

Article

A Game-Theoretic Approach for Electric Power Distribution during Power Shortage: A Case Study in Pakistan

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Abstract: Power supply is the cornerstone for the sustainable socio-economic development of any country. In a developing country like Pakistan, shortage of power supply is the main obstacle to its economic growth, making it a disputed and contested resource among different administrative units/provinces and socio-economic sectors. A key challenge is allocating the limited available power among provinces with conflicting and competing needs amid the supply-demand gap. In this research, the allocation of energy during a shortage is considered as a game-theoretic bankruptcy problem. Five bankruptcy rules namely the Proportional Rule, Constraint Equal Award Rule, Constraint Equal Loss Rule, Talmud Rule and Piniles Rule are used for power allocation among the provinces of Pakistan. Each province is characterized by its power demand. A new framework is also proposed for power allocation, which synthesizes the Nash bargaining solution concept with bankruptcy theory to resolve power-related disputes among the four provinces within Pakistan. Additionally, a new method is introduced in this study to compare and contrast the different allocation rules. The results suggest that the basic power demands of the provinces can be satisfied by the proposed disagreement points among the provinces, and the bargaining weights can highlight the role of different levels of power claims, lengths of transmission lines, and variations in population among provinces. The findings also suggest that, due to the lowest dispersion, the proportionate rule is the most suitable method for power allocation among the provinces. The paper combines relevant bankruptcy rules with Nash bargaining theory to propose an algorithm for addressing power sector supply-demand mismatches in Pakistan.

Keywords: bankruptcy problem; power demand; power allocation; Nash bargaining theory



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1. Introduction

The socio-economic development of any country is heavily dependent on power supply. Power distribution among socio-economic sectors in a fairer, efficient, and economical way is critical, and the strategic and careful distribution of power during a power shortage is vital to ensure that the allocation among users is equitable and reasonable.

During a power shortage, the power supply problem can be addressed using different approaches. The game-theoretic approach is one that can be beneficial for the management of electric power. This approach can impart strategic information to power management experts, helping them make the right decision by analyzing the power demands of different sectors and their strategic interaction.

The bankruptcy concept is a game theoretic approach that allocates scarce resources among various demanding sectors. Bankruptcy theory is a concept of economics in which the available resources are not adequate to satisfy the claims of all agents. During a shortage of power, the state is identical to the bankruptcy concept. Therefore, power allocation in times of power shortage can be treated as a typical bankruptcy problem. This resource sharing problem was described as the transferrable utility game by O' Neill [1], who approached this resource sharing problem as a coalition game. He determined the worth of the coalition of clients who have their respective demands on scarce common pool resources as the amount left after fully satisfying the claims of those who are not members of the coalition. Various researchers have applied this theory to the allocation of scarce resources. For example, in a study [2], the authors used the three bankruptcy rules for the water reallocation in Cyprus. To determine a fair resource allocation based on the legal status of the Caspian Sea, four bankruptcy rules were applied to reallocate oil and gas resources among the five littoral states [3]. Similarly, in another study [4] the authors applied ten bankruptcy methods to allocate the oil and natural gas reserves among Azerbaijan, Iran, Kazakhstan, Russia and Turkmenistan. Jarkeh et al. [5] applied seven bankruptcy rules for the water allocation among the three riparian countries of Iraq, Syria, and Turkey. The bankruptcy theory has been applied in various instances, and its application can be found in [5–9]. The most commonly used bankruptcy allocation rules are proportional rule (PRO), constrained equal award rule (CEA), constrained equal losses rule (CEL), Talmud rule, and Piniles rule. The aforementioned bankruptcy rules were applied to the power management sector in this study.

When the total claims (C) exceed the total available resources (E), the bankruptcy theory is applicable. This theory was used for multipurpose resource allocation situations [10], whereas Ansink and Marchiori used it for water resource management [6]. Auman and Maschler [11] and O'Neill [1] introduced bankruptcy theory, which was later studied by various researchers [12–15]. Several researchers have also used this theory for water allocation among riparian countries [10,16–19]. Bozorg-Haddad et al. [20] used bankruptcy theory for the allocation of water in Iran. In the previous studies, researchers solved the load shedding and power allocation problems by using a game theoretic (Bankruptcy) approach. These include the studies conducted by [21–27]. In the present research, in addition to applying simple bankruptcy rules, a synthesis of bankruptcy rules with a bargaining solution process is used to effectively address the power supply–demand mismatch. Various methods can solve the bargaining problems; however, much-desired properties, such as flexibility, invariance under changes in scale, unanimity, and Pareto optimality can be satisfied by the Nash bargaining solution [28,29]. Safari et al., [30], Houba [31], Sgobbi [32], Degefu, Dagmawi Mulugeta [21], Degefu, Dagmawi Mulugeta [33], and Qin et al. [34] used the Nash bargaining solution for the management and allocation of water resources. Five classical bankruptcy rules and the Nash bargaining theory are applied in this study for power allocation among the four provinces of Pakistan; then, a method is applied for the “selection of the best rule”.

The remainder of this paper is organized as follows. The background of the current condition of the power sector in Pakistan is described in Section 2. The bankruptcy allocation rules and the Nash bargaining theory are discussed in Section 3, and Section 4 summarizes the results. Finally, Section 5 concludes the paper.

2. Current Conditions of the Power Sector in Pakistan

In the last 15 years, the demand for energy in Pakistan has increased significantly. The supply has failed to cope with the growth, and as a result, the supply–demand gap has increased. As of 2020, the population of Pakistan has reached approximately 200 million. The country's economic growth has also been affected by this energy crisis [35]. Pakistan's power sector is managed by the Pakistan Water and Power Development Authority (WAPDA). The installed capacity of the WAPDA is approximately 20,921 megawatts (MW) [35].

According to the Energy Report by the State Bank of Pakistan [36], the peak generation of electricity was only 14,468 MW, while the demand was 18,511 MW, with an electricity shortage of 4043 MW. This supply–demand gap increases with the increase in demand and adversely affects the economy. In summer days, owing to poor load management, the load shedding in urban areas is for approximately 10 to 12 h a day, whereas the rural areas suffer around 12 to 18 h of load shedding. Of the total net energy generated, transmission and distribution (T&D) losses are very high, ranging between 13% and 37% [35].

According to [37], in terms of total peak consumption, among the five administrative units of the country, Punjab has the highest demand for electricity (11,347 MW), followed by Sindh (3943 MW), Khyber Pakhtunkhwa (KPK) (2054 MW), Baluchistan (962 MW), and Azad Jammu Kashmir (205 MW), respectively (Table 1). Due to its extremely low demand, the province of Azad Jammu Kashmir is not included in the allocation process. Consequently, the remaining peak power demand of 18,306 MW is divided among the remaining four provinces, against a total peak power generation of 14,263 MW. Therefore, a power deficit of 4043 MW is shared among the four provinces of Pakistan. The peak power demand, peak power generation, and total power deficit of the four provinces are listed in Table 1.

Table 1. Peak power demand, peak power generation and total power deficit of the four provinces of Pakistan.

Province	Peak Power Demand (MW)	Peak Power Demand (Total) (MW)	Peak Power Generation (MW)	Total Deficit (MW)
Punjab	11,347			
Sindh	3943			
KPK	2054	18,306	14,263	4043
Baluchistan	962			

3. Bankruptcy Rules and Nash Bargaining Theory: Methods for Managing the Allocation of Resources

3.1. Classical Bankruptcy Rules

The power allocation among the provinces of Pakistan was first performed using the five classical rules of bankruptcy, which are applied when the total resources or assets are insufficient to satisfy the demands of all agents (provinces, in this case). There are two reasons for using bankruptcy rules for power allocation among Pakistan's provinces. First, in real bankruptcy problems, the claims are also exceeded by total assets. Second, the bankruptcy rules are simple, easy to understand, and can be used by policymakers for power sharing [38]. In economics, bankruptcy rules are used when the total assets are not sufficient to satisfy the demands of all creditors. In this study, based on the theory of economics, we assume that the total power is insufficient to satisfy the power demands of all the provinces; therefore, the rules of bankruptcy can be used for power allocation among the provinces.

We assume that there are ' n ' number of claimants. The number of claimants is $n \geq 2$, whereas their claims are $C = (c_1, \dots, c_n)$. A bankruptcy problem for power allocation is defined as $F(N, E, C); i = 1, 2, \dots, n$. Here, N is the number of agents, E is the total resources, c_i is the claim of agent i . The bankruptcy problem aims to determine the allocation of each agent, denoted by $F(N, E, C) = x_i$, where $x_i \geq 0$; $x = (x_1, \dots, x_n)$. In addition, $0 \leq x_i \leq c_i$, where x is the allocation of an agent.

The five bankruptcy rules used to allocate power supply among the provinces during the times of power shortage are given as follows:

3.1.1. Proportional Rule (PRO)

The proportional rule (PRO) is given by:

$$p_i^{pro} = \rho c_i \text{ where } \rho = \frac{E}{C} \quad (1)$$

where E is total assets and C is the total amount of claims.

3.1.2. The Constrained Equal Award (CEA) Rule

This rule is given by:

$$x_i^{CEA} = \min(\lambda, c_i) \text{ where } \sum_{i \in N} \min(\lambda, c_i) = E \quad (2)$$

CEA assigns each agent an equal share λ of E , except that no creditor receives more than his or her claim.

3.1.3. The Constrained Equal Losses (CEL) Rule

This rule is defined as:

$$x_i^{CEL} = \max(0, c_i - \lambda) \text{ where } \sum_{i \in N} \max(0, c_i - \lambda) = E \quad (3)$$

In this rule, the losses of all agents are equal compared with their claims (λ), and no agent receives a negative allocation.

3.1.4. The Talmud Rule

The Talmud Rule is derived by combining the CEL and CEA rules, and is given by:

$$x_i^{TAL} = \begin{cases} CEA\left\{\frac{1}{2}c_i, E\right\} & \text{if } E \leq \frac{1}{2}C \\ \frac{1}{2}c_i + CEL\left\{\frac{1}{2}c_i, E - \frac{1}{2}C\right\} & \text{otherwise} \end{cases} \quad (4)$$

3.1.5. The Piniles Rule

For each c_i , x_i^{Pin} is calculated as follows [39]:

$$x_i^{Pin} = \begin{cases} x_i^{CEA}\left\{\frac{1}{2}c_i, E\right\} & \text{if } E \leq \frac{D}{2} \\ \frac{1}{2}c_i + x_i^{CEA}\left\{\frac{1}{2}c_i, E - \frac{D}{2}\right\} & \text{if } E \geq \frac{D}{2} \end{cases} \quad (5)$$

3.2. Power Allocation Using a Combination of the Asymmetric Nash Bargaining Theory and Power Bankruptcy Concept

Building on earlier work [31–36], we plan to use a power allocation framework that combines the asymmetric Nash bargaining solution concept with the bankruptcy theory to solve the power-sharing problem among the four provinces in Pakistan.

In the power bankruptcy case, the power allocation problem can be given as (N, E, c, x^-) , where N is the number of agents involved in a power dispute, E is the total amount of power available for sharing among the agents, c is the amount of power claimed by the agents, and x^- is the amount of power allocated to the agent. In this step, the asymmetric Nash bargaining theory is combined with the bankruptcy concept and applied to allocate power. While applying this methodology, the disagreement allocation points $(m_1, m_2, m_3, \dots, m_n)$ and bargaining weights $(w_1 = w_1, w_2, w_3, \dots, w_n)$ of the agents were also considered to ensure equity and self-enforceability in a closed and bounded space. Apart from providing a unique solution, such an optimization solution also satisfies a set of desirable properties. The solution maximizes the area between the disagreement point (mi) and the Pareto-optimal frontier (x^-).

The Nash equilibrium point can determine the disagreement points, the minimum benefit of each riparian, the maximum and minimum points, and other methods. In our case, the vector of disagreement points $(d_1, d_2, \dots, d_i, \dots, d_n)$ are defined as the benefits of minimum power allocation $(I_1, I_2, \dots, I_n)$ to riparians. This represents the minimum benefit that agents can accept. Therefore, the individual rationality requirements must be

reflected before the cooperation of the followers, so that the maximal and minimal solutions are satisfied. For each agent, the disagreement point formula is defined as follows:

$$d_i = u_i(m_i) \quad (6)$$

To solve the problem of minimal power allocation to each riparian, bankruptcy theory can be used when the total available power is less than the total power demand. The minimal power allocation formula for each riparian is given by:

$$m_i = \max(0, E - \sum_{k \neq i} c_k) \quad (7)$$

Subject to:

$$E < \sum_{i=1}^n c_i \quad (8)$$

The minimum power allocation to any province, especially to the provinces with more minor claims, may become zero if we use Equation (7) for the minimum power allocation. However, each province can demand a minimum amount of power λ_i in the power allocation process. Using the above theory of bankruptcy, the minimum power allocation may be less than the minimum power requirement for each province λ_i . Therefore, to avoid the case of an unreasonable minimum power allocation by bankruptcy theory, we propose the following formula, which determines the minimum power allocation and considers the minimum requirement for each riparian:

$$I_i = \max(\lambda_i, E - \sum_{k \neq i} c_k) \quad (9)$$

where λ_i is the minimum power requirement of each riparian, which, in our study, is considered half of the demand of any province.

For the optimization problem, the respective power claims of the provinces serve as the upper-bound core. According to [40], the optimization problem for the allocation of power under the bankruptcy scenario is given by:

$$\text{Maximize } N^{wi} = \left(x_1^- - \left(E - \sum_{i \in N/\{1\}} c_i \right) \right)^{w_1} \left(x_2^- - \left(E - \sum_{i \in N/\{2\}} c_i \right) \right)^{w_2} \left(x_3^- - \left(E - \sum_{i \in N/\{3\}} c_i \right) \right)^{w_3} \dots \left(x_n^- - \left(E - \sum_{i \in N/\{n\}} c_i \right) \right)^{w_n} \quad (10)$$

The model above is constrained by feasibility and individual rationality. The claims and disagreement points serve as the upper and lower bounds, respectively. Therefore, the power-sharing optimization problem can be formulated as:

$$\text{Maximize } N^{wi} = \left(x_p^- - \left(E - \sum_{i \in N/\{P\}} c_i \right) \right)^{w_p} \left(x_s^- - \left(E - \sum_{i \in N/\{S\}} c_i \right) \right)^{w_s} \left(x_b^- - \left(E - \sum_{i \in N/\{B\}} c_i \right) \right)^{w_b} \left(x_k^- - \left(E - \sum_{i \in N/\{K\}} c_i \right) \right)^{w_k} \quad (11)$$

Here, $\sum_{i=1}^n w_i = 1$.

In Equation (11),

x_p^- is the optimized power allocation for Punjab.

I_p is the lower core bound for Punjab.

x_s^- is the optimized power allocation for Sindh.

I_s is the lower core bound for Sindh.

x_b^- is the optimized power allocation for Baluchistan.

I_b is the lower core bound for Baluchistan.

x_K^- is the optimized power allocation for Khyber Pakhtunkhwa (KPK).

I_K is the lower core bound for Khyber Pakhtunkhwa (KPK).

The following constraints should be set for this allocation model:

The allocation of power to each agent (province) should be equal to or greater than its lower core bound.

$$x_i^- \leq I_i \quad i = 1, 2, \dots, n \quad (12)$$

The power allocation to each agent (province) should be more than its lower core bound and less than its claim.

$$I_i \leq x_i^- \leq c_i \quad (13)$$

The total power allocation should be less than or equal to the total available power.

$$\sum_{i=1}^n x_i^- \leq E \quad (14)$$

Determination of Bargaining Weights

The optimization model in Equation (11) was applied to the power-sharing problem in Pakistan. Three cases were analyzed in this study. In the first case, the bargaining weights of all provinces were assumed to be equal. According to [41], asymmetric Nash solutions induce symmetric Nash solutions; the converse is also true. In reality, all the provinces are different in terms of their socio-economic and power loss status; hence, they have different population and transmission losses. In the second case, therefore, the bargaining weights of the riparian provinces were taken according to their population, to demonstrate the importance of using different bargaining weights. According to the population of these provinces, the bargaining weights for the provinces of Punjab, Sindh, KPK, and Baluchistan are 0.55, 0.24, 0.15, and 0.06, respectively. These bargaining weights are directly proportional to the population; that is, the greater the population of the province, the greater the bargaining weight of the province.

In the third case, the bargaining weights of the provinces were taken in terms of the length of their transmission lines. In Pakistan, the improper maintenance of transmission lines results in T&D losses. The lengths of the transmission lines in kilometers for different provinces are listed in Table 2. The greater the length of transmission lines in the province, the higher the losses. Consequently, a greater length of the transmission line in the province will lead to a higher weight and hence more allocation. According to the length of the transmission lines for the provinces of Punjab, Sindh, KPK, and Baluchistan, the bargaining weights are 0.56, 0.16, 0.13, and 0.15, respectively. As noted above, these bargaining weights are also proportional to the length; that is, the greater the length of the transmission lines in any province, the more its bargaining weight will be. The length of the transmission lines (in kilometers) and population (in a million) for the four provinces in Pakistan are shown in Table 2.

Table 2. Length of transmission lines (in kilometers) and population (in a million) for the four provinces in Pakistan.

Province	Punjab	Sindh	Khyber Pakhtunkhwa (KPK)	Baluchistan	Total
Length of transmission lines (km)	28,921	8364	6954	7470	51,709
Population (in millions)	110	48	30	12	200

4. Results and Discussion

4.1. Results of the Bankruptcy Rules

The power allocation among the four provinces was determined using the five rules of bankruptcy. The results are shown in Table 3. A comparison of the results shows that the CEL rule favors agents with large claims, whereas the CEA rule prefers agents with small claims. The values of the power allocation using the PRO rule are located between the CEL and CEA rules. It is evident from the results that it is the most populous province

with the highest demand. The allocation received by Punjab is the highest, followed by Sindh, KPK, and Baluchistan.

Table 3. Power allocation under five “bankruptcy rules”.

Province	PRO (MW)	CEA (MW)	CEL (MW)	Talmud (MW)	Piniles (MW)
Punjab	8840	7304	10,321	10,079	7304
Sindh	3027	3943	2917	2675.5	3943
KPK	1600	2054	1025	1027	2054
Baluchistan	751	962	0	481	962

4.2. Results of the Nash Bargaining Theory

The results of the Nash bargaining theory are presented in Table 4. The Nash bargaining solution was applied under the three scenarios. The provinces are assigned equal weights in the first scenario, whereas in the second and third, the provinces are assigned weights according to the length of their transmission lines (greater length of transmission lines result in higher weights) and population (higher population results in higher weights), respectively. Table 5 shows the power allocation among the provinces as a percentage of the power demand under all rules. Table 5 shows that when power is allocated using homogenous weights, Punjab and Sindh receive 77 percent and 87 percent of its claims respectively, whereas KPK and Baluchistan receive 100 percent of their claims. However, when the power is allocated using heterogeneous weights, Punjab receives a higher proportion of its claims; this is because Punjab has the highest length of transmission lines and the highest population; because the highest weight is assigned to Punjab, it gets a higher percentage of its claims. Sindh and KPK have shorter transmission lines and a smaller population than Punjab, and therefore, the percentage of their claims is reduced when heterogeneous weights are applied.

Table 4. Power allocation using Nash bargaining theory.

Province	Power Allocation Using Homogeneous Weights (MW)	Power Allocation Using Heterogeneous Weights (Based on the Length of Transmission Lines) (MW)	Power Allocation Using Heterogeneous Weights (Based on the Population of the Province) (MW)
Punjab	8771	9913	9621
Sindh	3438	2717	2983
KPK	2054	1633	1658
Baluchistan	962	962	962

Table 5. Power allocation among the provinces as a percentage of power demand under the five bankruptcy rules and Nash bargaining solution.

Riparian	PRO (%)	CEA (%)	CEL (%)	Talmud (%)	Piniles (%)	Nash Bargaining (Homogenous Weights) (%)	Nash Bargaining Heterogenous Weights (Based on the Length of Transmission Lines) (%)	Nash Bargaining Heterogenous Weights (Based on the Population of the Province) (%)
Punjab	77	64	74	89	64	77	87	85
Sindh	77	100	50	68	74	87	69	75
KPK	77	100	50	50	100	100	79	80
Baluchistan	77	100	0	50	100	100	100	100

4.3. Selection of the Most Appropriate Rule

The selection of the most appropriate rule is essential, as it helps agents reach an agreement. The definition of equity in the distribution of resources is not clear. To select the most appropriate allocation rule, we apply the method proposed by [19]. This method

chooses a rule in which all stakeholders have the lowest dispersion of their total preferences on that rule. Therefore, the allocations are ranked in ascending order for each stakeholder separately. The priority vectors Ω are set for this reason, with elements w_i . Here, w_i is a vector with elements of ϑ_{ji} , where w_i is the preference vector, i is the number of rules, and j is the number of stakeholders. In the current study, $1 \leq i \leq 8$ and $1 \leq j \leq 4$.

The priority vector set for our study was $\Omega = \{w_1, w_2, \dots, w_n\}$ in which $w_1 = (4, 3, 4, 2)$, $w_2 = (7, 1, 1, 1)$, $w_3 = (6, 8, 6, 4)$, $w_4 = (1, 7, 5, 3)$, $w_5 = (7, 5, 1, 1)$, $w_6 = (5, 2, 1, 1)$, $w_7 = (2, 6, 3, 1)$ and $w_8 = (3, 4, 2, 1)$. The priority vector w_i was used for each bankruptcy rule. The priority vector with the lowest distance from the intermediate value is considered the best, denoted by $\bar{w}$. The dispersion around the mean of vector i , δ_i , is calculated using Equation (15) as follows:

$$\delta_i = \frac{\sum_{j=1}^n (\vartheta_{ji} - \bar{w})^2}{n} = \frac{\sum_{j=1}^n \left(\vartheta_{ji} - \frac{\sum_{j=1}^n \vartheta_{ji}}{n} \right)^2}{n} \quad (15)$$

As an example, for the CEA rule, we have,

$$\bar{w}_2 = \frac{7 + 1 + 1 + 1}{4} = 2.5$$

$$\delta_2 = \frac{(2.5 - 7)^2 + (2.5 - 1)^2 + (2.5 - 1)^2 + (2.5 - 1)^2}{4}$$

$$\delta_4 = 6.75$$

Values of δ_i for all rules is presented in Table 6. The rule with the lowest δ_i is considered the best allocation rule. The allocated power among the provinces as a percentage of their power demand is presented in Table 5. Table 6 shows that CEA and Piniles rules are ranked last, whereas the Proportionate rule is ranked first. It should be noted that the Proportionate rule is considered best here only according to the current scenario. Thus, with a change in the supply–demand gap, the best rule will also change.

Table 6. Ranking of the power allocation rules (bankruptcy and Nash bargaining solution).

Province	PRO	CEA	CEL	Talmud	Piniles	Nash Bargaining (Homogenous Weights)	Nash Bargaining Heterogenous Weights (Based on the Length of Transmission Lines)	Nash Bargaining Heterogenous Weights (Based on the Population of the Province)
δ_i	0.68	6.75	2.00	5.00	6.75	2.68	3.50	1.25
Rank	1	7	3	6	7	4	5	2

5. Conclusions

The five rules of bankruptcy and the Nash bargaining theory were applied in this study for power allocation among the provinces of Pakistan. The results demonstrate that for each of the five bankruptcy rules and the Nash bargaining theory, the power allocations are different. Therefore, different agents (provinces) may choose different allocation rules. In addition, other factors, such as the population and the length of the transmission lines in each province, are not taken into account using the simple bankruptcy rules. Because different agents (provinces) have different risks of exposure, the resource allocation rules should also take into account these factors so that allocation is deemed “equitable and reasonable” by all agents (provinces). Hence, in the Nash bargaining theory, power allocation is also accomplished using heterogeneous weights, which takes into account the “population of each province” and the “length of the transmission lines in each province.” Although these allocation rules provide an appropriate vision for power allocation, the distribution of power among the agents (provinces) is a complex task that cannot be solved solely by applying mathematical methods. Negotiations between the administrative units (provinces) of Pakistan are recommended, which would help the

provinces reach an agreement. The method applied for the selection of the best rule would help administrative units (provinces) reach a consensus. It is expected that the research findings will help resolve the increasing power disputes between the administrative units (provinces) of Pakistan.

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