Elastic Slow Dynamics in Polycrystalline Metal Alloys
Supplemental material

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(Dated: June 8, 2021)

I. THEORY

As discussed in the main text [1], velocity evolution during relaxation could be described as a multi-relaxation process (see Eq. 4) with unknown distribution of the relaxation times $F(\tau)$. Two approaches are then possible, as already discussed in the main text:

- derive a distribution $\hat{F}(\tau)$ fitting data as the sum of a finite number of exponentials equally spaced in the log space of relaxation times (from now on called discrete-log approach);

- make an hypotheses on the functional form of $F(\tau)$ and derive the parameters of the function fitting data using Eq. 4 (from now on called continuous-function approach).

The goal of this Section is to provide evidence that the approach of mixing the two strategies described and used in the main text is sufficiently robust and reliable.

A. Reliability of the discrete-log approach

Let us first consider that when fitting with a discrete basis of exponentials equally spaced in the log space of relaxation times, the dimension N of the basis (i.e. number of exponentials) is a free parameter (see Eq. 5). In Fig. 1a its influence on the solution is analysed comparing results for the Alu-C20 sample obtained for $B_1 = -4$, $B_N = 5$ and different values of N ($N = 10$ and $N = 8$). Provided that empirically the optimal choice is that of choosing an exponential basis with relaxation times every decade ($N = 10$), or differently said choosing $\Delta B = 1$, still the fact that the solution for the two cases considered looks significantly different is questioning the reliability of the approach. Note that the difference in amplitude could probably be accounted for by properly normalising the results, but still the peak position is considerably shifted.

When the correct spectra in the linear space $\hat{F}(\tau)$ are derived using Eq. 8, the two solutions look more similar (we recall that the amplitude difference is a partially removable artefact): see Fig. 1b. A significant difference in the width of the curves is however still present. We can thus conclude that the procedure is strongly influenced by the parameters chosen to describe the basis. Similar results are obtained for other samples and when changing B_1 or B_N.
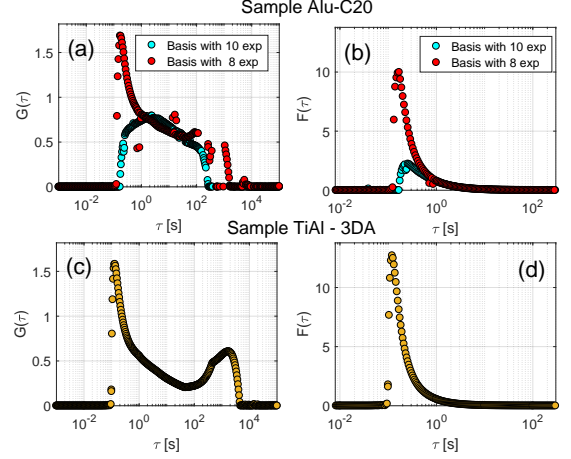

FIG. 1: Reliability of the discrete-log fit procedure. (a) Comparison of resulting spectra $G(\tau)$ of relaxation times (see Eq. 5) for sample Alu-C20 obtained fitting data with different dimension of the fitting basis; (b) continuous distribution $\hat{F}(\tau)$ derived from data of subplot (a) using Eq. 7; (c) and (d) the same for the titanium alloy sample.

An additional issue is affecting the efficiency of the procedure. When deriving the distribution in the linear space $\hat{F}(\tau)$, the function $G(\tau)$ is divided by the relaxation time (see Eq. 8). Given the wide range of variation of τ, the operation has the consequence of amplifying significantly small fitting errors for low values of τ and of loosing details at large values of τ. The result is that of providing misleading information, as in the case of the Titanium sample (subplots c and d of Fig. 1). While the distribution obtained from fitting $(G(\tau))$ looks presenting two well defined peaks, the one for larger relaxation times almost disappears when $\hat{F}(\tau)$ is derived (subplot d).

The conclusion of this analysis is two-fold:

- The distribution $G(\tau)$, i.e. obtained fitting with a discrete number of exponentials in the log space of relaxation times (subplots a and b of Fig. 1) is not fully reliable. As discussed in the main text it could be used to convey the main hypothesis about the distribution function to be used to define a suitable $F(\tau)$;

- The distribution $G(\tau)$ must be corrected using Eq. 7. However the procedure intrinsically smooths out the details of the distribution. As a consequence,

details of the function $\hat{F}(\tau)$ lose most of their meaning and the solution can be reasonably approximated in all cases as a $1/\tau$ distribution bounded in a given interval of allowed τ values, as in [2]. This approach is however not working for the titanium sample.

B.   Continuous-function approach

The function $G(\tau)$ obtained with the previous approach still provides a qualitatively useful suggestion for imposing constraints on the definition of the relaxation times spectrum. As discussed in the main text, the solution of the problem is a peaked and asymmetric distribution. Thus, a choice for the functional dependence of $F(\tau)$ could be adopted and fitting in the linear space could then be performed, thus avoiding the problems raised when pursuing the first discrete-log approach as discussed above.

The problem still remains ill-posed, since different solutions are possible. Besides the Weibull distribution used in our analysis (see Eq. 9), other functions could be used for fitting data, with equivalently good results. As an example, a possible alternative choice for the relaxation spectrum could be in the form:

$$\Phi(\tau) = \frac{a_1}{\tau^{a_2}} \frac{1}{e^{-a_3/\tau} - 1} \tag{1}$$

This is a possible function sharing the same asymmetry characteristics of the Weibull distribution, inspired by the black body radiation spectrum, which could thus be linked to the expected redistribution of stored elastic energy during relaxation.
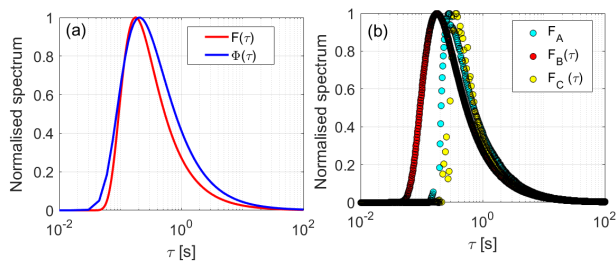

FIG. 2: Uniqueness of the relaxation spectrum solution. Sample is Alu-C20. (a) Relaxation spectra obtained using different functional dependencies for the distribution function. (b) Comparison of resulting spectra $F(\tau)$ and $\hat{F}(\tau)$ obtained from continuous function and discrete-log approaches.

Data for the Alu-C20 sample were fitted using the two functional forms for the distributions and the obtained relaxation spectra $F(\tau)$ and $\Phi(\tau)$ are shown in Fig. 2(a). The quality of the fits obtained with the two choices is comparable, with very slight differences which do not allow to establish one of the two distributions as the best

fitting one. Of course, the distributions obtained look different but still they share some of the main features. In particular the peak position is the same in the two cases. Furthermore, the level of asymmetry, the width at half height and the area are comparable. Similar results were obtained for other samples and amplitudes of conditioning as well.

Another issue arises concerning the uniqueness of the solution. The distributions $F(\tau)$ and $\hat{F}(\tau)$ derived with the continuous-function and the discrete-log approaches do not coincide. Besides being evident for the Titanium sample (two peaks distributions, compare Fig.s 6 and 5 in the main text), this is also the case for all materials: see Fig. 2b, where the two distributions obtained for sample Alu-C20 (red and cyan symbols) are compared. However, the shift in the peak position is resulting from the weakness of the discrete-log approach (perhaps due to the non centred position of the basis elements with respect to the interval Δ_B if it is considered in the linear space, as mentioned in the main text). To prove it, the following analysis was performed:

- experimental data for sample Alu-C20 are fit using the discrete-log approach and the relaxation spectrum $\hat{F}(\tau)$ derived. The solution is labelled as F_A in Fig. 2b.;

- the same data set was fit with the continuum function approach and the distribution $F(\tau)$ derived. The solution was used to generate a synthetic data set using Eq. 4. The solution is labelled as F_B in Fig. 2b.;

- the synthetic data set obtained from F_B was analysed using the discrete-log approach and the corresponding function $\hat{F}_S(\tau)$ derived. The solution is labelled as F_C in Fig. 2b.

Results, shown in Fig. 2b confirm our conclusions. The fit of synthetic data using the discrete-log approach (yellow symbols) differs from what theoretically expected and is well superimposed with the original fit of experimental data in the log-space.

Finally, for sake of completeness we present here a few results obtained on a Berea sample using data from literature [3], to prove the robustness of the approach. In Fig. 3a, we have analysed the full dataset available (i.e. including very early times in the evolution) and a reduced dataset (i.e. assuming early times were not detected). In the inset, the continuous function represent the full dataset while circles denote the reduced one. For both cases, fitting of the data was excellent without appreciable differences among the fitting curves (which are also shown in the inset but barely visible). The obtained relaxation spectra are very similar, thus confirming that determination of the peak position and width are very robust. Again amplitude of the distribution is not a reliable parameter.

Furthermore, a damaged Berea sample was also measured by us in different experimental configurations (not

discussed here for brevity). The derived relaxation spectrum was proven to be configuration independent (see subplot b in Fig. 3).
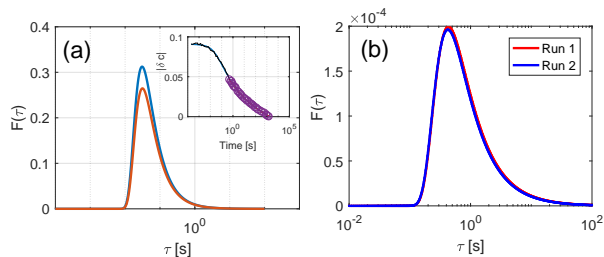

FIG. 3: Robustness of the procedure. (a) Comparison of resulting spectra of relaxation times for Berea obtained fitting data with a complete or incomplete dataset. (b) Robustness verified comparing the relaxation times spectra obtained for Berea using data measured performing experiments on the same sample in different conditions.

II. RESULTS

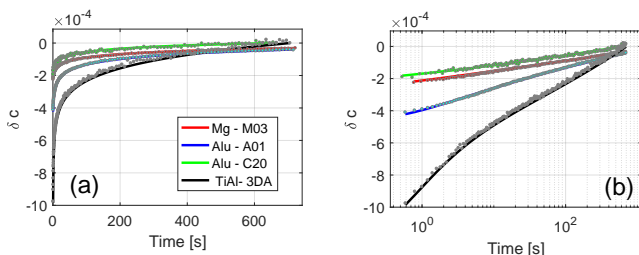In this Section, a few additional results are reported to support the conclusions in the main text.

FIG. 4: Fitting of experimental data using a continuous Weibull distribution of relaxation times for different samples. (a) linear scale; (b) log time scale.

First, in Fig. 4, the quality of the fitting procedure adopted is analysed. Experimental data for different samples have been fit using a continuous Weibull distributions of relaxation times using Eq. 9. The quality

of the fitting function (continuous line) in describing the experimental behavior (dots) is evident, with good agreement also for the early stages of the evolution (see subplot (b) where a log time scale is used). In all cases, also the intermediate times logarithmic recovery is obtained. In the case of the Titanium alloy sample (black curve) the need to fit the data using the superposition of two Weibull distributions is particularly evident in subplot (b), where, after a first logarithmic in time phase and a bending at early times (arising from the "saturation" of the first distribution), at about 10 seconds velocity starts
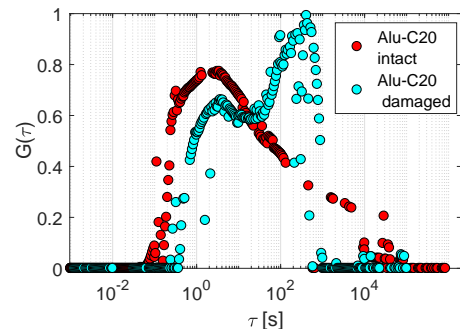

FIG. 5: Distribution of relaxation times obtained fitting data in the log space. Comparison of results for the intact and cracked Alu-C20 sample.

evolving rapidly again with a second logarithmic in time phase.

Finally, in the main text we have discussed the differences in the behavior of the intact and cracked Alu-C20 sample. As mentioned in the main text, the fit of experimental data using a log time basis gives distributions $G(\tau)$ which have a significantly different behavior. While for the intact specimen a one peak distributions is found (see red symbols in Fig. 5), evidences of a two peak distribution are obtained for the cracked sample (cyan symbols). The similarity of the distribution to that of Titanium (see Fig. 5 in the main text) is evident. The result thus supports the choice of fitting data in the continuous space with the superposition of two integrals, as done in the main text.

[1] Equations numbering refers to numbering in the main text.

[2] R. Snieder, C. Sens-Schnfelder, and R. Wu, The Time Dependence of Rock Healing as a Universal Relaxation Process, a Tutorial, Geophys. J. Int. **208**, 1 (2017).

[3] P. Shokouhi, J. Riviere, R. A. Guyer, and P. A. Johnson, Slow Dynamics of Consolidated Granular Systems: Multi-Scale Relaxation, Appl. Phys. Lett. **111**, 251604 (2017).