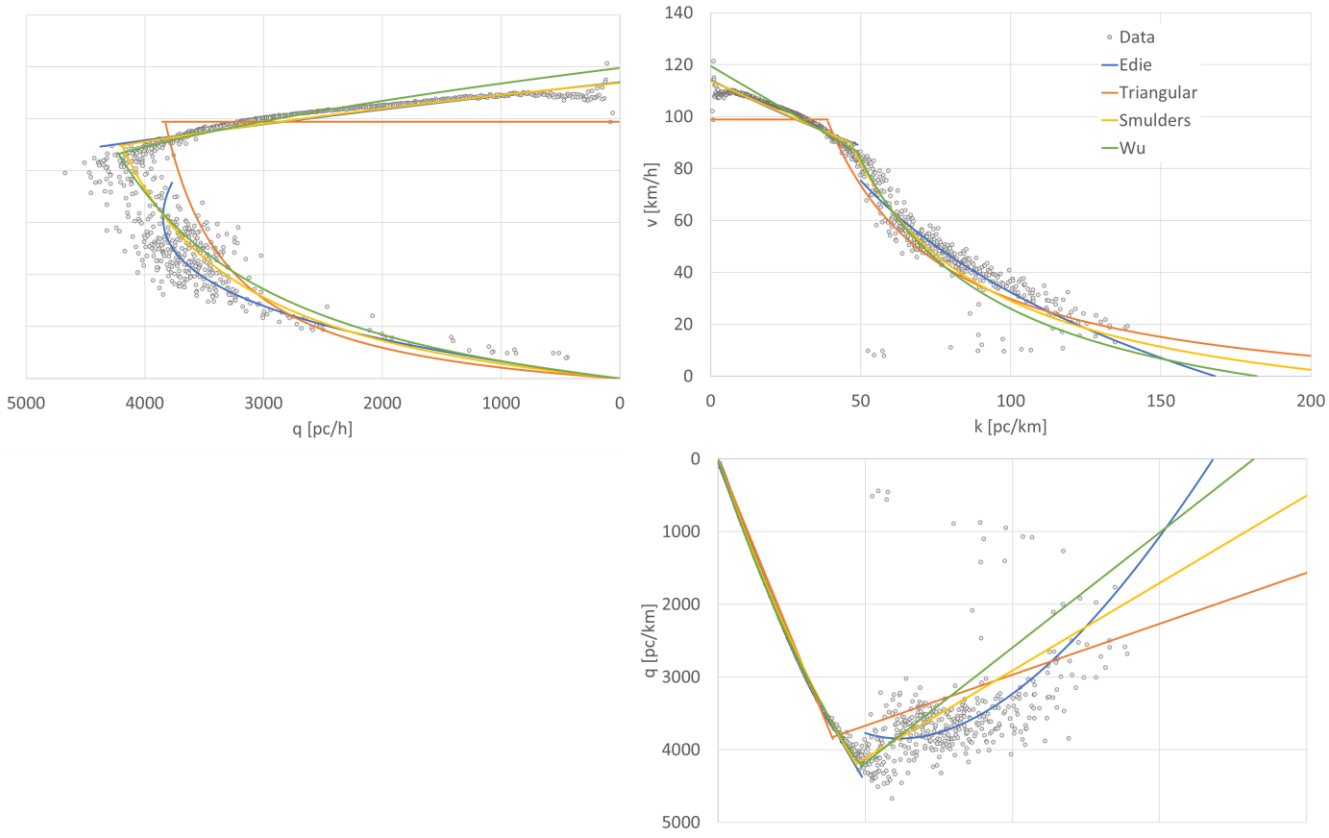


Figure 8. Fundamental diagram represented by two-regime models.

If we accept the failure to satisfy BC2, Kerner and Konhäuser's, Wang's, and Kucharski and Drabicki's models are worth considering. The first two perform relatively well on q_{max}, traffic parameter representation, which is a feature other BC2 models did not meet (Table 6). However, all come with some limitations. Kerner and Konhäuser's model carries a relatively big estimation error. Kucharski and Drabicki's model has one of the lowest estimation errors; however, it underestimates the values of q_{max} and v_{opt}. The problem with Wang's model is that traffic volume increases significantly for densities of the order of 150 pc/km and that speed does not tend to zero; instead, it tends to a certain minimal value which in the case if the field data is 12 km/h (Figure 7). In turn, this goes against Polish studies that have shown that for densities of the order of 50 pc/km/lane, vehicle stream speed does not exceed 10 km/h, and for densities of the order of 100 pc/km/lane, the speed is close to zero.

Table 6. Assessment and comparison of traffic flow models (unsatisfied BC2).

Assessment Criteria		Single-Regime Models—Group 1		Single-Regime Models—Group 2			
		Krystek	Underwood	Northwestern	Kerner & Konhäuser	Wang	Kucharski & Drabicki
1	*Satisfied boundary conditions*						
	BC1	+	+	+	+	+	+
	BC2	-	-	-	-	-	-
	BC3	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.
2	*Simplicity*						
	No. of parameters	2	2	2	2	4	4
	No. of equations	1	1	1	1	1	1
	ca_2	1.00	1.00	1.00	1.00	0.80	0.80
3	*Empirical accuracy*						
	RMSE (km/h)	9.16	7.74	6.53	8.58	6.47	6.02
	MAPE (%)	15.25	11.30	11.28	15.14	10.99	9.88
	ca_3	0.79	0.95	0.95	0.79	0.96	1.00
4	*Parameter value (expected value)*						
	v_{sw} (km/h)	127	141	111	107	110	109
	v_{opt} (km/h)	52	52	67	75	78	69
	k_{max} (pc/km)	-	-	-	-	-	-
	k_{opt} (pc/km)	78	79	61	59	50	49
	q_{max} (pc/h)	4060	4071	4095	4407	4116	4010
	ca_4	0.00	0.00	0.20	0.40	0.60	0.40
Final assessment		0.60	0.65	0.72	0.73	0.79	0.73

* The highlighted cells (criterion 4) mean that the estimated values of the parameters are consistent with the expected value ranges (Table 3). ** Weights for the criteria assessment were adopted as follows: $w_{21} = w_{22} = 0.5$, $w_2 = w_3 = w_4 = 0.33$.

5. Discussion

Considering the need to prepare the Polish Highway Capacity Manual, the objective of the article was to analyze existing models of the fundamental relationship, compare them, and select the best performing ones that could be used as a basis for the Polish method.

The problem that we encountered when reviewing the literature was that there is no agreement of researchers on how to assess and compare traffic flow models. Many comparisons have an expert basis rather than a quantitative approach [4,17,29]. The most widely used in quantitative assessment is the model fitness to empirical data [25–28]. The models' simplicity, compatibility with boundary conditions, and parameter values validity are rarely considered. On the other side, visual assessment of the models is hampered by the existence of the scatter in the real data explained by stochastic characteristics of traffic in real world [32–35], which is revealed by many speed values corresponding to the same density. Taking these issues into account, we proposed a universal and quantitative method for assessing models of the fundamental relationship. Based on the literature [4,9,18,24], we adopted four criteria for traffic flow models assessment and comparison: simplicity, empirical accuracy, correct estimation of model parameters, and meeting the boundary conditions; the criteria were quantified and assessed in a two-stage procedure. The final assessment is a weighted average of criteria assessment.

A detailed analysis using the method confirms the conclusions from the literature review, which is that no model fully meets the requirements the researchers [4,9,18] expected of the fundamental relationship models. This suggests and comes as a confirmation of Del Castillo and Benitez's conclusion [2] that the problem of finding the best model remains open. This is the most important conclusion from the study that encourages a search for a new model that could be applied to represent the empirical relations of traffic flow.

Analysis results show that the simplest two and three-parameter models do not represent traffic flow characteristics well. While two-regime models offer a much better empirical accuracy, they do so at the cost of model simplicity. The best representation of empirical relations with a relatively simple mathematical function is offered by models that represent the relation $v = f(k)$ with an S-shaped curve. This observation was picked by Wang [4], who proposed a model that uses a sigmoidal curve to represent the relation. Similarly, Drake et al. [14] proposed a model based on a bell-shaped curve. Thanks to the shape, it is possible to include the empirical data pattern where in the initial phase speed, it falls slowly until maximal flow is reached and starts falling much faster afterwards. The pace of the fall slows for low speeds. It also helps to take account of the break point where $k = k_{opt}$. However, most of these models fail to represent the area of maximum flow, and this is probably caused by the inevitable scatter in data that occurs at the highest flows and in congested traffic regime.

We observed that some of the models (with the S-shaped curve representing the relation $v = f(k)$) that fail to meet boundary conditions (Wang et al., Kerner and Konhäuser) seem to give a good representation of traffic flow, even in the intractable area of maximum flow. This raises some questions: Does the BC2 boundary condition have to be absolutely satisfied? Is failure to meet BC2 conditionally acceptable, and if so, under what condition? If our interest is solely in a free-flow regime before the maximum flow is reached (e.g., in methods for traffic condition assessment), is it crucial to meet BC in the congested regime? There are no obvious answers to these questions, and a lot will depend on the purpose of modeling. The results of a Polish study of maximal density [38] suggest that a conditional acceptance of a model that does not satisfy BC2 may be possible if within the expected range of maximal density (app. $100 \div 150$ pc/km/lane), speed is close to zero ($\leq$ 5 km/h) and diminishes as density continues to increase. This possibility is included in the proposed methodology (Figure 5).

The research comes with some limitations that we are aware of. The proposed methodology was tested on one site (permanent traffic counting station) and, thus, requires further investigation and validation with data from other sites. Another limitation is that the

models cannot be assessed if they have a different form and are not transformed to (2). One example is Van Aerde's four-parameter model [16], which is well received in the literature [17,40] and used as the basis for Germany's method for motorway traffic assessment [6]. This leads to the question: Would this model provide a better representation of the empirical relation compared with the analyzed models? Another limitation is that the assessment of a model's simplicity does not address its functional form. As a consequence, Greenshields' linear model and Greenberg's logarithmic model are given equal scores. How to include this issue in the assessment of model simplicity? All these issues should be considered in further work.

Author Contributions: Conceptualization, A.R. and K.J.; methodology, A.R.; validation, A.R. and K.J.; formal analysis, A.R.; investigation, A.R.; resources, A.R.; data curation, A.R.; writing—original draft preparation, A.R.; writing—review and editing, K.J.; visualization, A.R.; supervision, K.J. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: The data presented in this study are openly available in MostWiedzy repository at https://doi.org/10.34808/8xkq-7714 (accessed on 18 October 2021), reference number [30].

Acknowledgments: The study is part of a doctoral thesis [31] and was delivered under the RID 2B project which is designed to develop methods for estimating capacity and assessing traffic conditions on Poland's dual carriageways. The outcome of the project is the Polish Highway Capacity Manual, with procedures for assessing traffic conditions and identifying the capacity of rural and agglomeration roads [38].

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Greenshields, B.D. A study of traffic capacity. In Proceedings of the Fourteenth Annual Meeting of the Highway Research Board, Washington, DC, USA, 6–7 December 1934; Highway Research Board: Washington, DC, USA, 1935; pp. 448–477. Available online: http://pubsindex.trb.org/view.aspx?id=120649 (accessed on 6 September 2021).
2. Castillo, J.M.D.; Benítez, F.G. On the functional form of the speed-density relationship—I: General theory. *Transp. Res. Part B Methodol.* **1995**, *29*, 373–389. [CrossRef]
3. Wang, H.; Li, J.; Chen, Q.-Y.; Ni, D. Speed-Density Relationship: From Deterministic to Stochastic. In Proceedings of the 88th Transportation Research Board Annual Meeting, Washington, DC, USA, 11–15 January 2009.
4. Wang, H.; Li, J.; Chen, Q.-Y.; Ni, D. Logistic modeling of the equilibrium speed-density relationship. *Transp. Res. Part A* **2011**, *45*, 554–566. [CrossRef]
5. Transportation Research Board. *Highway Capacity Manual 6th Edition: A Guide for Multimodal Mobility Analysis*; Transportation Research Board of the National Academies: Washington, DC, USA, 2016.
6. Baier, M.M.; Brilon, W.; Hartkopf, G.; Lemke, K.; Maier, R.; Schmotz, M. *HBS2015 Handbuch für die Bemessung von Straßenverkehrsanlagen. Teil A - Autobahnen*; Forschungsgesellschaft für Straßen- und Verkehrswesen (FGSV), Kommission Bemessung von Straßenverkehrsanlagen: Köln, Germany, 2015.
7. Greenberg, H. An analysis of traffic flow. *Oper. Res.* **1958**, *7*, 79–85. Available online: http://links.jstor.org/sici?sici=0030-364X%28195901%2F02%297%3A1%3C79%3AAAOTF%3E2.0.CO%3B2-7 (accessed on 6 September 2021). [CrossRef]
8. Lighthill, M.J.; Whitham, G.B. On kinematic waves. II. A theory of traffic flow on long crowded roads. *Proc. R. Soc. A Math. Phys. Eng. Sci.* **1955**, *229*, 317–345. [CrossRef]
9. Underwood, R.T. Speed, volume and density relationships. In *Quality and Theory of traffic Flow*; Bureau of Highway Traffic, Yale University: New Haven, CT, USA, 1960; pp. 141–188. Available online: http://tft.eng.usf.edu/greenshields/docs/Greenshields_Quality_and_Theory_of_Traffic_Flow_1961.pdf (accessed on 14 August 2018).
10. Ardekani, S.A.; Ghandehari, M.; Nepal, S.M. Macroscopic speed-flow models for characterization of freeway and managed lanes. *Bul. Institutului Politeh. Din Iaşi* **2011**, *57*, 149. Available online: http://www.bipcons.ce.tuiasi.ro/Archive/222.pdf (accessed on 13 August 2018).
11. Pipes, L.A. Car following models and the fundamental diagram of road traffic. *Transp. Res.* **1967**, *1*, 21–29. [CrossRef]
12. Krystek, R. *Syntetyczny Wskaźnik Jakości Ruchu Ulicznego*; Politechnika Gdańska: Gdańsk, Poland, 1980.

13. Drew, D.R. *Deterministic Aspects of Freeway Operations and Control*; Research Report Number 24-4; Texas Transportation Institute: College Station, TX, USA, 1965. Available online: https://static.tti.tamu.edu/tti.tamu.edu/documents/24-4.pdf (accessed on 9 August 2017).
14. Drake, J.S.; Schofer, J.L.; May, A.D. A statistical analysis of speed density hypotheses. *Highw. Res. Rec.* 1966, 154, pp. 53–87. Available online: http://onlinepubs.trb.org/Onlinepubs/hrr/1967/154/154-004.pdf (accessed on 27 December 2017).
15. Kerner, B.S.; Konhäuser, P.; Schilke, M. Deterministic spontaneous appearance of traffic jams in slightly inhomogeneous traffic flow. *Phys. Rev. E* **1995**, *51*, 6243–6246. [CrossRef] [PubMed]
16. Van Aerde, M. A single regime speed-flow-density relationship for freeways and arterials. In Proceedings of the 74th Transportation Research Board Annual Meeting, Washington, DC, USA, 22–28 January 1995.
17. Rakha, H.; Crowther, B. Comparison of Greenshields, Pipes and Van Aerde car-following and traffic stream models. *Transp. Res. Rec. J. Transp. Res. Board* **2002**, *1802*, 248–262. [CrossRef]
18. MacNicholas, M.J. A simple and pragmatic representation of traffic flow. In *75 Years of the Fundamental Diagram for Traffic Flow Theory*; Greenshields Symposium; Transport Research Board: Woods Hole, MA, USA, 2011.
19. Kucharski, R.; Drabicki, A. Estimating macroscopic volume delay functions with the traffic density derived from measured speeds and flows. *J. Adv. Transp.* **2017**, *2017*, 4629792. [CrossRef]
20. Bureau of Public Roads. *Traffic Assignment Manual for Application with a Large, High Speed Computer*; Bureau of Public Roads: Washington, DC, USA, 1964.
21. Smulders, S.A. *Control of Freeway Traffic Flow*; Stichting Mathematisch Centrum: Amsterdam, The Netherlands, 1989.
22. Daganzo, C.F. *Fundamentals of Transportation and Traffic Operations*; Emerald: Bingley, UK, 1997.
23. Knoop, V.L. *Introduction to Traffic Flow Theory: An Introduction with Exercises*; Delft University of Technology: Delft, The Netherlands, 2017. Available online: http://www.victorknoop.eu/research/book/Knoop_Intro_traffic_flow_theory_edition1.pdf (accessed on 27 December 2017).
24. Myung, J.I.; Pitt, M.A. Model Comparison Methods. *Methods Enzymol.* **2004**, *383*, 351–366. [CrossRef] [PubMed]
25. Gaddam, H.K.; Rao, K.R. Speed–density functional relationship for heterogeneous traffic data: A statistical and theoretical investigation. *J. Mod. Transp.* **2019**, *27*, 61–74. [CrossRef]
26. May, A.D. *Traffic Flow Fundamentals*; Prentice-Hall, Inc.: Englewood Cliffs, NJ, USA, 1990; p. 464. Available online: http://books.google.com/books?id=JYJPAAAAMAAJ&pgis=1 (accessed on 11 September 2021).
27. Rakha, H. Validation of Van Aerde's simplified steadystate car-following and traffic stream model. *Transp. Lett.* **2009**, *1*, 227–244. [CrossRef]
28. Cheng, X.U. Analysis of Traffic Flow Speed-density Relation Model Characteristics. *J. Highw. Transp. Res. Dev.* **2014**, *8*, 104–110.
29. Ni, D.; Zheng, C.; Wang, H. A Unified Perspective on Traffic Flow Theory, Part III: Validation and Benchmarking. In Proceedings of the 11th International Conference of Chinese Transportation Professionals (ICCTP), Nanjing, China, 14–17 August 2011. [CrossRef]
30. Romanowska, A.; Kustra, W. *Permanent Traffic Counting Stations—Expressway S6 in Gdansk (Dataset Containing 5-min Aggregated Traffic Data and Weather Information)*, 2021. Available online: https://mostwiedzy.pl/en/open-research-data/permanent-traffic-counting-stations-expressway-s6-in-gdansk-dataset-containing-5-min-aggregated-traf,923120743943369-0 (accessed on 11 September 2021).
31. Romanowska, A. Macroscopic traffic Flow Models for basic Motorways and Express Roads Sections. Ph.D. Thesis, Gdansk University of Technology, Gdansk, Poland, 2019.
32. Coifman, B. *Advancing Traffic Flow Theory Using Empirical Microscopic Data*; NEXTRANS Project, No. 174OSUY2.2; The Ohio State University: Columbus, OH, USA, 2015.
33. Wang, H.; Ni, D.; Chen, Q.-Y.; Li, J. Stochastic modeling of the equilibrium speed-density relationship. *J. Adv. Transp.* **2013**, *47*, 126–150. [CrossRef]
34. van Wageningen-Kessels, F.; van Lint, H.; Vuik, K.; Hoogendoorn, S. Genealogy of traffic flow models. *EURO J. Transp. Logist.* **2014**, *4*, 445–473. [CrossRef]
35. Jabari, S.E.; Zheng, J.; Liu, H.X. A probabilistic stationary speed-density relation based on Newell's simplified car-following model. *Transp. Res. Part B Methodol.* **2014**, *68*, 205–223. [CrossRef]
36. Zheng, L.; He, Z.; He, T. A flexible traffic stream model and its three representations of traffic flow. *Transp. Res. Part C Emerg. Technol.* **2017**, *75*, 136–167. [CrossRef]
37. Rakha, H.; Arafeh, M. Calibrating steady-state traffic stream and car-following models using loop detector data. *Transp. Sci.* **2010**, *44*, 151–168. [CrossRef]
38. Olszewski, P.; Dybicz, T.; Kustra, W.; Romanowska, A.; Jamroz, K.; Ostrowski, K. Development of the New Polish Method for Capacity Analysis of Motorways and Expressways. *Arch. Civ. Eng.* **2020**, *66*, 453–470. [CrossRef]
39. Gavin, H.P. The Levenberg-Marquardt Algorithm for nonlinear Least Squares Curve-Fitting Problems. 2019. Available online: http://people.duke.edu/~{}hpgavin/ce281/lm.pdf (accessed on 24 May 2019).
40. Van Aerde, M.; Rakha, H. Multivariate calibration of single regime speed-flow-density relationships. In Proceedings of the 6th International VNIS Conference, Seattle, WA, USA, 30 July–2 August 1995; IEEE: New York, NY, USA, 1995; pp. 334–341.