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Optimal Versus Equal Dimensions of Round Bales of Agricultural Materials Wrapped with Plastic Film—Conflict or Compliance?

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Abstract: For the assumed bale volume, its dimensions (diameter, height), minimizing the consumption of the plastic film used for bale wrapping with the combined 3D method, depend on film and wrapping parameters. Incorrect selection of these parameters may result in an optimal bale diameter, which differs significantly from its height, while in agricultural practice bales with diameters equal or almost equal to the height dominate. The aim of the study is to formulate and solve the problem of selecting such dimensions of the bale with a given volume that the film consumption is minimal and, simultaneously, the bale diameter is equal or almost equal to its height. Necessary and sufficient conditions for such equilibria of the optimal bale dimensions are derived in the form of algebraic equations and inequalities. Four problems of the optimal bale dimension design guaranteeing assumed equilibrium of diameter and height are formulated and solved; both free and fixed bale volume are considered. Solutions of these problems are reduced to solving the sets of simple algebraic equations and inequalities with respect to two variables: integer number of film layers and continuous overlap ratio in bottom layers. Algorithms were formulated and examples regarding large bales demonstrate that they can handle the optimal dimensions' equilibria problems.

Keywords: 3D bale wrapping method; equal bale dimensions; mathematical model; minimal film consumption; optimal bale dimensions; round bales



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1. Introduction

Currently, the demand to limit consumption of the film used to wrap bales of agriculture materials has been receiving increasing attention to reduce both costs and damage to the environment caused by plastic waste [1–4]. For this purpose, both experimental studies comparing film usage for different wrapping conditions and methods [5–9] and model-based analytical approaches useful for estimation of the film usage [10–14] and its minimization [15–18] were used. In the last few years it has been shown that by appropriate optimal selection of the film width [15,16], overlaps between adjacent film strips [17], and film width and overlaps, together, [18] it is possible to reduce film consumption by up to 20%. The model-based optimization research concerned mainly conventional [15,17,18] and IntelliWrap [16] wrapping methods. The monograph [19] comprehends the issues of modelling of the film consumption for wrapping round bales and related optimization problems solved in the last few years.

Also, an appropriate choice of bale size dimensions (diameter, height) may guarantee decreasing film consumption [11,19]. In [11], where the dependence of the film consumption on the bale diameter for the conventional wrapping technique was investigated, the analytical analysis showed that the larger the bale diameter is, the lower is the film consumption per unit of bale volume, which led to the conclusion that the use of the bale with the largest permissible diameter ensures the smallest film consumption. These studies were based on a rough model which describes film usage as a continuous function of bale dimensions, film width and a number of wrapped film strips; however, the model did not

take into account mechanical properties of the stretch film and the direct relation between the number of wrapped film strips and bale and film parameters.

The concept of optimal (from the point of view of film consumption) selection of bale dimensions for the assumed bale volume was first introduced in [15] for the conventional wrapping method. A more accurate model was used that describes the consumption of the film as a function of the bale and film dimensions, mechanical parameters of the film (Poisson ratio, unit deformation), overlap ratio and the number of bale rotations [12]. The film consumption per unit of the bale volume is used as a measure of film usage. Since it is very difficult to find the optimal bale dimensions minimizing the original exact film usage index due to the discontinuity of this index, near-optimal parameters, being as important as optimal parameters for engineering applications, are sought. It was shown in [15] that in the case of using the conventional wrapping method, the optimal bale height is twice its optimal diameter. This optimality rule, which holds also for the IntelliWrap method [19], has only a theoretical character and is a consequence, among others, of multiple overlapping segments on the bale cylinder top and bottom, where there are 2–16 times more film layers than on the bale's lateral surface [10,19]. However, in the case of the combined 3D wrapping method [6,14], the optimal bale dimensions turn out to be useful and applicable from the engineering practice point of view.

In the 3D wrapping method, which offers the potential to minimize film usage [6,14,20] as well as to enhance the quality of silage [6], biaxial rotation of the film applicators results in two types of film layers wrapped perpendicularly: the bottom layers are wrapped around the bale's lateral surface and the upper layers are wrapped along the bale's longitudinal axis. For a detailed description of the wrapping process, see [6] and the producer's documentations [21–23]. The problem of the choice of the best bale dimensions to guarantee the minimal film consumption was solved in [20], where the optimality conditions were established in the form of algebraic cubic equations, which can easily be solved using both analytical and numerical methods. Analytical and numerical studies [19,20] have shown that the relation between the optimal diameter and height is not evident. The optimal diameter can be larger than the bale height [19] (Figure 6.5b), [20] (Figure 11c) or can be smaller than the bale height [19] (Figure 6.5d), [20] (Figure 11a), depending on the bale volume as well as film and wrapping parameters. But the optimal bale diameter can also be equal to its height [19] (Figure 6.5a,c), [20] (Figure 11b). The last case corresponds to typical large bales of 1.2 m diameter and height [10,24,25]. However, in agricultural practice typical bales are 1.2 to 1.3 m diameter and height [6,10,19], where diameter is equal [26–30] or almost equal [9,31,32], but not much bigger, than bale height. Other dimensions of large round bales, for example 1.2 m × 1.6 m Ø [33], or 1.2 m × 1.5 m Ø [34] or 1.5 m Ø [35], are used less frequently. Although round bales of the height greater than the diameter are also being investigated, for example 1.2 m × 0.9 m Ø [36], they are much less common in agriculture practice [37]. Standard 3D bale wrappers are designed to wrap cylindrical bales up to 1.6 m in diameter and up to 1.2 m in height [22], or up to 1.5 m diameter and 1.2 m height [21,23].

Therefore, the question becomes how to choose such film and wrapping process parameters for a round bale with a given volume wrapped using the 3D method so that the bale diameter minimizing film consumption is equal to or almost equal to its height. The aim of the paper is to solve the problem of the selection of the film and wrapping parameters (overlaps, numbers of bottom and upper film layers, etc.) that for a given bale volume, i.e., a given bale weight, its diameter and height are equal or near-equal and, at the same time, they minimize the film consumption. It is, in essence, the inverse problem in which the film and wrapping parameters are determined that the equilibria of optimal dimensions are guaranteed, while in the direct problem considered in [20] the optimal dimensions for a given parameters are sought. The thesis is that for any given bale volume and number of global film layers there exist film and wrapping parameters for which the optimal bale dimensions are equal or near-equal, with the pre-assumed proportions diameter/height. To prove the above, the necessary and sufficient conditions of equilibrium

and near-equilibrium of the optimal bale dimensions were derived in the form of algebraic equations. The equilibrium conditions are dichotomous, as some depend on the given bale volume and others apply to any bale volume. Then, the problems of selecting such bale wrapping parameters for which the optimal bale dimensions are equal or near-equal were formulated and solved, separately, for a given and an arbitrary bale volume. Suitable design algorithms were proposed and numerically verified for large bales.

2. Materials and Methods

In this section, the notions of equal and near-equal optimal dimensions of a round bale are introduced and the related necessary and sufficient conditions are derived. The research methodology is also described.

2.1. Equal and Near-Equal Optimal Bale Dimensions

A complete mathematical model describing stretch film consumption for wrapping a bale with p_b bottom and p_u upper film layers using the combined 3D technique was derived in [14]; the main formula describing film usage is recalled in Appendix A with the corresponding assumptions. Symmetry of the bale is assumed. Mechanical properties of the plastic film are described by the Poisson's ratio ν_f and unit deformation ε_{lf} ; thickness of the film is ignored, e.g., polyethylene film is 25 μm thick [38]. The main symbols are summarized in Nomenclature, Appendix C.

The problem of the choice of bale dimensions (diameter, height) minimizing the consumption of the film used to wrap a cylindrical bale by 3D method has been stated and solved for the first time in [20], where the necessary and sufficient condition of the existence of the unique optimal bale diameter D_b^* of the bale of pre-assumed volume V_{b0} was derived in the form of cubic equation with zero linear term coefficient [20]:

$$\frac{2\pi p_u}{\Omega(k_f)} (D_b^*)^3 + \frac{\pi p_b}{1 - k_{fb}} (2\delta - b_{fr} k_{fb}) (D_b^*)^2 - 4V_{b0} \left(\frac{p_b}{1 - k_{fb}} + \frac{p_u}{\Omega(k_f)} \right) = 0, \quad (1)$$

where b_{fr} is the width of stretched film described by Equation (A1), δ denotes the overlap of the extreme film strips in bottom film layers at the bases of the bale, k_{fb} and k_f are the overlap ratios determining the width of the contact between adjacent film strips in bottom and upper film layers; function $\Omega(k_f)$ is defined by Equation (A3). The coefficients in the first and second terms and the free term of Equation (1) depend on the pre-assumed bale volume as well as all the film and wrapping parameters. Based on the optimality condition, Equation (1), analytical and numerical analysis of the influence of film width, pre-assumed bale volume and numbers of bottom and upper film layers on optimal bale dimensions and optimal film consumption were carried out in [20], where many detailed conclusions regarding the impact of these parameters were formulated. Additionally, the influence of the bottom layers overlaps was studied in [19].

Equation (1) has one real positive root [20]. The corresponding optimal bale height H_b^* is [20]

$$H_b^* = \frac{4V_{b0}}{\pi (D_b^*)^2}. \quad (2)$$

For given V_{b0} the optimal D_b^* and H_b^* depend on film and wrapping parameters, they are linearly dependent and the difference between them is described by

$$D_b^* - H_b^* = \frac{1}{2} H_b^* \left[\frac{p_b \Omega(k_f)}{p_u (1 - k_{fb})} - 1 \right] - \frac{p_b \Omega(k_f)}{2p_u (1 - k_{fb})} (2\delta - b_{fr} k_{fb}). \quad (3)$$

From a quick inspection of Equation (3) it can be seen that the relation between the optimal diameter D_b^* and height H_b^* is not evident. The conducted research [19,20] has shown that the relations $D_b^* > H_b^*$ and $D_b^* < H_b^*$, as well as $D_b^* = H_b^*$ are possible, depending on the values of the film and wrapping parameters and volume V_{b0} ; compare [20] (Figure 11), where the proportions diameter/height are depicted for large bales. Only the relation $D_b^* \geq H_b^*$, desirable from the point of view of baling systems engineering, will be considered. The proportion diameter/height do not exceed 1.25 [22] (RW 1819, 150/120), or 1.2 [21] (BW 1850, 150/125), or [23] (WM 1851, 150/125); however, predominantly, it is of 1.05 order [21–23].

The goal of this paper is to study the selection of equal and nearly-equal optimal bale dimensions, therefore, the respective precise definitions are given.

Definition 1. (equal optimal dimensions). Bale diameter D_b^* and height H_b^* of bale with a volume V_{b0} are called equal optimal dimensions, if D_b^* accomplishes Equation (1) and H_b^* , given by Equation (2), is equal to D_b^* , i.e., $D_b^* = H_b^*$.

Definition 2. (near-equal optimal dimensions). Bale diameter D_b^* and height H_b^* of bale with a volume V_{b0} are called near-equal optimal dimensions, if D_b^* accomplishes Equation (1), H_b^* is given by Equation (2), and

$$\frac{D_b^*}{H_b^*} = 1 + \varepsilon_0, \quad (4)$$

where a sufficiently small $\varepsilon_0 > 0$. Then D_b^* and H_b^* are called nearly-equal with order ε_0 .

2.2. Methodology

In this paper, a model-based analytical approach was applied, which addressed the goals (derivation of the conditions for equal and near-equal optimal bale dimensions and solving the problems of wrapping parameter design guaranteeing such equilibria) by using mathematical tools. The solutions of the optimality conditions for the bale diameter were obtained by applying numerical tools.

Firstly, the necessary and sufficient conditions under which for the given bale volume the optimal bale diameter is near-equal to its optimal height with order ε_0 were derived using the optimality conditions expressed by Equations (1) and (2); Proposition 1 abstracts these results (Section 2.3.1). The conditions for optimal bale dimensions equilibria are given by algebraic equations. One of them uniquely relates bale volume to order ε_0 and film and wrapping parameters, i.e., this condition is satisfied for a given volume. The complementary condition determined by two simple equations relating only to ε_0 and film and wrapping parameter applies regardless of the bale volume. Based on these conditions, using differential calculus, the relationships between the volume of the bale with near-equal optimal dimensions and the order ε_0 , width of the film and overlaps were analyzed (Section 2.3.2). Next, laying $\varepsilon_0 = 0$ the conditions for exact equilibrium of the optimal bale dimensions was obtained directly from the conditions of their near-equilibrium (Section 2.3.3).

Knowing the conditions of equilibrium and near-equilibrium of optimal bale dimensions, the tasks of designing the wrapping process in such a way that these conditions would be satisfied were formulated and solved. Due to the dichotomous nature of the equilibrium conditions, design tasks for a given bale volume and those in which the bale volume is arbitrary were considered separately; these are covered in Section 3.1 and Section 3.2, respectively. Consequently, four design problems were solved; however, the solutions for equal optimal bale dimensions resulted directly from those for near-equilibrium by substituting $\varepsilon_0 = 0$ (Sections 3.3 and 3.4). It was assumed that the film parameters (width, mechanical parameters v_f , ε_{lf}) are known, i.e., a practically available plastic film can be used, e.g., a commercial PE film used traditionally due to its mechanical properties and low costs [26,27,39]. Also the global number of film layers p_l wrapped on the bale's lateral

surface was taken as a given. Many studies have investigated the number of desired film layers for baled silage preservation, for example [7,29,40,41]. Mostly, four, six or eight layers of film are applied [6,24,35]; however, ten, twelve, and even sixteen film layers in which the silages are wrapped are also considered [7,22,23,29]. Therefore no specific assumptions were taken concerning the number of global film layers; the numerical studies were conducted for p_l from four to sixteen. The standard overlaps 50%, 67% or 75% resulting in uniform film coverage [17] were assumed for upper film layers. Thus, the bottom layers overlaps δ , k_{fb} and film layers decomposition $p_l = p_b + p_u$ had to be selected. The upper and lower limits were taken for the overlaps, the selection of which should take into account the knowledge and experience of baling as well as the possibilities of the wrappers available. Consequently, the considered design problems consisted in solving a set of algebraic equations (resulting from the conditions of the equilibria of optimal bale dimensions) and inequalities (the constraints of variables) with respect to three decision variables. Two variables—the overlaps δ , k_{fb} —were continuous, while the third—the number of bottom film layers p_b —was integer. These mixed (hybrid) sets of equations and inequalities were solved analytically, separately for fixed and free bale volume. The solutions are abstracted by Propositions 4–7. For volume-free problems the solution is given by two inequalities directly related to the integer p_b and simple rules for computing the overlaps δ , k_{fb} . In the case of fixed bale volume the solution was derived in the form of two inequalities related to the overlap ratio k_{fb} , which must be verified for every considered p_b and unique algebraic rule for the determining of the overlap δ . If equal optimal bale dimensions are sought then these inequalities and formulas take particularly simple forms which are obtained by substituting $\varepsilon_0 = 0$. Computational algorithms were developed which enable designation of the sought wrapping parameters in a few steps. The examples illustrated how to use them (Excel is enough) as well as the effectiveness for optimal bale dimensions selection for a standard large bale.

In sum, in the multistage process several problems of optimal equal and near-equal bale dimensions design for cylindrical bales of a given volume were formulated and solved. The research framework is graphically shown in Figure 1, which also illustrates the relations between these problems.

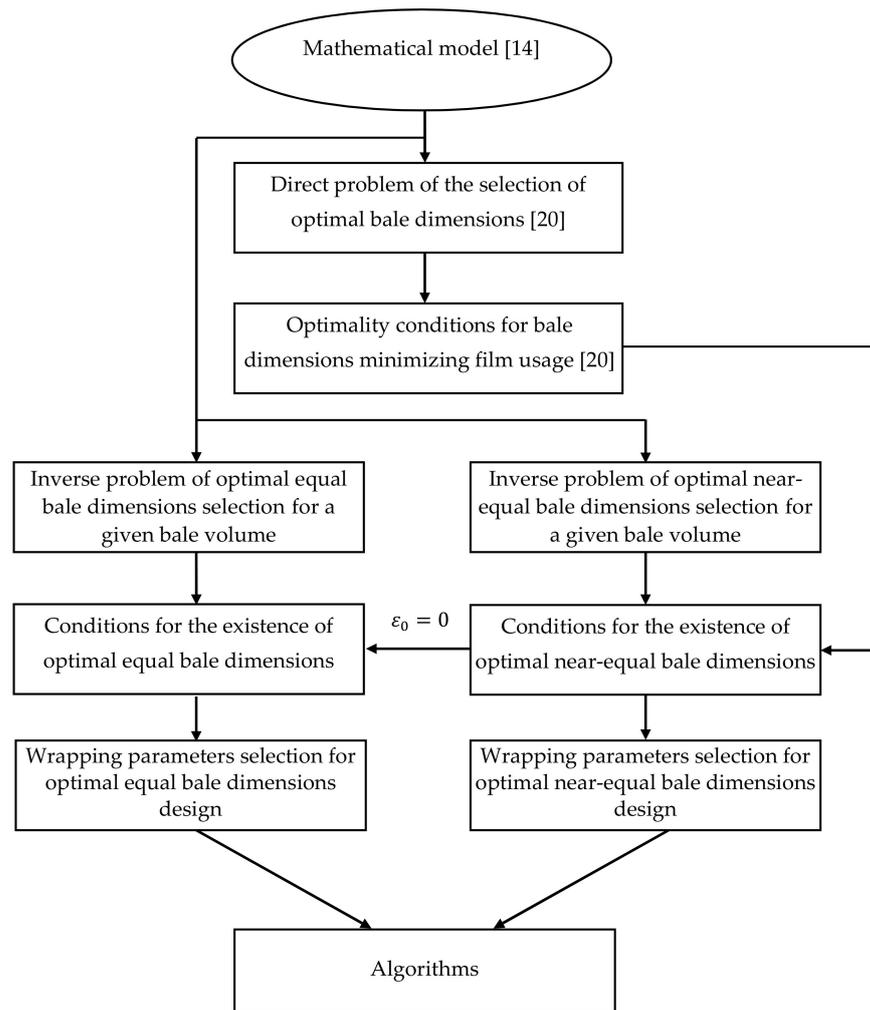


Figure 1. Schematic framework for the tasks of designing optimal equal and near-equal bale dimensions.

2.3. Necessary and Sufficient Conditions of Bale Dimensions Equilibrium and Near-Equilibrium

2.3.1. Near-Equal Optimal Dimensions

By Definition 2, the optimal bale dimensions are such that Equation (4) holds and the optimality condition expressed by Equation (1) is satisfied. For a given bale volume V_{b0} Equations (2) and (4) yield

$$D_b^* = \sqrt[3]{\frac{4(1 + \epsilon_0)V_{b0}}{\pi}} \tag{5}$$

and

$$H_b^* = \sqrt[3]{\frac{4V_{b0}}{\pi(1 + \epsilon_0)^2}} \tag{6}$$

The necessary and sufficient conditions of near-equal optimal dimensions are given by the following proposition, which is proved in Appendix B.

Proposition 1. For given: bale volume V_{b0} , width b_f of the film and its mechanical parameters v_f , ϵ_{1f} , numbers of film layers p_b , p_u and the overlaps δ , k_{fb} , k_f , such that the applicability condition expressed by Equation (A4) holds, the optimal bale diameter D_b^* is near-equal to its optimal height H_b^* with order ϵ_0 in the sense of Definition 2 if and only if one (and only one) of the conditions is satisfied:

(i) order ε_0 and the bale, film, and wrapping parameters are related by the equation

$$\frac{p_b \Omega(k_f) (2\delta - b_{fr} k_{fb}) (\sqrt[3]{1 + \varepsilon_0})^2}{p_b \Omega(k_f) - p_u (1 - k_{fb}) - 2p_u (1 - k_{fb}) \varepsilon_0} = \sqrt[3]{\frac{4V_{b0}}{\pi}}, \quad (7)$$

(ii) the equations

$$p_b \Omega(k_f) = p_u (1 - k_{fb}) + 2p_u (1 - k_{fb}) \varepsilon_0 \quad (8)$$

and

$$2\delta - b_{fr} k_{fb} = 0 \quad (9)$$

are satisfied, simultaneously. The optimal bale diameter D_b^* and height H_b^* are described by Equations (5) and (6), respectively; in case (i) for a given bale volume, in case (ii) for an arbitrary volume V_{b0} .

The necessary and sufficient conditions for near-equal optimal bale dimensions, case (ii), are independent of V_{b0} ; they are volume-free. However, those from case (i) hold for given V_{b0} , i.e., they are volume-fixed.

Note, that the inequality

$$\frac{2\delta - b_{fr} k_{fb}}{p_b \Omega(k_f) - p_u (1 - k_{fb}) - 2p_u (1 - k_{fb}) \varepsilon_0} > 0 \quad (10)$$

must hold to satisfy Equation (7); however, the expression $(2\delta - b_{fr} k_{fb})$ may be positive or negative, depending on the sign of the denominator. The necessary condition expressed by Equation (10) is independent on V_{b0} , while Equation (7) depends on the particular value of bale volume V_{b0} . Both conditions depend on ε_0 .

2.3.2. Bale Volume of Near-Equilibrium

From Equation (7), provided that inequality from Equation (10) is satisfied, we have

$$V_{b0} = \frac{\pi}{4} \left[\frac{p_b \Omega(k_f) (2\delta - b_{fr} k_{fb}) (\sqrt[3]{1 + \varepsilon_0})^2}{p_b \Omega(k_f) - p_u (1 - k_{fb}) - 2p_u (1 - k_{fb}) \varepsilon_0} \right]^3, \quad (11)$$

whence

$$\frac{\partial V_{b0}}{\partial \varepsilon_0} = \frac{\pi p_b \Omega(k_f) (2\delta - b_{fr} k_{fb})}{2\sqrt[3]{1 + \varepsilon_0}} V_{b0}^{2/3} \frac{[p_b \Omega(k_f) + 2p_u (1 - k_{fb}) + p_u (1 - k_{fb}) \varepsilon_0]}{[p_b \Omega(k_f) - p_u (1 - k_{fb}) - 2p_u (1 - k_{fb}) \varepsilon_0]^2}.$$

Thus, for given film and wrapping parameters, bale volume resulting in near-equal optimal dimensions is monotonically increasing function of ε_0 whenever

$$2\delta > b_{fr} k_{fb}, \quad (12)$$

and decreases with ε_0 , if $2\delta < b_{fr} k_{fb}$. If inequality from Equation (12) holds, then the volume V_{b0} decreases if the overlap k_{fb} grows, while in the opposite case the influence of k_{fb} is not so evident and depends on the sign of the expression in the square brackets of the numerator in the last fraction of the right hand side of the following equation

$$\frac{\partial V_{b0}}{\partial k_{fb}} = \frac{3\pi p_b \Omega(k_f) (\sqrt[3]{1 + \varepsilon_0})^2}{4} V_{b0}^{2/3} \frac{[-b_{fr} p_b \Omega(k_f) + b_{fr} p_u (1 - 2\delta) + 2p_u \varepsilon_0 (b_{fr} - 2\delta)]}{[p_b \Omega(k_f) - p_u (1 - k_{fb}) - 2p_u (1 - k_{fb}) \varepsilon_0]^2}.$$

If condition from Equation (12) holds, then V_{b0} given by Equation (11) grows with the increase of the overlap δ . In the opposite case the larger δ is, the smaller is the volume resulting in near-equal optimal dimensions. From the analysis of Equation (11) it also follows that the greater film width is, the smaller is V_{b0} , provided that inequality from Equation (12) holds. The influence of the bottom layers overlaps k_f and δ is illustrated in Figure 2a,b, where near-equilibrium volume V_{b0} is depicted as a function of ε_0 for plastic film (e.g., polyethylene, PE) characterized by Poisson’s ratio $\nu_f = 0.34$ [38] and unit deformation $\varepsilon_{lf} = 0.7(-)$ [7,12,14] of popular width $b_f = 0.75$ m [21–23,29,42]. Four bottom and upper film layers $p_b = p_u = 4$ and overlap ratio $k_f = 0.5$ are assumed. Figure 3a,b illustrate the effect of the film width b_f for commonly used overlap $\delta = 0.2$ m [21–23] and two overlap ratios $k_{fb} = 0.3, 0.35$; other parameters remain unchanged. The volume $V_{b0} = 1.357$ m³ corresponding to a standard large bale of diameter and height $D_b = H_b = 1.2$ m [24,25] is also marked on these pictures. Finally, note that for standard overlap $\delta = 0.2$ m [6,14,20–23], PE film and since $k_{fb} < \frac{1}{2}$, the condition from Equation (12) holds, in particular, if the film width $b_f < 1.052$ m. For $\delta = 0.15$ m this requirement is reduced to $b_f < 0.789$ m. For exemplary $k_{fb} = 0.4$ and $\delta = 0.2$ m inequality from Equation (12) is satisfied for $b_f < 1.31$ m; if $\delta = 0.15$ m this requirement is sharpened to $b_f < 0.99$ m.

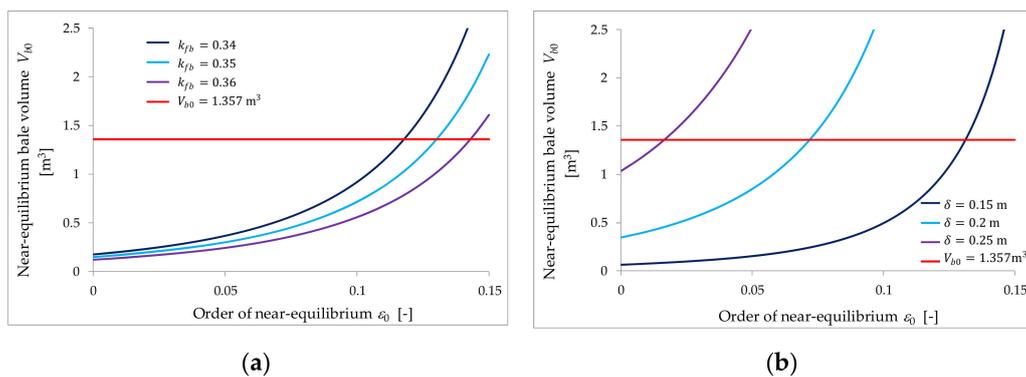


Figure 2. The volume V_{b0} , Equation (11), yielding the near-equal optimal bale dimensions as a function of the order ε_0 of near-equilibrium for $p_b = p_u = 4$ bottom and upper film layers, film width $b_f = 0.75$ m and wrapping parameters: (a) overlaps $\delta = 0.2$ m and $k_{fb} = 0.25, 0.3, 0.32$, (b) $k_{fb} = 0.3$ and $\delta = 0.15, 0.2, 0.25$ m.

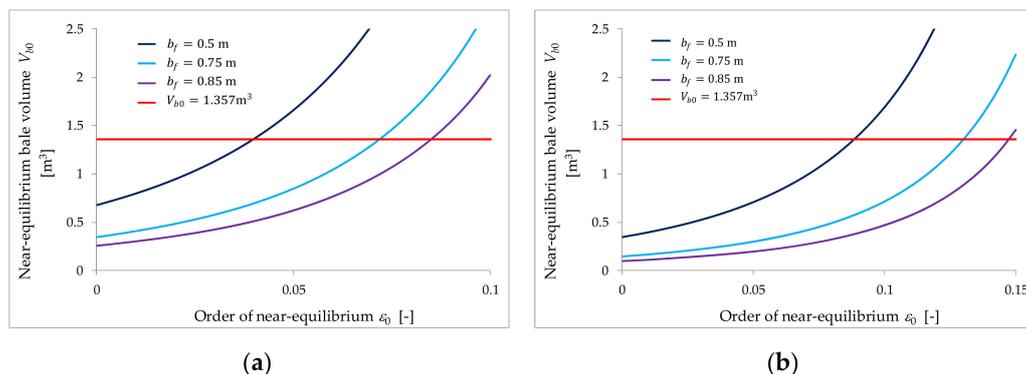


Figure 3. The volume V_{b0} , Equation (11), yielding the near-equal optimal bale dimensions as a function of the order ε_0 of near-equilibrium for $p_b = p_u = 4$ bottom and upper film layers, overlap $\delta = 0.2$ m and parameters: (a) overlap ratio $k_{fb} = 0.3$ and film widths $b_f = 0.5, 0.75, 0.85$ m, (b) overlap ratio $k_{fb} = 0.35$ and film widths $b_f = 0.5, 0.75, 0.85$ m.

Since in practically used wrapping systems the condition expressed by Equation (12) is usually satisfied and the near-equilibrium bale volume increases with the order ε_0 , a question arises about the conditions to be met by the film and wrapping parameters to guarantee for any $0 \leq \varepsilon \leq \varepsilon_0$ the existence of a bale volume such that the optimal

dimensions D_b^*, H_b^* are near-equal with order ε . The following result, derived in Appendix B based on Proposition 1, answers this question. The notation $V_{b0}(\varepsilon)$, $D_b^*(\varepsilon)$, and $H_b^*(\varepsilon)$ is introduced, locally, for near-equal bale parameters to emphasize the relationship between ε and V_{b0} , D_b^* , and H_b^* .

Proposition 2. For given film parameters $b_f, v_f, \varepsilon_{1f}$, numbers of film layers p_b, p_u and the overlaps δ, k_{fb}, k_f , such that the applicability condition expressed by Equation (A4) holds, for any $0 \leq \varepsilon \leq \varepsilon_0$, where a sufficiently small $\varepsilon_0 > 0$, there exists bale volume $V_{b0}(\varepsilon) > 0$ such that the optimal bale diameter D_b^* is near-equal to its optimal height H_b^* with order ε not greater than ε_0 , i.e.,

$$1 \leq \frac{D_b^*}{H_b^*} = 1 + \varepsilon \leq 1 + \varepsilon_0,$$

if and only if

$$p_b \Omega(k_f) - p_u(1 - k_{fb}) - 2p_u(1 - k_{fb})\varepsilon_0 \neq 0 \tag{13}$$

and one of the following conditions is satisfied:

- (a) inequality from Equation (12) holds,
- (b) two inequalities

$$2\delta - b_{fr}k_{fb} < 0, \tag{14}$$

$$p_b \Omega(k_f) - p_u(1 - k_{fb}) < 0, \tag{15}$$

hold, simultaneously. Then, for any $0 \leq \varepsilon \leq \varepsilon_0$ and volume $V_{b0}(\varepsilon)$ Equation (7) holds for $\varepsilon_0 = \varepsilon$. The optimal bale dimensions $D_b^*(\varepsilon)$, $H_b^*(\varepsilon)$ are expressed by equations:

$$D_b^*(\varepsilon) = \frac{p_b \Omega(k_f) (2\delta - b_{fr}k_{fb})(1 + \varepsilon)}{p_b \Omega(k_f) - p_u(1 - k_{fb}) - 2p_u(1 - k_{fb})\varepsilon}, \tag{16}$$

$$H_b^*(\varepsilon) = \frac{p_b \Omega(k_f) (2\delta - b_{fr}k_{fb})}{p_b \Omega(k_f) - p_u(1 - k_{fb}) - 2p_u(1 - k_{fb})\varepsilon}. \tag{17}$$

In case (a), $V_{b0}(0) \leq V_{b0}(\varepsilon) \leq V_{b0}(\varepsilon_0)$ and $D_b^*(\varepsilon)$, $H_b^*(\varepsilon)$ are monotonically increasing for $0 \leq \varepsilon \leq \varepsilon_0$. In case (b), $V_{b0}(0) \geq V_{b0}(\varepsilon) \geq V_{b0}(\varepsilon_0)$ and $D_b^*(\varepsilon)$, $H_b^*(\varepsilon)$ are monotonically decreasing for $0 \leq \varepsilon \leq \varepsilon_0$. If inequalities from Equations (13) and (14) hold, but inequality expressed by Equation (15) not, then there exists positive $\bar{\varepsilon}$ such that $0 < \bar{\varepsilon} < \varepsilon_0$ and

$$p_b \Omega(k_f) - p_u(1 - k_{fb}) - 2p_u(1 - k_{fb})\bar{\varepsilon} = 0. \tag{18}$$

For any $\bar{\varepsilon} < \varepsilon \leq \varepsilon_0$ bale volume $V_{b0}(\varepsilon)$ satisfying Equation (7) and optimal bale dimensions $D_b^*(\varepsilon)$, $H_b^*(\varepsilon)$ are near-equal in the sense of the inequalities

$$1 + \bar{\varepsilon} < \frac{D_b^*(\varepsilon)}{H_b^*(\varepsilon)} = 1 + \varepsilon \leq 1 + \varepsilon_0.$$

Both $V_{b0}(\varepsilon)$ and $D_b^*(\varepsilon)$, $H_b^*(\varepsilon)$ are monotonically decreasing for $\bar{\varepsilon} < \varepsilon \leq \varepsilon_0$.

Inequality expressed by Equation (12) holds for narrow films, while inequality form Equation (14) is satisfied for wider films.

2.3.3. Equal Optimal Dimensions

In this case the optimal bale diameter is equal to bale height, i.e., by Equation (2)

$$D_b^* = H_b^* = \sqrt[3]{\frac{4V_{b0}}{\pi}}, \tag{19}$$

and the optimality condition expressed by Equation (1) is satisfied, simultaneously. The necessary and sufficient conditions for the equilibrium are given by the following proposition, which results directly from Proposition 1 for $\epsilon_0 = 0$.

Proposition 3. For given: bale volume V_{b0} , width of the film b_f and its mechanical parameters v_f , ϵ_{1f} , numbers of film layers p_b , p_u and the overlaps δ , k_{fb} , k_f , such that the applicability condition expressed by Equation (A4) holds, the optimal bale diameter D_b^* is equal to its optimal height H_b^* if and only if one (and only one) of the conditions is satisfied:

(i) the bale, film and wrapping parameters are related by the equation

$$\frac{p_b \Omega(k_f) (2\delta - b_{fr} k_{fb})}{p_b \Omega(k_f) - p_u (1 - k_{fb})} = \sqrt[3]{\frac{4V_{b0}}{\pi}},$$

(ii) the equation

$$p_b \Omega(k_f) = p_u (1 - k_{fb})$$

and Equation (9) are satisfied, simultaneously. The optimal bale dimensions are given directly by Equation (19) for fixed volume V_{b0} in case (i) or for arbitrary volume in case (ii).

Note, that in case (i) the necessary condition for bale dimensions equilibrium expressed by Equation (10) takes the form

$$\frac{2\delta - b_{fr} k_{fb}}{p_b \Omega(k_f) - p_u (1 - k_{fb})} > 0.$$

3. Results and Discussion

In this section four problems of the selection of optimal bale dimensions being equal and near-equal are formulated, solved and illustrated by related examples.

It is known [17] that any overlap ratio of the form of irreducible fraction in which dividend is the divisor minus one, i.e.,

$$k_f = k_{f,u} = \frac{q - 1}{q}, \tag{20}$$

where $q \in \mathcal{N}$, \mathcal{N} denotes the set of positive integer numbers, results in the uniform film distribution on the bale's whole lateral surface and guarantee the same minimal film usage. An important special case for even p_u is $q = 2$, which means 50% overlap between the successive film strips [6,17,24,43]. For $k_f = k_{f,u}$ function $\Omega(k_f) = 1$ and since $p_u = p_l - p_b$, the applicability condition expressed by Equation (A4) takes the form

$$\frac{p_l - p_b}{q} = m, \quad m \in \mathcal{N}. \tag{21}$$

From the practical perspective, only four or five smallest such overlap ratios are useful [6,17,22,24,43,44], i.e., $2 \leq q \leq 5$ are worth considering.

Bearing in mind the dichotomous nature of the equilibrium conditions specified by Propositions 1 and 3, the problems of volume-free and volume-fixed near-equal and equal bale optimal dimensions design will be considered, separately.

3.1. Volume-Free Near-Equal Optimal Bale Dimensions Design

Let us consider the following problem of the optimal choice of bale dimensions.

Problem 1. Given film parameters b_f, v_f, ϵ_{1f} , upper layers overlap ratio k_f , Equation (20), number of global film layers p_l , and the order of near-equilibrium ϵ_0 . Find film layers decomposition (p_b, p_u) and the bottom layers overlaps

$$\delta_{min} \leq \delta \leq \delta_{max}, \tag{22}$$

where δ_{min} and δ_{max} are the smallest and largest admissible overlap δ , and

$$k_{fb,min} \leq k_{fb} \leq k_{fb,max} < \frac{1}{2}, \tag{23}$$

where $k_{fb,min}$ and $k_{fb,max}$ are the smallest and largest admissible k_{fb} , such that the applicability condition, Equation (21), holds and for any bale volume V_{b0} the optimal bale dimensions D_b^*, H_b^* are near-equal with given order ϵ_0 .

In view of Proposition 1, case (ii), the solution to the above problem exists if and only if there exist integer p_b, p_u and continuous parameters k_{fb}, δ satisfying Equations (8) and (9), inequalities expressed by Equations (22) and (23), and the applicability condition, Equation (21). This set of equations and inequalities is solved in the Appendix B, this solution is summarized in the next result.

Proposition 4. *If inequality*

$$b_{fr}(2 + 2\epsilon_0) > 2\delta_{max}(1 + 2\epsilon_0) \tag{24}$$

holds, then the solution to the Problem 1 exists if and only if there exists an integer p_b such that inequalities:

$$p_l \frac{(1 - k_{fb,max})(1 + 2\epsilon_0)}{(2 + 2\epsilon_0) - k_{fb,max}(1 + 2\epsilon_0)} \leq p_b \leq p_l \frac{(1 - k_{fb,min})(1 + 2\epsilon_0)}{(2 + 2\epsilon_0) - k_{fb,min}(1 + 2\epsilon_0)}, \tag{25}$$

$$p_l \frac{(b_{fr} - 2\delta_{max})(1 + 2\epsilon_0)}{b_{fr}(2 + 2\epsilon_0) - 2\delta_{max}(1 + 2\epsilon_0)} \leq p_b \leq p_l \frac{(b_{fr} - 2\delta_{min})(1 + 2\epsilon_0)}{b_{fr}(2 + 2\epsilon_0) - 2\delta_{min}(1 + 2\epsilon_0)} \tag{26}$$

are satisfied together with the applicability condition, Equation (21). For any p_b solving Problem 1 the overlap ratio k_{fb} is given by equation

$$k_{fb} = \frac{p_l(1 + 2\epsilon_0) - p_b(2 + 2\epsilon_0)}{(p_l - p_b)(1 + 2\epsilon_0)}, \tag{27}$$

while by Equation (9) the overlap $\delta = \frac{1}{2}b_{fr}k_{fb}$. For the assumed bale volume optimal bale dimensions D_b^* and H_b^* are given by Equations (5) and (6), respectively.

Problem 1 is a mixed decision problem with integer variable p_b and continuous variables δ, k_{fb} . From Proposition 4 the following algorithm follows.

3.1.1. Algorithm 1

Assume the inequality expressed by Equation (24) is satisfied and film parameters b_f, v_f, ϵ_{1f} , upper layers overlap ratio k_f , Equation (20), number of global film layers p_l , and the order of near-equilibrium ϵ_0 are given. Take minimal $\delta_{min}, k_{fb,min}$ and maximal $\delta_{max}, k_{fb,max}$ values of the overlaps δ, k_{fb} based on the knowledge and experience of baling to guarantee an appropriate tightness of the wrappings.

1. Determine the set \mathcal{P}_1 of all integer p_b defined by the inequalities

$$\max \left\{ p_l \frac{1-k_{fb,max}}{2+2\varepsilon_0-k_{fb,max}}, p_l \frac{b_{fr}-2\delta_{max}}{1+2\varepsilon_0 b_{fr}-2\delta_{max}} \right\} \leq p_b \leq \min \left\{ p_l \frac{1-k_{fb,min}}{2+2\varepsilon_0-k_{fb,min}}, p_l \frac{b_{fr}-2\delta_{min}}{1+2\varepsilon_0 b_{fr}-2\delta_{min}} \right\}, \quad (28)$$

for which there exists $2 \leq q \leq 5$ satisfying applicability condition, Equation (21).

2. If the set \mathcal{P}_1 is empty, then the solution to Problem 1 does not exist—go to step 3. Otherwise, go to step 4.
3. Change the lower $k_{fb,min}$, δ_{min} or upper $k_{fb,max}$, δ_{max} bounds of wrapping parameters, or the order ε_0 , or the film width b_f and repeat the computations starting from step 1.
4. For any $p_b \in \mathcal{P}_1$ compute the overlap ratio k_{fb} according to Equation (27) and, next, the overlap $\delta = \frac{1}{2} b_{fr} k_{fb}$.
5. If a bale volume V_{b0} is assumed, then for any $p_b \in \mathcal{P}_1$ compute film usage FC obtained for optimal D_b^* , H_b^* , Equations (5) and (6), using Equation (A2) and choose that p_b which yields the minimal film consumption.

Application of Proposition 4 and Algorithm 1 is illustrated by an example.

3.1.2. Example 1

Plastic film of the parameters $v_f = 0.34$, $\varepsilon_{lf} = 0.7(-)$ and $b_f = 0.75$ m is assumed. Even numbers of global film layers $4 \leq p_l \leq 16$ and “uniform” overlap ratios k_f , Equation (20), are considered. Firstly, the following bounds for bottom layers overlaps are assumed: $k_{fb,min} = 0.2$, $k_{fb,max} = 0.45$, $\delta_{min} = 0.15$ m, $\delta_{max} = 0.3$ m. Three orders $\varepsilon_0 = 0.05, 0.1, 0.15$ of near-equilibrium are considered. Inequality from Equation (24) is satisfied. Unfortunately, the set \mathcal{P}_1 of integer solutions p_b of the inequalities expressed by Equation (28) is empty for all ε_0 considered. When the lower constraint of δ is changed into $\delta_{min} = 0.1$ m, then for each ε_0 the set \mathcal{P}_1 is composed of a few p_b . They are listed in Table 1 together with respective overlaps k_{fb} and δ . Bale volume $V_{b0} = 1.357$ m³ is assumed, film usage for the optimal bale dimensions is computed and given in the last column of Table 1; for all p_l for which the set \mathcal{P}_1 is nonempty, there is only one $p_b \in \mathcal{P}_1$. Note, that for given p_l film usage obtained for D_b^* , H_b^* and decomposition (p_b, p_u) are the same for all ε_0 considered. Thus, the smallest ε_0 can be chosen. For example, for $V_{b0} = 1.357$ m³ taking $\varepsilon_0 = 0.05$ we obtain near-equal optimal dimensions $D_b^* = 1.22$ m and $H_b^* = 1.162$ m. The same regularity holds for wider film $b_f = 0.9$ m, for which the solutions to Problem 1 are given in Table 2.

Table 1. The numbers of global p_l and bottom p_b film layers and bottom layers overlaps k_{fb} , δ for which Problem 1 of volume-free near-equal optimal bale dimensions design has solution for the assumed order of near-equilibrium ε_0 and bale from Example 1; film width $b_f = 0.75$ m, the bounds for constraints from Equations (22) and (23): $k_{fb,min} = 0.2$, $k_{fb,max} = 0.45$, $\delta_{min} = 0.1$ m, $\delta_{max} = 0.3$ m, $4 \leq p_l \leq 16$ were considered. Film usage FC , Equation (A2), for the optimal bale dimensions for bale volume $V_{b0} = 1.357$ m³.

| ε_0 | p_l | p_b | k_{fb} [-] | δ [m] | FC [m ⁻¹] |
|-----------------|-------|-------|--------------|--------------|-------------------------|
| 0.05 | 10 | 4 | 0.394 | 0.113 | 50.815 |
| | 12 | 5 | 0.351 | 0.100 | 61.958 |
| 0.1 | 10 | 4 | 0.444 | 0.127 | 50.815 |
| | 12 | 5 | 0.405 | 0.116 | 61.958 |
| | 14 | 6 | 0.375 | 0.107 | 71.541 |
| | 16 | 7 | 0.352 | 0.101 | 81.124 |
| 0.15 | 14 | 6 | 0.423 | 0.121 | 71.541 |
| | 16 | 7 | 0.402 | 0.115 | 81.124 |

Table 2. The numbers of global p_l and bottom p_b film layers and bottom layers overlaps k_{fb} , δ for which Problem 1 of volume-free near-equal optimal bale dimensions design has solution for given order of near-equilibrium ϵ_0 and bale from Example 1; film width $b_f = 0.9$ m, the bounds for constraints from Equations (22) and (23): $k_{fb,min} = 0.2$, $k_{fb,max} = 0.45$, $\delta_{min} = 0.1$ m, $\delta_{max} = 0.3$ m, $4 \leq p_l \leq 16$ were considered. Film usage FC , Equation (A2), for the optimal bale dimensions for bale volume $V_{b0} = 1.357$ m³.

| ϵ_0 | p_l | p_b | k_{fb} [-] | δ [m] | FC [m ⁻¹] |
|--------------|-------|-------|--------------|--------------|-------------------------|
| 0.05 | 10 | 4 | 0.394 | 0.135 | 49.478 |
| | 12 | 5 | 0.351 | 0.120 | 59.507 |
| | 14 | 6 | 0.318 | 0.109 | 67.664 |
| | 16 | 7 | 0.293 | 0.100 | 77.693 |
| 0.1 | 10 | 4 | 0.444 | 0.152 | 55.360 |
| | 12 | 5 | 0.405 | 0.139 | 59.507 |
| | 14 | 6 | 0.375 | 0.129 | 67.664 |
| | 16 | 7 | 0.352 | 0.121 | 77.693 |
| 0.15 | 14 | 6 | 0.423 | 0.145 | 67.664 |
| | 16 | 7 | 0.402 | 0.138 | 77.693 |

3.1.3. Effect of the Non-Equilibrium Order ϵ_0

Parameter ϵ_0 influences the constraints expressed by Equations (25) and (26). If ϵ_0 grows, then for given p_l both the lower and upper bounds for p_b expressed by Equations (25) and (26) increase provided that the inequality from Equation (24) holds; however, these changes are not significant (see Table 1).

3.1.4. Effect of the Film Width

Film width b_f , or equivalently b_{fr} , influence the inequality from Equation (24); if b_{fr} grows, then this inequality is still satisfied. Also, the lower and upper bounds from inequalities expressed by Equation (26) grow with b_{fr} . Thus, the increase of the film width can extend the set of integer p_b which solve Problem 1. If, for example, the width $b_f = 0.9$ m, then for $\delta_{min} = 0.15$ m the set of solutions to Problem 1 for $\epsilon_0 = 0.1$ is not empty; for $p_l = 10$ integer $p_b = 4$ and the bottom layers overlaps $k_{fb} = 0.444$, $\delta = 0.152$ m solve this task. The solutions to Problem 1 for $b_f = 0.9$ m and $\delta_{min} = 0.1$ m are summarized in Table 2.

3.2. Volume-Fixed Near-Equal Optimal Bale Dimensions Design

Based on Proposition 1, case (i), the following problem of bale dimensions optimal design was formulated.

Problem 2. Given: film parameters b_f , v_f , ϵ_{lf} , upper layers overlap ratio k_f , Equation (20), number of global film layers p_l , bale volume V_{b0} , and order of near-equilibrium ϵ_0 . Find film layers decomposition (p_b, p_u) and the bottom layers overlaps δ , k_{fb} satisfying constraints expressed by Equations (22) and (23) such that the optimal bale dimensions D_b^* , H_b^* yielding bale volume V_{b0} are near-equal with order ϵ_0 and results in the minimal film usage.

Based on Proposition 1 and the properties of film consumption index FC , Equation (A2), with non-continuous ceiling function the solution of the above problem is derived in Appendix B. The next proposition abstracts this solution.

Proposition 5. The solution to Problem 2 exists if and only if there exists an integer p_b and k_{fb} such that inequalities:

$$\check{k}_{fb}(p_b) \leq k_{fb} \leq \min \left\{ \hat{k}_{fb}(p_b), \bar{k}_{fb}(p_b), k_{fb,max} \right\}, \tag{29}$$

$$k_{fb} \neq \frac{(p_l - p_b)(1 + 2\varepsilon_0) - p_b}{(p_l - p_b)(1 + 2\varepsilon_0)} = k_{fb,ne}(p_b) \tag{30}$$

are satisfied together with the applicability condition, Equation (21), where

$$k_{fb}(p_b) = \frac{2\delta_{max}(\sqrt[3]{1 + \varepsilon_0})^2 - \sqrt[3]{\frac{4V_{b0}}{\pi}} + (1 + 2\varepsilon_0)\left(\frac{p_l}{p_b} - 1\right)\sqrt[3]{\frac{4V_{b0}}{\pi}}}{(1 + 2\varepsilon_0)\left(\frac{p_l}{p_b} - 1\right)\sqrt[3]{\frac{4V_{b0}}{\pi}} + b_{fr}(\sqrt[3]{1 + \varepsilon_0})^2}, \tag{31}$$

$$\check{k}_{fb}(p_b) = \max \left\{ \frac{2\delta_{min}(\sqrt[3]{1 + \varepsilon_0})^2 - \sqrt[3]{\frac{4V_{b0}}{\pi}} + (1 + 2\varepsilon_0)\left(\frac{p_l}{p_b} - 1\right)\sqrt[3]{\frac{4V_{b0}}{\pi}}}{(1 + 2\varepsilon_0)\left(\frac{p_l}{p_b} - 1\right)\sqrt[3]{\frac{4V_{b0}}{\pi}} + b_{fr}(\sqrt[3]{1 + \varepsilon_0})^2}, k_{fb,min} \right\}, \tag{32}$$

and $\hat{k}_{fb}(p_b)$ is given by

$$\hat{k}_{fb}(p_b) = 1 - \frac{2}{\frac{b_{fr}(\sqrt[3]{1 + \varepsilon_0})^2}{\sqrt[3]{\frac{4V_{b0}}{\pi}}} \left[\frac{\sqrt[3]{\frac{4V_{b0}}{\pi}} \left[\frac{2}{1 - \check{k}_{fb}(p_b)} - \left(\frac{p_l}{p_b} - 1\right)(1 + 2\varepsilon_0) \right]}{b_{fr}(\sqrt[3]{1 + \varepsilon_0})^2} \right] + \left(\frac{p_l}{p_b} - 1\right)(1 + 2\varepsilon_0)}. \tag{33}$$

For any such p_b and k_{fb} the overlap δ is given by equation

$$\delta = \frac{\sqrt[3]{\frac{4V_{b0}}{\pi}}}{2(\sqrt[3]{1 + \varepsilon_0})^2} \left[1 - \left(\frac{p_l}{p_b} - 1\right)(1 - k_{fb})(1 + 2\varepsilon_0) \right] + \frac{b_{fr}k_{fb}}{2}, \tag{34}$$

the optimal bale dimensions are described by Equations (5) and (6) and film usage

$$FC(p_b) = \frac{4b_f\sqrt[3]{1 + \varepsilon_0}}{(\varepsilon_{lf} + 1)\left[\sqrt[3]{\frac{4V_{b0}}{\pi}}\right]^2} \left\{ p_b \left[\frac{\sqrt[3]{\frac{4V_{b0}}{\pi}} \left[\frac{2}{1 - \check{k}_{fb}(p_b)} - \left(\frac{p_l}{p_b} - 1\right)(1 + 2\varepsilon_0) \right]}{b_{fr}(\sqrt[3]{1 + \varepsilon_0})^2} \right] + \frac{2(2 + \varepsilon_0)}{\pi(1 + \varepsilon_0)} \left[\frac{\pi\sqrt[3]{\frac{4V_{b0}}{\pi}}(1 + \varepsilon_0)(p_l - p_b)}{2b_{fr}(\sqrt[3]{1 + \varepsilon_0})^2} \right] \right\}, \tag{35}$$

where $[x]$ denotes ceiling function [45]. Integer p_b^* solving Problem 2 is such that

$$FC(p_b^*) = \min_{p_b} FC(p_b). \tag{36}$$

It may be easily shown that if the following inequality

$$b_{fr} \left[1 - \frac{1}{\left(\frac{p_l}{p_b} - 1\right)(1 + 2\varepsilon_0)} \right] < 2\delta_{min} \tag{37}$$

holds, then

$$k_{fb,ne}(p_b) < \bar{k}_{fb}(p_b) \tag{38}$$

and condition from Equation (30) can be neglected. In all the tested examples the above inequality has been satisfied, compare Table 3 below. The inequality from Equation (37), similar to the condition expressed by Equation (12), holds especially if the number of bottom film layers is such that

$$p_b > \frac{1 + 2\varepsilon_0}{2 + 2\varepsilon_0} p_l.$$

The inequalities specified in the above proposition must be solved for two variables: integer p_b and continuous k_{fb} . For any fixed p_b the overlap δ given by Equation (34) increases linearly with growing k_{fb} .

3.2.1. Algorithm 2

Assume film parameters $b_f, v_f, \varepsilon_{lf}$, overlap ratio k_f , Equation (20), number of global film layers p_l , the order of near-equilibrium ε_0 , and bale volume V_{b0} are given. Take minimal $\delta_{min}, k_{fb,min}$ and maximal $\delta_{max}, k_{fb,max}$ values of the overlaps δ, k_{fb} to guarantee the appropriate tightness of the wrappings.

1. Determine the set \mathcal{P}_2 of all integer p_b for which there exists $2 \leq q \leq 5$ satisfying applicability condition, Equation (21), and the set of overlap ratios $\mathcal{K}_2(p_b)$ defined by the inequalities expressed by Equations (29) and (30) is nonempty.
2. If the set \mathcal{P}_2 is empty, then the solution to Problem 2 does not exist—go to step 3. Otherwise, go to step 4.
3. Change the lower $k_{fb,min}, \delta_{min}$ or upper $k_{fb,max}, \delta_{max}$ bounds of wrapping parameters, or the order ε_0 , or the film width b_f and go to step 1.
4. Solve in p_b^* the integer programming task expressed by Equation (36) for $p_b \in \mathcal{P}_2$.
5. For the best p_b^* choose practically reasonable $k_{fb} \in \mathcal{K}_2(p_b^*)$, compute the overlap $\delta = \delta(p_b^*)$ according to Equation (34) and optimal D_b^*, H_b^* using Equations (5) and (6). The minimal film consumption is equal to $FC(p_b^*)$ computed in step 4.

3.2.2. Example 2

Film parameters and overlaps constraints from Example 1 are taken, again. Bale volume $V_{b0} = 1.357 \text{ m}^3$ is assumed together with orders of near-equilibrium $\varepsilon_0 = 0.05$ and $\varepsilon_0 = 0.1$. Integer p_b , bottom $\check{k}_{fb}(p_b)$ and upper $\hat{k}_{fb}, \bar{k}_{fb}$ bounds, parameters $k_{fb,ne}$ and closed intervals \mathcal{K}_2 of k_{fb} are given in Table 3. In all tested examples $k_{fb,ne} < \check{k}_{fb}(p_b)$. The upper bound of the closed interval of k_{fb} defined by Equation (29) is determined by $\hat{k}_{fb}(p_b)$, or $\bar{k}_{fb}(p_b)$, or $k_{fb,max}$ —there is no rule here. For any $k_{fb} \in \mathcal{K}_2$ the formulas from Equation (34) for computing δ are presented in the penultimate column. In the last column film usage $FC(p_b)$, Equation (35), is given. For most $p_b \in \mathcal{P}_2$ the closed intervals $\mathcal{K}_2(p_b)$ are wide enough to select a practically convenient overlap ratio k_{fb} . In some cases this interval is very narrow, e.g., for $p_l = 10, p_b = 6$, we have $\mathcal{K}_2 = [0.200, 0.204]$; however, $k_{fb} = 0.2$ is acceptable from the engineering point of view. For $\varepsilon_0 = 0.05$ and $p_l = 4, 6, 8$ set \mathcal{P}_2 is composed of only one number p_b of bottom film layers and this p_b solves Problem 2. For $p_l = 10, 12, 14, 16$ there are at least two p_b for which the set $\mathcal{K}_2(p_b)$ is nonempty, the optimal film usage $FC(p_b^*)$ defined in Equation (36) is marked by bold in Table 3. For $\varepsilon_0 = 0.05$, according to Equations (5) and (6), the optimal bale dimensions $D_b^* = 1.22 \text{ m}$, $H_b^* = 1.162 \text{ m}$, for $\varepsilon_0 = 0.1$ we have $D_b^* = 1.239 \text{ m}$, $H_b^* = 1.126 \text{ m}$. If $\varepsilon_0 = 0.05$ is assumed, then for $p_l = 4, 6, 8$ the overlaps $k_{fb} = 0.3$ and $\delta = 0.2195 \cong 0.22$ can be applied. For $p_l = 10$ Problem 2 is solved for the same overlaps for $p_b = 5$ yielding minimal film usage. Similarly, for $p_l = 16$ and $p_b = 8$. For $p_l = 12$ solution to Problem 2 is composed by $p_b = 7$ and, for example, $k_{fb} = 0.2$ and $\delta = 0.272$. For $p_l = 14$ we have: $p_b = 6$ and $k_{fb} = 0.4$ and $\delta = 0.184$. Similarly, based on the data from Table 3, the solutions to Problem 2 can be found for the greater order $\varepsilon_0 = 0.1$.

Table 3. The numbers of global p_l and bottom p_b film layers, lower $\check{k}_{fb}(p_b)$, Equation (32), and upper \hat{k}_{fb} , \bar{k}_{fb} bounds defined in Equations (33) and (31), parameters $k_{fb,ne}$, Equation (30), the non-empty sets of overlap ratios $\mathcal{K}_2(p_b)$ defined by the inequalities expressed by Equations (29) and (30) and the linear functions, Equation (34), describing overlap δ for $k_{fb} \in \mathcal{K}_2(p_b)$ determined to find the solution to Problem 2 of near-equal optimal bale dimensions design for fixed volume $V_{b0} = 1.357 \text{ m}^3$, bale from Example 2; orders of near-equilibrium $\epsilon_0 = 0.05, 0.1$, film width $b_f = 0.75 \text{ m}$, bounds for the constraints from Equations (22) and (23): $k_{fb,min} = 0.2, k_{fb,max} = 0.45, \delta_{min} = 0.15 \text{ m}, \delta_{max} = 0.3 \text{ m}, 4 \leq p_l \leq 16$. Film usage $FC(p_b)$, Equation (35), for $p_b \in \mathcal{P}_2$ and $k_{fb} \in \mathcal{K}_2(p_b)$; the optimal $FC(p_b^*)$ defined in Equation (36) is marked by bold.

| ϵ_0 | p_l | p_b | $k_{fb,ne} [-]$ | $\check{k}_{fb}(p_b) [-]$ | $\hat{k}_{fb} [-]$ | $\bar{k}_{fb} [-]$ | \mathcal{K}_2 | $\delta [m]$ | $FC(p_b)$ |
|--------------|-------|-------|-----------------|---------------------------|--------------------|--------------------|-----------------|------------------------------|------------------------------|
| 0.05 | 4 | 2 | 0.091 | 0.225 | 0.348 | 0.387 | [0.225, 0.348] | $0.925 \cdot k_{fb} - 0.058$ | 20.804 |
| | 6 | 3 | 0.091 | 0.225 | 0.348 | 0.387 | [0.225, 0.348] | $0.925 \cdot k_{fb} - 0.058$ | 31.979 |
| | 8 | 4 | 0.091 | 0.225 | 0.348 | 0.387 | [0.225, 0.348] | $0.925 \cdot k_{fb} - 0.058$ | 41.607 |
| | 10 | 4 | 0.394 | 0.424 | 0.447 | 0.545 | [0.424, 0.447] | $1.244 \cdot k_{fb} - 0.378$ | 52.445 |
| | 10 | 5 | 0.091 | 0.225 | 0.348 | 0.387 | [0.225, 0.348] | $0.925 \cdot k_{fb} - 0.058$ | 51.235 |
| | 10 | 6 | -0.364 | 0.200 | 0.260 | 0.204 | [0.200, 0.204] | $0.712 \cdot k_{fb} + 0.155$ | 51.573 |
| | 12 | 5 | 0.351 | 0.393 | 0.430 | 0.520 | [0.393, 0.430] | $1.180 \cdot k_{fb} - 0.314$ | 62.073 |
| | 12 | 6 | 0.091 | 0.225 | 0.348 | 0.387 | [0.225, 0.348] | $0.925 \cdot k_{fb} - 0.058$ | 62.411 |
| | 12 | 7 | -0.273 | 0.200 | 0.274 | 0.237 | [0.200, 0.237] | $0.742 \cdot k_{fb} + 0.124$ | 61.200 |
| | 14 | 6 | 0.318 | 0.370 | 0.418 | 0.502 | [0.370, 0.418] | $1.138 \cdot k_{fb} - 0.271$ | 71.700 |
| | 14 | 7 | 0.091 | 0.225 | 0.348 | 0.387 | [0.225, 0.348] | $0.925 \cdot k_{fb} - 0.058$ | 72.038 |
| | 14 | 8 | -0.212 | 0.200 | 0.284 | 0.259 | [0.200, 0.259] | $0.765 \cdot k_{fb} + 0.102$ | 72.376 |
| | 16 | 7 | 0.293 | 0.353 | 0.409 | 0.488 | [0.353, 0.409] | $1.107 \cdot k_{fb} - 0.241$ | 82.876 |
| | 16 | 8 | 0.091 | 0.225 | 0.348 | 0.387 | [0.225, 0.348] | $0.925 \cdot k_{fb} - 0.058$ | 81.666 |
| | 16 | 9 | -0.169 | 0.200 | 0.292 | 0.276 | [0.200, 0.276] | $0.783 \cdot k_{fb} + 0.084$ | 82.004 |
| | 0.1 | 4 | 2 | 0.167 | 0.273 | 0.381 | 0.429 | [0.273, 0.381] | $0.961 \cdot k_{fb} - 0.113$ |
| 6 | | 3 | 0.167 | 0.273 | 0.381 | 0.429 | [0.273, 0.381] | $0.961 \cdot k_{fb} - 0.113$ | 32.096 |
| 8 | | 4 | 0.167 | 0.273 | 0.381 | 0.429 | [0.273, 0.381] | $0.961 \cdot k_{fb} - 0.113$ | 41.769 |
| 8 | | 5 | -0.389 | 0.200 | 0.273 | 0.206 | [0.200, 0.206] | $0.691 \cdot k_{fb} + 0.158$ | 42.217 |
| 10 | | 5 | 0.167 | 0.273 | 0.381 | 0.429 | [0.273, 0.381] | $0.961 \cdot k_{fb} - 0.113$ | 52.980 |
| 10 | | 6 | -0.250 | 0.200 | 0.293 | 0.255 | [0.200, 0.255] | $0.736 \cdot k_{fb} + 0.113$ | 51.890 |
| 12 | | 5 | 0.405 | 0.433 | 0.461 | 0.554 | [0.433, 0.450] | $1.232 \cdot k_{fb} - 0.383$ | 62.206 |
| 12 | | 6 | 0.167 | 0.273 | 0.381 | 0.429 | [0.273, 0.381] | $0.961 \cdot k_{fb} - 0.113$ | 62.654 |
| 12 | | 7 | -0.167 | 0.200 | 0.307 | 0.286 | [0.200, 0.286] | $0.768 \cdot k_{fb} + 0.080$ | 63.102 |
| 14 | | 6 | 0.375 | 0.411 | 0.449 | 0.538 | [0.411, 0.449] | $1.187 \cdot k_{fb} - 0.338$ | 73.417 |
| 14 | | 7 | 0.167 | 0.273 | 0.381 | 0.429 | [0.273, 0.381] | $0.961 \cdot k_{fb} - 0.113$ | 72.327 |
| 14 | | 8 | -0.111 | 0.200 | 0.317 | 0.308 | [0.200, 0.308] | $0.792 \cdot k_{fb} + 0.056$ | 72.775 |
| 16 | | 7 | 0.352 | 0.395 | 0.440 | 0.525 | [0.395, 0.440] | $1.154 \cdot k_{fb} - 0.306$ | 83.091 |
| 16 | | 8 | 0.167 | 0.273 | 0.381 | 0.429 | [0.273, 0.381] | $0.961 \cdot k_{fb} - 0.113$ | 83.538 |
| 16 | | 9 | -0.071 | 0.200 | 0.325 | 0.324 | [0.200, 0.324] | $0.811 \cdot k_{fb} + 0.038$ | 82.448 |
| 16 | | 10 | -0.389 | 0.200 | 0.273 | 0.206 | [0.200, 0.206] | $0.691 \cdot k_{fb} + 0.158$ | 82.896 |

3.3. Volume-Free Optimal Equal Bale Dimensions Design

In this section the problem of volume-free design of equal optimal bale dimensions is considered being, in fact, a special case of Problem 1 for the order of near-equilibrium $\epsilon_0 = 0$.

Problem 3. Given: film parameters b_f, v_f, ϵ_{1f} , overlap ratio k_f , Equation (20), and number of global film layers p_l . Find film layers decomposition (p_b, p_u) and overlaps δ, k_{fb} satisfying constraints expressed by Equations (22) and (23) such that the applicability condition, Equation (21), hold and for any bale volume V_{b0} the optimal dimensions D_b^*, H_b^* are equal.

The solution to the above problem is identical with that of Problem 1 for $\epsilon_0 = 0$. Thus, the next result follows directly from Proposition 4 by lying $\epsilon_0 = 0$ in inequalities expressed by Equations (25) and (26) and in Equations (5), (6) and (27). Inequality from Equation (24), here $b_{fr} > \delta_{max}$, is identity.

Proposition 6. *The solution to Problem 3 exists if and only if there exists an integer p_b such that inequalities*

$$\max\left\{\frac{b_{fr}-2\delta_{max}}{2(b_{fr}-\delta_{max})}p_l, \frac{1-k_{fb,max}}{2-k_{fb,max}}p_l\right\} \leq p_b \leq \min\left\{\frac{p_l}{2}, \frac{b_{fr}-2\delta_{min}}{2(b_{fr}-\delta_{min})}p_l, \frac{1-k_{fb,min}}{2-k_{fb,min}}p_l\right\} \quad (39)$$

are satisfied together with the applicability condition, Equation (21). For any such p_b the bottom layers overlaps are uniquely determined by

$$k_{fb} = \frac{p_l - 2p_b}{p_l - p_b}, \quad \delta = \frac{1}{2}b_{fr}k_{fb}. \quad (40)$$

For a given bale volume optimal bale dimensions D_b^ and H_b^* are given by Equation (19).*

Thus, to solve Problem 3 an integer p_b fulfilling inequalities from Equation (39) must be found. The fractions dependent on b_{fr} in “max” and “min” functions, which define the lower and upper bounds in this inequalities, increase with the film width. However, the upper and lower bounds may be determined by other arguments of “max” or “min”, thus a larger b_{fr} does not necessarily imply a greater p_b . Proposition 6 yielded the following algorithm.

3.3.1. Algorithm 3

Assume film parameters $b_f, v_f, \varepsilon_{lf}$, overlap ratio k_f , Equation (20), and the number of global film layers p_l are given. Take minimal $\delta_{min}, k_{fb,min}$ and maximal $\delta_{max}, k_{fb,max}$ values of the overlaps δ, k_{fb} .

1. Determine the set \mathcal{P}_3 of all integer p_b defined by the inequalities from Equation (39) for which there exists $2 \leq q \leq 5$ satisfying applicability condition, Equation (21).
2. If the set \mathcal{P}_3 is empty, then the solution to Problem 3 does not exist—go to step 3. Otherwise, go to step 4.
3. Change the lower $k_{fb,min}, \delta_{min}$ or upper $k_{fb,max}, \delta_{max}$ bounds of wrapping parameters or the film width b_f and go to step 1.
4. For any $p_b \in \mathcal{P}_3$ compute the overlap ratio k_{fb} and, next, overlap δ according to Equation (40).
5. For any $p_b \in \mathcal{P}_3$ and assumed bale volume V_{b0} compute film usage obtained for the optimal D_b^*, H_b^* , Equation (19), using Equation (A2) and choose that p_b which yields the minimal film consumption.

3.3.2. Example 3

The same mechanical parameters of the film and wrapping parameters constraints as in the previous examples are assumed. Film widths $b_f = 0.75, 0.9$ m and $4 \leq p_l \leq 16$ are considered. For $b_f = 0.75$ m, similarly as in Example 1, the set of integer solutions p_b of inequalities expressed by Equation (39) is empty for all p_l considered, while for the wider film for $p_l = 14$ solution to Problem 3 exists. When the lower constraint expressed by Equation (22) is changed into $\delta_{min} = 0.1$ m, then for some p_l Problem 3 has a solution. These solutions are listed in Table 4. For every p_l , nonempty \mathcal{P}_3 is a singleton (a unit set), thus step 5 of the selection of best film layers composition $p_l = p_b + p_u$ can be omitted here; however film usage FC is added in the last column.

Table 4. The numbers of global p_l and bottom p_b film layers and bottom layers overlaps k_{fb} , δ for which Problem 3 of volume-free equal optimal bale dimensions design has a solution for bale from Example 3; film width $b_f = 0.75, 0.9$ m, the bounds from Equations (22) and (23): $k_{fb,min} = 0.2, k_{fb,max} = 0.45, \delta_{min} = 0.1, 0.15$ m, $\delta_{max} = 0.3$ m, $4 \leq p_l \leq 16$. Film usage FC , Equation (A2) for the assumed bale volume $V_{b0} = 1.357$ m³.

| δ_{min} [m] | b_f [m] | p_l | p_b | k_{fb} [-] | δ [m] | FC [m ⁻¹] |
|--------------------|-----------|-------|-------|--------------|--------------|-------------------------|
| 0.1 | 0.75 | 8 | 3 | 0.400 | 0.114 | 41.047 |
| | | 14 | 5 | 0.444 | 0.127 | 72.524 |
| | | 16 | 6 | 0.400 | 0.114 | 82.095 |
| | 0.9 | 8 | 3 | 0.400 | 0.137 | 39.449 |
| | | 10 | 4 | 0.333 | 0.114 | 49.478 |
| | | 14 | 5 | 0.444 | 0.152 | 76.222 |
| 0.15 | 0.9 | 16 | 6 | 0.400 | 0.137 | 78.898 |
| | | 14 | 5 | 0.444 | 0.152 | 76.222 |

3.4. Volume-Fixed Optimal Equal Bale Dimensions Design

Problem 4. Given: film parameters b_f, v_f, ϵ_{1f} , overlap ratio k_f , Equation (20), number of global film layers p_l and bale volume V_{b0} . Find film layers decomposition (p_b, p_u) and overlaps δ, k_{fb} satisfying constraints from Equations (22) and (23) such that the optimal bale dimensions D_b^*, H_b^* resulting in the volume V_{b0} are equal and film usage is minimal.

Lying $\epsilon_0 = 0$ in inequalities expressed by Equations (29) and (30) and in Equations (34) and (35) based on Proposition 5 the following solution results.

Proposition 7. The solution to Problem 4 exists if and only if there exist k_{fb} and an integer p_b such that inequalities

$$\bar{k}_{fb}(p_b) = \max\{\check{k}_{fb}(p_b), k_{fb,min}\} \leq k_{fb} \leq \min\left\{\tilde{k}_{fb}(p_b), \frac{2\delta_{max} + \left(\frac{p_l}{p_b} - 2\right)\sqrt[3]{\frac{4V_{b0}}{\pi}}}{\left(\frac{p_l}{p_b} - 1\right)\sqrt[3]{\frac{4V_{b0}}{\pi}} + b_{fr}}, k_{fb,max}\right\} \quad (41)$$

and inequality

$$k_{fb} \neq \frac{p_l - 2p_b}{p_l - p_b} \quad (42)$$

are satisfied together with the applicability condition, Equation (21), where

$$\check{k}_{fb}(p_b) = \frac{2\delta_{min} + \left(\frac{p_l}{p_b} - 2\right)\sqrt[3]{\frac{4V_{b0}}{\pi}}}{\left(\frac{p_l}{p_b} - 1\right)\sqrt[3]{\frac{4V_{b0}}{\pi}} + b_{fr}}, \quad (43)$$

and $\tilde{k}_{fb}(p_b)$ is defined by the next equation with $\bar{k}_{fb}(p_b)$ introduced in Equation (41)

$$\tilde{k}_{fb}(p_b) = 1 - \frac{2}{\frac{b_{fr}}{\sqrt[3]{\frac{4V_{b0}}{\pi}}} \left[\frac{\sqrt[3]{\frac{4V_{b0}}{\pi}} \left[\frac{2}{1 - \bar{k}_{fb}(p_b)} - \left(\frac{p_l}{p_b} - 1\right) \right]}{b_{fr}} \right] + \left(\frac{p_l}{p_b} - 1\right)}. \quad (44)$$

For any such p_b and k_{fb} the overlap δ is given by

$$\delta = \frac{\sqrt[3]{\frac{4V_{b0}}{\pi}}}{2} \left[1 - \left(\frac{p_l}{p_b} - 1\right) (1 - k_{fb}) \right] + \frac{b_{fr} k_{fb}}{2}, \quad (45)$$

the optimal bale dimensions are given by Equation (19) and film usage

$$FC(p_b) = \frac{4b_f}{(\varepsilon_{lf}+1) \left[\sqrt[3]{\frac{4V_{b0}}{\pi}} \right]^2} \left\{ \left[p_b \frac{\sqrt[3]{\frac{4V_{b0}}{\pi}} \left[\frac{2}{1-k_{fb}(p_b)} - \left(\frac{p_l}{p_b} - 1 \right) \right]}{b_{fr}} \right] + \frac{4}{\pi} \left[\frac{\pi \sqrt[3]{\frac{4V_{b0}}{\pi}} (p_l - p_b)}{2b_{fr}} \right] \right\} \quad (46)$$

Integer p_b^* solving Problem 4 is defined by optimization task expressed by Equation (36).

Thus, the inequalities specified in the proposition must be solved for two variables: integer p_b and continuous k_{fb} . Next optimization task, Equation (36), must be solved and the overlap δ can be computed.

Example 4

Film parameters and overlap ratio k_f as in the previous examples are given. Film widths $b_f = 0.75, 0.9$ m, even p such that $4 \leq p_l \leq 16$ and bale volume $V_{b0} = 1.357 \text{ m}^3$ are assumed. For any p_l the sets of overlap ratios k_{fb} satisfying inequalities from Equations (41) and (42) are given in the fourth column of Table 5. As previously, for any k_{fb} from these closed intervals the linear formulas from Equation (45) for computing δ are given in the next column, in the last column film consumptions $FC(p_b)$, Equation (46), are enclosed. For these p_l for which there are more than one p_b such that inequalities from Equations (41) and (42) are satisfied the minimal film usage, solving optimization task from Equation (36), is marked by bold. Note that in most cases film usage for $b_f = 0.9$ m is smaller from that for $b_f = 0.75$ m; this confirms the analysis of the film width influence on the optimal film usage from [14,20]—the broader film width is applied, the smaller minimal film usage is achieved.

Table 5. The numbers of global p_l and bottom p_b film layers, bottom layers overlaps k_{fb} , δ fulfilling inequalities from Equations (41) and (42), and respective film usage FC , Equation (46), for $V_{b0} = 1.357 \text{ m}^3$ and bale from Example 4; film widths $b_f = 0.75, 0.9$ m, the bounds from Equations (22) and (23): $k_{fb,min} = 0.2, k_{fb,max} = 0.45, \delta_{min} = 0.15 \text{ m}, \delta_{max} = 0.3 \text{ m}, 4 \leq p_l \leq 16$. Minimal film usage solving Problem 4 of volume-fixed design of equal optimal bale dimensions is marked by bold.

| b_f [m] | p_l | p_b | k_{fb} [-] | δ [m] | FC [m^{-1}] |
|-----------|-------|-------|----------------|------------------------------|--------------------------|
| 0.75 | 4 | 2 | [0.200, 0.311] | $0.886 \cdot k_{fb}$ | 20.726 |
| | 6 | 3 | [0.200, 0.311] | $0.886 \cdot k_{fb}$ | 30.309 |
| | 8 | 3 | [0.428, 0.439] | $1.286 \cdot k_{fb} - 0.4$ | 41.232 |
| | 8 | 4 | [0.200, 0.311] | $0.886 \cdot k_{fb}$ | 41.453 |
| | 10 | 4 | [0.380, 0.412] | $1.186 \cdot k_{fb} - 0.3$ | 50.815 |
| | 10 | 5 | [0.200, 0.311] | $0.886 \cdot k_{fb}$ | 51.036 |
| | 12 | 5 | [0.346, 0.394] | $1.126 \cdot k_{fb} - 0.240$ | 61.958 |
| | 12 | 6 | [0.200, 0.311] | $0.886 \cdot k_{fb}$ | 60.619 |
| | 14 | 6 | [0.322, 0.382] | $1.086 \cdot k_{fb} - 0.2$ | 71.541 |
| | 14 | 7 | [0.200, 0.311] | $0.886 \cdot k_{fb}$ | 71.762 |
| | 14 | 8 | [0.200, 0.204] | $0.736 \cdot k_{fb} + 0.15$ | 70.423 |
| | 16 | 6 | [0.428, 0.439] | $1.286 \cdot k_{fb} - 0.4$ | 80.903 |
| | 16 | 7 | [0.304, 0.372] | $1.057 \cdot k_{fb} - 0.171$ | 81.124 |
| | 16 | 8 | [0.200, 0.311] | $0.886 \cdot k_{fb}$ | 81.345 |
| | 16 | 9 | [0.200, 0.222] | $0.752 \cdot k_{fb} + 0.133$ | 81.566 |

Table 5. Cont.

| b_f [m] | p_l | p_b | k_{fb} [–] | δ [m] | FC [m ⁻¹] |
|-----------|-------|-------|----------------|------------------------------|-----------------------|
| 0.9 | 4 | 2 | [0.200, 0.263] | $0.943 \cdot k_{fb}$ | 20.058 |
| | 6 | 3 | [0.200, 0.263] | $0.943 \cdot k_{fb}$ | 30.087 |
| | 8 | 3 | [0.410, 0.450] | $1.343 \cdot k_{fb} - 0.4$ | 43.861 |
| | 8 | 4 | [0.200, 0.263] | $0.943 \cdot k_{fb}$ | 38.244 |
| | 10 | 4 | [0.362, 0.377] | $1.243 \cdot k_{fb} - 0.3$ | 49.478 |
| | 10 | 5 | [0.200, 0.263] | $0.943 \cdot k_{fb}$ | 48.273 |
| | 12 | 5 | [0.330, 0.357] | $1.183 \cdot k_{fb} - 0.240$ | 59.507 |
| | 12 | 6 | [0.200, 0.263] | $0.943 \cdot k_{fb}$ | 58.302 |
| | 14 | 5 | [0.443, 0.450] | $1.423 \cdot k_{fb} - 0.480$ | 76.222 |
| | 14 | 6 | [0.306, 0.343] | $1.143 \cdot k_{fb} - 0.2$ | 67.664 |
| | 14 | 7 | [0.200, 0.263] | $0.943 \cdot k_{fb}$ | 68.331 |
| | 16 | 6 | [0.410, 0.450] | $1.343 \cdot k_{fb} - 0.4$ | 87.722 |
| | 16 | 7 | [0.288, 0.333] | $1.114 \cdot k_{fb} - 0.171$ | 77.693 |
| | 16 | 8 | [0.200, 0.263] | $0.943 \cdot k_{fb}$ | 76.487 |
| | 16 | 9 | [0.200, 0.206] | $0.810 \cdot k_{fb} + 0.133$ | 90.389 |

3.5. Optimal Film Usage

For the combined 3D wrapping technique, the analysis of the optimal film consumption reached for D_b^* , H_b^* was carried out in [19,20], where it was shown, in particular, that for the bottom film layers $p_b \gg p_u$ the optimal selection of bale diameter and height may result in a 3.37% to even 23.13% reduction in film usage depending on the number of global film layers and wrapping parameters, which means up to 23% film cost savings. Many conclusions regarding the impact of film width, pre-assumed bale volume and numbers of bottom and upper film layers on near-optimal bale dimensions and near-optimal film consumption were formulated [19,20].

4. Conclusions

It has been shown that for bale wrapping by a 3D combined method for given film parameters and given a global number of film layers it is possible, both for assumed and for an arbitrary bale volume, to select such wrapping parameters that guarantee the pre-assumed equilibria of optimal bale dimensions. Thus, the “compliance” is the answer to the question from the title.

The derived necessary and sufficient conditions for the balance of optimal bale dimensions allowed for the formulating of four practical problems of selecting the wrapping parameters, in particular the overlaps in the bottom film layers and decomposition of the global number of layers into bottom and upper layers. The mathematical form of the conditions of optimal bale dimensions equilibrium indicated the legitimacy of considering two separate design strategies, one for fixed and one for free bale volume. Design algorithms were derived. These algorithms only require a solution with respect to the number of film layers and overlap ratio of the bottom film layers of two or four simple algebraic inequalities. Then, on the basis of very simple formulas, the remaining wrapping parameters and the optimal bale diameter and height can be determined. Examples regarding standard large bales demonstrate that the algorithms can handle the optimal dimensions' equilibria problems.

Optimal dimensions mean a reduction in film consumption by up to 23%. Simultaneously, equal and near-equal bale dimensions are essential from the agriculture engineering practice and bale wrapping technique point of view. The algorithms developed can be applied for design of wrapping processes using a 2D method for cylindrical bales of arbitrary agricultural materials, e.g., lignocellulosic agricultural residues [46,47], bearing in mind the minimisation of the film usage and the desired equilibrium between bale dimensions. Potential applications of the algorithms include the baling of waste, such as municipal solid waste [48,49], for example.

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Appendix A

Appendix A.1. Film Consumption

A mathematical model describing the consumption of the stretch film to wrap a cylindrical bale using combined 3D technique was derived in [14]. It was assumed that the subsequent film strips, which are wrapped in p_b bottom layers on the bale’s lateral surface, overlap one another, creating the overlap $k_{fb}b_{fr}$, where k_{fb} is a dimensionless relative ratio determining the width of the contact between adjacent film strips and the film width after stretching b_{fr} is described by [12]:

$$b_{fr} = b_f(1 - \nu_f \varepsilon_{lf}), \tag{A1}$$

where b_f is the width of un-stretched film, ν_f and ε_{lf} are the Poisson’s ratio and unit deformation of the film. The extreme film strips are overlapped at the bases of the bale for δ , as shown in [14] (Figure 3). It was assumed that $0 < k_{fb} < \frac{1}{2}$, for which only one film layer results for one wrapping cycle. For the wrappings of p_u upper layers it was assumed that the subsequent film strips are wrapped along the bale’s longitudinal axis with the overlap ratio k_f [14] (Figure 2).

The global film consumption measured by the index FC defined as the ratio of the surface area of un-stretched film used to wrap the bale to bale volume, is described by the function [14]:

$$FC = \frac{4p_b b_f}{D_b H_b (\varepsilon_{lf} + 1)} \left\lceil \frac{H_b + 2\delta - b_{fr} k_{fb}}{b_{fr} (1 - k_{fb})} \right\rceil + \frac{8(D_b + H_b) b_f}{\pi D_b^2 H_b (\varepsilon_{lf} + 1)} \left\lceil \frac{\pi D_b p_u}{2b_{fr} \Omega(k_f)} \right\rceil, \tag{A2}$$

where $\lceil x \rceil$ denotes ceiling function [45], function [20]

$$\Omega(k_f) = (1 - k_f) \left\lfloor \frac{1}{1 - k_f} \right\rfloor, \tag{A3}$$

is introduced for brevity of the notation; $\lfloor x \rfloor$ is the floor function [45]. The above formula indicates the dependence of the film usage on film parameters ε_{lf} , ν_f , b_f , bale diameter D_b and height H_b , the overlaps δ , k_{fb} and k_f of the bottom and upper film layers, and the numbers of film layers p_u , p_b . Second summand of FC describe film used for upper layers wrappings provided that the following applicability condition [14]:

$$\frac{p_u}{\left\lfloor \frac{1}{1 - k_f} \right\rfloor} = m, \quad m \in \mathcal{N}, \tag{A4}$$

holds, where \mathcal{N} denotes the set of positive integer numbers. Global number of basic film layers wrapped on the bale’s lateral surface is $p_l = p_b + p_u$.

Appendix A.2. Film Consumption Minimization

As the non-continuity and non-differentiability of the original film usage index FC undermines our ability to directly analytically solve the problem of film usage minimization, special attention has been given to the simpler case of near-optimal bale dimensions design, this being as important as the optimal parameters for engineering applications. The problem of the selection of near-optimal bale dimensions has been constructed by

minimizing continuous lower bound of the original film usage index FC , where the goal function of a non-linear optimization problem is convex and differentiable. The necessary and sufficient optimality condition for near-optimal bale diameter, called “optimal” in [20] and here, was established in the form of a standard cubic equation, Equation (1), which can easily be solved using both analytical and numerical methods. The results of the numerical experiments [19,20] demonstrated that for any four to sixteen even layers of the film there are such compositions of bottom and upper film layers that the relative near-optimality errors do not exceed 0.01% whenever the optimal bale dimensions are used.

Appendix B

Proof of Proposition 1. The optimal bale diameter D_b^* is near-equal to the optimal bale height H_b^* with order ε_0 if and only if Equations (1) and (5) are satisfied, simultaneously. For D_b^* given by Equation (5) the optimality condition, Equation (1), takes the form

$$\frac{2\pi p_u}{\Omega(k_f)} \frac{4(1+\varepsilon_0)V_{b0}}{\pi} + \frac{\pi p_b}{1-k_{fb}} (2\delta - b_{fr}k_{fb}) \left(\sqrt[3]{\frac{4(1+\varepsilon_0)V_{b0}}{\pi}} \right)^2 - 4V_{b0} \left[\frac{p_b}{1-k_{fb}} + \frac{p_u}{\Omega(k_f)} \right] = 0$$

and is equivalent to the next equation

$$\frac{2p_u(1+\varepsilon_0)}{\Omega(k_f)} \sqrt[3]{\frac{4V_{b0}}{\pi}} + \frac{p_b}{1-k_{fb}} (2\delta - b_{fr}k_{fb}) (\sqrt[3]{1+\varepsilon_0})^2 - \sqrt[3]{\frac{4V_{b0}}{\pi}} \left[\frac{p_b}{1-k_{fb}} + \frac{p_u}{\Omega(k_f)} \right] = 0,$$

which can be rewritten as follows

$$\frac{p_b}{1-k_{fb}} (2\delta - b_{fr}k_{fb}) (\sqrt[3]{1+\varepsilon_0})^2 \sqrt[3]{\frac{4V_{b0}}{\pi}} \left[\frac{p_b}{1-k_{fb}} - \frac{p_u}{\Omega(k_f)} - \frac{2p_u}{\Omega(k_f)} \varepsilon_0 \right]. \tag{A5}$$

To show the Proposition two cases must be considered separately:

- (i) the wrapping parameters and order ε_0 are such that

$$\frac{p_b}{1-k_{fb}} \neq \frac{p_u}{\Omega(k_f)} + \frac{2p_u}{\Omega(k_f)} \varepsilon_0, \tag{A6}$$

- (ii) for the wrapping parameters and ε_0 the following equality holds

$$\frac{p_b}{1-k_{fb}} = \frac{p_u}{\Omega(k_f)} + \frac{2p_u}{\Omega(k_f)} \varepsilon_0. \tag{A7}$$

Provided that inequality from Equation (A6) is satisfied, we have

$$\frac{\frac{p_b}{1-k_{fb}} (2\delta - b_{fr}k_{fb}) (\sqrt[3]{1+\varepsilon_0})^2}{\frac{p_b}{1-k_{fb}} - \frac{p_u}{\Omega(k_f)} - \frac{2p_u}{\Omega(k_f)} \varepsilon_0} = \sqrt[3]{\frac{4V_{b0}}{\pi}}, \tag{A8}$$

whence Equation (7) follows. The optimal bale diameter and height are described by Equations (5) and (6) and for D_b^* and H_b^* Equations (1) and (5) are satisfied; case (i) is proved.

In case (ii), when Equation (A7) holds, Equation (A5) is satisfied if and only if, simultaneously,

$$2\delta - b_{fr}k_{fb} = 0.$$

Then, Equation (1) takes especially simple form

$$\frac{2\pi p_u}{\Omega(k_f)} (D_b^*)^3 - 4V_{b0} \left[\frac{p_b}{1 - k_{fb}} + \frac{p_u}{\Omega(k_f)} \right] = 0,$$

whence the optimal bale diameter

$$D_b^* = \sqrt[3]{\frac{2V_{b0}}{\pi} \left[\frac{p_b \Omega(k_f)}{p_u(1 - k_{fb})} + 1 \right]}. \tag{A9}$$

Simultaneously, from Equation (A7) it follows that

$$\frac{p_b \Omega(k_f)}{p_u(1 - k_{fb})} = 1 + 2\varepsilon_0.$$

Thus Equations (A9) and (5) are identical and, by Equation (4), formula from Equation (6) follows for any V_{b0} , which completes the proof of case (ii) since Equations (A7) and (8) are equivalent.

Proof of Proposition 2. By Proposition 1 the optimal bale dimensions are near-equal with order ε_0 if:

- in case (i) inequality expressed by Equation (13) is satisfied,
- in case (ii)

$$p_b \Omega(k_f) - p_u(1 - k_{fb}) - 2p_u(1 - k_{fb})\varepsilon_0 = 0 \tag{A10}$$

and, simultaneously, Equation (9) holds. In the second case any change of ε_0 to $\varepsilon < \varepsilon_0$ results in a loss of equality in Equation (A10). Then the inequality expressed by Equation (13) holds, i.e., case (i) occurs, and since according to Equation (9) $2\delta - b_{fr}k_{fb} = 0$, we immediately conclude that Equation (7) is satisfied only for volume $V_{b0} = 0$. Thus, if for ε_0 Equation (A10) holds, the optimal bale dimensions cannot be near-equal for any $\varepsilon < \varepsilon_0$.

Assume now, that for ε_0 the inequality expressed by Equation (13) is satisfied. Let us consider bale volume $V_{b0}(\varepsilon_0)$ uniquely given by Equation (7). Differentiating Equation (7) on both sides with respect to ε_0 results in

$$\frac{4}{3\pi \left(\sqrt[3]{\frac{4V_{b0}(\varepsilon_0)}{\pi}} \right)^2} \cdot \frac{dV_{b0}(\varepsilon_0)}{d\varepsilon_0} = p_b \Omega(k_f) (2\delta - b_{fr}k_{fb}) \frac{\frac{2}{3} [p_b \Omega(k_f) - p_u(1 - k_{fb}) - 2p_u(1 - k_{fb})\varepsilon_0] + 2p_u(1 - k_{fb})(1 + \varepsilon_0)}{\sqrt[3]{1 + \varepsilon_0} [p_b \Omega(k_f) - p_u(1 - k_{fb}) - 2p_u(1 - k_{fb})\varepsilon_0]^2},$$

whence, after algebraic manipulations, we obtain

$$\frac{dV_{b0}(\varepsilon_0)}{d\varepsilon_0} = \frac{\pi p_b \Omega(k_f) (2\delta - b_{fr}k_{fb}) [p_b \Omega(k_f) + 2p_u(1 - k_{fb}) + p_u(1 - k_{fb})\varepsilon_0]}{2 \sqrt[3]{1 + \varepsilon_0} [p_b \Omega(k_f) - p_u(1 - k_{fb}) - 2p_u(1 - k_{fb})\varepsilon_0]^2} \left[\sqrt[3]{\frac{4V_{b0}(\varepsilon_0)}{\pi}} \right]^2. \tag{A11}$$

Thus, monotonicity of the function $V_{b0}(\varepsilon_0)$ depends on the sign of the expression $(2\delta - b_{fr}k_{fb})$. First, consider case (a) of the proposition assuming that inequality from Equation (12) holds. Then the denominator of the left hand side of Equation (7) is also positive, and if inequality given by Equation (13) holds for ε_0 , then it holds also for any $0 \leq \varepsilon \leq \varepsilon_0$. Function $V_{b0}(\varepsilon)$ is monotonically increasing for $0 \leq \varepsilon \leq \varepsilon_0$. For $\varepsilon_0 = \varepsilon$, by Equations (5), (6) and (A8), the optimal bale dimensions $D_b^*(\varepsilon)$, $H_b^*(\varepsilon)$ can be equivalently expressed by Equations (16) and (17). Thus, $D_b^*(\varepsilon)$, $H_b^*(\varepsilon)$ are also monotonically increasing for $0 \leq \varepsilon \leq \varepsilon_0$.

In case (b), when inequality expressed by Equation (14) is satisfied, function $V_{b0}(\varepsilon_0)$ decreases with ε_0 and, by Equation (7), inequality from Equation (13) takes the form

$$p_b \Omega(k_f) - p_u(1 - k_{fb}) - 2p_u(1 - k_{fb})\varepsilon_0 < 0.$$

If, simultaneously, inequality expressed by Equation (15) holds, then for any $0 \leq \varepsilon \leq \varepsilon_0$ there exists $V_{b0}(\varepsilon)$ fulfilling Equation (7). Otherwise, there exists $0 < \bar{\varepsilon} < \varepsilon_0$ such that Equation (18) holds and only for $\bar{\varepsilon} < \varepsilon \leq \varepsilon_0$ Equation (7) is satisfied for some $V_{b0}(\varepsilon)$. In both cases, for $0 \leq \varepsilon \leq \varepsilon_0$ or $\bar{\varepsilon} < \varepsilon \leq \varepsilon_0$, by Equation (16) we have

$$\frac{dD_b^*(\varepsilon)}{d\varepsilon} = p_b \Omega(k_f) (2\delta - b_{fr} k_{fb}) \frac{[p_b \Omega(k_f) + p_u(1 - k_{fb})]}{[p_b \Omega(k_f) - p_u(1 - k_{fb}) - 2p_u(1 - k_{fb})\varepsilon]^2}.$$

Thus, in view of the inequality from Equation (14), optimal bale diameter decreases with increasing ε . The analysis of Equation (17) yields analogous property of $H_b^*(\varepsilon)$. Proposition 2 is proved. \square

Proof of Proposition 4. For k_f expressed by Equation (20) and $p_u = p_l - p_b$ Equation (8) takes the form

$$p_b [1 + (1 - k_{fb})(1 + 2\varepsilon_0)] = p_l (1 - k_{fb})(1 + 2\varepsilon_0),$$

whence the bottom layers overlap ratio is uniquely determined by Equation (27). The overlaps δ and k_{fb} are related by Equation (9). Substituting k_{fb} , Equation (27), into constraints described by Equation (23) leads to inequalities

$$k_{fb,min} \leq \frac{p_l(1 + 2\varepsilon_0) - p_b(2 + 2\varepsilon_0)}{(p_l - p_b)(1 + 2\varepsilon_0)} \leq k_{fb,max},$$

which can be rewritten in the equivalent form of constraints expressed with respect to integer p_b by Equation (25).

Simultaneously, on the basis of Equations (9) and (27), the constraints expressed by Equation (22) are satisfied if and only if

$$\delta_{min} \leq \frac{b_{fr}}{2} \cdot \frac{p_l(1 + 2\varepsilon_0) - p_b(2 + 2\varepsilon_0)}{(p_l - p_b)(1 + 2\varepsilon_0)} \leq \delta_{max},$$

which is equivalent to the inequalities expressed by Equation (26), provided that inequality described by Equation (24) is satisfied. Proposition 4 is derived. \square

Proof of Proposition 5. According to Proposition 1 the solution to Problem 2 exists if and only if there exist integer p_b and overlaps δ, k_{fb} satisfying Equation (7), constraints from Equations (22) and (23) and the applicability condition, Equation (21), provided that the denominator of the left hand side of Equation (7) is non-zero, i.e., that inequality from Equation (13) holds, which for $k_f = k_{f,u}$ and $p_u = p_l - p_b$ can be rewritten as

$$p_b - (p_l - p_b)(1 - k_{fb}) - 2(p_l - p_b)(1 - k_{fb})\varepsilon_0 \neq 0.$$

The above condition can be expressed directly with respect to the overlap ratio as inequality described by Equation (30). By Equation (7), having in mind that $\Omega(k_f) = 1$ and $p_u = p_l - p_b$, the overlap δ is uniquely given by equation

$$p_b(2\delta - b_{fr} k_{fb}) (\sqrt[3]{1 + \varepsilon_0})^2 = \sqrt[3]{\frac{4V_{b0}}{\pi}} [p_b - (p_l - p_b)(1 - k_{fb}) - 2(p_l - p_b)(1 - k_{fb})\varepsilon_0],$$

whence direct formula from Equation (34) follows. Thus, the inequalities from Equation (22) take the form

$$\delta_{min} \leq \frac{\sqrt[3]{\frac{4V_{b0}}{\pi}}}{2(\sqrt[3]{1+\epsilon_0})^2} \left[1 - \left(\frac{p_l}{p_b} - 1 \right) (1 - k_{fb}) (1 + 2\epsilon_0) \right] + \frac{b_{fr}k_{fb}}{2} \leq \delta_{max}$$

and can be unravelled with respect to k_{fb} as follows

$$\bar{k}_{fb}(p_b) = \frac{2\delta_{min}(\sqrt[3]{1+\epsilon_0})^2 - \sqrt[3]{\frac{4V_{b0}}{\pi}} + (1+2\epsilon_0)\left(\frac{p_l}{p_b} - 1\right)\sqrt[3]{\frac{4V_{b0}}{\pi}}}{(1+2\epsilon_0)\left(\frac{p_l}{p_b} - 1\right)\sqrt[3]{\frac{4V_{b0}}{\pi}} + b_{fr}(\sqrt[3]{1+\epsilon_0})^2} \leq k_{fb} \leq \frac{2\delta_{max}(\sqrt[3]{1+\epsilon_0})^2 - \sqrt[3]{\frac{4V_{b0}}{\pi}} + (1+2\epsilon_0)\left(\frac{p_l}{p_b} - 1\right)\sqrt[3]{\frac{4V_{b0}}{\pi}}}{(1+2\epsilon_0)\left(\frac{p_l}{p_b} - 1\right)\sqrt[3]{\frac{4V_{b0}}{\pi}} + b_{fr}(\sqrt[3]{1+\epsilon_0})^2} = \bar{k}_{fb}(p_b). \tag{A12}$$

Simultaneously, for the overlap ratio k_{fb} inequalities from Equation (23) must be satisfied.

It has been proved above that there exist decomposition (p_b, p_u) and the bottom layers overlaps δ, k_{fb} satisfying constraints expressed by Equations (22) and (23) such that the optimal D_b^*, H_b^* of the bale of volume V_{b0} are near-equal with order ϵ_0 if and only if there exist an integer p_b and continuous parameter k_{fb} such that inequalities expressed by Equations (A12) and (23), i.e., jointly

$$\check{k}_{fb}(p_b) = \max \left\{ \bar{k}_{fb}(p_b), k_{fb,min} \right\} \leq k_{fb} \leq \min \left\{ \bar{k}_{fb}(p_b), k_{fb,max} \right\}, \tag{A13}$$

are satisfied together with the applicability condition, Equation (21), and inequality from Equation (30). For any such p_b and k_{fb} the overlap δ is given by Equation (34). By Proposition 1 the optimal bale dimensions are given by Equations (5) and (6). Since by Equation (34), we have

$$2\delta - b_{fr}k_{fb} = \frac{\sqrt[3]{\frac{4V_{b0}}{\pi}}}{(\sqrt[3]{1+\epsilon_0})^2} \left[1 - \left(\frac{p_l}{p_b} - 1 \right) (1 - k_{fb}) (1 + 2\epsilon_0) \right],$$

having in mind that $\Omega(k_f) = 1$ it can be proved that for the optimal D_b^*, H_b^* film usage index FC , Equation (A2), is described by

$$FC = \frac{4b_f\sqrt[3]{1+\epsilon_0}}{(\epsilon_{lf}+1)\left[\sqrt[3]{\frac{4V_{b0}}{\pi}}\right]^2} \left\{ p_b \left[\frac{\sqrt[3]{\frac{4V_{b0}}{\pi}} \left[\frac{2}{1-k_{fb}} - \left(\frac{p_l}{p_b} - 1 \right) (1+2\epsilon_0) \right]}{b_{fr}(\sqrt[3]{1+\epsilon_0})^2} \right] + \frac{2(2+\epsilon_0)}{\pi(1+\epsilon_0)} \left[\frac{\pi\sqrt[3]{\frac{4V_{b0}}{\pi}}(1+\epsilon_0)(p_l-p_b)}{2b_{fr}(\sqrt[3]{1+\epsilon_0})^2} \right] \right\}, \tag{A14}$$

where only numerator of the fraction in the argument of the ceiling function in the first sum in curly brackets depends on k_{fb} and is non-decreasing left-continuous function of k_{fb} , piecewise constant in the intervals determined by its discontinuity points. If $\hat{k}_{fb}(p_b)$ is discontinuity point being direct right neighbourhood of $\check{k}_{fb}(p_b)$ defined in Equation (A13), i.e., for $\hat{k}_{fb}(p_b)$ the following equation holds

$$\left[\frac{\sqrt[3]{\frac{4V_{b0}}{\pi}} \left[\frac{2}{1-\hat{k}_{fb}(p_b)} - \left(\frac{p_l}{p_b} - 1 \right) (1+2\epsilon_0) \right]}{b_{fr}(\sqrt[3]{1+\epsilon_0})^2} \right] = \frac{\sqrt[3]{\frac{4V_{b0}}{\pi}} \left[\frac{2}{1-\check{k}_{fb}(p_b)} - \left(\frac{p_l}{p_b} - 1 \right) (1+2\epsilon_0) \right]}{b_{fr}(\sqrt[3]{1+\epsilon_0})^2}, \tag{A15}$$

then for any

$$\check{k}_{fb}(p_b) \leq k_{fb} \leq \hat{k}_{fb}(p_b), \tag{A16}$$

film usage FC , Equation (A14), being left-continuous in discontinuity points is identical. Thus, combining inequalities from Equations (A12), (23), and (A16) result in the inequalities from Equation (29). Direct formula expressed by Equation (33) follows from Equation (A15), while Equation (35) results directly from Equation (A14). Integer minimization task in Equation (36) makes it possible to find the best film layers decomposition. Proposition 5 is true. \square

Appendix C

Appendix C.1. Nomenclature

| | |
|--|---|
| b_f | width of un-stretched film, m |
| b_{fr} | width of stretched film, Equation (A1), m |
| D_b, H_b | bale diameter and height, m |
| D_b^* | optimal bale diameter, solution of Equation (1), m |
| H_b^* | optimal bale height given by Equation (2), m |
| FC | film consumption index, Equation (A2), m^{-1} |
| k_{fb}, k_f | overlap ratios applied to wrap bottom and upper film layers |
| $k_{fb,min}, k_{fb,max}$ | the smallest and largest admissible k_{fb} , Equation (23) |
| $\check{k}_{fb}(p_b)$ | lower bound of k_{fb} solving Problem 2, Equation (32) |
| $\hat{k}_{fb}(p_b), \bar{k}_{fb}(p_b)$ | upper bounds of k_{fb} solving Problem 2, Equations (33) and (31) |
| $\underline{k}_{fb}(p_b)$ | lower bound of k_{fb} solving Problem 4, Equation (41) |
| $\check{k}_{fb}(p_b), \tilde{k}_{fb}(p_b)$ | bounds of k_{fb} solving Problem 4, Equations (43) and (44) |
| $\mathcal{K}_2(p_b)$ | set of overlap ratios k_{fb} defined for $p_b \in \mathcal{P}_2$ by Equations (29) and (30) |
| m | integer number |
| \mathcal{N} | set of all positive integer numbers |
| p_l | global number of basic film layers |
| p_b, p_u | numbers of basic film layers in bottom and upper layers |
| p_b^* | optimal p_b solving Problems 2 and 4 defined in Equation (36) |
| \mathcal{P}_1 | set of p_b for which Problem 1 has solution, defined in Equation (28) |
| $\mathcal{P}_2, \mathcal{P}_3$ | sets of p_b for which Problems 2,3 have solution |
| ν_f | Poisson's ratio of the stretch film |
| V_{b0} | pre-assumed bale volume, m^3 |
| ε_0 | order of nearly-equal bale dimensions introduced in Equation (4) |
| ε_{1f} | unit deformation of the stretch film |
| Ω | function of the overlap ratio k_f defined by Equation (A3) |
| δ | overlap of extreme film strips at base of bale, m |
| $\delta_{min}, \delta_{max}$ | the smallest and largest admissible overlap δ , Equation (22), m |

Appendix C.2. Mathematical Terminology

| | |
|---------------------|---|
| $\lceil x \rceil$ | the smallest integer not less than x , ceiling function |
| $\lfloor x \rfloor$ | the largest integer not greater than x , floor function |

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