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Tensile Buckling of a Rod with an End Moving along a Circular Guide: Improved Experimental Investigation Based on a Dynamic Approach

Boris Blostotsky, Elia Efraim * and Yuri Ribakov

Department of Civil Engineering, Ariel University, Ariel 40700, Israel; bx@ariel.ac.il (B.B.); ribakov@ariel.ac.il (Y.R.)

* Correspondence: efraime@ariel.ac.il

Abstract: Investigation of buckling under tension is highly important from theoretical and practical viewpoints to ensure safety and the proper performance of mechanical systems. In the present work, tensile buckling is investigated experimentally, and the critical force is measured in systems where one end of an elastic tensile rod slides along a straight guide, while the other slides along a curve. An experimental setup is proposed and developed for determining the critical tensile load of the elastic rod by a dynamic method. This setup allows measuring free vibrations and frequency with the required accuracy. Improvement of the critical load accuracy is achieved by approaching the maximum load to the critical one. Limitations in selecting the test parameters are found according to the required extrapolation accuracy of the dominant natural vibration frequency dependence on tensile load. Theoretical analysis and tests are performed for the rod connection schemes pinned–rigid, rigid–pinned, and rigid–rigid, considering imperfections in the fixation of the rod ends. It is experimentally shown that the system buckling at tensile load is possible and that experimental and theoretical values of the critical load are in good agreement. The achieved accuracy, estimated by the discrepancy between the calculated and the experimental values, is 2.1–3.5%.

Keywords: tensile buckling; tensile rod; experimental; dynamic method; curved slider

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1. Introduction

The phenomenon of tensile buckling is of considerable scientific interest, and numerous theoretical studies have been devoted to it, especially in the last decade. For example, tensile buckling is studied for elastic rods [1,2], elastomeric bearings [3], which lose stability due to shear deformation, bars with sliding connections [4] installed in separate sections, allowing elastic transverse movement in the beam sections. Various theoretical models of elements and mathematical models of their analysis are used. It is shown that the results of theoretical analyses depend on the models and methods for their analysis [5]. As an example, contradictory analysis results of this phenomenon for a stretched hinged supported beam have been provided. In this regard, an experimental study is of particular importance in order to verify the existence of the phenomenon and compare the calculated and experimental values of the critical force.

Theoretical and practical issues, related to an elastic tensile rod with one end sliding along a straight guide and the other one along a curve, have been widely investigated in the recent years [6–8]. The possibility of buckling at some critical tensile force value was studied theoretically. The post-buckling behavior of the system was described, and calculation dependences, allowing to find the critical load, were proposed [6,7]. Hinged and clamped connection schemes between rod and curved guide were considered. However, the investigated system was studied experimentally only after buckling. A thorough literature review on the topic was recently presented by Simão and Silva [8].

The present study was aimed at an experimental investigation of the buckling phenomenon for an elastic rod with an end sliding along a circular guide and with the following connection schemes: pinned–rigid, rigid–pinned, and rigid–rigid. To decrease the influence of friction in the guides, a dynamic method was used to study buckling and obtain the critical load [9,10]. With this method, the critical force value was assumed to correspond to the zero frequency of natural vibrations [10,11]. Since the zero frequency cannot be reached, the critical load was determined by extrapolating the results measured with a certain error at lower values of the axial force. It was shown theoretically [11] and experimentally [12] that for a compressed rod with rigid, pinned, and free end conditions, the buckling phenomenon is indicated by a monotonic decrease in the natural vibration frequency as the compression load increases. The same indicator for buckling at tension was used in the current research. The novelties of this study applying the dynamic method include:

- developing a testing stand that allows reducing the influence of resistance to the movement of the rod end along a circular guide;
- the methodology of the experiment, which allows determining the critical load with sufficient accuracy, including the choice of rod–curved guide system parameters, taking into account the deformation of the rod at the connection nodes.

To obtain the critical load by extrapolating the experimental values of squared natural frequencies, a linear dependence between frequency and axial load is expedient, since it includes a minimum of parameters and therefore it is less sensitive to frequency and force measurement errors. For the above-mentioned connection schemes, this dependence was proved previously [13]. It was shown that as the axial force becomes higher, the frequency increases and asymptotically approaches to that of a cable under tension. There are no similar available data for a system in which the rod end moves along a circular trajectory, like the case that is investigated in the present study.

In known devices for testing rods with an end moving along a curved guide, roller guides are used [7,14,15]. The main direction for reducing the friction resistance of the rod end to movement in the nodes connecting the rod ends to the supporting structure is the use of bearing units, such as in the case of an axial rotary attachment [16] or the rod's end movement along a cylindrical surface [14], when the rod is loaded by a followed force, directed towards the positive pole. In both cases, the friction resistance during rotation and movement creates significant damping and interference during the rod free vibrations. This resistance limits the possibility of increasing the axial force in the rod to values close to critical and distorts the dependence of the vibration frequency on the axial force. Both factors reduce the accuracy of determining the critical force by extrapolating the measured values to zero frequency.

To improve the extrapolation accuracy, the maximum test load should be brought closer to the critical one. At the same time, the possibility of measuring the natural vibration frequency with sufficient accuracy is correspondingly reduced, due to its small value. Since the minimum frequency is limited by the rod movable end's friction resistance to movement, the direction for the test device improvement should be a decrease in the resistance to movement. To plan the experiment and define the parameters limits that can be used for extrapolation, dependence between the squared natural vibration frequency and tensile load was theoretically investigated in the present study, considering the lumped mass of the sliding mechanism at the rod end.

In the present study, that was aimed at an experimental investigation of buckling in systems including a tensile rod with an end sliding along a circular guide, the following problems are solved:

- finding a dependence of the natural vibration frequency of the rod with a mass at the end that moves along a curved guide on the tensile force;
- finding a range of the rod parameters in the form of L/R (L —rod length, R —radius of curvature), allowing a linear extrapolation to obtain the buckling load, using squared frequencies;

- experimental verification of rod buckling at tension, finding the buckling load using experimental data and comparing the obtained value with the calculated one.

The scientific novelty of the present study includes:

- developing a scientific method for the experimental determination of the critical load for a rod that has an end moving along a circular guide. The method includes an originally developed stand, the idea and design of which allow reducing the influence of resistance to the movement of the rod's end that moves along a circular guide and ensuring an accurate determination of the critical load;
- experimentally proving the phenomenon of buckling under tension of a rod, the end of which moves along a curved guide under different end conditions, considering imperfections in its end's fixation;
- experimental determination of the critical load and proven correspondence of the critical load value to theoretical calculations.

2. Analytical Investigation of the Experimental Parameters Range

The stand for testing (Figure 1) consists of a wheel 1, placed on bearing 2, an elastic rod 3, a sliding guide 4 that moves straight along a line, passing through the wheel center (line OC in the figure). One end of the rod is connected to the wheel, and the other moves along the direction slide. The rod connection with the wheel and the direction slide enables changing the deformable rod length. It also allows pinned–rigid, rigid–rigid, or rigid–pinned connections. The axial load in the rod was applied by a weight connected to the clamp through a block by a cable.

Figure 1. (a) Scheme of a stand for testing systems, including a tensile rod with an end sliding along a circular guide: 1—curved guide, placed on bearing 2; 3—elastic rod; 4—sliding guide; (b) forces in the rod section and forces acting on the curved guide. Note that V_A and M_A are internal forces acting in the rod at its connection to the curved guide, N_A is the tensile force in the rod, F is the external force applied to obtain the transverse stiffness of the system.

According to the Rayleigh method [17], it was assumed that the system's mode shape corresponds to that under a static force [18] applied to the mass. The dominant modal frequency of the system was obtained as follows:

$$\omega^2 = \frac{K}{m(\alpha + \delta)} \quad (1)$$

where K is the rod static stiffness in transverse direction, $\delta = M/m$, M is the equivalent mass at the end of the rod, m is the elastic rod mass, α is equivalence coefficient of the elastic rod mass and that of the lumped mass at the rod end:

$$\alpha = \frac{1}{L} \int_0^L y^2(x) dx \quad (2)$$

Here, $y(x)$ is the shape of the elastic rod due to unit static displacement at its end, corresponding to the trajectory; L is the length of the deformed part of the rod.

According to possible rod end displacement, the rod stiffness in transverse direction is defined as a force F that yields a unit displacement at the rod end along the circle with a radius R (see Figure 1). According to the possible displacement at the end A of the rod, the above-mentioned force direction is tangent to the trajectory (perpendicular to the radius, connecting point A with the rotation center).

From the equilibrium of the curved guide (Figure 1), the load F , required for deviation $\varphi = 1/R$, corresponding to rigid connection, is

$$F = -V_A + M_A/R, \quad (3)$$

where V_A and M_A are shear force and bending moment in the rod, respectively. At $F = 0$, Equation (3) corresponds to the equilibrium of a curved clamp [7,14].

For a pinned connection between the wheel and the rod end

$$F = -V_A + N_A(\varphi - \gamma), \quad (4)$$

where N_A is an axial force in the rod, γ is the angle between the rod axis and the line OC, and φ is the wheel rotation angle, corresponding to unit displacement of the rod end along the circular trajectory (see Figure 1).

To obtain the dominant vibration frequency of the system, it was assumed that the equivalent mass of the rod was lumped at its end [19]. The general equivalent mass of the system includes that of the rod and that of the curved guiding elements. The shape of the elastic rod and the stiffness in transverse direction are obtained by integrating the differential equation of the rod deformation under tension

$$d^4y/dz^4 - d^2y/dz^2 = 0, \quad (5)$$

where $z = u x/L$, $u = L(N/EI)^{0.5}$, E is the modulus of elasticity, I is the moment of inertia of the rod section, N is the axial force. As

$$N = u^2 EI/L^2, \quad (6)$$

then u^2 is a non-dimensional parameter of the axial force in the rod.

The general solution of Equation (5) for tensile force is

$$y = C_1 \sinh z + C_2 \cosh z + C_3 z + C_4, \quad (7)$$

For all cases that are considered in the present study (rigid–rigid, pinned–rigid and rigid–pinned)

$$y(0) = 0; y(x = L) = 1. \quad (8)$$

Additionally, for the rigid–rigid connection, considering the sign of the curvature

$$y_x(0) = 0; y_x(x = L) = -1/R. \quad (9)$$

For the pinned–rigid connection

$$y_{xx}(0) = 0; y_x(x = L) = -1/R, \quad (10)$$

and for the rigid–pinned connection

$$y_x(0) = 0; y_{xx}(x = L) = 0. \quad (11)$$

According to the boundary conditions, the coefficients of particular solutions, C_1 , C_2 , C_3 and C_4 can be found as follows:

For the rigid–rigid connection

$$C_1 = \frac{\sinh u + (\cosh u - 1)L/uR}{(\sinh u - u)\sinh u - (\cosh u - 1)^2}, \quad (12a)$$

$$C_2 = \frac{-(\cosh u - 1) - (\sinh u - u)L/uR}{(\sinh u - u)\sinh u - (\cosh u - 1)^2}, \quad (12b)$$

$$C_3 = -uC_1/L, \quad (12c)$$

$$C_4 = -C_2, \quad (12d)$$

for the pinned–rigid connection

$$C_1 = \frac{1 + L/R}{\sinh u - u \cosh u}, \quad (13a)$$

$$C_3 = -\left(\frac{C_1}{L}u \cosh u + \frac{1}{R}\right), \quad (13b)$$

$$C_2 = C_4 = 0, \quad (13c)$$

for the rigid–pinned connection

$$C_1 = \frac{1}{\tanh u - u}, \quad (14a)$$

$$C_2 = -C_4 = -C_1 \tanh u, \quad (14b)$$

$$C_3 = C_1 u/L. \quad (14c)$$

The elastic rod shapes for various connection schemes at $N = 0$ within a diapason of $1.5 \leq L/R \leq 3$ used in the experiments are given in Figure 2. For comparison, shapes corresponding to a linear displacement of the rod end ($L/R = 0$) are also shown in the Figure.

Figure 3 presents the dependence of $\alpha(0)$ and $\alpha(u_{cr})$ on L/R for various connection schemes. Here, u_{cr} is a non-dimensional parameter that corresponds to the buckling load, and its value is obtained later. A numerical analysis of the data showed that for $0 \leq u \leq u_{cr}$, the dependence of α on u can be approximated as a linear function of u^2 :

$$\alpha = \alpha(0) + [\alpha(u_{cr}) - \alpha(0)](u/u_{cr})^2. \quad (15)$$

Following Equations (3) and (4) and using known dependences for moments and shear forces, the rod stiffness in transverse direction is obtained as follows:

- for the rigid connection between the rod and the wheel

$$K = -EI(y_{xxx} + y_{xx}/R), \quad (16)$$

- for the pinned connection between the rod and the wheel

$$K = -EIy_{xxx} + N(y_x + 1/R). \quad (17)$$

Figure 2. Elastic rod shapes for various connection schemes: (a) rigid-pinned, (b) pinned-rigid, (c) rigid-rigid.

Figure 3. The dependence of $\alpha(0)$ and $\alpha(u_{cr})$ on L/R for various connection schemes.

The expression for $y(x)$ in these equations is a particular solution of Equation (7) corresponding to the boundary conditions combination at the rod ends. For all connection types, the transverse stiffness can be represented as $K = K'EI/L^3$, where K' is non-dimensional stiffness:

for the rigid-rigid connection

$$K' = u \frac{\sinh u [(1 + R/L)(R/L)u^2 - 1] + u \cosh u}{2 + u \sinh u - 2 \cosh u} (L/R)^2, \quad (18)$$

for the pinned–rigid connection

$$K' = u^3 \frac{(1 + L/R)(\cosh u + L/(uR)\sinh u)}{u \cosh u - \sinh u}, \quad (19)$$

for the rigid–pinned connection

$$K' = u^3/(u - \tanh u) + u^2 R/L. \quad (20)$$

The non-dimensional stiffness depends on the non-dimensional parameters u and L/R . The values of u range from 0 to u_{cr} , which corresponds to the buckling load. The value of u_{cr} is obtained from $K' = 0$ [12] and the critical load N_{cr} according to Equation (6):

$$N_{cr} = u_{cr}^2 EI/L^2. \quad (21)$$

Table 1 presents the values of u_{cr} for the investigated end connection conditions. Figure 4 shows the dependence of u_{cr} on R/L . For the rigid–rigid and rigid–pinned cases, the values of u correspond to those obtained by Misseroni et al. [7] and Bigoni et al. [6], respectively. The symmetry of the rigid–rigid case relative $R/L = 0.5$ was demonstrated previously [7]. The values of the rigid–pinned case are symmetric to those of the pinned–rigid, also with relative $R/L = 0.5$.

Table 1. Values of u_{cr} for the investigated end connection conditions.

		u_{cr}		
		Rigid–Pinned	Pinned–Rigid	Rigid–Rigid
L/R	1.500	2.985	1.288	3.041
	1.667	2.464	1.512	2.622
	2.000	1.915	1.915	2.400
	2.500	1.512	2.464	2.622
	3.000	1.288	2.985	3.041

Figure 4. Dependence of roots u_{cr} on the R/L ratio.

The present study analyzed the dependence of K' on $0 \leq u \leq u_{cr}$ using the normalized non-dimensional values K'/K'_0 and u^2/u_{cr}^2 . Here, K'_0 is non-dimensional stiffness at $N = 0$. This approach is convenient to investigate the deviation from linear dependence

$$K'_{lin}/K'_0 = 1 - u^2/u_{cr}^2. \quad (22)$$

For all connection cases, the graphs of normalized dependences pass through points (1, 0) and (0, 1). For the rigid–rigid connection (Figure 5), the graphs become close to the

linear dependence, as the ratio L/R is closer to 2. For other values of L/R , the dependence is nonlinear. The effect of L/R for the pinned–rigid and rigid–pinned cases is similar.

Figure 5. Dependences of normalized values of non-dimensional stiffness K'/K_0 on normalized values u^2/u_{cr}^2 for the rigid–rigid connection.

In a similar way, we analyzed the dependence of ω^2/ω_0^2 on u^2/u_{cr}^2 , where ω_0 is the dominant vibration frequency at $N = 0$. According to Equation (1),

$$\omega^2/\omega_0^2 = K'/K_0 \frac{\alpha(0) + \delta}{\alpha + \delta}. \quad (23)$$

For $1.5 \leq L/R \leq 3$, in the experiments with constant R and variable L , the coefficient δ varies from 6.8 to 22.7. The coefficient α for the same L/R and for $u \leq u_{cr}$ varies within a diapason between 0.14 and 0.28. As the values of α are rather small, the nonlinearity of ω^2/ω_0^2 vs. tensile force N is determined by that of K' .

There is a range of L/R for which the dependence is close to linear with a certain accuracy. Considering a linear dependence of ω^2 on the rod tension,

$$\omega_{lin}^2/\omega_0^2 = 1 - \frac{u^2}{u_{cr}^2}, \quad (24)$$

the dependence of maximum deviation on ω^2/ω_0^2 is analyzed according to Equations (23) and (24) in a range of $0 \leq u \leq u_{cr}$:

$$D\% = \max(\omega^2/\omega_0^2 - \omega_{lin}^2/\omega_0^2)\%. \quad (25)$$

The value of ω^2/ω_0^2 according to Equation (23) is obtained considering the dependence $\alpha(u)$ using Equation (15) for the given values of δ . The maximum deviation value occurs about at the middle of the interval $0 \leq u \leq u_{cr}$. Table 2 and Figure 6 present the results of this analysis.

The range of allowed L/R depends on the selected accuracy of the correspondence between the dependence and its linear approximation. As the accuracy of the experiment, dependent on measuring the frequency and using the squared frequency for extrapolation, was 2%, and the accuracy of force measuring was 0.5%, the total maximum error was assumed to be 2.5%. In this case, the range of the allowed L/R values was $1.33 \leq L/R \leq 4$ for the rigid–rigid connection, $1 \leq L/R \leq 3$ for the pinned–rigid case, and $L/R \geq 1.5$ for the rigid–pinned one. The common range for all connections was $1.5 \leq L/R \leq 3$. Therefore, this interval was selected for further experiments. For all end connection schemes, the

following values were selected from this interval, which are symmetric to $R/L = 0.5$: 0.33; 0.4; 0.5; 0.6 and 0.67.

Table 2. Maximum deviation of ω^2/ω_0^2 from linear dependence in the range of $0 \leq u \leq u_{cr}$ for various L/R ratios.

R/L	L/R	$D\%$		
		Rigid-Rigid	Pinned-Rigid	Rigid-Pinned
	0.80	1.250	4.71	0.05
	0.75	1.333	2.31	0.06
	0.67	1.500	0.63	0.11
	0.60	1.667	0.11	0.23
	0.50	2.000	0.04	0.51
	0.40	2.500	0.11	1.23
	0.33	3.000	0.63	2.29
	0.25	4.000	2.29	5.47
	0.80	1.250	4.71	0.05

Figure 6. Variation of maximum deviation of ω^2/ω_0^2 from linear dependency in the range of $0 \leq u \leq u_{cr}$ vs. the R/L ratio.

3. Experimental Investigation

3.1. Experimental Program and Setup

According to the main aim of the research—experimental investigation of critical tensile force and its comparison with an analytical value—the following problems were solved experimentally:

- measuring the moment of inertia of rotating masses in order to obtain the equivalent mass M ;
- finding the modulus of elasticity of the rod;
- studying the influence of semi-rigidity in the rod ends connection, caused by the elasticity of the connection elements at the rod ends to the guide and slider;
- obtaining the buckling load for all investigated types of end connections.

To solve the above-mentioned problems, a stand, corresponding to a general scheme shown in Figure 1, was used. The view of the stand, presenting its construction, is shown in Figure 7a. The axis of wheel 1 with bearings and the linear guiding elements of the direction slide 2 are connected to the basic rod 3. Joint 4 that connects the elastic rod 5 to the wheel allows a rigid or a pinned connection. Connection joint 6 between elastic rod and

direction slide has a screw stopper that allows a rigid or a pinned connection. According to the construction of the elastic rod connection joint to the wheel, the radius r of the rod end displacement was 305 and 295 mm for the rigid and pinned connections, respectively. The elastic rod was made of a hardened steel strip with a section of $19.5 \times 1.47 \text{ mm}^2$; the working length L was controlled by measuring, according to the selected L/R ratio. For the rigid connection, the working distance was measured from contact A (Figure 7b,c), and for the pinned one, from the rotation axis (Figure 7d). The extension of the traction cable 7 (pulley block and weights at the end) are not shown in Figure 7a and were used as a typical solution for loading.

Figure 7. (a) Upper view on the horizontal stand for testing systems, including a tensile rod with an end moving along a circular trajectory: 1—wheel, 2—direction slide, 3—basic rod, 4—connection joint to the wheel, 5—elastic rod, 6—connection joint to the direction slide, 7—traction cable; (b) rigid joint connection of the elastic rod to the wheel; (c) connection joint between elastic rod and direction slide with a screw stopper that allows a rigid or a pinned connection; (d) pinned connection of the elastic rod to the wheel.

To obtain the vibration frequency, the wheel circular velocity was measured by an inductive gage, including a permanent magnet located on the wheel and a stationary coil. The signal was transmitted from a data logger to a PC each 0.01 s. This transmission allowed transferring the data at the maximum measured frequency with an error of 0.8%. For linear measurements of static displacements, the error for a standard gage was 0.05 mm.

In the proposed testing stand design, the reduction of the rod movable end resistance to movement was achieved by the increased movement radius and structural design of mobility by wheel rotation. If the radius of the rolling balls in the central bearing assembly is 12 mm and the wheel radius is 295 mm, the decrease in friction resistance to rod end movement is 24.6 times.

The design of the test setup and the methods for determining the investigated system parameters and the critical force allow testing of other systems which include a tensile or compressed element and an end that slides along a curved trajectory [8,20]. A theoretically investigated system that includes a rigid tensile rod and rotation springs at its ends [8] can be tested without changing the design of the bench. To measure the rotation spring stiffness, the guide element with a spring at one end is fixed on the wheel rim, and the second end of the rod is pivotally fixed on the guide element.

The theoretically investigated system [21], the so-called Ziegler's beam [20], includes a stretchable elastic element on two hinged supports, one of which can move in the longitudinal direction. An additional rigid rod is rigidly fixed to the movable end of the elastic rod and is loaded by a force that has a constant direction when the rod is rotated. To test this system, the wheel axle is mounted on a linear sliding element 2 (Figure 7), one end of the elastic rod is fixed rigidly to the wheel axle, and the second one is hinged to the basis rod 3 in the figure. The wheel rim is loaded by a constant direction force from the elastic rod side. With this arrangement, the wheel spokes act as a rigid rod. This avoids the possibility of contact between the compressed curved element and the rigid rod [20], since they are in different planes. To create a force that acts in a constant direction, a weight load can be used. With this aim, the basis rod 3 should be installed vertically.

3.2. Obtaining the Moment of Inertia of the Wheel's Rotating Mass

The experiment was carried out for a vertical location of the wheel (Figure 8). The elastic rod (1) was made of a $\varnothing 2$ mm high-strength wire; its length was 600 mm, and its mass 15 g. Two equal additional masses (2) ($M_b/2$ each one) were connected to a flexible belt (3) placed on the wheel, as shown in Figure 8.

Figure 8. Experimental setup for measuring the moment of inertia of rotating masses: 1—elastic rod, 2—additional masses, 3—flexible belt.

From the expression of the frequency of a system

$$M_b R_b^2 = -I_0 + D / (2\pi f)^2, \quad (26)$$

where I_0 is the moment of inertia of the wheel's rotating mass, D is the rotational stiffness, f is the free vibrations frequency. The measurements were carried out for three cases: 1—without additional mass; 2—with an additional mass of 1164 g; 3—with an additional mass of 2272 g. The experimental results at $R_b = 315$ mm are shown in Figure 9.

Figure 9. Linear approximation of $M_b R_b^2$ vs. $1/f^2$ for experimental values of the wheel's rotating mass moment of inertia.

From a linear approximation of $M_b R_b^2$ vs. $1/f^2$ by the least squares method, it follows that $I_0 = 0.131 \text{ kg}\cdot\text{m}^2$. Correspondingly, for radius $R = 305$ mm and $R = 295$ mm for the rigid and pinned connections between the rod and the wheel, the equivalent masses were 1.40 kg and 1.51 kg, respectively.

For the selected range of L/R and rod cross sections, the ratio $\delta = M/m$ is 7 ... 15. This estimation illustrates a negligible influence of the rod mass on the calculated deviation of ω^2 vs. N and the corresponding error, caused by assumptions regarding the rod modal shape for vibrations with dominant frequency.

3.3. Measuring the Flexural Rigidity of the Rod

An experimental investigation of the rod flexural rigidity is necessary due to the following reasons:

- steel class, its heat, and mechanical treatment;
- influence of the ductility of the rigid connections at the ends on the rod's lateral stiffness.

The modulus of elasticity of the steel strip was measured using a standard procedure [22] with a confidence level of 90% and a mean value depreciation area corresponding to $204 \pm 2 \text{ GPa}$.

The ductility of rigid connections is caused by contact deformations at the contact area of the rod and the elastic compliance of the attachment screws. It consists in a decrease in the rod's lateral stiffness [23–25]. In the present research, this decrease was taken into account considering the stiffness reduction coefficient λ , which was determined experimentally by comparing the experimental and calculated values of the transverse stiffness at $N = 0$

$$\lambda = K_0 / K_0^*, \quad (27)$$

where K_0 is the experimental value of the rod flexural rigidity at $N = 0$, and K_0^* is the calculated value of the rod flexural rigidity. Following Equations (18)–(20) at $N \rightarrow 0$, for the rigid–rigid connection:

$$K_0^* = 12[1 - L/R + (L/R)^2/3]EI/L^3, \quad (28)$$

for the pinned–rigid connection:

$$K_0^* = 3(1 - L/R)^2EI/L^3, \quad (29)$$

and the for rigid–pinned connection:

$$K_0^* = 3EI/L^3. \quad (30)$$

The experimental values of the rod flexural rigidity were obtained by static testing according to the scheme shown in Figure 8 by adding a weight at one of the cables ends and measuring the corresponding displacement of the weight. The tests were carried out for all end connection types and specimens' lengths planned for obtaining the buckling load. The dependence of displacement X on the load at the end of the rod for the rigid–rigid connection and rod length of 610 mm ($L/R = 2$) is shown in Figure 10. The experimental values of the rod flexural rigidity were obtained by the least-squares method [26]. It was found that, for the investigated length range of 443–915 mm, just the end connection combination has an essential effect on the end factors. Independent of the length, the following values can be used: 0.89 for rigid–rigid, 0.96 for pinned–rigid, and 0.94 for rigid–pinned connections.

Figure 10. Dependence of the displacement X on the load F at the end of the rod for the rigid–rigid connection and rod length of 610 mm ($L/R = 2$).

Following the determined values of the stiffness reduction coefficient, from Equations (28)–(30), the effective rod's length should be increased by 1.039, 1.013, and 1.02 times, relative to the measured values. The value of N_{cr} was calculated using Equation (21), considering the influence of ductility on increasing the rod's effective length. Accordingly, the values of the critical load, calculated correspondingly to the measured rod's length, should be decreased by 8%, 3%, and 4%, respectively.

3.4. Experimental Verification of the Design Scheme

The mode shapes and dominant modal frequencies for the selected structural scheme (see Figure 1) depend only on the rod and rotating masses inertia forces, rod flexural rigidity, and natural damping. Free vibrations should be harmonic, and their spectrum has

one dominant frequency. Additionally, the values of dominant vibration frequencies for the selected end connection schemes should correspond to the calculated values (Equation (1)).

To verify the above-mentioned requirements, free vibration tests were carried out for rigid–rigid connection schemes, and further spectral analysis of the measured data was performed. The L/R ratio was 2, and $N = 0$. The ratio δ between the equivalent mass of the wheel and the rod was 20.5. It allowed decreasing the error due to approximation in calculating the coefficient α (Equation (2)).

Figure 11 presents the vibration time histories of the structure and the corresponding spectra (obtained using FFT for 100 vibration periods). The calculated value (according to Equation (1)) was 0.62 Hz. Differences between experimental and calculated values were up to 3%. The measured damping ratio was 3%.

Figure 11. (a) Vibration time history of the tested structure; (b) corresponding spectra, obtained using FFT for 100 vibration periods.

Thus, the experiments confirmed the selected design scheme and the validity of the assumptions that were used for the calculation of modal frequencies.

3.5. Experimental Verification of the Proposed Dependencies

The possibilities of experimental verification are limited by the minimal natural vibration frequency of the system. As the tensile force becomes higher, the resistance of the bearing increases, the rod stiffness in transverse direction decreases, which yields high damping, and vibrations significantly decrease during a very short time that is lower than one natural vibration period; this prevents finding the natural frequency. It is experimentally shown that to obtain the natural vibration frequency with an accuracy of 1%, this frequency for the investigated system should be at minimum 0.3 Hz. A corresponding value of the maximal tensile load achieved in the experiment, N_{max} , is presented in Table 3.

It is shown in the experiments that, in order to obtain the critical load by extrapolating the measured values up to zero, the minimal number of experimental values of frequency should be 5. If the number is lower, extrapolation accuracy decreases. Increasing the number of the measured values does not increase the accuracy due to the errors in single measurements.

To evaluate the accuracy of dependence of N on f^2 and to find the value of N_{cr} , linear extrapolation of the measured values was used:

$$N = N_{cre} - \mu f^2, \quad (31)$$

where N_{cre} is the extrapolated value of the critical force. The parameters N_{cre} and μ were obtained by approximating Equation (31) by the least squares method [26]. Figure 12 presents the dependence of N on f^2 for rigid–rigid end connections and L/R ratio of 1.5 and a corresponding linear approximation.

Estimation of the linear dependence error was carried out according to the absolute value of the deviation:

$$D_{lin}\% = \text{abs} \left[\frac{N - N_{lin}}{N} \right] \%, \quad (32)$$

where N_{lin} is calculated following Equation (31), corresponding to the obtained values of N_{cr} and r and the experimental frequency values. The maximum deviation values are shown in Table 3.

Table 3. Maximal experimental and theoretical values of the tensile load and extrapolated experimental values of critical tensile loads for different L/R ratios.

L/R	N_{max} , [N]	N_{cr} , [N]	N_{max}/N_{cr}	N_{cre} , [N]	D_N , %	D_{lin} , %
rigid–rigid						
1.50	35.2	41.4	0.85	40.5	2.1	2.3
1.67	20.2	24.9	0.81	24.4	2.2	1.7
2.00	11.2	14.5	0.77	14.1	2.7	1.3
2.50	7.6	11.1	0.69	10.7	3.1	2.3
3.00	6.9	10.4	0.67	10.1	3.5	3.2
pinned–rigid						
1.50		8.0	<0.6			
1.67	5.5	8.9	0.62	8.7	3.2	2.3
2.00	6.5	10.0	0.65	9.7	2.8	2.8
2.50	7.1	10.6	0.67	10.3	2.7	3.1
3.00	7.3	10.8	0.68	10.5	2.4	3.3
rigid–pinned						
1.50	38.7	45.0	0.86	44.1	2.3	2.2
1.67	20.1	24.9	0.81	24.2	2.8	2.5
2.00	7.2	10.4	0.69	10.7	3.1	2.8
2.50		4.2	<0.6			
3.00		2.1	<0.6			

Note: N_{max} —experimentally achieved values of the tensile load; N_{cr} —theoretical critical tensile loads values; N_{cre} —extrapolated experimental values of the critical tensile load.

Figure 12. Dependence of the squared natural vibration frequency on load N for $L/R = 1.5$ and rigid–rigid end connections.

Evaluation of the error in obtaining N_{cr} was performed using the following dependence:

$$D_N\% = \text{abs}[(N_{cr} - N_{cre})/N_{cr}]\%. \quad (33)$$

The calculated and experimental values of the critical load as well as estimation of measurements' accuracy are given in Table 3. The experimental values of N_{cre} are given only for cases in which the maximal tensile force is at least 60% of the calculated critical value. From the analysis of the error values given in Table 3, it follows that:

- the linear approximation model that was used in this study corresponds to the experimental dependence of f^2 on N ;
- the calculation model of the system and buckling conditions of the tensile rod correspond to the physical system.

4. Conclusions

The buckling of a system, including a tensile rod with one end moving along a linear sliding guide and the second end connected to a circular sliding guide, was studied theoretically and experimentally for pinned, rigid, and semi-rigid connection schemes. A device for experimental investigation of the rod critical tensile load is proposed. The device can provide higher accuracy compared to known alternative devices. This is achieved by approaching the maximum load to the critical one. The achieved accuracy, estimated by the discrepancy between the calculated and the experimental values depending on the L/R ratio, was 2.1–3.5%.

In the frame of the study, the following characteristics were determined:

- dependency of natural vibration frequencies on tensile load, cross-sectional dimensions of the elastic rod, and mass on its end;
- values of the critical load for different connection schemes, considering elastic compliance in the attachment nodes;
- range of the ratio between the rod length L and the radius of the rod end trajectory R that yields a linear dependence between the transverse stiffness of the system and the tensile load, allowing the application of a dynamic method to obtain the critical load with minimal error.

It was experimentally confirmed that:

- an elastic rod with various types of end connection schemes buckles under a tensile load;
- the free vibrations feasibility is subjected to changing the tension load from 0 to a value that is close to the critical load;
- it is possible to use the linear approximation model to obtain the dependence of the squared natural vibration frequency f^2 on the tensile load N ;
- the calculation model of the system, buckling conditions, and critical load value correspond to the physical model.

The results of this study can be used to design mechanisms including elements with ends moving along curved sliding guides.

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