Article

Influence of Bending Vibration on the Vertical Vibration Behaviour of Railway Vehicles Carbody

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Abstract: The topic of reducing structural vibrations in the case of flexible carbodies of railway vehicles has been intensively studied, but it is still an active research topic thanks to the importance of the perspective of improving the ride comfort. However, no study has been identified in the specialty literature to feature the contribution of the vibration structural modes upon the vibration behaviour of the railway vehicle carbody. The structural vibration modes of the flexible carbodies are particularly complex; however, the first vertical bending mode holds great significance in terms of the ride comfort. This paper analyses the influence of the first vertical bending mode on the vibration behaviour in three reference points of the railway vehicle carbody in correlation with the carbody flexibility, the vehicle velocity and the suspension damping. This study relies on comparisons between the results of the numerical simulations obtained for a ‘flexible carbody’ type model of the vehicle and the ones obtained for a ‘rigid carbody’ type model. The first part of this study analyses the characteristics of the vertical vibrations behaviour of the flexible carbody based on the dynamic response of the vehicle and expressed as the acceleration power spectral density. In the second part, the influence of the vertical bending on the vertical vibrations level of the carbody is analysed using the root mean square of the vertical acceleration.

Keywords: railway vehicle; flexible carbody; bending vibration; ‘rigid carbody’ model; ‘flexible carbody’ model

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1. Introduction

The railway vehicle is a complex oscillating system that has vibration behaviour with specific characteristics, which can have unfavourable effects on ride quality, comfort of passengers, cargo integrity and traffic safety. The vibrations of railway vehicles are mainly caused by the geometric deviations of the track and by the irregularities of the rolling surfaces of the rails [1–8]. They come from the construction imperfections, track exploitation and change in the infrastructure due to the action of the environment factors or soil movements [9]. These above, along with the constructive discontinuities of the rail-joints, switches and crossings, represent the major causes of railway vehicle vibration. Wheel defects such as eccentricity, ovality, polygonal profile, corrugation, flat wheel, flattening or the defects of rolling surfaces are also causes of railway vehicle vibration [10–14].

What defines the vibrations in the railway vehicle is that they are low frequency vibrations that occur both vertically and horizontally in the shape of simple vibration modes—rigid modes—and complex vibration modes—structural vibration modes. The construction of the railway vehicle usually complies with the rules of geometric symmetry, inertial and elastic; hence, the vertical vibrations can be regarded as decoupled from the ones in the horizontal vibrations and, for that reason, separately dealt with.

The issue of reducing the vibrations of the railway vehicle is open to research, mainly for the high velocity vehicles. Reducing the weight of the vehicle, in particular the carbody, is one of the basic criteria for the design of high-speed rail vehicles. Reducing the weight

of the vehicle contributes to increasing speed and reducing energy consumption, ground vibrations and manufacturing costs [15].

The design of the lightweight carbody involves the use of lightweight materials and the modification of mechanical structures. This often leads to a lower structural stiffness of the carbody and therefore to a decrease of the eigenfrequencies. The lighter the vehicle carbody, the greater its flexibility, and this facilitates an easy excitation of the carbody structural vibrations that have a negative effect on the ride comfort [16].

Although the carbody structural vibrations are particularly complex, the first vertical bending mode has the greatest influence on ride comfort. The natural frequency of this mode of vibration is found in the range of 6–12 Hz, where the human body has a deeper sensitivity to the vertical vibrations.

The structural vibrations of flexible carbodies have been the object of research in a large number of studies [15,17–21], and the reduction or control of these vibration so as to improve the ride comfort have been dealt with in numerous research [22–40] that can be grouped as follows: approaches concerning the isolation of vibrations, developed on passive, semi-active and active concepts; and approaches regarding the damping of the carbody vibrations, developed in turn on passive or active concepts [39,40]. Nevertheless, to the best of the authors' knowledge, there is no study that features the contribution of the vibration structural modes upon the vibration behaviour of the railway vehicle carbody. Such study is the subject of this paper, which examines the influence of the first vertical bending mode upon the vibration behaviour of the carbody. To show the impact of the vertical bending upon the carbody vibration behaviour, the results derived from the flexible carbody model and the ones obtained using the rigid carbody model need to be compared. Consequently, the study presented in the paper is based on comparisons between the results from the numerical simulations, developed on two models of a four-axle vehicle with two-stage suspension. In the first model, the carbody is represented by a rigid body—the rigid carbody' model—and, in the second model, the carbody is represented by a Euler-Bernoulli equivalent beam—the, flexible carbody' model. Both models include an original model of the secondary suspension [18,24,26], which considers both the elements of the vertical suspension and the transmission system of the longitudinal forces between the bogie and the carbody. Via this system, the bogie pitch vibrations are transmitted to the carbody and excite the first vertical bending mode.

The basic characteristics of the vertical vibrations behaviour of flexible carbodies are analysed by means of the dynamic response of the vehicle, expressed as the power spectral density of the vertical acceleration in three reference points of the carbody—at the centre and against the two bogies. These points are considered critical points in terms of the vertical vibrations level of the carbody and of the ride comfort [22,41]. The influence of the vertical bending upon the vertical vibrations level of the carbody is highlighted through the root mean square of acceleration in correlation with the carbody flexibility, velocity and the damping ratio of the vehicle suspension.

2. Model of the Railway Vehicle

In this section, the two models of the railway vehicle are presented—the, 'rigid carbody' model and the 'flexible carbody' model. The 'rigid carbody' model is a simple model of the railway vehicle, usually used to study the basic characteristics of the carbody vibrations. The 'flexible carbody' model is used on a regular basis to examine the carbody bending vertical vibrations thanks to the fact that it displays a good agreement between the numerical simulations and the field tests [23].

2.1. General Model Description

For the proposed study, a four-axle vehicle with two suspension stages travelling at a constant velocity V on a track with vertical irregularities is considered. The two vehicle models—the 'rigid carbody' model and the 'flexible carbody' model, shown in Figure 1 [42]—have several elements in common. Both models comprise seven bodies

representing the carbody, the suspended masses of the two bogies and the four axles connected to each other by Kelvin-Voigt type systems, helping to model the two stages of suspension. Both bogies and axles are represented by rigid bodies.

Figure 1. Model of the vehicle: ‘rigid carbody’ model; ‘flexible carbody’ model.

The carbody inertia in relation to the rigid vibration modes is described by mass m_c and the inertia moment J_c . The carbody length is L_c , and the carbody wheelbase is $2a_c$. The rigid vibration modes of the bogies are bounce z_{bi} and pitch θ_{bi} , with $i = 1, 2$. Each bogie has the mass m_b and inertia moment J_b . The bogie wheelbase is $2a_b$.

Each axle is connected to the bogie chassis through a Kelvin-Voigt system, with the elastic constant $2k_{zb}$ and the damping constant $2c_{zb}$. This system models the primary suspension corresponding to an axle.

The bogie is connected to the carbody through the secondary suspension, which is modelled by an assembly consisting of three Kelvin-Voigt systems, of which two are for translation (vertical and longitudinal) and one is for rotation. The Kelvin-Voigt system for longitudinal translation models the transmission system of the longitudinal forces between the carbody and the bogie, and the elements of the Kelvin-Voigt system for rotation take over the relative angular displacement between the carbody and the bogie. The plan of the secondary suspension is located at the distance h_c from the medium fiber of the carbody and at the distance h_b from the gravity centre of the bogie. The characteristics of the secondary suspension are represented by the elastic constants on the vertical direction $2k_{zc}$, the longitudinal direction $2k_{xc}$ and the pitch angular stiffness $2k_{\theta c}$, as well as the damping constants $2c_{zc}$, $2c_{xc}$ and $2c_{\theta c}$.

As for the track model, the rigid track hypothesis is adopted, an approach justified by the fact that the eigenfrequencies of the vertical vibrations of the vehicle carbody considered in this study—bounce, pitch and vertical bending—are much smaller than the frequencies in the wheelset-track system. The track vertical irregularities impose vertical displacements of the axles, noted with $z_{w1} \dots 4$.

2.2. The ‘Rigid Carbody’ Model

For the ‘rigid carbody’ model, the vertical vibrations modes in the carbody are bounce z_c and pitch θ_c , and the motion equations of that describe these movements are of the form

$$m_c \ddot{z}_c + 2c_{zc} [2\dot{z}_c - (\dot{z}_{b1} + \dot{z}_{b2})] + 2k_{zc} [2z_c - (z_{b1} + z_{b2})] = 0, \quad (1)$$

$$J_c \ddot{\theta}_c + 2c_{zc} a_c [2a_c \dot{\theta}_c - (\dot{z}_{b1} - \dot{z}_{b2})] + 2k_{zc} a_c [2a_c \theta_c - (z_{b1} - z_{b2})] + 2c_{xc} h_c [2h_c \dot{\theta}_c + h_b (\dot{\theta}_{b1} + \dot{\theta}_{b2})] + 2k_{xc} h_c [2h_c \theta_c + h_b (\theta_{b1} + \theta_{b2})] + 2c_{\theta c} [2\dot{\theta}_c - (\dot{\theta}_{b1} + \dot{\theta}_{b2})] + 2k_{\theta c} [2\theta_c - (\theta_{b1} + \theta_{b2})] = 0. \quad (2)$$

The equations of the bounce and pitch motions are as follows:

$$m_b \ddot{z}_{b1} + 2c_{zb} [2\dot{z}_{b1} - (\dot{z}_{w1} + \dot{z}_{w2})] + 2k_{zb} [2z_{b1} - (z_{w1} + z_{w2})] + 2c_{zc} (\dot{z}_{b1} - \dot{z}_c - a_c \dot{\theta}_c) + 2k_{zc} (z_{b1} - z_c - a_c \theta_c) = 0 \quad (3)$$

$$m_b \ddot{z}_{b2} + 2c_{zb} [2\dot{z}_{b2} - (\dot{z}_{w3} + \dot{z}_{w4})] + 2k_{zb} [2z_{b2} - (z_{w3} + z_{w4})] + 2c_{zc} (\dot{z}_{b2} - \dot{z}_c + a_c \dot{\theta}_c) + 2k_{zc} (z_{b2} - z_c + a_c \theta_c) = 0 \quad (4)$$

$$J_b \ddot{\theta}_{b1} + 2c_{zb} a_b [2a_b \dot{\theta}_{b1} - (\dot{z}_{w1} - \dot{z}_{w2})] + 2k_{zb} a_b [2a_b \theta_{b1} - (z_{w1} - z_{w2})] + 2c_{xc} h_b (h_b \dot{\theta}_{b1} + h_c \dot{\theta}_c) + 2k_{xc} h_b (h_b \theta_{b1} + h_c \theta_c) + 2c_{\theta c} (\dot{\theta}_{b1} - \dot{\theta}_c) + 2k_{\theta c} (\theta_{b1} - \theta_c) = 0. \quad (5)$$

$$J_b \ddot{\theta}_{b2} + 2c_{zb} a_b [2a_b \dot{\theta}_{b2} - (\dot{z}_{w3} - \dot{z}_{w4})] + 2k_{zb} a_b [2a_b \theta_{b2} - (z_{w3} - z_{w4})] + 2c_{xc} h_b (h_b \dot{\theta}_{b2} + h_c \dot{\theta}_c) + 2k_{xc} h_b (h_b \theta_{b2} + h_c \theta_c) + 2c_{\theta c} (\dot{\theta}_{b2} - \dot{\theta}_c) + 2k_{\theta c} (\theta_{b2} - \theta_c) = 0. \quad (6)$$

2.3. The ‘Flexible Carbody’ Model

In the case of the ‘flexible carbody’ model, the vehicle carbody is represented by a free-free equivalent beam, with a constant section and uniformly distributed mass, Euler-Bernoulli type. This model allows for the consideration of the vertical bending modes of the carbody.

The beam parameters are: $\rho_c = m_c/L$ —beam mass per length unit; μ —coefficient of structural damping; and EI —bending modulus, where E is longitudinal elastic modulus and I is the moment of inertia in the beam cross-section.

The carbody vibration modes are bounce z_c , pitch θ_c and the first vertical bending mode. The carbody displacement $w_c(x, t)$ comes from the overlap of these three vibration modes of the carbody,

$$w_c(x, t) = z_c(t) + \left(x - \frac{L_c}{2}\right) \theta_c(t) + X_c(x) T_c(t), \quad (7)$$

where $T_c(t)$ is the coordinate of the carbody vertical bending and $X_c(x)$ its eigenfunction

$$X_c(x) = \sin \beta x + \sinh \beta x - \frac{\sin \beta L - \sinh \beta L}{\cos \beta L - \cosh \beta L} (\cos \beta x + \cosh \beta x), \quad (8)$$

with

$$\beta = \sqrt[4]{\omega_c^2 \rho_c / (EI)} \text{ and } \cos \beta L_c \cosh \beta L_c - 1 = 0, \quad (9)$$

where ω_c is the natural angular frequency of the carbody vertical bending.

The vertical motions of the vehicle are described by the equations of the rigid vibration modes of the carbody and bogies—bounce and pitch—as well as by the equation of the carbody vertical bending.

The carbody motion equation has the general form

$$EI \frac{\partial^4 w_c(x, t)}{\partial x^4} + \mu I \frac{\partial^5 w_c(x, t)}{\partial x^4 \partial t} + \rho_c \frac{\partial^2 w_c(x, t)}{\partial t^2} = \sum_{i=1}^2 F_{zci} \delta(x - l_i) + \sum_{i=1}^2 (M_{ci} - h_c F_{xci}) \frac{d\delta(x - l_i)}{dx}, \quad (10)$$

where $\delta(\cdot)$ is Dirac's delta function, distances l_i define the position of the carbody support points on the secondary suspension and F_{zci} , F_{xci} and M_{ci} are the forces and the moments, respectively, due to the secondary suspension of the bogie i

$$F_{zci} = -2c_{zc} \left(\frac{\partial w_c(l_i, t)}{\partial t} - \dot{z}_{bi} \right) - 2k_{zc} [w_c(l_i, t) - z_{bi}] \quad (11)$$

$$F_{xci} = 2c_{xc} \left(h_c \frac{\partial^2 w_c(l_i, t)}{\partial x \partial t} + h_b \dot{\theta}_{bi} \right) + 2k_{xc} \left(h_c \frac{\partial w_c(l_i, t)}{\partial x} + h_b \theta_{bi} \right) \quad (12)$$

$$M_{ci} = -2c_{\theta c} \left(\frac{\partial^2 w_c(l_i, t)}{\partial x \partial t} - \dot{\theta}_{bi} \right) - 2k_{\theta c} \left[\frac{\partial w_c(l_i, t)}{\partial x} - \theta_{bi} \right] \quad (13)$$

With the help of the modal analysis method and considering the orthogonality property of the eigenfunction of the carbody vertical bending, the motion Equation (8) is transformed into three second order differential equations with ordinary derivatives, describing the three vibration modes of the carbody—bounce, pitch and the vertical bending.

$$m_c \ddot{z}_c = \sum_{i=1}^2 F_{zci} \quad (14)$$

$$J_c \ddot{\theta}_c = \sum_{i=1}^2 F_{zci} \left(l_i - \frac{L_c}{2} \right) - \sum_{i=1}^2 (M_{ci} - h_c F_{xci}) \quad (15)$$

$$m_{mc} \ddot{T}_c + c_{mc} \dot{T}_c + k_{mc} T_c = \sum_{i=1}^2 F_{zci} X_c(l_i) + \sum_{i=1}^2 (M_{ci} - h_c F_{xci}) \frac{dX_c(l_i)}{dx} \quad (16)$$

In Equation (16) the carbody modal mass is noted with m_{mc} , the carbody modal damping with c_{mc} and the carbody modal stiffness with k_{mc} , as they are defined as follows:

$$m_{mc} = \rho_c \int_0^L X_c^2 dx, \quad c_{mc} = \mu I \int_0^L \left(\frac{d^2 X_c}{dx^2} \right)^2 dx, \quad k_{mc} = EI \int_0^L \left(\frac{d^2 X_c}{dx^2} \right)^2 dx. \quad (17)$$

Based on the symmetry properties of the eigenfunction $X_c(x)$, the following notations are introduced:

$$X_c(l_1) = X_c(l_2) = \varepsilon, \quad (18)$$

$$\frac{dX_c(l_1)}{dx} = -\frac{dX_c(l_2)}{dx} = \lambda. \quad (19)$$

After replacing (9)–(11) in the carbody motion Equations (14)–(16), in which the notations (18) and (19) are inserted, the carbody motion equations are brought to the following form:

$$m_c \ddot{z}_c + 2c_{zc} [2\dot{z}_c + 2\varepsilon \dot{T}_c - (\dot{z}_{b1} + \dot{z}_{b2})] + 2k_{zc} [2z_c + 2\varepsilon T_c - (z_{b1} + z_{b2})] = 0 \quad (20)$$

$$J_c \ddot{\theta}_c + 2c_{zc} a_c [2a_c \dot{\theta}_c - (\dot{z}_{b1} - \dot{z}_{b2})] + 2k_{zc} a_c [2a_c \theta_c - (z_{b1} - z_{b2})] + 2c_{xc} h_c [2h_c \dot{\theta}_c + h_b (\dot{\theta}_{b1} + \dot{\theta}_{b2})] + 2k_{xc} h_c [2h_c \theta_c + h_b (\theta_{b1} + \theta_{b2})] + 2c_{\theta c} [2\dot{\theta}_c - (\dot{\theta}_{b1} + \dot{\theta}_{b2})] + 2k_{\theta c} [2\theta_c - (\theta_{b1} + \theta_{b2})] = 0 \quad (21)$$

$$m_{mc} \ddot{T}_c + c_{mc} \dot{T}_c + k_{mc} T_c + 2c_{zc} \varepsilon [2\dot{z}_c + 2\varepsilon \dot{T}_c - (\dot{z}_{b1} + \dot{z}_{b2})] + 2k_{zc} \varepsilon [2z_c + 2\varepsilon T_c - (z_{b1} + z_{b2})] + 2c_{xc} h_c \lambda [2h_c \lambda \dot{T}_c + h_b (\dot{\theta}_{b1} - \dot{\theta}_{b2})] + 2k_{xc} h_c \lambda [2h_c \lambda T_c + h_b (\theta_{b1} - \theta_{b2})] + 2c_{\theta c} \lambda [2\lambda \dot{T}_c - (\dot{\theta}_{b1} - \dot{\theta}_{b2})] + 2k_{\theta c} \lambda [2\lambda T_c - (\theta_{b1} - \theta_{b2})] = 0 \quad (22)$$

For this model, the bounce and pitch equations of the bogies are as follows:

$$m_b \ddot{z}_{b1} + 2c_{zb}[2\dot{z}_{b1} - (\dot{z}_{w1} + \dot{z}_{w2})] + 2k_{zb}[2z_{b1} - (z_{w1} + z_{w2})] + 2c_{zc}(\dot{z}_{b1} - \dot{z}_c - a_c \dot{\theta}_c - \varepsilon \dot{T}_c) + 2k_{zc}(z_{b1} - z_c - a_c \theta_c - \varepsilon T_c) = 0 \quad (23)$$

$$m_b \ddot{z}_{b2} + 2c_{zb}[2\dot{z}_{b2} - (\dot{z}_{w3} + \dot{z}_{w4})] + 2k_{zb}[2z_{b2} - (z_{w3} + z_{w4})] + 2c_{zc}(\dot{z}_{b2} - \dot{z}_c + a_c \dot{\theta}_c - \varepsilon \dot{T}_c) + 2k_{zc}(z_{b2} - z_c + a_c \theta_c - \varepsilon T_c) = 0 \quad (24)$$

$$J_b \ddot{\theta}_{b1} + 2c_{zb}a_b[2a_b \dot{\theta}_{b1} - (\dot{z}_{w1} - \dot{z}_{w2})] + 2k_{zb}a_b[2a_b \theta_{b1} - (z_{w1} - z_{w2})] + 2c_{xc}h_b[h_b \dot{\theta}_{b1} + h_c(\dot{\theta}_c + \lambda \dot{T}_c)] + 2k_{xc}h_b[h_b \theta_{b1} + h_c(\theta_c + \lambda T_c)] + 2c_{\theta c}(\dot{\theta}_{b1} - \dot{\theta}_c - \lambda \dot{T}_c) + 2k_{\theta c}(\theta_{b1} - \theta_c - \lambda T_c) = 0 \quad (25)$$

$$J_b \ddot{\theta}_{b2} + 2c_{zb}a_b[2a_b \dot{\theta}_{b2} - (\dot{z}_{w3} - \dot{z}_{w4})] + 2k_{zb}a_b[2a_b \theta_{b2} - (z_{w3} - z_{w4})] + 2c_{xc}h_b[h_b \dot{\theta}_{b2} + h_c(\dot{\theta}_c - \lambda \dot{T}_c)] + 2k_{xc}h_b[h_b \theta_{b2} + h_c(\theta_c - \lambda T_c)] + 2c_{\theta c}(\dot{\theta}_{b2} - \dot{\theta}_c + \lambda \dot{T}_c) + 2k_{\theta c}(\theta_{b2} - \theta_c + \lambda T_c) = 0 \quad (26)$$

3. The Dynamic Response of the Railway Vehicle to the Track Vertical Irregularities

To analyse the vertical vibrations behaviour of the vehicle carbody, we utilised the power spectral density of acceleration in three important points considered critical points of the vibrations level of the carbody and the ride comfort in the railway vehicles [22,41]. These reference points are marked in Figure 1 and found at the centre of the carbody—point m —and against the support points of the carbody onto the secondary suspension, above the bogie 1—point b_1 —and above the bogie 2—point b_2 .

To calculate the power spectral density of the vertical acceleration of the carbody, the track vertical irregularities are considered to represent a stochastic stationary process, which can be described via the power spectral density. For the average statistical properties of the European railways, the theoretical curve of the power spectral density described by the relation [43]

$$S(\Omega) = \frac{A\Omega_c^2}{(\Omega^2 + \Omega_r^2)(\Omega^2 + \Omega_c^2)}, \quad (27)$$

is considered representative, where Ω is the wavenumber, $\Omega_c = 0.8246$ rad/m, $\Omega_r = 0.0206$ rad/m and A is a constant depending on the track quality. For a high-quality track, $A = 2.119 \times 10^{-7}$ radm, and A is 6.124×10^{-7} radm for a low-quality track.

As a function of the angular frequency $\omega = V\Omega$, the power spectral density of the track irregularities can be expressed as in the general equation

$$G(\omega) = \frac{A\Omega_c^2 V^3}{[\omega^2 + (V\Omega_c)^2][\omega^2 + (V\Omega_r)^2]}. \quad (28)$$

The power spectral density of the carbody vertical acceleration $G_c(\omega)$ is calculated differently for the two models of the vehicle, starting from the power spectral density of the track vertical irregularities $G(\omega)$ and from the response functions corresponding to the vibration modes of the carbody—bounce and pitch $\bar{H}_{zc}(\omega)$, $\bar{H}_{\theta c}(\omega)$ —and the vertical bending $\bar{H}_{Tc}(\omega)$.

In the point m located at the centre of the carbody, the power spectral density of the acceleration is calculated with the following equations:

- for the, rigid carbody' model,

$$G_{cm}(\omega) = G(\omega) \left| \omega^2 \bar{H}_{zc}(\omega) \right|^2. \quad (29)$$

- for the, flexible carbody' model,

$$G_{cm}(\omega) = G(\omega) \left| \omega^2 \left[\bar{H}_{zc}(\omega) + X_c \left(\frac{L_c}{2} \right) \bar{H}_{Tc}(\omega) \right] \right|^2. \quad (30)$$

In the points b_1 and b_2 , located against the two bogies, the power spectral density of the vertical acceleration is given by the equations below:

- for the, rigid carbody' model,

$$G_{cb1,2}(\omega) = G(\omega) \left| \omega^2 [\overline{H}_{z_c}(\omega) \pm a_c \overline{H}_{\theta_c}(\omega)] \right|^2. \quad (31)$$

- for the, flexible carbody' model,

$$G_{cb1,2}(\omega) = G(\omega) \left| \omega^2 [\overline{H}_{z_c}(\omega) \pm a_c \overline{H}_{\theta_c}(\omega) + X_c(l_{1,2}) \overline{H}_{T_c}(\omega)] \right|^2. \quad (32)$$

To study the influence of the vertical bending upon the level of vertical vibrations in the vehicle carbody, the root mean square of the vertical acceleration in the three reference points of the carbody will be utilised. This is calculated based on the dynamic response of the vehicle expressed in the form of the power spectral density of the carbody acceleration

- at the centre of the carbody,

$$a_{cm} = \sqrt{\frac{1}{\pi} \int_0^\infty G_{cm}(\omega) d\omega}; \quad (33)$$

- above the two bogies,

$$a_{cb1,2} = \sqrt{\frac{1}{\pi} \int_0^\infty G_{cb1,2}(\omega) d\omega}. \quad (34)$$

4. Results and Discussion

This section analyses the influence of the vertical bending upon the vertical vibrations behaviour of the railway vehicle carbody using the results from the numerical simulations developed based on the two previously presented models—the 'rigid carbody' model and the 'flexible carbody' model. The basic characteristics of the vibration behaviour of the vehicle carbody are analysed using the power spectral density of the vertical acceleration in the reference points of the carbody—at the centre of the carbody and against the two bogies. The influence of the vertical bending upon the vertical vibrations level in the reference points of the carbody are analysed based on the root mean square of the vertical acceleration, in correlation with the velocity and damping ratio of the vehicle suspension, for four values of the flexibility of the carbody, corresponding to the bending frequencies of 6 Hz, 8 Hz, 10 Hz and 12 Hz.

The reference parameters of the vehicle model used in the numerical simulations are shown in Table 1. The eigenfrequencies of the vertical vibration modes of the carbody corresponding to the reference parameters of the model are the bounce eigenfrequency $f_b = 1.17$ Hz, pitch eigenfrequency $f_p = 1.46$ Hz and vertical bending eigenfrequency $f_{vb} = 8$ Hz. It is considered that the vehicle is running on a low-quality track characterised by the coefficient $A = 1.080 \times 10^{-6}$ rad/m.

For the reference parameters of the 'flexible carbody' model, the diagrams in Figure 2 show the power spectral density of the vertical acceleration in the reference points of the carbody, depending on the velocity. The bending vibration is observed as mainly affecting the vibration behaviour at the carbody centre (diagram a) and less above the two bogies (diagrams b and c). At the centre carbody, the power spectral density of the acceleration is dominated by the bounce vibration (at 1.17 Hz), but the contribution of the bending vibration (at 8 Hz) is also noticeable at high velocities. In the reference points above the bogies, the vibration behaviour of the carbody is dominated by the pitch vibration (at 1.46 Hz), whereas the bending weight is not significant.

Table 1. Reference parameters of the vehicle model.

Symbol	Definition	Symbol	Definition
Joint parameters for the ‘rigid carbody’ model and the ‘flexible carbody’ model			
The parameters of the carbody and bogie			
$m_c = 34,000 \text{ kg}$	carbody mass	$L_c = 26.4 \text{ m}$	carbody length
$m_b = 3200 \text{ kg}$	bogie suspended mass	$2a_b = 19 \text{ m}$	carbody wheelbase
$J_c = 1,963,840 \text{ kg}\cdot\text{m}^2$	carbody inertia moment	$2a_b = 2.56 \text{ m}$	bogie wheelbase
$J_b = 2048 \text{ kg}\cdot\text{m}^2$	bogie inertia moment	$h_c = 1.3 \text{ m}/h_b = 0.2 \text{ m}$	elevations of the secondary suspension plane
The parameters of the secondary suspension per bogie			
$2c_{zc} = 34.28 \text{ kNs/m}$	vertical damping	$2k_{zc} = 1.2 \text{ MN/m}$	vertical stiffness
$2c_{xc} = 50 \text{ kNs/m}$	longitudinal damping	$2k_{xc} = 4 \text{ MN/m}$	longitudinal stiffness
$2c_{\theta c} = 2 \text{ kNm}$	angular damping	$2k_{\theta c} = 256 \text{ kN/m}$	angular stiffness
The parameters of the primary suspension per wheelset			
$4c_{zb} = 52.21 \text{ kNs/m}$	vertical damping	$4k_{zb} = 4.4 \text{ MN/m}$	vertical stiffness
Parameters specific to the ‘flexible carbody’ model			
$EI = 3.158 \times 10^9 \text{ Nm}^2$	bending modulus	$c_{mc} = 53.117 \text{ kNm/s}$	carbody modal damping
$m_{mc} = 35,224 \text{ kg}$	carbody modal mass	$k_{mc} = 88.998 \text{ MN/m}$	carbody modal stiffness

Figure 2. The power spectral density of the carbody acceleration: (a) at the carbody centre; (b) against the front bogie; (c) against the rear bogie.

To evaluate the influence of the vertical bending upon the of the vertical vibration behaviour of the carbody, the diagrams in Figure 3 are utilised, showing the power spectral density of the vertical acceleration at the velocity of 180 km/h for both the ‘rigid carbody’ model and the ‘flexible carbody’ model. For the ‘flexible carbody’ model, four values of the carbody bending frequency were considered, namely 6 Hz, 8 Hz, 10 Hz and 12 Hz. What can be noticed in the diagrams are the peaks for the four resonance frequency of bending and the ones corresponding to the vibration rigid modes—the resonance frequency of the carbody bounce and the resonance frequency of the carbody pitch. Apart from these resonance peaks, a series of anti-resonance frequencies can be identified due to the geometric filtering effect [21,44–47].

Figure 3. Influence of the vertical bending upon the power spectral density of the carbody acceleration: (a) at the carbody centre; (b) against the front bogie; (c) against the rear bogie.

As shown in Figure 2, the bending vibration is visibly influencing the vibration behaviour of the carbody, mainly at its centre (diagram a) and less above the two bogies (diagrams b and c). The contribution of the bending vibration is all the more important as the carbody flexibility is higher, which can be clearly noticed at the carbody centre, where the spectrum of the power density of the acceleration is dominated by the bending vibration of the carbody at the frequency of 6 Hz. The more the bending vibration frequency

increases, the less important the weight of this vibration mode is, the dominant vibration mode being the bounce in the reference points of the carbody above the bogies and the spectrum of the power density of the accelerations dominated by the pitch vibration, both for the 'rigid carbody' and for the 'flexible carbody' models. In these points, the bending weight is very low (see diagrams (b') and (c')), irrespective of the frequency of this vibration mode.

Based on the diagrams in Figure 4, the influence of the vertical bending upon the level of vertical vibrations in the carbody reference points in correlation with velocity is analysed. This analysis is performed by comparing the values of the root mean square of acceleration for the 'flexible carbody' model with those corresponding to the 'rigid carbody' model. The vertical bending causes the increase of the level of vibrations at the centre of carbody, with this increase being important under certain conditions—high flexibility of the carbody and high velocities (diagram a). This is visible in Figure 5, which features the percentage growth of the root mean square of acceleration for the 'flexible carbody' model versus the 'rigid carbody' model. As an example, the root mean square of acceleration for the flexible carbody goes up by 93.6% for $f_{vb} = 6$ Hz at a velocity of 200 km/h compared to the root mean square of acceleration for the rigid carbody. For higher values of the carbody bending frequency, the increase in the root mean square of acceleration does not exceed 20%. As seen in the diagrams (b) and (c), the carbody vertical bending does not sensibly affect the level of vibrations above the bogies, which is valid irrespective of the carbody bending frequency.

Figure 4. Influence of the carbody vertical bending upon the root mean square of acceleration in correlation with the velocity: (a) at the carbody centre; (b) against the front bogie; (c) against the rear bogie.

Figure 5. Increase of root mean square of acceleration at the carbody centre versus the rigid carbody.

Another interesting aspect to be analysed is related to the manner in which the vertical bending changes the position of the point where the vibrations level is the highest, hereafter called the critical point of the carbody. This can be done based on the diagrams in Figure 6, which show the root mean square of acceleration in the reference points of the rigid carbody (diagram a) and the root mean square of acceleration in the reference points of the flexible carbody for $f_{vb} = 6$ Hz (diagram b). It is observed that the bending vibration does not change the position of the carbody critical point. For both models, the carbody critical point is located above one of the two bogies, and its position changes as a function of velocity. At velocities smaller than 92 km/h, the carbody critical points alternate either above the rear bogie or the front bogie. A large velocity interval follows afterwards, which extends up to 190 km/h, where the carbody critical point is the reference point situated above the front bogie.

Figure 6. The position of the carbody critical point: (a) 'rigid carbody' model; (b) 'flexible carbody' model ($f_{vb} = 6$ Hz).

Figure 7 highlights the influence of the carbody vertical bending upon the root mean square of acceleration at a velocity of 200 km/h, in correlation with the damping ratio of the secondary suspension and defined according to the equation

$$\zeta_c = \frac{4c_{zc}}{2\sqrt{4k_{zc}m_c}}. \quad (35)$$

Figure 7. Influence of the carbody vertical bending upon the root mean square of acceleration in correlation with the damping ratio of secondary suspension: (a) at the carbody centre; (b) against the front bogie; (c) against the rear bogie.

In accordance with the reference parameters of the vehicle model (see Table 1), the reference value of the damping ratio of the secondary suspension is $\zeta_c = 0.12$.

For both models, it is noticed that the root mean square of acceleration continuously decreases until it reaches a minimum value, while the damping ratio of the secondary suspension increases to a certain value. The minimum value of the root mean square of acceleration depends on the carbody reference point and then on the model type—the ‘rigid carbody’ model or the ‘flexible carbody’ model. In the case of the ‘rigid carbody’ model, the damping ratio that minimises the root mean square of acceleration at the carbody centre is 0.2, while the root mean square of acceleration is minimum for $\zeta_c = 0.32$ and $\zeta_c = 0.43$, respectively (see Figure 8), above the bogies.

Figure 8. The damping ratio of secondary suspension minimizing the root mean square of acceleration: (a) at the carbody centre; (b) against the front bogie; (c) against the rear bogie.

The value of the damping ratio minimizing the root mean square of acceleration at the carbody centre declines for the flexible carbody (Figure 8, diagram a); the more flexible the carbody, the smaller this value (for instance, $\zeta_c = 0.10$ for $f_{vb} = 6$ Hz; $\zeta_c = 0.18$ for $f_{vb} = 12$ Hz). In the points located above the two bogies, the value of the damping ratio that minimises the root mean square of acceleration coincides with the value identified for the rigid carbody, $\zeta_c = 0.32$ —above the front bogie, $\zeta_c = 0.43$ —above the rear bogie, for $f_{vb} = 8$ Hz, $f_{vb} = 10$ Hz and $f_{vb} = 12$ Hz (Figure 8, diagrams b and c). For a very flexible carbody ($f_{vb} = 6$ Hz), the damping ratio that minimises the root mean square of acceleration is smaller, $\zeta_c = 0.30$ and $\zeta_c = 0.40$, respectively.

The Figure 9 is used to analyse the influence of the carbody vertical bending upon the root mean square of acceleration at a velocity of 200 km/h, in correlation with the damping ratio of primary suspension. The damping ratio of primary suspension is calculated with the relation:

$$\zeta_b = \frac{4c_{zb}}{2\sqrt{4k_{zb}m_b}}. \quad (36)$$

For the reference parameters of the vehicle model (see Table 1), the reference value of the damping ratio for the primary suspension is $\zeta_b = 0.22$.

Figure 9. Influence of the carbody vertical bending upon the root mean square of acceleration in correlation with the damping ratio of primary suspension: (a) at the carbody centre; (b) against the front bogie; (c) against the rear bogie.

The results show that, for both models, the root mean square of acceleration continuously decreases with the increase of the damping of the primary suspension. The influence of the carbody vertical bending is visible in the decrease rate of the root mean square of acceleration, as shown in Figure 10.

Figure 10. The decrease rate of the root mean square of acceleration with the damping ratio of primary suspension: (a) at the carbody centre; (b) against the front bogie; (c) against the rear bogie.

For the rigid carbody, the decrease rate of the root mean square of acceleration is higher above the two bogies than at the carbody centre. Instead, for the flexible carbody, the decrease rate of the root mean square of acceleration is higher at the carbody centre for any of the four situations being considered.

Comparing the results from the flexible carbody with those corresponding to the rigid carbody, the decrease rate of the root mean square of acceleration is observed to always be significantly higher for the flexible carbody at the carbody centre. Conversely, in

the reference points of the carbody above the bogies, the decrease rates of the root mean square of acceleration for the flexible carbody can be close to or even lower than the ones corresponding to the rigid carbody (for $f_{vb} = 8$ Hz and $f_{vb} = 12$ Hz).

5. Conclusions

This paper analyses the effects of the first vertical bending mode upon the vibration behaviour of the railway vehicle carbody. This is done by comparing the results from the numerical simulations developed based on a ‘flexible carbody’ vehicle model with the ones from the ‘rigid carbody’ model.

The basic characteristics of the vibration behaviour of the carbody for various carbody flexibilities have been analysed by means of the power spectral density of the vertical acceleration in three reference points of the carbody situated at the carbody centre and against the two bogies. The bending vibration has been shown to influence the vibration behaviour of the carbody, mainly at its centre and less against the two bogies. At the carbody centre, the contribution of this vibration mode is more important at high velocities, to the point when the power spectral density of the vertical acceleration dominates in the case of very flexible carbodies. In the carbody reference points above the bogies, the power spectral density of the vertical acceleration is dominated by the pitch vibration, the bending weight being very low, even for a high flexibility in the carbody.

To highlight the influence of the vertical bending upon the vibrations level in the carbody reference points, the values of the root mean square of vertical acceleration for the various flexibility of the carbody in the ‘flexible carbody’ model have been compared with the ones in the ‘rigid carbody’ model, in correlation with the velocity and the damping ratios of the two suspension stages of the vehicle.

The results correlated with the velocity have shown an important increase in the vibrations level at the carbody centre due to the bending vibrations in the range of high velocities. For the most adverse conditions—high flexibilities of the carbody and high velocities—the root mean square of acceleration is almost double the root mean square of acceleration in the rigid carbody. Still, based on the results correlated with the velocity, it has been shown the bending vibrations do not change the position of the critical point of the vibrations level of the carbody. Both for the ‘rigid carbody’ model and the ‘flexible carbody’ model, the critical point of the carbody is located above one of the bogies, and its position modifies as a function of velocity.

In accordance with the results correlated with the damping ratio of secondary suspension, the value of the damping ratio minimizing the root mean square of acceleration at the carbody centre has been proven to be lower for the flexible carbody than the rigid carbody. This value decreases along with the bending frequency of the carbody. In the points above the two bogies, the value of the damping ratio minimizing the root mean square of acceleration is the same for both models.

For both models, the results correlated with the damping ratio of primary suspension have showed that the level of vibrations continuously decreases with the increase of the damping of primary suspension, yet the influence of the carbody vertical bending is visible in the decrease rate of the root mean square of acceleration in the carbody reference points. In case of the rigid carbody, the decrease rate of the root mean square of acceleration is higher above the two bogies than at the carbody centre. Instead, for the flexible carbody, the decrease rate of the root mean square of acceleration is higher at the carbody centre.

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