applied sciences

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Adaptive Control for a Biological Process under Input Saturation and Unknown Control Gain via Dead Zone Lyapunov Functions

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Abstract: In this work, substrate control of a biological process with unknown varying control gain, input saturation, and uncertain reaction rate is addressed. A novel adaptive controller is proposed, which tackles the combined effect of input saturation and unknown varying control gain with unknown upper and lower bounds. The design is based on dead zone radially unbounded Lyapunov-like functions, with the state backstepping as control framework. The convergence of the modified tracking error and the boundedness of the updated parameters are ensured by means of the Barbalat's lemma. As the first distinctive feature, a new second-order auxiliary system is proposed that tackles the effect of saturated input and the unknown varying control gain with unknown upper and lower bounds. As the second distinctive feature, the modified tracking error converges to a compact set whose width is user-defined, so that it does not depend on bounds of either external disturbances, model terms, or model coefficients. The convergence region of the current tracking error is determined for the closed loop system subject to the formulated controller and the proposed auxiliary system. Finally, numerical simulation illustrates the performance of the proposed controller.

Keywords: input saturation; uncertain nonlinear system; adaptive control; unknown control gain; backstepping control; dead zone Lyapunov function

Citation: Rincón, A.; Hoyos, F.E.; Candelo-Becerra, J.E. Adaptive Control for a Biological Process under Input Saturation and Unknown Control Gain via Dead-Zone Lyapunov Functions. *Appl. Sci.* **2021**, *11*, 251. https://doi.org/10.3390/app11010251

Received: 27 November 2020
Accepted: 23 December 2020
Published: 29 December 2020

1. Introduction

Automatic control of biological processes based on non-adaptive schemes is commonly affected by model uncertainty: (i) uncertain time varying coefficients of the reaction rates, (ii) uncertain time varying reaction yields, (iii) uncertain concentrations, and (iv) varying and uncertain or noisily measured inflow substrate concentration [1–3]. Adaptive control can achieve output stabilization despite these model uncertainties. Indeed, it can guarantee asymptotic convergence of the tracking error and boundedness of its updated parameters [4–7].

In addition to the effects of model uncertainty, control of biological process can be affected by actuator saturation in the case that integral action is used, for instance, the case with update laws [8,9]. Indeed, in adaptive control design, updated parameters may change excessively [10]. One strategy for tackling the effect of input saturation in control design of nonlinear systems is the augmented error signal (AES). In adaptive controllers with AES strategy, it is ensured that (i) closed loop signals are bounded and excessive increase of updated parameters is avoided and (ii) the modified tracking error asymptotically converges to a compact set of small size [10–12]. An early incorporation of the AES strategy to Lyapunov-based adaptive control is presented in [10]. The tracking error signal is substituted by the modified tracking error, which is the sum of the tracking error and a filter in terms of the input error Δu, which is the difference between the constrained and unconstrained input signals [10–12]. In the case of high order nonlinear systems, the backstepping strategy is commonly used, and the auxiliary system is defined as a nth order filter in terms of Δu [11–13].

In adaptive backstepping control design, the input saturation is usually tackled by using the AES strategy [12,14]. However, accounting for unknown varying control gain is not common in these designs. In [12], a general nonlinear time-delay system of nth order is considered, and an output feedback backstepping is designed. Moreover, the model of a two-stage chemical reactor with recirculation is considered as a second-order particular case. However, the controller design assumes the control gain as constant and perfectly known. In [15], a CSTRconsisting of a second-order SISO model is considered, in which only the output is known. The used anti-windup compensator amounts to the auxiliary system of the AES strategy. However, the controller design assumes that the control gain is perfectly known. In [14], a nth order system with unknown nonlinear control gain is considered. The unknown nonlinear nature of the control gain is tackled by using the Nussbaum gain strategy. However, the width of the convergence region of the modified tracking error depends on unknown model coefficients and terms. In summary, in the aforementioned AES-based robust adaptive backstepping control designs, the modified tracking error converges to a compact set whose width depends on the bounds of either external disturbances, model terms or model parameters [11,12,14]. This implies that such bounds must be known to achieve a user-defined magnitude of the steady value of the modified tracking error.

In this study, a modified/new robust adaptive backstepping controller is developed for a second-order SISO nonlinear system, tackling the effect of input saturation and unknown varying control gain with unknown upper and lower bounds. The design is based on dead zone radially unbounded forms, and a new auxiliary system is proposed. The asymptotic convergence of the modified tracking error is proved by using the Barbalat's lemma, accounting for the unknown varying control gain, the saturated input, and the formulated controller. It is ensured that the regular tracking error converges to a residual set of user-defined width, for the case that the input saturation eventually ceases. The main contributions of this study with respect to adaptive backstepping control designs for systems with input saturation are listed below.

- The proposed auxiliary system is robust against varying and unknown control gain with unknown upper and lower bounds. This is in contrast to common adaptive backstepping control designs (see in [12,15]) where the auxiliary system considers the control gain as constant and unknown.
- The modified tracking error converges to a compact set whose width is user-defined, so that it does not depend on the bounds of either external disturbances, model terms, system states, or model parameters. This is in contrast to common adaptive backstepping control designs (see in [12,15]) and also those that use the Nussbaum gain strategy (see in [14]) where the width of the convergence region of the modified tracking error depends on such kind of bounds.
- The convergence region of the tracking error is determined for the closed loop system under the formulated controller with the proposed auxiliary system.

The work is organized as follows. In Section 2, the model of the biological process, the reference model, and the statement of the control goal are presented. In Section 3, the controller is designed and the stability properties of the closed loop states are determined. In Section 4, a simulation example is presented. In Section 5, the conclusions are drawn.

2. Model Description, Reference Model and Control Goal

2.1. Model Description

The second order nonlinear SISO model of an hydroponic culture described in Appendix A is

$$\frac{dx_1}{dt} = a_1x_2 - a_2x_1 - r_{x1} \tag{1}$$
$$\frac{dx_2}{dt} = a_3\frac{1}{x_3}(x_1 - x_2) + \frac{a_4}{x_3}x_2 + bv \tag{2}$$
$$\frac{dx_3}{dt} = Q_e - Q_i - Q_{loss} + v \tag{3}$$

where

$$b = \frac{P_{ad} - x_2}{x_3},\ a_1 = Q_i/V_u,\ a_2 = Q_e/V_u,\ a_3 = Q_e,\ a_4 = Q_{loss}$$
$$x_1 = P_e,\ x_2 = P_i,\ x_3 = V_l,\ v = Q_{ad},$$

P_e is the nutrient concentration in the upper CSTR, P_i is the nutrient concentration in the lower CSTR, V_l is the volume of the lower CSTR, V_u is the volume of the upper CSTR, and Q_e is the flow that leaves the upper CSTR and enters the lower CSTR; Q_i is the flow that leaves the lower CSTR and enters the upper CSTR; Q_{ad} is the flow of addition of fresh nutrient solution to the mixing tank; and P_{ad} is the nutrient concentration of the Q_{ad} flow. The state x_1 is the output to be controlled, whereas the inlet flowrate Q_{ad} is chosen as control input, it is non-negative and its upper bound is determined by the operational limit of the pump. Therefore, the relationship between the constrained control signal (denoted as v) and the unconstrained control signal (denoted as u) is

$$v = \begin{cases} u_{max} & \text{if } u > u_{max} \\ u & \text{if } u \in [u_{min}\ \ u_{max}] \\ u_{min} & \text{if } u < u_{min} \end{cases} \tag{4}$$

The following assumptions are considered:

Assumption 1. *The state variables x_1 and x_2, x_3 are bounded for v bounded, and satisfy $x_1 \in \mathbb{R}^+$, $x_2 \in \mathbb{R}^+$, $x_3 \in \mathbb{R}^+$.*

Assumption 2. *a_1, a_2, a_3, a_4 are constant, a_2, a_3, a_4 are unknown whereas a_1 is known.*

Assumption 3. *The reaction rate r_{x1} satisfies one of the following: (i) it is unknown, non-negative, and $r_{x1} \leq \bar{\mu}_1\bar{r}_{x1}$, where $\bar{\mu}_1$ is unknown, positive, constant, whereas $\bar{r}_{x1}$ is a known continuous function of x_1 with well-defined $d\bar{r}_{x1}/dx_1$; (ii) it is a known continuous function of x_1, with well-defined dr_{x1}/dx_1; in this case, $r_{x1} = \bar{\mu}_1\bar{r}_{x1}$ with $\bar{r}_{x1} = r_{x1}$, $\bar{\mu}_1 = 1$ holds true.*

Assumption 4. *The values of x_1, x_2 are known, whereas x_3 is noisily measured: $x_{3m} = x_3 + \delta_{x3}$, where x_{3m} is the noisy measurement and δ_{x3} is the measurement noise.*

Assumption 5. *There is lack of knowledge on the control gain b according to one of the following conditions: (i) $b = b_\delta b_m$, where b_m is known, and possibly varying, and bounded for x_1, x_2, x_3 bounded, whereas b_δ is unknown, varying, and satisfies: $\bar{\mu}_{lb} \leq |b_\delta| \leq \bar{\mu}_{ub}$, where $\bar{\mu}_{lb}$, $\bar{\mu}_{ub}$ are constant, positive and unknown; (ii) b is unknown, varying, and satisfies $\bar{\mu}_{lb} \leq |b| \leq \bar{\mu}_{ub}$, where $\bar{\mu}_{lb}$, $\bar{\mu}_{ub}$ are unknown positive constants; in this case, b can be expressed as $b = b_\delta b_m$, $b_m = 1$, $b_\delta = b$, so that b_δ is unknown, varying and satisfies $\bar{\mu}_{lb} \leq |b_\delta| \leq \bar{\mu}_{ub}$. In both conditions, $b = b_\delta b_m$, where b_m is known, whereas b_δ is unknown, varying and satisfies $\bar{\mu}_{lb} \leq |b_\delta| \leq \bar{\mu}_{ub}$; so that $|b| \leq \bar{\mu}_{ub}|b_m|$.*

2.2. Reference Model

The use of a reference model allows to obtain the expected transient plant response (rise time, settling time, overshoot). The reference model is defined as [16,17]

$$y_d = \frac{a_{m1}/2}{(p + a_{m1}/2)} \frac{a_{m1}/2}{(p + a_{m1}/2)} W_{ref} \tag{5}$$

where y_d is the desired output; W_{ref} is the command signal, which is user-defined and bounded; a_m is a positive constants defined by the user, which determine the speed of convergence of the signal y_d towards W_{ref}; and $p = d/dt$ is the differential operator. Due to the above characteristics, the signals y_d, $\dot{y}_d$, $\ddot{y}_d$ are bounded and known. The reference model (5) can be rewritten as

$$\frac{d^2 y_d}{dt^2} + a_{m1} \frac{dy_d}{dt} + \frac{a_{m1}^2}{4} y_d = \frac{a_{m1}^2}{4} W_{ref}$$

or, equivalently,

$$\begin{aligned} \frac{dy_d}{dt} &= -\frac{a_{m1}}{2} y_d + \frac{a_{m1}}{2} y_{do} \\ \frac{dy_{do}}{dt} &= -\frac{a_{m1}}{2} y_{do} + \frac{a_{m1}}{2} W_{ref} \end{aligned}$$

2.3. Control Goal

Consider (i) the plant model (1) to (3), subject to input constraint (4) and Assumptions 1 to 5; (ii) the tracking error $e = x_1 - y_d$, where x_1 is the output, and y_d is the desired output, whose characteristics are mentioned in Section 2.2; (iii) the residual set $\Omega_{eo} = \{e : |e| \leq C_b\}$, whose width C_b is positive, constant and user-defined. The goal of the controller design is to formulate a control law for v such that (Gi) the tracking error e converges asymptotically to the residual set Ω_{eo}, (Gii) the control law and the update laws are bounded under closed loop operation, so that excessive parameter increase is avoided; (Giii) the control law, the update law and the auxiliary system involve no discontinuous signals.

Remark 1. *The condition Giii is stated because the presence of discontinuous signals in the control law may lead to input chattering, and problems of existence and uniqueness of closed loop trajectories, and consequently, Filippov theory is needed, as discussed in [18,19].*

3. Control Design and Stability Analysis

In this section, the proposed robust adaptive controller is developed for the model described in Section 2, which involves input constraint, unknown varying control gain, and unknown model parameters.

3.1. Controller Design

The controller design uses Lyapunov theory, with the adaptive backstepping strategy as framework. New states z_1 and z_2 are defined as function of x_1, x_2, and updated parameters. Furthermore, an overall Lyapunov function V is defined as the sum of V_{z1}, the dead zone Lyapunov function for z_1; V_{z2}, the dead zone Lyapunov function for z_2; and a Lyapunov function for each parameter updating error. Differential equations are defined for the new states, and the time derivatives of the Lyapunov functions are defined. The mechanisms for the updated parameters and the control input u are chosen such that dV/dt is negative semi-definite, thus implying the asymptotic convergence of z_1. Other important features of the developed procedure are

- a new auxiliary system of second order is proposed, whose input includes the control signal error Δu, which is the difference between the constrained and the unconstrained control signals;
- a modified tracking error z_1 is defined as the sum of the regular tracking error and the state of the auxiliary system;
- the definition the z_i states is based on the adaptive state backstepping method;
- dead zone radially unbounded quadratic forms are used instead of current quadratic forms; and
- a new treatment of the $b\Delta u$ term is proposed, including a new parameterization of the unknown model parameters, and the formulation of a new auxiliary system.

In a basic adaptive backstepping control design, the z_1 state would be defined as the tracking error, $z_1 = x_1 - y_d$. In contrast, we define z_1 by adding $-\psi_1$ to the tracking error:

$$z_1 = x_1 - y_d - \psi_1 \tag{6}$$

where the ψ_1 state is the output of a stable second order linear filter whose input contains the input error Δu, the difference between the non-saturated and the saturated input signals. The controller design is aimed at driving z_1 to Ω_{z1}, $\Omega_{z1} = \{z_1 : |z_1| \le C_b\}$. The advantages of the definition of z_1 (6) and the controller design based on z_1 instead of the tracking error e are (i) excessive increase of the updated parameters is avoided and (ii) the convergence region of the tracking error $e = x_1 - y_d$ in presence of input saturation can be determined. The time derivative of Equation (6) is

$$\dot{z}_1 = \dot{x}_1 - \dot{y}_d - \dot{\psi}_1 \tag{7}$$

Incorporating the $\dot{x}_1$ expression (1) yields

$$\dot{z}_1 = a_1 x_2 - a_2 x_1 - r_{x1} - \dot{y}_d - \dot{\psi}_1 \tag{8}$$

Let

$$V_{z1} = \begin{cases} (1/3)(z_1 - C_b)^3 \ \text{ for } \ z_1 \ge C_b \\ 0 \ \text{ for } \ z_1 \in (-C_b, C_b) \\ (1/3)(-1)(z_1 + C_b)^3 \ \text{ for } \ z_1 \le -C_b \end{cases} \tag{9}$$

This truncated Lyapunov function is inspired on that of [20,21]. Early versions of dead zone Lyapunov functions are presented in [17,22,23]; versions for backstepping-based controllers in [21,24], and other versions in [20,25,26]. The use of the dead zone Lyapunov function (9) allows designing the adaptive controller, tackling the presence of unknown varying terms or parameters, and avoiding the use of discontinuous signals. The main properties of V_{z1} (9) are

$$\begin{aligned} &V_{z1} = 0 \text{ for } z_1 \in [-C_b,\ C_b] \\ &V_{z1} > 0 \text{ for } z_1 \notin [-C_b,\ C_b] \\ &V_{z1} \text{ is continuous with respect to } z_1, \text{ and it is bounded for } z_1 \text{ bounded} \end{aligned}$$

The above properties and a stable dynamics of V_{z1} imply the convergence of z_1 to Ω_{z1}, $\Omega_{z1} = \{z_1 : |z_1| \le C_b\}$, as is shown in the convergence theorem in Section 3.2.

Differentiating (9) with respect to time yields

$$\dot{V}_{z1} = f_{z1}\dot{z}_1 \tag{10}$$

$$f_{z1} = \frac{dV_{z1}}{dz_1} = \begin{cases} (z_1 - C_b)^2 \ \text{ for } \ z_1 \ge C_b \\ 0 \ \text{ for } \ z_1 \in (-C_b, C_b) \\ (-1)(z_1 + C_b)^2 \ \text{ for } \ z_1 \le -C_b \end{cases} \tag{11}$$

Incorporating the expression for $\dot{z}_1$ (8) into Equation (10) and arranging yields

$$\dot{V}_{z1} = -k_{1b}|f_{z1}| - k_1 f_{z1}^2 + f_{z1}a_1x_2 \\ +f_{z1}(k_1 f_{z1} - \dot{y}_d - \dot{\psi}_1 - a_2x_1 - r_{x_1} + k_{1b}sign(f_{z1})) \tag{12}$$

where $-k_{1b}|f_{z1}|$ was incorporated in order to provide robustness and $|f_{z1}|$ was expressed as $|f_{z1}| = f_{z1}sign(f_{z1})$, which would imply the presence of the signal $sign(f_{z1})$ in the definition of z_2 and would hamper the determination of $\dot{z}_2$. Thus, notice that from the definition of f_{z1} (11) it follows that

$$|f_{z1}| = f_{z1}sat_{z1} \tag{13}$$

where

$$sat_{z1} = \begin{cases} \frac{z_1}{C_b}\left(2 - \frac{|z_1|}{C_b}\right) & \text{for } z_1 \in (-C_b, C_b) \\ sgn(z_1) & \text{otherwise} \end{cases} \tag{14}$$

$$\frac{dsat_{z1}}{dz_1} = \begin{cases} \frac{2}{C_b}\left(1 - \frac{|z_1|}{C_b}\right) & \text{for } z_1 \in (-C_b, C_b) \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

Using (13) in (12) instead of $|f_{z1}| = f_{z1}sign(f_{z1})$, yields

$$\dot{V}_{z1} = -k_{1b}|f_{z1}| - k_1 f_{z1}^2 + f_{z1}a_1x_2 \\ +f_{z1}(k_1 f_{z1} - \dot{y}_d - \dot{\psi}_1 - a_2x_1 - r_{x_1} + k_{1b}sat_{z1}) \tag{16}$$

To obtain the required right hand side of dV_{z1}/dt, the effect of the term $f_{z1}a_1x_2 + f_{z1}(k_1f_{z1} - \dot{y}_d - \dot{\psi}_1 - a_2x_1 - r_{x_1} + k_{1b}sat_{z1})$ should be tackled by adequate definition of the new state z_2. However, a_2 is unknown and r_{x1} is unknown, and its upper bound comprises and unknown constant $\bar{\mu}_1$. Therefore, the term $f_{z1}(-a_2x_1 - r_{x1})$ should be expressed in terms of updated parameters and parameter estimation error. Recall from Assumption 3 that $r_{x1} \leq \bar{\mu}_1\bar{r}_{x1}$ or $r_{x1} = \bar{\mu}_1\bar{r}_{x1}$, therefore

$$f_{z1}(-r_{x1}) \leq \bar{\mu}_1\bar{r}_{x1}|f_{z1}| \tag{17}$$

The term $f_{z1}(-a_2x_1 - r_{x1})$ can be parameterized accounting for (17) and (13):

$$f_{z1}(-a_2x_1 - r_{x1}) \leq \varphi_1^\top \theta_1 f_{z1} \tag{18}$$

$$\varphi_1 = [-x_1,\ sat_{z1}\bar{r}_{x1}]^\top,\ \theta_1 = [a_2,\ \bar{\mu}_1]^\top \tag{19}$$

As the parameter vector θ_1 is unknown, we express it in terms of an updated parameter vector and a parameter updating error as follows. Let $\tilde{\theta}_1 = \hat{\theta}_1 - \theta_1$, where $\tilde{\theta}_1$, $\hat{\theta}_1$ are the parameter estimation error and the updated parameter, being $\hat{\theta}_1$ provided by an updating mechanism defined later. Thus, θ_1 can be expressed as $\theta_1 = \hat{\theta}_1 - \tilde{\theta}_1$. Substituting into Equation (18) yields $f_{z1}(-a_2x_1 - r_{x1}) \leq \varphi_1^\top\hat{\theta}_1 f_{z1} - \varphi_1^\top\tilde{\theta}_1 f_{z1}$. Substituting this into Equation (16) yields

$$\dot{V}_{z1} \leq -k_{1b}|f_{z1}| - k_1 f_{z1}^2 + f_{z1}a_1x_2 \\ +f_{z1}\left(\varphi_1^\top\hat{\theta}_1 + k_1 f_{z1} - \dot{y}_d - \dot{\psi}_1 + k_{1b}sat_{z1}\right) - \varphi_1^\top\tilde{\theta}_1 f_{z1} \tag{20}$$

The dynamics of V_{z1} is affected by the following terms: (i) the term $-k_{1b}|f_{z1}| - k_1 f_{z1}^2$ is negative and it provides stability; (ii) the term $-\varphi_1^\top\tilde{\theta}_1 f_{z1}$, which is later tackled by defining a quadratic form for $\tilde{\theta}_1$, as can be noticed in the boundedness and convergence theorems in Section 3.2; and (iii) the term $f_{z1}a_1x_2 + f_{z1}(\varphi_1^\top\hat{\theta}_1 + k_1f_{z1} - \dot{y}_d - \dot{\psi}_1 + k_{1b}sat_{z1})$, which is later tackled by adequate definition of the new state z_2, so that this term equals $f_{z1}z_2$ and (20) yields

$$\dot{V}_{z1} \leq -k_{1b}|f_{z1}| - k_1 f_{z1}^2 + f_{z1}z_2 - \varphi_1^\top \tilde{\theta}_1 f_{z1} \tag{21}$$

where

$$z_2 = a_1 x_2 + (k_1 f_{z1} - \dot{y}_d - \dot{\psi}_1) + k_{1b} sat_{z1} + \varphi_1^\top \hat{\theta}_1 \tag{22}$$

The effect of the term $f_{z1}z_2$ can be tackled by adequate dynamics of V_{z1} and V_{z2}, such that $\dot{V}_{z1} + \dot{V}_{z2}$ involves the term $-kf_{z1}^2 - kf_{z2}^2$, being f_{z2} a saturation function of z_2 and V_{z2} a quadratic function of f_{z2}. To this end, the required dynamics of z_2 is generated next.

Differentiating z_2 (22) with respect to time yields

$$\dot{z}_2 = a_1 \dot{x}_2 + k_1 \frac{df_{z1}}{dz_1}\dot{z}_1 - \ddot{y}_d - \ddot{\psi}_1 + k_{1b}\frac{dsat_{z1}}{dz_1}\dot{z}_1 + \dot{\varphi}_1^\top \hat{\theta}_1 + \varphi_1^\top \dot{\hat{\theta}}_1 \tag{23}$$

The auxiliary state ψ_1 and its time derivatives are provided by the auxiliary system, whose general structure consists of a second order linear filter whose input W_ψ involves the input error Δu:

$$\dot{\psi}_1 = -K_{\psi 1}\psi_1 + \psi_2 \tag{24}$$

$$\dot{\psi}_2 = -K_{\psi 2}\psi_2 + W_\psi \tag{25}$$

where the term W_ψ, which will be defined later, is used to cancel the effect of the Δu term, which is the difference between the saturated and non-saturated input signals. From (24), (25) it follows that $\ddot{\psi}_1 = K_{\psi 1}^2\psi_1 - (K_{\psi 1} + K_{\psi 2}\psi_2) + W_\psi$. Substituting this expression and the expressions for $\dot{z}_1$ (7), $\dot{x}_1$ (1) and $\dot{x}_2$ (3) into Equation (23) and arranging yields

$$\begin{aligned}\dot{z}_2 = &-k_2 f_{z2} + a_1 a_3 \frac{x_1 - x_2}{x_3} + a_1 a_4 \frac{x_2}{x_3} \\ &+ B_{1d}(-a_2 x_1 - r_{x1}) + B_{1e} + a_1 b v + (-1)W_\psi\end{aligned} \tag{26}$$

where

$$B_{1d} = k_1 \frac{df_{z1}}{dz_1} + k_{1b}\frac{dsat_{z1}}{dz_1} + B_{1b} \tag{27}$$

$$\begin{aligned}B_{1e} = &B_{1d}a_1 x_2 + \left(k_1 \frac{df_{z1}}{dz_1} + k_{1b}\frac{dsat_{z1}}{dz_1}\right)(-1)(\dot{y}_d + \dot{\psi}_1) \\ &+(-1)K_{\psi 1}^2\psi_1 + (K_{\psi 1} + K_{\psi 2})\psi_2 + B_{10} + \varphi^\top \dot{\hat{\theta}}_1 + k_2 f_{z2}\end{aligned} \tag{28}$$

$$B_{1b} = (-1)\hat{\theta}_{1,1} + sat_{z1}\frac{d\bar{r}_{x1}}{dx_1}\hat{\theta}_{1,2} + \bar{r}_{x1}\frac{dsat_{z1}}{dz_1}\hat{\theta}_{1,2}$$

$$B_{10} = \bar{r}_{x1}\frac{dsat_{z1}}{dz_1}(-\dot{y}_d - \dot{\psi}_1)\hat{\theta}_{1,2}$$

where

$$\frac{df_{z1}}{dz_1} = \begin{cases} 2(z_1 - C_b) \text{ for } z_1 \geq C_b \\ 0 \text{ for } z_1 \in (-C_b, C_b) \\ (-2)(z_1 + C_b) \text{ for } z_1 \leq -C_b \end{cases}$$

and $dsat_{z1}/dz_1$ is defined in Equation (15). Let

$$V_z = V_{z1} + V_{z2} \tag{29}$$

$$V_{z2} = \begin{cases} (1/2)(z_2 - C_b)^2 \text{ for } z_2 \geq C_b \\ 0 \text{ for } z_2 \in (-C_b, C_b) \\ (1/2)(z_2 + C_b)^2 \text{ for } z_2 \leq -C_b \end{cases} \tag{30}$$

The main properties of V_{z2} are

$$V_{z2} = 0 \text{ for } z_2 \in [-C_b,\ C_b]$$
$$V_{z2} > 0 \text{ for } z_2 \notin [-C_b,\ C_b]$$
$$V_{z2} \text{ is continuous with respect to } z_2, \text{ and it is bounded for } z_2 \text{ bounded}$$

The above properties and a stable dynamics of V_{z2} imply the convergence of z_2 to Ω_{z2}, $\Omega_{z2} = \{z_2 : |z_2| \leq C_b\}$, as shown in the convergence theorem in Section 3.2.

Differentiating V_{z2} with respect to time yields

$$\dot{V}_{z2} = f_{z2}\frac{dz_2}{dt}, \tag{31}$$

$$f_{z2} = \frac{dV_{z2}}{dz_2} = \begin{cases} z_2 - C_b \ \text{ for } \ z_2 \geq C_b \\ 0 \ \text{ for } \ z_2 \in (-C_b, C_b) \\ z_2 + C_b \ \text{ for } \ z_2 \leq -C_b \end{cases} \tag{32}$$

Incorporating the expression for $\dot{z}_2$ (26) into Equation (31) and arranging yields

$$\begin{aligned} \dot{V}_{z2} = &-k_2 f_{z2}^2 + f_{z2}\left(a_1 a_3 \frac{x_1 - x_2}{x_3} + a_1 a_4 \frac{x_2}{x_3} + B_{1d}(-a_2 x_1 - r_{x1}) + B_{1e}\right) \\ &+ f_{z2}(a_1 b v + (-1) W_\psi) \end{aligned} \tag{33}$$

Differentiating V_z in (29) with respect to time yields $\dot{V}_z = \dot{V}_{z1} + \dot{V}_{z2}$. Substituting the expressions for $\dot{V}_{z1}$ (21) and $\dot{V}_{z2}$ (33) yields

$$\begin{aligned} \dot{V}_z \leq &-C_b|f_{z1}| + f_{z1}z_2 - k_1 f_{z1}^2 - k_2 f_{z2}^2 - \varphi_1^\top \tilde{\theta}_1 f_{z1} \\ &+ f_{z2}\left(a_1 a_3 \frac{x_1 - x_2}{x_3} + a_1 a_4 \frac{x_2}{x_3} + B_{1d}(-a_2 x_1 - r_{x1}) + B_{1e}\right) \\ &+ f_{z2}(a_1 b v + (-1) W_\psi) \end{aligned} \tag{34}$$

To obtain the required right hand side of dV_z/dt, the terms

$$f_{z1}z_2 + f_{z2}\left(a_1 a_3 \frac{x_1 - x_2}{x_3} + a_1 a_4 \frac{x_2}{x_3} + B_{1d}(-a_2 x_1 - r_{x1}) + B_{1e}\right)$$

should be rewritten and tackled by means of proper definition of W_ψ and the control law.

In the $f_{z1}z_2$ term, the z_2 signal must be expressed in terms of f_{z2}, because the controller design is based on the dead zone functions f_{z1} and f_{z2} rather than z_1 and z_2. From definition (32), it follows that $f_{z2} = z_2 + \delta$,

$$\delta = \begin{cases} -C_b \ \text{ for } \ z_2 \geq C_b \\ 0 \ \text{ for } \ z_2 \in (-C_b, C_b) \\ C_b \ \text{ for } \ z_2 \leq -C_b \end{cases}$$

therefore, $z_2 = f_{z2} - \delta$ and $|\delta| \leq C_b$. Therefore, $f_{z1}z_2 = f_{z1}f_{z2} - \delta f_{z1}$. Substituting into Equation (34) yields

$$\begin{aligned} \dot{V}_z \leq &-C_b|f_{z1}| - \delta f_{z1} - k_1 f_{z1}^2 - k_2 f_{z2}^2 - \varphi_1^\top \tilde{\theta}_1 f_{z1} \\ &+ f_{z2}\left(f_{z1} + a_1 a_3 \frac{x_1 - x_2}{x_3} + a_1 a_4 \frac{x_2}{x_3} + B_{1d}(-a_2 x_1 - r_{x1}) + B_{1e}\right) \\ &+ f_{z2}(a_1 b v + (-1) W_\psi) \end{aligned} \tag{35}$$

so that the error term δ leads to the undesired uncertainty term $-\delta f_{z1}$ in Equation (35). The property $|\delta| \leq C_b$ implies that $-\delta f_{z1} \leq C_b|f_{z1}|$, which is canceled by the already existing term $-C_b|f_{z1}|$, so that $-C_b|f_{z1}| - \delta f_{z1} \leq 0$, and Equation (35) yields

$$\begin{aligned}\dot{V}_z \leq & -k_1 f_{z1}^2 - k_2 f_{z2}^2 - \varphi_1^\top \tilde{\theta}_1 f_{z1} \\ & + f_{z2}\left(f_{z1} + a_1 a_3 \frac{x_1 - x_2}{x_3} + a_1 a_4 \frac{x_2}{x_3} + B_{1d}(-a_2 x_1 - r_{x1}) + B_{1e}\right) \\ & + f_{z2}\left(a_1 b v + (-1) W_\psi\right) \end{aligned} \tag{36}$$

Therefore, the term $-C_b|f_{z1}|$ incorporated in Equation (12) is necessary for counteracting the effect of the error term δ resulting from $f_{z2} = z_2 + \delta$.

In order to facilitate the design of the control law for u, the constrained input signal v is expressed in terms of the unconstrained input signal u and the input error Δu, and the effect of Δu is later canceled by the input of the auxiliary system, W_ψ. Let

$$\Delta u = v - u \tag{37}$$

hence

$$v = u + \Delta u \tag{38}$$

Substituting (38) into (36), yields

$$\begin{aligned}\dot{V}_z \leq & -k_1 f_{z1}^2 - k_2 f_{z2}^2 - \varphi_1^\top \tilde{\theta}_1 f_{z1} \\ & + f_{z2}\left(f_{z1} + a_1 a_3 \frac{x_1 - x_2}{x_3} + a_1 a_4 \frac{x_2}{x_3} + B_{1d}(-a_2 x_1 - r_{x1}) + B_{1e}\right) \\ & + a_1 f_{z2} b u + a_1 f_{z2} b \Delta u + f_{z2}(-1) W_\psi \end{aligned} \tag{39}$$

The effect of the term $a_1 f_{z2} b \Delta u$ should be canceled by an adequate choice of W_ψ, but this is hampered by the uncertainty on the control gain b. To this end, b is expressed in terms of an updated parameter and a parameter updating error. Recall from assumption 5 that $|b| \leq \bar{\mu}_{ub}|b_m|$. Therefore,

$$a_1 f_{z2} b \Delta u \leq \mu_{ub} a_1 |f_{z2} b_m \Delta u| \tag{40}$$

As the upper bound $\bar{\mu}_{ub}$ is unknown, it is expressed in terms of updated parameter and parameter updating error. Let $\tilde{\theta}_{ub} = \hat{\theta}_{ub} - \bar{\mu}_{ub}$, where $\tilde{\theta}_{ub}$, $\hat{\theta}_{ub}$ are a parameter estimation error and an updated parameter, and $\hat{\theta}_{ub}$ is provided by an updating mechanism defined later. From the above definition it follows that $\bar{\mu}_{ub} = \hat{\theta}_{ub} - \tilde{\theta}_{ub}$. Substituting into Equation (40) and arranging, yields

$$a_1 f_{z2} b \Delta u \leq \hat{\theta}_{ub} a_1 |f_{z2} b_m \Delta u| - a_1 |f_{z2} b_m \Delta u| \tilde{\theta}_{ub} \tag{41}$$

Substituting into Equation (39), yields

$$\begin{aligned}\dot{V}_z \leq & -k_1 f_{z1}^2 - k_2 f_{z2}^2 - \varphi_1^\top \tilde{\theta}_1 f_{z1} \\ & + f_{z2}\left(f_{z1} + a_1 a_3 \frac{x_1 - x_2}{x_3} + a_1 a_4 \frac{x_2}{x_3} + B_{1d}(-a_2 x_1 - r_{x1}) + B_{1e}\right) \\ & + a_1 f_{z2} b u + \hat{\theta}_{ub} a_1 |f_{z2} b_m \Delta u| - a_1 |f_{z2} b_m \Delta u| \tilde{\theta}_{ub} + f_{z2}(-1) W_\psi \end{aligned} \tag{42}$$

Thus, the right hand side of dV_z/dt is affected by the following terms: (i) the term $\hat{\theta}_{ub} a_1 |f_{z2} b_m \Delta u|$, which is later tackled by properly choosing W_ψ, and (ii) the term $-a_1 |f_{z2} b_m \Delta u| \tilde{\theta}_{ub}$, which is later tackled by defining the quadratic form for $\tilde{\theta}_{ub}$. To this end, if $|f_{z2}|$ is expressed as $|f_{z2}| = f_{z2} sign(f_{z2})$, the resulting expression of W_ψ would contain the $sign(f_{z2})$ signal, so that chattering might occur. Therefore, we notice from definition (32) that

$$|f_{z2}| = f_{z2} sat_{z2} \tag{43}$$

where

$$sat_{z2} = \begin{cases} +1 \text{ for } z_2 \geq C_b \\ (1/C_b)z_2 \text{ for } z_2 \in (-C_b, C_b) \\ -1 \text{ for } z_2 \leq -C_b \end{cases} \tag{44}$$

Therefore, (41) leads to

$$a_1 f_{z2} b \Delta u + f_{z2}(-1)W_\psi \leq f_{z2}\left(\hat{\theta}_{ub} sat_{z2} a_1 |b_m \Delta u| - W_\psi\right) - a_1 |f_{z2} b_m \Delta u| \tilde{\theta}_{ub} \tag{45}$$

we choose

$$W_\psi = a_1 \hat{\theta}_{ub} sat_{z2} |b_m \Delta u| \tag{46}$$

substituting into Equation (24), (25) gives the auxiliary system

$$\dot{\psi}_1 = -K_{\psi 1}\psi_1 + \psi_2 \tag{47a}$$

$$\dot{\psi}_2 = -K_{\psi 2}\psi_2 + a_1 \hat{\theta}_{ub} sat_{z2} |b_m \Delta u| \tag{47b}$$

$$K_{\psi 1} > 1/2, \;\; K_{\psi 2} > 1/2 \tag{47c}$$

where sat_{z2} is defined in (44). Substituting (46) into Equation (45) yields $a_1 f_{z2} b \Delta u + f_{z2}(-1)W_\psi \leq -a_1 |f_{z2} b_m \Delta u| \tilde{\theta}_{ub}$. Substituting this into (42) gives

$$\begin{aligned} \dot{V}_z \leq & -k_1 f_{z1}^2 - k_2 f_{z2}^2 - \varphi^\top \tilde{\theta}_1 f_{z1} \\ & + f_{z2}\left(a_1 a_3 \frac{x_1 - x_2}{x_3} + a_1 a_4 \frac{x_2}{x_3} + B_{1d}(-r_{x1} - a_2 x_1) + f_{z1} + B_{1e}\right) \\ & + a_1 f_{z2} b_\delta b_m u - |a_1 f_{z2} b_m \Delta u| \tilde{\theta}_{ub} \end{aligned} \tag{48}$$

The term

$$f_{z2}\left(a_1 a_3 \frac{x_1 - x_2}{x_3} + a_1 a_4 \frac{x_2}{x_3} + B_{1d}(-r_{x1} - a_2 x_1) + f_{z1} + B_{1e}\right)$$

should be canceled by the input signal u appearing in the term $a_1 f_{z2} b_\delta b_m u$, but this is hampered by the lack of knowledge on (i) a_3, μ_{lb}, a_4, a_2 and (ii) the control gain term b_δ and its lower bound $\bar{\mu}_{lb}$. Therefore, we need to express these terms as function of b_δ, updated parameters and parameter updating errors. To begin, the term comprising $x_1 - x_2$ can be expressed as

$$f_{z2} a_1 a_3 \frac{x_1 - x_2}{x_3} \leq \mu_{lb} a_1 \frac{|f_{z2}(x_1 - x_2)|}{x_{3m}} \theta_3, \;\; \theta_3 = \frac{a_3}{\mu_{lb}}\left(1 + \frac{max(\delta_{x3})}{min(x_3)}\right) \tag{49}$$

Let $\tilde{\theta}_3 = \hat{\theta}_3 - \theta_3$, where $\tilde{\theta}_3$ and $\hat{\theta}_3$ are a parameter updating error and an updated parameter, respectively, and $\hat{\theta}_3$ is provided by an updating mechanism defined later. From the above definition it follows that θ_3 can be expressed as $\theta_3 = \hat{\theta}_3 - \tilde{\theta}_3$. Substituting into Equation (49) and using property $\mu_{lb} \leq |b_\delta|$ from Assumption 5, yields

$$f_{z2} a_1 a_3 \frac{x_1 - x_2}{x_3} \leq a_1 \frac{|b_\delta f_{z2}(x_1 - x_2)|}{x_{3m}} \hat{\theta}_3 - \mu_{lb} a_1 \frac{|f_{z2}(x_1 - x_2)|}{x_{3m}} \tilde{\theta}_3 \tag{50}$$

In a similar way

$$f_{z2}a_1\frac{a_4}{x_3}x_2 \leq a_1\frac{|b_\delta x_2 f_{z2}|}{x_{3m}}\hat{\theta}_4 - \mu_{lb}a_1\frac{|x_2 f_{z2}|}{x_{3m}}\tilde{\theta}_4 \tag{51}$$
where $\tilde{\theta}_4 = \hat{\theta}_4 - \theta_4$

$$f_{z2}B_{1d}(-r_{x1}) \leq |b_\delta f_{z2}B_{1d}|\bar{r}_{x1}\hat{\theta}_{rx1} - \mu_{lb}|f_{z2}B_{1d}|\bar{r}_{x1}\tilde{\theta}_{rx1} \tag{52}$$
where $\tilde{\theta}_{rx1} = \hat{\theta}_{rx1} - \theta_{rx1}$

$$f_{z2}B_{1d}(-a_2x_1) \leq |b_\delta f_{z2}B_{1d}x_1|\hat{\theta}_2 - \mu_{lb}|f_{z2}B_{1d}x_1|\tilde{\theta}_2 \tag{53}$$
where $\tilde{\theta}_2 = \hat{\theta}_2 - \theta_2$

$$f_{z2}(f_{z1}+B_{1e}) \leq |b_\delta f_{z2}(f_{z1}+B_{1e})|\hat{\theta}_{ilb} - \mu_{lb}|f_{z2}(f_{z1}+B_{1e})|\tilde{\theta}_{ilb} \tag{54}$$
where $\tilde{\theta}_{ilb} = \hat{\theta}_{ilb} - 1/\mu_{lb}$

Notice that the expressions (50)–(54) contain $|f_{z2}|$. If it is expressed as $|f_{z2}| = f_{z2}sign(f_{z2})$, the resulting control law would contain the $sign(f_{z2})$ signal, so that input chattering might occur. Thus, we use the expression (43), so that

$$|b_\delta f_{z2}| = b_\delta f_{z2}sgn(b_\delta)sat_{z2} \tag{55}$$

Substituting (50)–(54) into Equation (48) and using (55), yields

$$\begin{aligned}
\dot{V}_z \leq{}& -k_1 f_{z1}^2 - k_2 f_{z2}^2 - \varphi_1^\top\tilde{\theta}_1 f_{z1} \\
&+ b_\delta f_{z2}\left(sgn(b_\delta)sat_{z2}\left(a_1\frac{|x_1-x_2|}{x_{3m}}\hat{\theta}_3 + a_1\frac{x_2}{x_{3m}}\hat{\theta}_4 + |B_{1d}|\bar{r}_{x1}\hat{\theta}_{rx} + |B_{1d}x_1|\hat{\theta}_2 + |f_{z1}+B_{1e}|\hat{\theta}_{ilb}\right) + a_1 b_m u\right) \\
&- \mu_{lb}|f_{z2}|a_1\frac{|x_1-x_2|}{x_{3m}}\tilde{\theta}_3 - \mu_{lb}a_1|f_{z2}|\frac{|x_2|}{x_{3m}}\tilde{\theta}_4 - \mu_{lb}|f_{z2}B_{1d}|\bar{r}_{x1}\tilde{\theta}_{rx1} \\
&- \mu_{lb}|f_{z2}B_{1d}x_1|\tilde{\theta}_2 - \mu_{lb}|f_{z2}(f_{z1}+B_{1e})|\tilde{\theta}_{ilb} - |a_1 f_{z2}b_m\Delta u|\tilde{\theta}_{ub}
\end{aligned} \tag{56}$$

Thus, the right hand side of dV_z/dt is affected by the following terms: (i) the term $-k_1 f_{z1}^2 - k_2 f_{z2}^2$ which leads to convergence of f_{z1}^2 and f_{z2}^2 to zero, as shown in the convergence theorem in Section 3.2; (ii) the terms involving $\tilde{\theta}_1$, $\tilde{\theta}_3$, $\tilde{\theta}_4$, $\tilde{\theta}_{rx1}$, $\tilde{\theta}_2$, $\tilde{\theta}_{ilb}$, $\tilde{\theta}_{ub}$, which are later tackled by properly defining their quadratic forms and the update laws, what is shown later in the analysis of dV/dt; and (iii) the remaining term

$$b_\delta f_{z2}\left(sgn(b_\delta)sat_{z2}\left(a_1\frac{|x_1-x_2|}{x_{3m}}\hat{\theta}_3 + a_1\frac{x_2}{x_{3m}}\hat{\theta}_4 + |B_{1d}|\bar{r}_{x1}\hat{\theta}_{rx} + |B_{1d}x_1|\hat{\theta}_2 + |f_{z1}+B_{1e}|\hat{\theta}_{ilb}\right)\right)$$

which is tackled by properly defining the control law, so that the u signal cancels it:

$$\begin{aligned}
&b_\delta f_{z2}\left(sgn(b_\delta)sat_{z2}\left(a_1\frac{|x_1-x_2|}{x_{3m}}\hat{\theta}_3 + a_1\frac{x_2}{x_{3m}}\hat{\theta}_4 + |B_{1d}|\bar{r}_{x1}\hat{\theta}_{rx} + |B_{1d}x_1|\hat{\theta}_2 + |f_{z1}+B_{1e}|\hat{\theta}_{ilb}\right) + a_1 b_m u\right) \\
&\quad = 0
\end{aligned} \tag{57}$$

Solving (57) for u yields

$$u = \frac{-1}{b_m a_1} sgn(b_\delta) sat_{z2} \left(a_1 \frac{|x_1 - x_2|}{x_{3m}} \hat{\theta}_3 + |B_{1d}| \bar{r}_{x1} \hat{\theta}_{rx1} + |B_{1d} x_1| \hat{\theta}_2 + |f_{z1} + B_{1e}| \hat{\theta}_{ilb} + a_1 \frac{|x_2|}{x_{3m}} \hat{\theta}_4 \right) \quad (58)$$

with $k_{1b} = C_b$

Remark 2. *From the control law (58), it follows that the value of the control signal u depends on (i) the nutrient concentration in the upper CSTR, that is, $x_1 = P_e$; (ii) the nutrient concentration in the lower CSTR, that is, $x_2 = P_i$, (iii) the measurement of the liquid volume in the lower CSTR, that is, $x_{3m} = V_{lm}$; (iv) the reaction rate term $\bar{r}_{x1}$ and $d\bar{r}_{x1}/dx_1$, which are functions of x_1; and (v) the desired output y_d and its time derivative dy_d/dt, provided by Equation (5). Therefore, the input signal v also depends on x_1, x_2, x_3, y_d, as it is a saturation function of u according to expression (4).*

Remark 3. *In practical implementation of the developed controller, the flow valve manipulates the flow of nutrient solution Q_{ad}, using the signal v (4) and the control law (58), so as to drive z_1 to Ω_{z1}, $\Omega_{z1} = \{z_1 : |z_1| \leq C_b\}$ and z_2 to Ω_{z2}, $\Omega_{z2} = \{z_2 : |z_2| \leq C_b\}$. Also, as v depends on x_1, x_2, x_3 according to Remark 2, the flow valve uses their measurement.*

Remark 4. *A closed loop is generated by the application of the developed controller, because the saturated signal v depends on x_1, x_2, x_3 according to remark 2, and x_1, x_2 and x_3 depend on v according to model (1)–(3).*

Substituting u (58) into Equation (57) yields

$$\dot{V}_z \leq -k_1 f_{z1}^2 - k_2 f_{z2}^2 - \varphi_1^\top \tilde{\theta}_1 f_{z1} - \mu_{lb} |f_{z2}| a_1 \frac{|x_1 - x_2|}{x_{3m}} \tilde{\theta}_3 - \mu_{lb} a_1 |f_{z2}| \frac{|x_2|}{x_{3m}} \tilde{\theta}_4 - \mu_{lb} |f_{z2} B_{1d}| \bar{r}_{x1} \tilde{\theta}_{rx1} - \mu_{lb} |f_{z2} B_{1d} x_1| \tilde{\theta}_2 - \mu_{lb} |f_{z2} (f_{z1} + B_{1e})| \tilde{\theta}_{ilb} - |a_1 f_{z2} b_m \Delta u| \tilde{\theta}_{ub} \quad (59)$$

In order to tackle the effect of the parameter updating errors in $\dot{V}_z$, the overall Lyapunov function is defined as

$$V(\bar{x}) = V_z + V_\theta, \quad (60)$$

where V_θ is the sum of the quadratic forms for the parameter updating errors, and it is defined as

$$V_\theta = (1/2) \tilde{\theta}_1^\top \Gamma_1^{-1} \tilde{\theta}_1 + (1/2) \mu_{lb} \gamma_3^{-1} \tilde{\theta}_3^2 + (1/2) \mu_{lb} \gamma_4^{-1} \tilde{\theta}_4^2 + (1/2) \mu_{lb} \gamma_{rx1}^{-1} \tilde{\theta}_{rx1}^2 + (1/2) \mu_{lb} \gamma_2^{-1} \tilde{\theta}_2^2 + (1/2) \mu_{lb} \gamma_{ilb}^{-1} \tilde{\theta}_{ilb}^2 + (1/2) \gamma_{ub}^{-1} \tilde{\theta}_{ub}^2 \quad (61)$$

The vector of closed loop state variables is $\bar{x} = \left[z_1, \ z_2, \ \tilde{\theta}_1, \ \tilde{\theta}_3, \ \tilde{\theta}_4, \ \tilde{\theta}_{rx1}, \ \tilde{\theta}_2, \ \tilde{\theta}_{ilb}, \ \tilde{\theta}_{ub} \right]^\top$. Differentiating (60) with respect to time yields

$$\dot{V} = \dot{V}_z + \dot{V}_\theta \quad (62)$$

Differentiating V_θ (61) with respect to time yields

$$\dot{V}_\theta = \tilde{\theta}_1^\top \Gamma^{-1} \dot{\hat{\theta}}_1 + \mu_{lb} \gamma_3^{-1} \tilde{\theta}_3 \dot{\hat{\theta}}_3 + \mu_{lb} \gamma_4^{-1} \tilde{\theta}_4 \dot{\hat{\theta}}_4 + \mu_{lb} \gamma_{rx1}^{-1} \tilde{\theta}_{rx1} \dot{\hat{\theta}}_{rx1} + \mu_{lb} \gamma_2^{-1} \tilde{\theta}_2 \dot{\hat{\theta}}_2 + \mu_{lb} \gamma_{ilb}^{-1} \tilde{\theta}_{ilb} \dot{\hat{\theta}}_{ilb} + \gamma_{ub}^{-1} \tilde{\theta}_{ub} \dot{\hat{\theta}}_{ub} \quad (63)$$

Substituting (63) and (59) into Equation (62) yields

$$\dot{V} \le -k_1 f_{z1}^2 - k_2 f_{z2}^2 - \varphi_1^\top \tilde{\theta}_1 f_{z1} - \mu_{lb}|f_{z2}|a_1\frac{|x_1-x_2|}{x_{3m}}\tilde{\theta}_3 - \mu_{lb}a_1|f_{z2}|\frac{|x_2|}{x_{3m}}\tilde{\theta}_4 - \mu_{lb}|f_{z2}B_{1d}|\bar{r}_{x1}\tilde{\theta}_{rx1}$$
$$-\mu_{lb}|f_{z2}B_{1d}x_1|\tilde{\theta}_2 - \mu_{lb}|f_{z2}(f_{z1}+B_{1e})|\tilde{\theta}_{ilb} - |a_1 f_{z2} b_m \Delta u|\tilde{\theta}_{ub}$$
$$+\tilde{\theta}_1^\top \Gamma^{-1}\dot{\hat{\theta}}_1 + \mu_{lb}\gamma_3^{-1}\tilde{\theta}_3\dot{\hat{\theta}}_3 + \mu_{lb}\gamma_4^{-1}\tilde{\theta}_4\dot{\hat{\theta}}_4 + \mu_{lb}\gamma_{rx1}^{-1}\tilde{\theta}_{rx1}\dot{\hat{\theta}}_{rx1} + \mu_{lb}\gamma_2^{-1}\tilde{\theta}_2\dot{\hat{\theta}}_2$$
$$+\mu_{lb}\gamma_{ilb}^{-1}\tilde{\theta}_{ilb}\dot{\hat{\theta}}_{ilb} + \gamma_{ub}^{-1}\tilde{\theta}_{ub}\dot{\hat{\theta}}_{ub} \quad (64)$$

In order to cancel the effect of the term

$$-\varphi_1^\top \tilde{\theta}_1 f_{z1} - \mu_{lb}|f_{z2}|a_1\frac{|x_1-x_2|}{x_{3m}}\tilde{\theta}_3 - \mu_{lb}a_1|f_{z2}|\frac{|x_2|}{x_{3m}}\tilde{\theta}_4 - \mu_{lb}|f_{z2}B_{1d}|\bar{r}_{x1}\tilde{\theta}_{rx1}$$
$$-\mu_{lb}|f_{z2}B_{1d}x_1|\tilde{\theta}_2 - \mu_{lb}|f_{z2}(f_{z1}+B_{1e})|\tilde{\theta}_{ilb} - |a_1 f_{z2} b_m \Delta u|\tilde{\theta}_{ub}$$

the update laws are chosen as

$$\dot{\hat{\theta}}_1 = \Gamma_1 \varphi_1 f_{z1} \quad (65)$$
$$\dot{\hat{\theta}}_{rx1} = \gamma_{rx1}|B_{1d}|\bar{r}_{x1}|f_{z2}| \quad (66)$$
$$\dot{\hat{\theta}}_3 = \gamma_3 a_1 \frac{|x_1-x_2|}{x_{3m}}|f_{z2}| \quad (67)$$
$$\dot{\hat{\theta}}_{ilb} = \gamma_{ilb}|f_{z1}+B_{1e}||f_{z2}| \quad (68)$$
$$\dot{\hat{\theta}}_{ub} = \gamma_{ub}a_1|b_m\Delta u||f_{z2}| \quad (69)$$
$$\dot{\hat{\theta}}_4 = \gamma_4 a_1 \frac{|x_2|}{x_{3m}}|f_{z2}| \quad (70)$$
$$\dot{\hat{\theta}}_2 = \gamma_2|B_{1d}||x_1 f_{z2}| \quad (71)$$

where Γ_1 is 2×2 diagonal matrix whose diagonal entries are user-defined, positive, and constant, whereas γ_{rx1}, γ_3, γ_{ilb}, γ_{ub}, γ_4, and γ_2 are user-defined positive constants.

Substituting the update laws (65) to (71) into Equation (64) and arranging yields

$$\dot{V} \le -k_1 f_{z1}^2 - k_2 f_{z2}^2 \quad (72)$$

Remark 5. *The formulated controller comprises (i) the control law (58); (ii) the update laws (65) to (71); and (iii) the auxiliary system (47a), (47b). The signals involved therein are (i) z_1 (6), f_{z1} (11), sat_{z1} (14), z_2 (22), f_{z2} (32), sat_{z2} (44), φ_1 (19), B_{1d} (27), B_{1e} (28); (ii) the desired output y_d, provided by model (5), according to subsection 2.2; (iii) the input error Δu (37), which involves u (58) and v (4), (iv) the constants k_1, k_2, $K_{\psi 1} > 1/2$, $K_{\psi 2} > 1/2$, which are user-defined and positive; (v) the user-defined positive constant C_b, which is the width of the residual set Ω_{eo} defined in Section 2.3; and (vi) the constant $k_{1b} = C_b$.*

Remark 6. *In the controller development, a new treatment of the $b\Delta u$ term is proposed, and the main tasks of this treatment are (i) the term $b\Delta u f_{z2}$ is expressed in terms of its upper bound (40); (ii) as such upper bound is unknown, it is expressed in terms of parameter updating error and update parameter (41); and (iii) the update law (69) is defined so as to obtain adequate time derivative of the overall Lyapunov-like function.*

Remark 7. *The resulting auxiliary system (47a), (47b) is quite different with respect to the current ones (see [11–13]): it involves a saturation-like function of the z_2 signal; it involves the updated parameter $\hat{\theta}_{ub}$, which is function of f_{z2} and Δu; the Δu signal is in absolute value.*

Remark 8. *The modified tracking error z_1 asymptotically converges to the compact set Ω_{z1}, whose width C_b is user-defined, so that it does not depend on the bonds of external disturbances,* *model coefficients, or model terms. Consequently, the convergence of z_1 is achieved without requiring knowledge on these bounds. This is in contrast to common robust adaptive backstepping designs (see [11,12,14]), where the convergence region depends on such kind of bounds, so that the convergence of the modified error to a compact set of user-defined size requires the knowledge on such bounds.*

Remark 9. *Some remarkable features of the formulated controller are (i) the control law, the update laws and the auxiliary system are function of modified error z_1 instead of the regular tracking error $e = x_1 - y_d$, and (ii) saturation functions of the tracking error are used instead of discontinuous functions, in order to avoid undesired chattering.*

3.2. Boundedness and Convergence Analysis

Theorem 1 (Boundedness of the closed loop signals)**.** *Consider the model (1) to (3), subject to input constraint (4) and Assumptions 1 to 5. If the control law (58), update laws (65) to (71) and auxiliary system (47a), (47b) are applied, then (Ti) the signals z_1, z_2, $\hat{\theta}_1$, $\hat{\theta}_3$, $\hat{\theta}_4$, $\hat{\theta}_{rx1}$, $\hat{\theta}_2$, $\hat{\theta}_{ilb}$, $\hat{\theta}_{ub}$ are bounded; (Tii) the signals $b_m u$, $b_m \Delta u$ are bounded.*

Proof. Arranging and integrating Equation (72), yields

$$V + k_1 \int_{to}^{t} f_{z1} dt + k_2 \int_{to}^{t} f_{z2} dt \leq V(\bar{x}_{to}) \tag{73}$$

Therefore, $V(\bar{x}) \leq V(\bar{x}_{to})$, so that $V \in L_\infty$. In view of (60), one further obtains $V_z \in L_\infty$, $V_\theta \in L_\infty$. Further, considering definitions of V_z (29), V_{z1} (9), V_{z2} (30) one obtains $z_1 \in L_\infty$, $z_2 \in L_\infty$. Further, considering $V_\theta \in L_\infty$ and definition (61), it follows that $\hat{\theta}_1$, $\hat{\theta}_3$, $\hat{\theta}_4$, $\hat{\theta}_{rx1}$, $\hat{\theta}_2$, $\hat{\theta}_{ilb}$, $\hat{\theta}_{ub}$ are bounded. This completes the proof of Ti.

Considering Equation (58), and the boundedness of all the closed loop signals involved therein, one obtains that $b_m v$ is bounded. Further, considering definition (37), the bounded nature of v and b_m stated in assumption 2.5, it follows that $b_m \Delta u$ is bounded. This completes the proof of Tii. □

Theorem 2 (Convergence of signals z_1, z_2 and e)**.** *Consider the model (1) to (3), subject to input constraint (4) and Assumptions 1–5. If the control law (58), update laws (65) to (71) and auxiliary system (47a), (47b) are applied, then (Ti) the signal z_1 converges asymptotically to Ω_{z1}, $\Omega_{z1} = \{z_1 : |z_1| \leq C_b\}$; (Tii) the signal z_2 converges asymptotically to Ω_{z2}, $\Omega_{z2} = \{z_2 : |z_2| \leq C_b\}$; (Tiii) if Δu vanishes, then $e = x_1 - y_d$ converges asymptotically to Ω_{eo}, $\Omega_{eo} = \{e : |e| \leq C_b\}$; (Tiv) $e = x_1 - y_d$ converges asymptotically to Ω_e,*

$$\Omega_e = \left\{ e : |e| \leq C_b + \frac{1}{\sqrt{2k_o \beta_2 (K_{\psi 2} - 1/2)}} \sqrt{\sup_{0 \leq s \leq t} (W_\psi^2)} \right\} \tag{74}$$

$$W_\psi = \hat{\theta}_{ub} b_m \Delta u$$

$$k_o = 2 \min\{K_{\psi 1} - 1/2,\ \beta_1 (K_{\psi 2} - 1/2)\} \tag{75}$$

where β_1, β_2 are constants that satisfy $1 = \beta_1 + \beta_2$, $\beta_1 \in (0,\ 1)$, $\beta_2 \in (0,\ 1)$.

Proof. From Equation (73) it follows that

$$k_1 \int_{to}^{t} f_{z1} dt \leq V(\bar{x}_{to}),\ k_2 \int_{to}^{t} f_{z2} dt \leq V(\bar{x}_{to}) \tag{76}$$

so that $f_{z1} \in L_1$. Applying the Barbalat's lemma [27], yields $\lim_{t \to +\infty} f_{z1}^2 = 0$. Furthermore, considering the definition of f_{z1} (11), it follows that z_1 converges asymptotically to Ω_{z1}. This completes the proof of Ti. From Equation (76) if follows that $f_{z2} \in L_1$. Applying the

Barbalat's lemma [27], yields $\lim_{t\to+\infty} f_{z2}^2 = 0$. Furthermore, considering the definition of f_{z2} (32), it follows that z_2 converges asymptotically to Ω_{z2}. This completes the proof of Tii.

From the definition of z_1 (6), it follows that $e = x_1 - y_d$ can be expressed as

$$e = z_1 + \psi_1 \tag{77}$$

From Equations (47a) and (47b), it follows that if Δu vanishes, then ψ_1 and ψ_2 converge to zero. From (77), accounting for the convergence of z_1 to Ω_{z1}, it follows that e converges asymptotically to Ω_{eo}. This completes the proof of Tiii.

We choose the quadratic form

$$V_\psi = (1/2)\psi_1^2 + (1/2)\psi_2^2 \tag{78}$$

Differentiating with respect to time, yields $\dot{V}_\psi = \psi_1\dot{\psi}_1 + \psi_2\dot{\psi}_2$. Substituting the auxiliary system (47a), (47b) and arranging, yields $\dot{V}_\psi = -K_{\psi1}\psi_1^2 + \psi_1\psi_2 - K_{\psi2}\psi_2^2 + \psi_2 W_\psi$. Factorizing, yields

$$\dot{V}_\psi \le -(K_{\psi1} - 1/2)\psi_1^2 - \beta_1(K_{\psi2} - 1/2)\psi_2^2 + \frac{1}{4\beta_2(K_{\psi2} - 1/2)}W_\psi^2$$

Arranging yields

$$\dot{V}_\psi \le -k_o V_\psi + \frac{1}{4\beta_2(K_{\psi2} - 1/2)}\sup_{0\le s\le t}(W_\psi^2)$$

where k_o is a positive constant (75). Therefore,

$$V_\psi \le V_{\psi o}e^{-k_o t} + \frac{1}{4k_o\beta_2(K_{\psi2} - 1/2)}\sup_{0\le s\le t}(W_\psi^2)$$

Using the definition of V_ψ (78), we get

$$|\psi_1| \le \sqrt{2V_{\psi o}e^{-k_o t} + \frac{1}{2k_o\beta_2(K_{\psi2} - 1/2)}\sup_{0\le s\le t}(W_\psi^2)}$$

From this it follows that ψ_1 converges asymptotically to $\Omega_{\psi1}$,

$$\Omega_{\psi1} = \left\{\psi_1 : |\psi_1| \le \frac{1}{\sqrt{2k_o\beta_2(K_{\psi2} - 1/2)}}\sqrt{\sup_{0\le s\le t}(W_\psi^2)}\right\}$$

From Equation (77), the above result and result Ti, it follows that $e = x_1 - y_d$ converges asymptotically to the compact set Ω_e (74). This completes the proof of Tiv. □

Remark 10. *The parameter updating errors are bounded despite input saturation, so that excessive increase of updated parameters is avoided.*

Remark 11. *From result Tiv of Theorem 2, it can be observed that the bound of the steady tracking error can be made small by choosing large values of $K_{\psi1}$, $K_{\psi2}$.*

4. Simulation Example

Consider the aeroponic system described in Appendix A whose model is given by Equations (1)–(3), with input constraint (4), Assumptions 1–5, and control goal and desired output y_d stated in Sections 2.2 and 2.3. The control law, the update laws, the auxiliary system, and their parameters and signals are stated in Remark 5. At what follows, the values of x_1, x_2 and x_3 are generated through the model (1) to (3) with specific parameter values. These values of x_1, x_2 and x_3 are used by the controller, but the model parameters and upper or lower bounds are not.

The input saturation values are $u_{min} = 0$ and

$$u_{max} = \begin{cases} 350 \text{ L/day for } t \leq 7 \text{ days} \\ 10.944 \text{ L/day otherwise} \end{cases}. \tag{79}$$

The input value $Q_{ad} = 0$ is used until x_1 reaches the value 70 mg/L, so that the controller is started at $t = 4$ days. The parameters of the reference model (5) are chosen as $a_{m1} = 40$ and

$$W_{ref} = \begin{cases} 75 \text{ mg/L for } t \leq 7 \text{ days} \\ 80 \text{ mg/L otherwise} \end{cases}, \tag{80}$$

whereas the desired width of the convergence region is chosen to be $C_b = 0.4$. The user-defined parameters of the control law, update laws and auxiliary system are chosen as

$$k_1 = \begin{cases} 8 \text{ for } t \leq 7 \text{ days} \\ 2 \text{ otherwise} \end{cases}, \tag{81}$$

$k_2 = 0.005$, $k_{\psi 1} = 20 > 1/2$, $k_{\psi 2} = 20 > 1/2$, $\gamma_{1,1} = 8 \times 10^{-4}$, $\gamma_{1,2} = 0.008$, $\gamma_{rx1} = 4 \times 10^{-7}$, $\gamma_3 = 0.004$, $\gamma_{ilb} = 4 \times 10^{-7}$, $\gamma_{ub} = 4 \times 10^{-9}$, $\gamma_4 = 4 \times 10^{-4}$.

We consider the measurement noise for x_3 in the control gain b, such that b satisfies the first condition of assumption 2.5, with

$$b_m = \frac{P_{ad} - x_2}{x_{3m}}, \quad b_\delta = \frac{x_{3m}}{x_3}$$

Due to the controller starting at $t = 4$ days and the change of W_{ref} at $t = 7$ days, the system behavior is separated in the time intervals $[4 \quad 7)$ and $[7 \quad \infty)$ days. For $t \in [4 \quad 7)$ days:

- all the closed loop signals are bounded (see Figures 1–3).
- the signal z_1 is near Ω_{z1} at initial time ($z_{1to} \approx -0.7$), it enters Ω_{z1} at 6.72 days and it remains inside until $t = 7$ days (Figure 1d).
- the updated parameters remain bounded, and its change is not excessive; $\hat{\theta}_{1,1}$, $\hat{\theta}_{1,2}$ change when $z_1 \notin \Omega_{z1}$, and remain constant otherwise (Figure 3).
- input signal v: for $t \in [4 \quad 5.09] \cup [6.52 \quad 6.6]$ days it exhibits reiterated saturation at its lower bound, with only one moment of saturation at its upper bound (at $t = 4.89$ days approx); during other moments it exhibits changing behavior (Figure 2c,d).

For $t \geq 7$ days:

- all the closed loop signals are bounded (see Figures 1–3).
- the signal z_1 is inside Ω_{z1} at $t = 7$ days ($z_{1to} \approx -0.39$), it leaves, it enters Ω_{z1} at 7.55 days approx. and it remains inside afterwards (Figure 1d).
- the updated parameters remain bounded, $\hat{\theta}_{1,1}$, $\hat{\theta}_{1,2}$ are constant when $z_1 \in \Omega_{z1}$, and the other updated parameters are constant when $z_2 \in \Omega_{z2}$ (Figure 3).
- input signal v: for $t \in [7.0 \quad 7.57]$ days, it remains saturated at its upper bound; for $t \in (7.57 \quad 8.44]$ days, it exhibits saturation at its lower bound with some few saturation

at its upper bounds; for $t > 8.44$ days, it exhibits reiterated saturation at both its upper and lower bounds (Figure 2c,d).

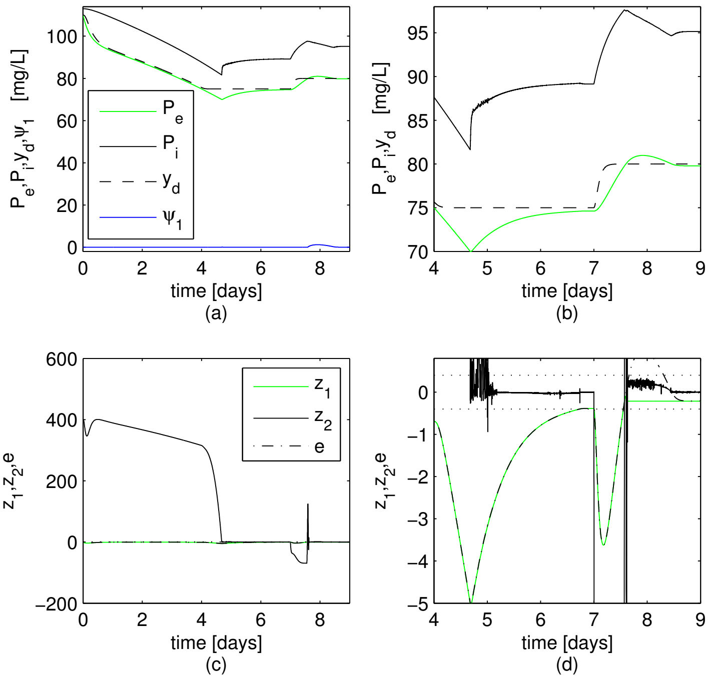

Figure 1. Simulation results for the system signals under the formulated controller. (**a**) Time course of the states P_e, P_i, desired output y_d, and auxiliary signal ψ_1. (**b**) Detail of the time course of P_e, P_i, and y_d. (**c**) Time course of the modified tracking error z_1, signal z_2 and tracking error e. (**d**) Detail of the signals z_1, z_2, and e; the horizontal dotted lines represent C_b and $-C_b$.

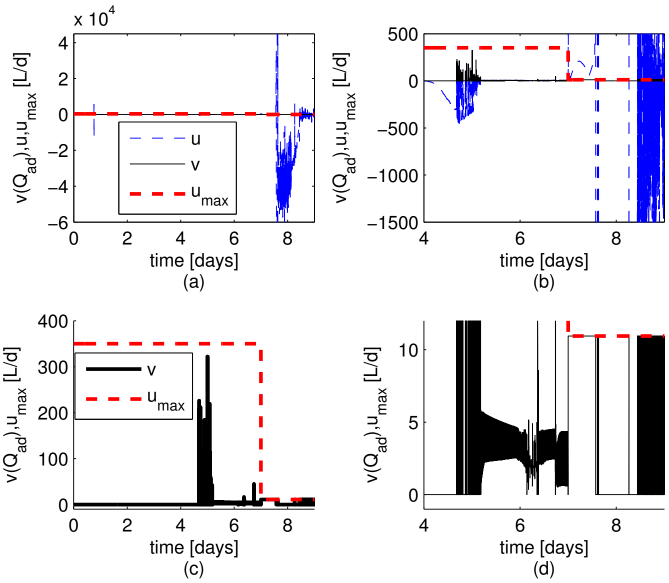

Figure 2. Simulation results for the system signals under the formulated controller. (**a**) Time course of the non-saturated input u and the saturated input v. (**b**) Detail of the time course of signals u and v. (**c**) Time course of the saturated input v. (**d**) Detail of the signal v.

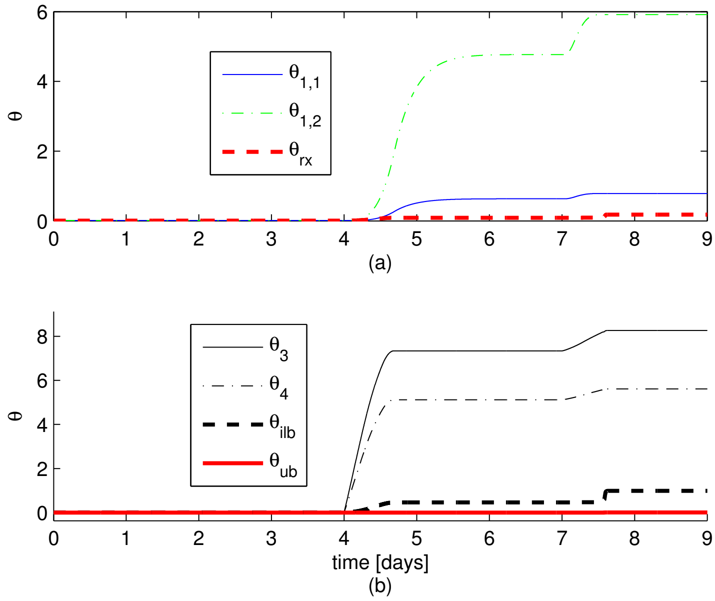

Figure 3. Simulation results for the system signals under the formulated controller. (**a**) Time course of the updated parameters $\hat{\theta}_{1,1}$, $\hat{\theta}_{1,2}$, $\hat{\theta}_{rx}$. (**b**) Time course of the updated parameters $\hat{\theta}_3$, $\hat{\theta}_4$, $\hat{\theta}_{ilb}$, $\hat{\theta}_{ub}$.

5. Conclusions

In this paper, an adaptive backstepping controller was developed for a second order plant model subject to unknown model parameters, unknown reaction rate, unknown varying control gain, and input saturation. The controller provides important contributions to adaptive control design for second-order models with input saturation:

- It tackles the combined effect of constrained control input and unknown varying control gain with unknown bounds. To this end, a new auxiliary system is proposed.
- The modified tracking error asymptotically converges to a compact set whose width is user-defined and it does not depend on bounds of either external disturbances, model terms or parameters. Recall that in common robust backstepping designs, the tracking error converges to a compact set whose width depends on such kind of bounds, so that these bounds are required in order to obtain the expected width.

Other important features of the controller and closed loop system are

- the model coefficients, and upper and lower bounds of model terms are not required to be known, except a_1;
- the exact value of the reaction rate term r_{x1} is not required to be known;
- the control gain b is varying and unknown, although it can be expressed as $b = b_\delta b_x$, where b_x is known and b_δ is unknown;
- discontinuous functions are not used in the control law, update laws and auxiliary system; instead, saturation type functions are used; and
- the boundedness of the updated parameters is ensured in the presence of input saturation, so that excessive parameter increase is avoided.

Significant improvements were made to the control design in order to tackle the unknown varying nature of the control gain b and the input saturation. Dead zone radially unbounded functions were used. As the gain b appears in the $b\Delta$ term, the design of the auxiliary system must be modified.

The developed controller design can be applied to other second order nonlinear systems as the mathematical manipulations are provided.

Author Contributions: Conceptualization, A.R.; methodology, A.R.; validation, F.E.H. and J.E.C.-B.; formal analysis, A.R., F.E.H., and J.E.C.-B.; investigation, A.R.; writing—original draft preparation, A.R.; writing–review and editing, A.R., F.E.H., and J.E.C.-B.; visualization, F.E.H. and J.E.C.-B. All authors have read and agreed to the published version of the manuscript.

Funding: The work of A. Rincón was supported by Universidad Católica de Manizales. The work of F.E. Hoyos and J.E. Candelo-Becerra was supported by Universidad Nacional de Colombia—Sede Medellín.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A. Hydroponic System and Formulation of the Mass Balance Model

The hydroponic system of the work in [28] comprises three vertical cultivation beds with grown tomatoes, a nutrient solution tank (mixing tank), a closed nutrient solution circulation system, and a lighting system. An electric pump drives a fraction of the nutrient solution from the mixing tank to the cultivation beds, and the drainage solution is conveyed back to the mixing tank. The mixing tank is eventually replenished with new nutrient solution in order to maintain a high EC, near 1.5 mS/cm. The nutrient solution in the mixing tank exhibits decrease of electrical conductivity (EC) and concentration of major ions during time periods with no addition of fresh nutrient solution, whereas there is an increase of EC during addition of nutrient solution [28].

We consider the control of concentration of some major ion in the cultivation bed, by manipulation of the flow of fresh nutrient solution to the mixing tank (Q_{ad}), considering constant flows Q_i and Q_e. To this end, the mass balance model is developed for the concentration of some general nutrient ion, but in the simulation example the $NO_3 - N$ ion is considered, because its behavior is quite similar to that of the electrical conductivity (EC). We consider the mixing tank, the cultivation beds with plants, the flow of addition of fresh nutrient solution to the mixing tank, and the flows of nutrient solution between the mixing tank and the cultivation beds. We assume that the system can be represented by two linked continuous stirred tank reactors (CSTR), see Figure A1:

- The upper CSTR corresponds to the nutrient solution in the cultivation beds. The nutrient concentration is denoted as P_e, the water volume is denoted as V_u, the rate of nutrient removal is denoted as r_{x1}, and the evapotranspiration rate is denoted as Q_{ET}. Nutrient removal occurs via sorption and plant uptake. We assume that the water volume V_u is constant.
- The lower CSTR corresponds to the nutrient solution in the mixing tank. The nutrient concentration is denoted as P_i and the water volume is denoted as V_l. The nutrient solution mixes with the incoming flow, which is in turn the flow leaving the upper CSTR. We assume that V_l is varying because of water evaporation losses and varying nature of flow Q_{ad}.

In addition, Q_i is the flowrate that leaves the lower CSTR and enters the upper CSTR, and Q_e is the flowrate that leaves the upper CSTR and enters the lower CSTR. The outflow (Q_e) is lower than the inflow (Q_i), due to evapotranspiration and constant nature of volume V_u. We assume that flows Q_i and Q_e are constant. The development of the mass balance model gives as result the model (1) to (3).

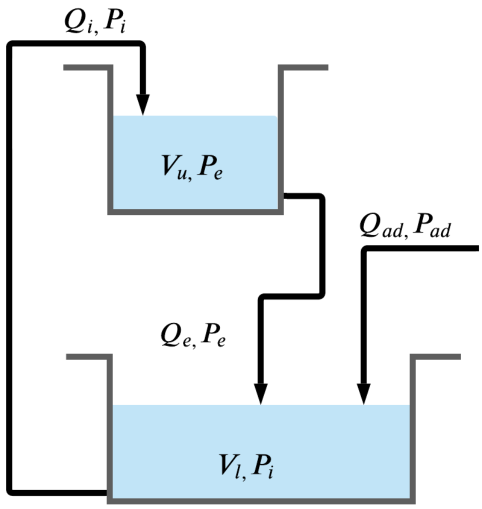

Figure A1. Schematic diagram of the CSTR-based model for the hydroponic system.

This model with the reaction rate expression

$$r_{x1} = \mu_{mx} \frac{P_e}{K + P_e} + \mu_o \tag{A1}$$

was fitted to the experimental data of $NO_3 - N$ shown in [28], time interval 20–33 days, which corresponds to stage II (adaptation). This was performed by minimization of the squares of the errors between experimental and simulated values of $NO_3 - N$ (P_i) [29]. The obtained model parameters are $a_1 = 3.55\,\text{days}^{-1}$, $a_2 = 3.485\,\text{days}^{-1}$, $a_3 = 3.254$ L/day, $a_4 = 0.1066$ L/day, $P_{ad} = 110$ mg/L, $\mu_{mx} = 1051.6$ mg/(Ld), $K = 1509.8$ mg/L, and $\mu_o = 6.903$ mg/(Ld).

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