






## Article

# A Self-Adjoint Coupled System of Nonlinear Ordinary Differential Equations with Nonlocal Multi-Point Boundary Conditions on an Arbitrary Domain

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**Abstract:** The main object of this paper is to investigate the existence of solutions for a self-adjoint coupled system of nonlinear second-order ordinary differential equations equipped with nonlocal multi-point coupled boundary conditions on an arbitrary domain. We apply the Leray–Schauder alternative, the Schauder fixed point theorem and the Banach contraction mapping principle in order to derive the main results, which are then well-illustrated with the aid of several examples. Some potential directions for related further researches are also indicated.

**Keywords:** system of ordinary differential equations; nonlocal multi-point boundary conditions; Sturm–Liouville problems; existence results; uniqueness results; Schauder fixed point theorem; Leray–Schauder alternative; Banach contraction mapping principle

## 1. Introduction

The topic of boundary value problems is an important area of investigation in view of its applications in a variety of disciplines such as modern fluid mechanics [1], nano boundary layer fluid flows [2], conservation laws [3], cellular systems and aging models [4], magnetohydrodynamic flow of a second grade nanofluid over a nonlinear stretching sheet [5] and magneto Maxwell nano-material by a surface of variable thickness [6]. For the application of self-adjoint differential equations, for instance, see [7,8]. Much of the literature on boundary value problems deals with classical boundary conditions. However, these conditions fail to cater the complexities of the physical and chemical processes occurring within the domain. In order to cope with this situation, the concept of nonlocal boundary conditions serves as an excellent tool. Such conditions involve the values of the unknown function at some interior positions as well as at the end points of the domain. It is imperative to note that the measurement provided by a nonlocal condition is regarded as more accurate than the one described by a local (fixed) condition. For some recent works on nonlocal nonlinear boundary value problems, see [9–22] and the references cited therein. Furthermore, in a recent article [23], coupled nonlinear third-order ordinary

differential equations with nonlocal multi-point anti-periodic type boundary conditions was investigated.

Modern tools (variational and topological methods) of functional analysis play an important role in establishing the existence theory for nonlinear boundary value problems [24,25]. For the application of the fixed-point theory to single-valued and multi-valued boundary value problems of ordinary differential equations, for instance, see [26,27] and the references cited therein.

Motivated by the aforementioned recent work [23], here we introduce and study the following self-adjoint coupled system of nonlinear second-order ordinary differential equations on an arbitrary domain:

$$\begin{cases} \left( p(t)u'(t) \right)' = f(t, u(t), v(t)), & t \in [a, b], \\ \left( q(t)v'(t) \right)' = g(t, u(t), v(t)), & t \in [a, b], \end{cases} \quad (1)$$

subject to nonlocal multi-point coupled boundary conditions of the form:

$$u'(a) = 0, \quad v'(a) = 0, \quad u(b) = \sum_{j=1}^m \alpha_j v(\eta_j), \quad v(b) = \sum_{k=1}^n \beta_k u(\xi_k), \quad (2)$$

where  $f, g : [a, b] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  are given continuous functions,  $a < \eta_1 < \dots < \eta_m < \xi_1 < \dots < \xi_n < b$ ,  $\alpha_j \in \mathbb{R}^+$  ( $j = 1, 2, \dots, m$ ),  $\beta_k \in \mathbb{R}^+$  ( $k = 1, 2, \dots, n$ ), and  $p, q \in C([a, b], \mathbb{R}^+)$ . Here, we emphasize that the parameters  $\eta_j$  and  $\xi_k$  indicate the nonlocal positions within the interval  $[a, b]$ . In general, we can write these nonlocal positions as  $a < \eta_j, \xi_k < b$ . However, for practical purpose (see examples in Section 4), it is necessary to fix the nonlocal positions within the given domain.

In order to study the existence and uniqueness of solutions for the problem (1) and (2) via fixed point theory, we transform it into a system of integral equations (see Lemma 1 in the next section) to define a fixed point problem associated with the problem (1) and (2). This idea is indeed important from application point of view, for example, see [28] and the references therein.

The rest of the paper is organized as follows. In Section 2, we prove an auxiliary lemma related to the linear variant of the problem (1) and (2), which plays a key role in obtaining the existence and uniqueness results for the problem (1) and (2). The main results are established in Section 3, while the illustrative examples are presented in Section 4. Finally, in Section 5 on conclusions, some potential directions for related further researches are also indicated.

## 2. An Auxiliary Lemma

In this section, we prove a lemma for a linear variant of the problem (1) and (2), which plays a key role in the forthcoming analysis.

**Lemma 1.** For  $f_1, g_1 \in C([a, b], \mathbb{R})$  and  $B \neq 0$ , the solution of the following linear system of differential equations:

$$\left( p(t)u'(t) \right)' = f_1(t), \quad \left( q(t)v'(t) \right)' = g_1(t), \quad t \in [a, b], \quad (3)$$

subject to the boundary conditions (2), can be expressed in the following formulas:

$$\begin{aligned}
u(t) = & \int_a^t \left( \frac{1}{p(s)} \int_a^s f_1(\tau) d\tau \right) ds \\
& + \frac{1}{B} \left[ - \int_a^b \left( \frac{1}{p(s)} \int_a^s f_1(\tau) d\tau \right) ds + \sum_{j=1}^m \alpha_j \int_a^{\eta_j} \frac{1}{q(s)} \left( \int_a^s g_1(\tau) d\tau \right) ds \right. \\
& \left. - \int_a^b \left( \frac{\sum_{j=1}^m \alpha_j}{q(s)} \int_a^s g_1(\tau) d\tau \right) ds + \left( \sum_{j=1}^m \alpha_j \right) \left( \sum_{k=1}^n \beta_k \int_a^{\xi_k} \frac{1}{p(s)} \left( \int_a^s f_1(\tau) d\tau \right) ds \right) \right]
\end{aligned} \quad (4)$$

and

$$\begin{aligned}
v(t) = & \int_a^t \left( \frac{1}{q(s)} \int_a^s g_1(\tau) d\tau \right) ds \\
& + \frac{1}{B} \left[ - \int_a^b \left( \frac{1}{q(s)} \int_a^s g_1(\tau) d\tau \right) ds + \sum_{k=1}^n \beta_k \int_a^{\xi_k} \frac{1}{p(s)} \left( \int_a^s f_1(\tau) d\tau \right) ds \right. \\
& \left. - \int_a^b \left( \frac{\sum_{k=1}^n \beta_k}{p(s)} \int_a^s f_1(\tau) d\tau \right) ds + \left( \sum_{k=1}^n \beta_k \right) \left( \sum_{j=1}^m \alpha_j \int_a^{\eta_j} \frac{1}{q(s)} \left( \int_a^s g_1(\tau) d\tau \right) ds \right) \right],
\end{aligned} \quad (5)$$

where

$$B = \left[ 1 - \left( \sum_{j=1}^m \alpha_j \right) \left( \sum_{k=1}^n \beta_k \right) \right]. \quad (6)$$

**Proof.** Integrating the linear differential Equation (3) twice from  $a$  to  $t$ , and using the conditions  $u'(a) = 0$ ,  $v'(a) = 0$ , we get

$$u(t) = u(a) + \int_a^t \left( \frac{1}{p(s)} \int_a^s f_1(\tau) d\tau \right) ds \quad (7)$$

and

$$v(t) = v(a) + \int_a^t \left( \frac{1}{q(s)} \int_a^s g_1(\tau) d\tau \right) ds. \quad (8)$$

Using the coupled boundary conditions given by (2) in (7) and (8), we obtain a system of equations:

$$u(a) - \left( \sum_{j=1}^m \alpha_j \right) v(a) = - \int_a^b \left( \frac{1}{p(s)} \int_a^s f_1(\tau) d\tau \right) ds + \sum_{j=1}^m \alpha_j \int_a^{\eta_j} \frac{1}{q(s)} \left( \int_a^s g_1(\tau) d\tau \right) ds \quad (9)$$

and

$$- \left( \sum_{k=1}^n \beta_k \right) u(a) + v(a) = - \int_a^b \left( \frac{1}{q(s)} \int_a^s g_1(\tau) d\tau \right) ds + \sum_{k=1}^n \beta_k \int_a^{\xi_k} \frac{1}{p(s)} \left( \int_a^s f_1(\tau) d\tau \right) ds. \quad (10)$$

Solving (9) and (10) for  $u(a)$  and  $v(a)$ , together with the notation (6), we find that

$$u(a) = \frac{1}{B} \left[ - \int_a^b \left( \frac{1}{p(s)} \int_a^s f_1(\tau) d\tau \right) ds + \sum_{j=1}^m \alpha_j \int_a^{\eta_j} \frac{1}{q(s)} \left( \int_a^s g_1(\tau) d\tau \right) ds \right. \\ \left. - \int_a^b \left( \frac{\sum_{j=1}^m \alpha_j}{q(s)} \int_a^s g_1(\tau) d\tau \right) ds + \left( \sum_{j=1}^m \alpha_j \right) \left( \sum_{k=1}^n \beta_k \int_a^{\xi_k} \frac{1}{p(s)} \left( \int_a^s f_1(\tau) d\tau \right) ds \right) \right] \quad (11)$$

and

$$v(a) = \frac{1}{B} \left[ - \int_a^b \left( \frac{1}{q(s)} \int_a^s g_1(\tau) d\tau \right) ds + \sum_{k=1}^n \beta_k \int_a^{\xi_k} \frac{1}{p(s)} \left( \int_a^s f_1(\tau) d\tau \right) ds \right. \\ \left. - \int_a^b \left( \frac{\sum_{k=1}^n \beta_k}{p(s)} \int_a^s f_1(\tau) d\tau \right) ds + \left( \sum_{k=1}^n \beta_k \right) \left( \sum_{j=1}^m \alpha_j \int_a^{\eta_j} \frac{1}{q(s)} \left( \int_a^s g_1(\tau) d\tau \right) ds \right) \right]. \quad (12)$$

Inserting the values of  $u(a)$  and  $v(a)$  in (7) and (8) respectively, we obtain the solutions (4) and (5). By direct computation, one can obtain the converse of the lemma. This completes the proof.  $\square$

Now we state the fixed point theorems used in establishing the existence theory for the system (1).

**Lemma 2** (Banach fixed point theorem [29]). *Let  $X$  be a Banach space,  $D \subset X$  closed and  $F : D \rightarrow D$  a strict contraction, i.e.,  $|Fx - Fy| \leq k|x - y|$  for some  $k \in (0, 1)$  and all  $x, y \in D$ . Then  $F$  has a fixed point in  $D$ .*

**Lemma 3** (Leray–Schauder alternative [30]). *Let  $\Psi$  be a Banach space, and  $Y : \Psi \rightarrow \Psi$  be a completely continuous operator (i.e., a map restricted to any bounded set in  $\Psi$  is compact). Let  $\Theta(Y) = \{y \in \Psi : y = \epsilon Y(y) \text{ for some } 0 < \epsilon < 1\}$ . Then either the set  $\Theta(Y)$  is unbounded or  $Y$  has at least one fixed point.*

**Lemma 4** (Schauder fixed point theorem [30]). *Let  $C$  be a convex (not necessary closed) subset of a normed linear space  $E$ . Then each compact map  $F : C \rightarrow C$  has at least one fixed point.*

### 3. Main Results

Let  $(\mathcal{P}, \|\cdot\|)$  be a Banach space, where

$$\mathcal{P} = \{u(t) | u(t) \in C([a, b], \mathbb{R})\} \quad \text{and} \quad \|u\| = \sup\{|u(t)|, t \in [a, b]\}.$$

Obviously, the product space  $(\mathcal{P} \times \mathcal{P}, \|(u, v)\|)$  is a Banach space with the norm given by

$$\|(u, v)\| = \|u\| + \|v\|$$

for  $(u, v) \in \mathcal{P} \times \mathcal{P}$ .

Using Lemma 1, we transform the problem (1) and (2) into an equivalent fixed point problem as follows:

$$(u, v) = \mathcal{T}(u, v), \quad (13)$$

where  $\mathcal{T} : \mathcal{P} \times \mathcal{P} \rightarrow \mathcal{P} \times \mathcal{P}$  is defined as

$$\mathcal{T}(u, v)(t) := (\mathcal{T}_1(u, v)(t), \mathcal{T}_2(u, v)(t)), \quad (14)$$

$$\mathcal{T}_1(u, v)(t) = \int_a^t \left( \frac{1}{p(s)} \int_a^s f(\tau, u(\tau), v(\tau)) d\tau \right) ds$$

$$\begin{aligned}
& + \frac{1}{B} \left[ - \int_a^b \left( \frac{1}{p(s)} \int_a^s f(\tau, u(\tau), v(\tau)) d\tau \right) ds \right. \\
& + \sum_{j=1}^m \alpha_j \int_a^{\eta_j} \frac{1}{q(s)} \left( \int_a^s g(\tau, u(\tau), v(\tau)) d\tau \right) ds \\
& - \int_a^b \left( \frac{\sum_{j=1}^m \alpha_j}{q(s)} \int_a^s g(\tau, u(\tau), v(\tau)) d\tau \right) ds \\
& \left. + \left( \sum_{j=1}^m \alpha_j \right) \left( \sum_{k=1}^n \beta_k \int_a^{\xi_k} \frac{1}{p(s)} \left( \int_a^s f(\tau, u(\tau), v(\tau)) d\tau \right) ds \right) \right]
\end{aligned} \tag{15}$$

and

$$\begin{aligned}
\mathcal{T}_2(u, v)(t) &= \int_a^t \left( \frac{1}{q(s)} \int_a^s g(\tau, u(\tau), v(\tau)) d\tau \right) ds \\
& + \frac{1}{B} \left[ - \int_a^b \left( \frac{1}{q(s)} \int_a^s g(\tau, u(\tau), v(\tau)) d\tau \right) ds \right. \\
& + \sum_{k=1}^n \beta_k \int_a^{\xi_k} \frac{1}{p(s)} \left( \int_a^s f(\tau, u(\tau), v(\tau)) d\tau \right) ds \\
& - \int_a^b \left( \frac{\sum_{k=1}^n \beta_k}{p(s)} \int_a^s f(\tau, u(\tau), v(\tau)) d\tau \right) ds \\
& \left. + \left( \sum_{k=1}^n \beta_k \right) \left( \sum_{j=1}^m \alpha_j \int_a^{\eta_j} \frac{1}{q(s)} \left( \int_a^s g(\tau, u(\tau), v(\tau)) d\tau \right) ds \right) \right].
\end{aligned} \tag{16}$$

We need the following assumptions in the forthcoming analysis:

(A<sub>1</sub>) (Growth conditions) There exist real constants  $\sigma_i, \theta_i \geq 0$  ( $i = 1, 2$ ), and  $\sigma_0 > 0$ ,  $\theta_0 > 0$ , such that  $\forall u, v \in \mathbb{R}$ , we have

$$|f(t, u, v)| \leq \sigma_0 + \sigma_1|u| + \sigma_2|v|, \quad |g(t, u, v)| \leq \theta_0 + \theta_1|u| + \theta_2|v|.$$

(A<sub>2</sub>) (Sub-growth conditions) There exist nonnegative functions  $\omega(t), \lambda(t) \in L(a, b)$  such that

$$|f(t, u, v)| \leq \omega(t) + \mu_1|u|^{c_1} + \mu_2|v|^{c_2}, \quad u, v \in \mathbb{R}, \quad \mu_1, \mu_2 > 0, \quad 0 < c_1, c_2 < 1,$$

$$|g(t, u, v)| \leq \lambda(t) + \nu_1|u|^{k_1} + \nu_2|v|^{k_2}, \quad u, v \in \mathbb{R}, \quad \nu_1, \nu_2 > 0, \quad 0 < k_1, k_2 < 1.$$

(A<sub>3</sub>) (Lipschitz conditions) For all  $t \in [a, b]$  and  $u_i, v_i \in \mathbb{R}$  ( $i = 1, 2$ ), there exist  $\ell_i > 0$  ( $i = 1, 2$ ) such that

$$|f(t, u_1, v_1) - f(t, u_2, v_2)| \leq \ell_1(|u_1 - u_2| + |v_1 - v_2|)$$

and

$$|g(t, u_1, v_1) - g(t, u_2, v_2)| \leq \ell_2(|u_1 - u_2| + |v_1 - v_2|).$$

For the sake of computational convenience, we set

$$\mathcal{O}_1 = L_1 + L_3, \quad \mathcal{O}_2 = L_2 + L_4, \tag{17}$$

where

$$\left\{ \begin{array}{l} L_1 = \frac{1}{|B\bar{p}|} \left[ \frac{(b-a)^2}{2} (|B|+1) + \left( \sum_{j=1}^m \alpha_j \right) \left( \sum_{k=1}^n \beta_k \frac{(\xi_k - a)^2}{2} \right) \right], \\ L_2 = \frac{\sum_{j=1}^m \alpha_j}{|B\bar{q}|} \left[ \frac{(\eta_j - a)^2}{2} + \frac{(b-a)^2}{2} \right], \quad L_3 = \frac{\sum_{k=1}^n \beta_k}{|B\bar{p}|} \left[ \frac{(b-a)^2}{2} + \frac{(\xi_k - a)^2}{2} \right], \\ L_4 = \frac{1}{|B\bar{q}|} \left[ \frac{(b-a)^2}{2} (|B|+1) + \left( \sum_{k=1}^n \beta_k \right) \left( \sum_{j=1}^m \alpha_j \frac{(\eta_j - a)^2}{2} \right) \right], \\ \bar{p} = \inf_{s \in [a,b]} |p(s)|, \quad \bar{q} = \inf_{s \in [a,b]} |q(s)|, \end{array} \right. \quad (18)$$

$$\mathcal{O} = \min\{1 - (\mathcal{O}_1\sigma_1 + \mathcal{O}_2\theta_1), 1 - (\mathcal{O}_1\sigma_2 + \mathcal{O}_2\theta_2)\}, \quad \sigma_i, \theta_i \text{ are given in } (A_1). \quad (19)$$

### 3.1. Existence Results

The first existence result for the problem (1) and (2) is based on the Leray–Schauder alternative (Lemma 3).

**Theorem 1.** Assume that the condition  $(A_1)$  holds and that

$$\mathcal{O}_1\sigma_1 + \mathcal{O}_2\theta_1 < 1 \text{ and } \mathcal{O}_1\sigma_2 + \mathcal{O}_2\theta_2 < 1, \quad (20)$$

where  $\mathcal{O}_i$  ( $i = 1, 2$ ) are given by (17). Then there exists at least one solution for the problem (1) and (2) on  $[a, b]$ .

**Proof.** In the first step, we show that the operator  $\mathcal{T} : \mathcal{P} \times \mathcal{P} \rightarrow \mathcal{P} \times \mathcal{P}$  defined by (14) is completely continuous. From the continuity of the functions  $f$  and  $g$ , it follows that  $\mathcal{T}_1$  and  $\mathcal{T}_2$  are continuous and hence the operator  $\mathcal{T}$  is continuous. Let  $\Lambda \subset \mathcal{P} \times \mathcal{P}$  be bounded. Then, there exist positive constants  $Q_f$  and  $Q_g$  such that  $|f(t, u(t), v(t))| \leq Q_f$ , and  $|g(t, u(t), v(t))| \leq Q_g$ ,  $\forall (u, v) \in \Lambda$ . Then, for any  $(u, v) \in \Lambda$ , we obtain

$$\begin{aligned} |\mathcal{T}_1(u, v)(t)| &= \left| \int_a^t \left( \frac{1}{p(s)} \int_a^s f(\tau, u(\tau), v(\tau)) d\tau \right) ds \right. \\ &\quad + \frac{1}{B} \left[ - \int_a^b \left( \frac{1}{p(s)} \int_a^s f(\tau, u(\tau), v(\tau)) d\tau \right) ds \right. \\ &\quad + \sum_{j=1}^m \alpha_j \int_a^{\eta_j} \frac{1}{q(s)} \left( \int_a^s g(\tau, u(\tau), v(\tau)) d\tau \right) ds \\ &\quad \left. - \int_a^b \left( \frac{\sum_{j=1}^m \alpha_j}{q(s)} \int_a^s g(\tau, u(\tau), v(\tau)) d\tau \right) ds \right. \\ &\quad \left. + \left( \sum_{j=1}^m \alpha_j \right) \left( \sum_{k=1}^n \beta_k \int_a^{\xi_k} \frac{1}{p(s)} \left( \int_a^s f(\tau, u(\tau), v(\tau)) d\tau \right) ds \right) \right] \Big| \\ &\leq Q_f \left\{ \frac{1}{|B\bar{p}|} \left[ |B| \frac{(t-a)^2}{2} + \frac{(b-a)^2}{2} + \left( \sum_{j=1}^m \alpha_j \right) \left( \sum_{k=1}^n \beta_k \frac{(\xi_k - a)^2}{2} \right) \right] \right\} \\ &\quad + Q_g \left\{ \frac{\sum_{j=1}^m \alpha_j}{|B\bar{q}|} \left[ \frac{(\eta_j - a)^2}{2} + \frac{(b-a)^2}{2} \right] \right\}, \end{aligned}$$

which, on taking the norm for  $t \in [a, b]$ , yields  $\|\mathcal{T}_1(u, v)\| \leq Q_f L_1 + Q_g L_2$ . Similarly, we have  $\|\mathcal{T}_2(u, v)\| \leq Q_f L_3 + Q_g L_4$ , where  $L_i$  ( $i = 1, \dots, 4$ ) are given by (18). In consequence, we get

$$\|\mathcal{T}(u, v)\| \leq Q_f \mathcal{O}_1 + Q_g \mathcal{O}_2,$$

where  $\mathcal{O}_i$  ( $i = 1, 2$ ) are given by (17). Hence, we deduce that the operator  $\mathcal{T}$  is uniformly bounded. Next, we prove that  $\mathcal{T}$  is an equicontinuous operator. For  $t_1, t_2 \in [a, b]$  with  $t_1 < t_2$ , we have

$$\begin{aligned} & |\mathcal{T}_1(u, v)(t_2) - \mathcal{T}_1(u, v)(t_1)| \\ &= \left| \int_a^{t_2} \left( \frac{1}{p(s)} \int_a^s f(\tau, u(\tau), v(\tau)) d\tau \right) ds - \int_a^{t_1} \left( \frac{1}{p(s)} \int_a^s f(\tau, u(\tau), v(\tau)) d\tau \right) ds \right| \\ &= \left| \int_a^{t_1} \left( \frac{1}{p(s)} \int_a^s f(\tau, u(\tau), v(\tau)) d\tau \right) ds + \int_{t_1}^{t_2} \left( \frac{1}{p(s)} \int_a^s f(\tau, u(\tau), v(\tau)) d\tau \right) ds \right. \\ &\quad \left. - \int_a^{t_1} \left( \frac{1}{p(s)} \int_a^s f(\tau, u(\tau), v(\tau)) d\tau \right) ds \right| \\ &\leq Q_f \frac{|(t_2 - a)^2 - (t_1 - a)^2|}{2\bar{p}} \rightarrow 0 \text{ as } t_2 - t_1 \rightarrow 0 \text{ independent of } (u, v). \end{aligned}$$

In a similar manner, one can find that

$$|\mathcal{T}_2(u, v)(t_2) - \mathcal{T}_2(u, v)(t_1)| \rightarrow 0 \text{ as } t_2 - t_1 \rightarrow 0 \text{ independent of } (u, v).$$

Thus, the operator  $\mathcal{T}$  is equicontinuous.

Finally, we verify that the set  $\Theta = \{(u, v) \in \mathcal{P} \times \mathcal{P} | (u, v) = \epsilon \mathcal{T}(u, v), 0 < \epsilon < 1\}$  is bounded. Let  $(u, v) \in \Theta$ . Then  $(u, v) = \epsilon \mathcal{T}(u, v)$ , and for any  $t \in [a, b]$ , we have

$$u(t) = \epsilon \mathcal{T}_1(u, v)(t), \quad v(t) = \epsilon \mathcal{T}_2(u, v)(t).$$

Then, using the growth conditions  $(A_1)$ , we obtain

$$\begin{aligned} |u(t)| &\leq L_1(\sigma_0 + \sigma_1|u| + \sigma_2|v|) + L_2(\theta_0 + \theta_1|u| + \theta_2|v|) \\ &\leq L_1\sigma_0 + L_2\theta_0 + (L_1\sigma_1 + L_2\theta_1)\|u\| + (L_1\sigma_2 + L_2\theta_2)\|v\| \end{aligned}$$

and

$$\begin{aligned} |v(t)| &\leq L_3(\sigma_0 + \sigma_1|u| + \sigma_2|v|) + L_4(\theta_0 + \theta_1|u| + \theta_2|v|) \\ &\leq L_3\sigma_0 + L_4\theta_0 + (L_3\sigma_1 + L_4\theta_1)\|u\| + (L_3\sigma_2 + L_4\theta_2)\|v\|. \end{aligned}$$

From the foregoing inequalities, we get

$$\begin{aligned} \|u\| + \|v\| &\leq (L_1 + L_3)\sigma_0 + (L_2 + L_4)\theta_0 + [(L_1 + L_3)\sigma_1 + (L_2 + L_4)\theta_1]\|u\| \\ &\quad + [(L_1 + L_3)\sigma_2 + (L_2 + L_4)\theta_2]\|v\|, \end{aligned}$$

which, in view of (19) and (20), implies that

$$\|(u, v)\| \leq \frac{\mathcal{O}_1\sigma_0 + \mathcal{O}_2\theta_0}{\mathcal{O}}.$$

This shows that the set  $\Theta$  is bounded. Thus, the hypotheses of Lemma 4 are satisfied and hence its conclusion implies that the operator  $\mathcal{T}$  has at least one fixed point. Therefore, the problem (1) and (2) has at least one solution on  $[a, b]$ . This completes the proof.  $\square$

In the following result, we apply the Schauder fixed point theorem (Lemma 4) to prove the existence of solutions for the problem (1) and (2).

**Theorem 2.** Assume that the condition  $(A_2)$  holds true. Then there exists at least one solution for the problem (1) and (2) on  $[a, b]$ .

**Proof.** Fixing

$$\delta \geq \max \left\{ 6\mathcal{O}_1 \|\omega\|, 6\mathcal{O}_2 \|\lambda\|, (6\mu_1 \mathcal{O}_1)^{\frac{1}{1-c_1}}, (6\mu_2 \mathcal{O}_1)^{\frac{1}{1-c_2}}, (6\nu_1 \mathcal{O}_2)^{\frac{1}{1-k_1}}, (6\nu_2 \mathcal{O}_2)^{\frac{1}{1-k_2}} \right\},$$

we introduce a set given by

$$\Delta = \{(u, v) \in \mathcal{P} \times \mathcal{P} : \|(u, v)\| \leq \delta\}$$

and consider the operator  $\mathcal{T} : \Delta \rightarrow \Delta$ . For any  $(u, v) \in \Delta$ , we have

$$\begin{aligned} |\mathcal{T}_1(u, v)(t)| &= \left| \int_a^t \left( \frac{1}{p(s)} \int_a^s f(\tau, u(\tau), v(\tau)) d\tau \right) ds \right. \\ &\quad + \frac{1}{B} \left[ - \int_a^b \left( \frac{1}{p(s)} \int_a^s f(\tau, u(\tau), v(\tau)) d\tau \right) ds \right. \\ &\quad + \sum_{j=1}^m \alpha_j \int_a^{\eta_j} \frac{1}{q(s)} \left( \int_a^s g(\tau, u(\tau), v(\tau)) d\tau \right) ds \\ &\quad \left. - \int_a^b \left( \frac{\sum_{j=1}^m \alpha_j}{q(s)} \int_a^s g(\tau, u(\tau), v(\tau)) d\tau \right) ds \right. \\ &\quad \left. + \left( \sum_{j=1}^m \alpha_j \right) \left( \sum_{k=1}^n \beta_k \int_a^{\xi_k} \frac{1}{p(s)} \left( \int_a^s f(\tau, u(\tau), v(\tau)) d\tau \right) ds \right) \right] \Big| \\ &\leq \left( \omega(t) + \mu_1 |u|^{c_1} + \mu_2 |v|^{c_2} \right) \left\{ \frac{1}{|B\bar{p}|} \left[ |B| \frac{(t-a)^2}{2} + \frac{(b-a)^2}{2} \right. \right. \\ &\quad \left. \left. + \left( \sum_{j=1}^m \alpha_j \right) \left( \sum_{k=1}^n \beta_k \frac{(\xi_k - a)^2}{2} \right) \right] \right\} \\ &\quad + \left( \lambda(t) + \nu_1 |u|^{k_1} + \nu_2 |v|^{k_2} \right) \left\{ \frac{\sum_{j=1}^m \alpha_j}{|B\bar{q}|} \left[ \frac{(\eta_j - a)^2}{2} + \frac{(b-a)^2}{2} \right] \right\}, \end{aligned}$$

which, on taking the norm for  $t \in [a, b]$ , yields

$$\|\mathcal{T}_1(u, v)\| \leq \left( \|\omega\| + \mu_1 \|u\|^{c_1} + \mu_2 \|v\|^{c_2} \right) L_1 + \left( \|\lambda\| + \nu_1 \|u\|^{k_1} + \nu_2 \|v\|^{k_2} \right) L_2.$$

Similarly, one can find that

$$\|\mathcal{T}_2(u, v)\| \leq \left( \|\omega\| + \mu_1 \|u\|^{c_1} + \mu_2 \|v\|^{c_2} \right) L_3 + \left( \|\lambda\| + \nu_1 \|u\|^{k_1} + \nu_2 \|v\|^{k_2} \right) L_4,$$



where  $L_i$  ( $i = 1, \dots, 4$ ) are given by (18). Consequently, we obtain

$$\|\mathcal{T}(u, v)\| \leq \left( \|\omega\| + \mu_1 \|u\|^{c_1} + \mu_2 \|v\|^{c_2} \right) \mathcal{O}_1 + \left( \|\lambda\| + \nu_1 \|u\|^{k_1} + \nu_2 \|v\|^{k_2} \right) \mathcal{O}_2 \leq \delta,$$

where  $\mathcal{O}_1$  and  $\mathcal{O}_2$  are given by (17). Thus, we deduce that  $\mathcal{T} : \Delta \rightarrow \Delta$ .

Following the arguments used in the proof of Theorem 1, it is easy to show that the operator  $\mathcal{T}$  is completely continuous. So, by the Schauder fixed point theorem, there exists a solution for the problem (1) and (2) on  $[a, b]$ . The proof is now completed.  $\square$

### 3.2. Uniqueness Results

In this subsection, we apply Banach's contraction mapping principle (Lemma 2) to establish the uniqueness of solutions for the problem (1) and (2).

**Theorem 3.** Assume that  $(A_3)$  holds. In addition, we suppose that

$$\mathcal{O}_1 \ell_1 + \mathcal{O}_2 \ell_2 < 1, \quad (21)$$

where  $\mathcal{O}_i$  ( $i = 1, 2$ ) are given by (17). Then the problem (1) and (2) has a unique solution on  $[a, b]$ .

**Proof.** Define  $N_1 = \sup_{t \in [a, b]} |f(t, 0, 0)|$  and  $N_2 = \sup_{t \in [a, b]} |g(t, 0, 0)|$ , and fix

$$\kappa \geq \frac{N_1 \mathcal{O}_1 + N_2 \mathcal{O}_2}{1 - (\ell_1 \mathcal{O}_1 + \ell_2 \mathcal{O}_2)}.$$

Consider a set  $B_\kappa = \{(u, v) \in \mathcal{P} \times \mathcal{P} : \|(u, v)\| \leq \kappa\}$ , and show that  $\mathcal{T}B_\kappa \subset B_\kappa$ . For any  $(u, v) \in B_\kappa$ ,  $t \in [a, b]$ , it follows by the condition  $(A_3)$  that

$$\begin{aligned} |f(t, u(t), v(t))| &= |f(t, u(t), v(t)) - f(t, 0, 0) + f(t, 0, 0)| \\ &\leq |f(t, u(t), v(t)) - f(t, 0, 0)| + |f(t, 0, 0)| \\ &\leq \ell_1 (\|u\| + \|v\|) + N_1 \leq \ell_1 \|(u, v)\| + N_1 \leq \ell_1 \kappa + N_1. \end{aligned}$$

Similarly,  $|g(t, u(t), v(t))| \leq \ell_2 \|(u, v)\| + N_2 \leq \ell_2 \kappa + N_2$ . Then, for  $(u, v) \in B_\kappa$ , we obtain

$$\begin{aligned} |\mathcal{T}_1(u, v)(t)| &= \left| \int_a^t \left( \frac{1}{p(s)} \int_a^s f(\tau, u(\tau), v(\tau)) d\tau \right) ds \right. \\ &\quad + \frac{1}{B} \left[ - \int_a^b \left( \frac{1}{p(s)} \int_a^s f(\tau, u(\tau), v(\tau)) d\tau \right) ds \right. \\ &\quad + \sum_{j=1}^m \alpha_j \int_a^{\eta_j} \frac{1}{q(s)} \left( \int_a^s g(\tau, u(\tau), v(\tau)) d\tau \right) ds \\ &\quad \left. - \int_a^b \left( \frac{\sum_{j=1}^m \alpha_j}{q(s)} \int_a^s g(\tau, u(\tau), v(\tau)) d\tau \right) ds \right. \\ &\quad \left. + \left( \sum_{j=1}^m \alpha_j \right) \left( \sum_{k=1}^n \beta_k \int_a^{\tilde{\xi}_k} \frac{1}{p(s)} \left( \int_a^s f(\tau, u(\tau), v(\tau)) d\tau \right) ds \right) \right] \Big| \\ &\leq (\ell_1 \kappa + N_1) \left\{ \frac{1}{|B\bar{p}|} \left[ |B| \frac{(t-a)^2}{2} + \frac{(b-a)^2}{2} + \left( \sum_{j=1}^m \alpha_j \right) \left( \sum_{k=1}^n \beta_k \frac{(\tilde{\xi}_k - a)^2}{2} \right) \right] \right\} \end{aligned}$$

$$+ (\ell_2 \kappa + N_2) \left\{ \frac{\sum_{j=1}^m \alpha_j}{|B\bar{q}|} \left[ \frac{(\eta_j - a)^2}{2} + \frac{(b - a)^2}{2} \right] \right\},$$

which implies that

$$\|\mathcal{T}_1(u, v)\| \leq (\ell_1 \kappa + N_1)L_1 + (\ell_2 \kappa + N_2)L_2.$$

In the same fashion, one can obtain  $\|\mathcal{T}_2(u, v)\| \leq (\ell_1 \kappa + N_1)L_3 + (\ell_2 \kappa + N_2)L_4$ , where  $L_i$  ( $i = 1, \dots, 4$ ) are defined by (18). Consequently, we have,

$$\|\mathcal{T}(u, v)\| \leq (\ell_1 \kappa + N_1)\mathcal{O}_1 + (\ell_2 \kappa + N_2)\mathcal{O}_2 \leq \kappa.$$

Therefore,  $\mathcal{T}B_\kappa \subset B_\kappa$ .

Next, we show that the operator  $\mathcal{T}$  is a contraction. Let  $(u_1, v_1), (u_2, v_2) \in \mathcal{P} \times \mathcal{P}$ , then

$$\begin{aligned} & |\mathcal{T}_1(u_1, v_1)(t) - \mathcal{T}_1(u_2, v_2)(t)| \\ & \leq \left\{ \int_a^t \left( \frac{1}{|p(s)|} \int_a^s \left| f(\tau, u_1(\tau), v_1(\tau)) - f(\tau, u_2(\tau), v_2(\tau)) \right| d\tau \right) ds \right. \\ & \quad + \frac{1}{|B|} \left[ \int_a^b \left( \frac{1}{|p(s)|} \int_a^s \left| f(\tau, u_1(\tau), v_1(\tau)) - f(\tau, u_2(\tau), v_2(\tau)) \right| d\tau \right) ds \right. \\ & \quad + \sum_{j=1}^m \alpha_j \int_a^{\eta_j} \frac{1}{|q(s)|} \left( \int_a^s \left| g(\tau, u_1(\tau), v_1(\tau)) - g(\tau, u_2(\tau), v_2(\tau)) \right| d\tau \right) ds \\ & \quad + \int_a^b \left( \frac{\sum_{j=1}^m \alpha_j}{|q(s)|} \int_a^s \left| g(\tau, u_1(\tau), v_1(\tau)) - g(\tau, u_2(\tau), v_2(\tau)) \right| d\tau \right) ds \\ & \quad \left. + \left( \sum_{j=1}^m \alpha_j \right) \left( \sum_{k=1}^n \beta_k \int_a^{\xi_k} \frac{1}{|p(s)|} \left( \int_a^s \left| f(\tau, u_1(\tau), v_1(\tau)) - f(\tau, u_2(\tau), v_2(\tau)) \right| d\tau \right) ds \right) \right] \Big\} \\ & \leq \ell_1 (|u_1 - u_2| + |v_1 - v_2|) \left\{ \frac{1}{|B\bar{p}|} \left[ |B| \frac{(t-a)^2}{2} + \frac{(b-a)^2}{2} + \left( \sum_{j=1}^m \alpha_j \right) \left( \sum_{k=1}^n \beta_k \frac{(\xi_k - a)^2}{2} \right) \right] \right\} \\ & \quad + \ell_2 (|u_1 - u_2| + |v_1 - v_2|) \left\{ \frac{\sum_{j=1}^m \alpha_j}{|B\bar{q}|} \left[ \frac{(\eta_j - a)^2}{2} + \frac{(b-a)^2}{2} \right] \right\}, \end{aligned}$$

which leads to the following estimate:

$$\|\mathcal{T}_1(u_1, v_1) - \mathcal{T}_1(u_2, v_2)\| \leq (\ell_1 L_1 + \ell_2 L_2) (\|u_1 - u_2\| + \|v_1 - v_2\|). \quad (22)$$

Analogously, we have

$$\|\mathcal{T}_2(u_1, v_1) - \mathcal{T}_2(u_2, v_2)\| \leq (\ell_1 L_3 + \ell_2 L_4) (\|u_1 - u_2\| + \|v_1 - v_2\|). \quad (23)$$

From (22) and (23), we obtain

$$\|\mathcal{T}(u_1, v_1) - \mathcal{T}(u_2, v_2)\| \leq (\mathcal{O}_1 \ell_1 + \mathcal{O}_2 \ell_2) (\|u_1 - u_2\| + \|v_1 - v_2\|), \quad (24)$$

where  $\mathcal{O}_i$  ( $i = 1, 2$ ) are given by (17). In view of the assumption (21), it follows from (24) that the operator  $\mathcal{T}$  is a contraction. Hence, by Banach contraction mapping principle, the operator  $\mathcal{T}$  has a fixed point, which corresponds to a unique solution of the problem (1) and (2) on  $[a, b]$ . The proof is completed.  $\square$

#### 4. Illustrative Examples

**Example 1.** Consider the following coupled system of second-order ordinary differential equations

$$\begin{cases} \left( \left( \frac{9t+5}{6+t} \right)^3 u'(t) \right)' = f(t, u, v) = \frac{e^{-t}}{39} + \frac{2}{t^3 + 105} u(t) + \frac{|v(t)|^2}{82(1 + |v(t)|)}, & t \in [0, 2], \\ \left( \sqrt{2t^2 + 1} v'(t) \right)' = g(t, u, v) = \frac{\cos(t)}{16} + \frac{1}{272\pi} \sin(4\pi u) + \left( \frac{2}{t+9} \right)^2 v(t), & t \in [0, 2], \end{cases} \quad (25)$$

supplemented with the following boundary conditions:

$$\begin{aligned} u'(0) &= 0, \quad u(2) = \frac{3}{8}v\left(\frac{1}{3}\right) + \frac{1}{2}v\left(\frac{2}{3}\right) + \frac{5}{8}v(1), \\ v'(0) &= 0, \quad v(2) = \frac{2}{7}u\left(\frac{6}{5}\right) + \frac{4}{7}u\left(\frac{7}{5}\right) + \frac{6}{7}u\left(\frac{8}{5}\right) + \frac{8}{7}u\left(\frac{9}{5}\right). \end{aligned} \quad (26)$$

Here,

$$p(t) = \left( \frac{9t+5}{6+t} \right)^3 \quad \text{and} \quad q(t) = \sqrt{2t^2 + 1},$$

$$a = 0, \quad b = 2, \quad \eta_1 = 1/3, \quad \eta_2 = 2/3, \quad \eta_3 = 1, \quad \xi_1 = 6/5, \quad \xi_2 = 7/5, \quad \xi_3 = 8/5, \quad \xi_4 = 9/5,$$

$$\alpha_1 = 3/8, \quad \alpha_2 = 1/2, \quad \alpha_3 = 5/8, \quad \beta_1 = 2/7, \quad \beta_2 = 4/7, \quad \beta_3 = 6/7, \quad \beta_4 = 8/7.$$

Using the given data, it follows that  $|B| \approx 3.285715 \neq 0$  ( $B$  is given by (6)),  $\bar{p} \approx 0.578704$ ,  $\bar{q} = 1$ ,  $L_1 \approx 7.437340$ ,  $L_2 \approx 1.048309$ ,  $L_3 \approx 4.958609$ ,  $L_4 \approx 2.995171$ ,  $[\bar{p}, \bar{q}]$  and  $L_i$  ( $i = 1, \dots, 4$ ) are defined in (18),  $\mathcal{O}_1 \approx 12.395949$  and  $\mathcal{O}_2 \approx 4.043479$  [ $\mathcal{O}_1$  and  $\mathcal{O}_2$  are given by (17)]. Obviously,

$$|f(t, u, v)| \leq \frac{1}{39} + \frac{2}{105} \|u\| + \frac{1}{82} \|v\|, \quad |g(t, u, v)| \leq \frac{1}{16} + \frac{1}{68} \|u\| + \frac{4}{81} \|v\|,$$

with  $\sigma_0 = 1/39$ ,  $\sigma_1 = 2/105$ ,  $\sigma_2 = 1/82$ ,  $\theta_0 = 1/16$ ,  $\theta_1 = 1/68$ ,  $\theta_2 = 4/81$ . Moreover,  $\mathcal{O}_1\sigma_1 + \mathcal{O}_2\theta_1 \approx 0.295576 < 1$  and  $\mathcal{O}_1\sigma_2 + \mathcal{O}_2\theta_2 \approx 0.350848 < 1$ , which imply that (20) is satisfied. Clearly the hypotheses of Theorem 1 are satisfied. Therefore, by the conclusion of Theorem 1, the problems (25) and (26) has at least one solution on  $[0, 2]$ .

**Example 2.** Consider the following system:

$$\begin{cases} \left( \left( \frac{9t+5}{6+t} \right)^3 u'(t) \right)' = \frac{3}{5} + \left( t^2 + \frac{7}{12} \right) \left[ (u(t))^{\frac{2}{9}} + (v(t))^{\frac{5}{7}} \right], & t \in [0, 2], \\ \left( \sqrt{2t^2 + 1} v'(t) \right)' = \frac{9}{4} + \left( t + \frac{2}{11} \right)^4 \left[ (u(t))^{\frac{2}{5}} + (v(t))^{\frac{3}{8}} \right], & t \in [0, 2], \end{cases} \quad (27)$$

subject to the coupled boundary conditions of Example 1.

Here  $c_1 = 2/9$ ,  $c_2 = 5/7$ ,  $k_1 = 2/5$ ,  $k_2 = 3/8$ . Evidently, the condition  $(A_2)$  is satisfied. Thus, the conclusion of Theorem 2 applies to the system (27) with boundary conditions (26). So, there exists a solution of the problem (27) with the coupled boundary conditions (26) on  $[0, 2]$ .

**Example 3.** Consider the following system:

$$\begin{cases} \left( \left( \frac{9t+5}{6+t} \right)^3 u'(t) \right)' = \frac{3}{12\sqrt{t^3+121}} \left( \tan^{-1} u(t) + \frac{|v(t)|}{1+|v(t)|} \right), & t \in [0, 2], \\ \left( \sqrt{2t^2+1} v'(t) \right)' = \frac{9}{7} + \frac{2}{180\pi} \sin(5\pi x) + \frac{1}{18(t+1)} v(t), & t \in [0, 2], \end{cases} \quad (28)$$

with the coupled boundary conditions (26).

Obviously, we have

$$|f(t, u_1, v_1) - f(t, u_2, v_2)| \leq \frac{1}{44}(|u_1 - u_2| + |v_1 - v_2|)$$

and

$$|g(t, u_1, v_1) - g(t, u_2, v_2)| \leq \frac{1}{18}(|u_1 - u_2| + |v_1 - v_2|),$$

with  $\ell_1 = 1/44$  and  $\ell_2 = 1/18$ . From Example 1, we conclude that

$$\mathcal{O}_1 \approx 12.395949, \quad \mathcal{O}_2 \approx 4.043479 \quad \text{and} \quad \mathcal{O}_1 \ell_1 + \mathcal{O}_2 \ell_2 \approx 0.506364 < 1.$$

Thus, by Theorem 3, the problem (28) equipped with the boundary conditions (26) has a unique solution on  $[0, 2]$ .

## 5. Conclusions

We have presented the sufficient criteria for the existence and uniqueness of solutions for a coupled system of self-adjoint nonlinear second-order ordinary differential equations supplemented with nonlocal multi-point coupled boundary conditions on an arbitrary domain. The given boundary value problem is converted into an equivalent fixed point operator equation, which is solved by applying the standard fixed point theorems. We have demonstrated the application of the obtained results by constructing examples. As a special case, our results correspond to a coupled system of self-adjoint nonlinear second-order ordinary differential equations with mixed boundary conditions  $[u'(a) = 0, v'(a) = 0, u(b) = 0, v(b) = 0]$  if we fix  $\alpha_j = 0$  and  $\beta_k = 0$  for all  $j = 1, \dots, m$  and  $k = 1, \dots, n$ . Our results are new in the given configuration and contribute to the theory of Sturm-Liouville problems.

It is hoped that several recent works (see, for example, [31–36]) will provide incentive and motivation for making further advances along the lines of the demonstrated applications of fixed point results and various operators of fractional calculus in the areas of differential equations and their associated boundary value problems which we have presented in this article.

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