applied sciences

MDPI

Article

An Enhanced Auxiliary Information-Based EWMA-t Chart for Monitoring the Process Mean

Jen-Hsiang Chen [1] and Shin-Li Lu [2,*]

[1] Department of Information Management, Shih Chien University Kaohsiung Campus, Kaohsiung City 845, Taiwan; jhchen@g2.usc.edu.tw
[2] Department of Industrial Management and Enterprise Information, Aletheia University, New Taipei City 251, Taiwan
* Correspondence: shinlilu@mail.au.edu.tw; Tel.: +886-2-2621-2121

Received: 20 February 2020; Accepted: 22 March 2020; Published: 26 March 2020

Abstract: The exponentially weighted moving average t chart using auxiliary information (AIB-EWMA-t chart) is an effective approach for monitoring small process mean shifts when the process standard deviation is unstable or poorly estimated. To further enhance the sensitivity of the AIB-EWMA-t chart, in this study, we propose an AIB generally weighted moving average (GWMA) t chart (AIB-GWMA-t chart) to monitor the process mean. The existing EWMA-t, GWMA-t, and AIB-EWMA-t charts are special cases of the AIB-GWMA-t chart. Numerical simulation studies indicate that the AIB-GWMA-t chart performs uniformly and substantially better than the EWMA-t and GWMA-t charts in terms of average run length. Moreover, the AIB-GWMA-t chart with large design and adjustment parameters also outperforms the AIB-EWMA-t chart when the correlation coefficients are within a certain range. An illustrative example is provided to highlight the efficiency of the proposed AIB-GWMA-t chart in detecting small process mean shifts.

Keywords: average run length; auxiliary information; EWMA-t chart; GWMA-t chart; statistical process control

1. Introduction

Memory-type control charts, which utilize both current and past information to calculate their plotting statistics, are well known to be sensitive tools in detecting small process mean shifts in statistical process control. One memory-type control chart is the exponentially weighted moving average (EWMA) chart introduced by Roberts [1] to effectively improve the detection ability of the Shewhart control chart when monitoring small process mean shifts. Since then, numerous studies aiming to improve the sensitivity and detection ability of EWMA charts have been proposed, such as Crowder [2], Ng and Case [3], Lucas and Saccucci [4], Steiner [5], and Capizzi and Masarotto [6]. In pioneering work, adopting design and adjustment parameters, Sheu and Lin [7] developed a generally weighted moving average (GWMA) chart and found that it outperforms the classical Shewhart and EWMA charts in detecting small process mean shifts.

Subsequently, the popularity of GWMA-type charts has risen rapidly owing to their various quality characteristics. Sheu and Chiu [8] developed a GWMA chart for monitoring Poisson observations. Sheu and Lu [9] proposed a GWMA chart for which observational data show significant autocorrelation. Lu [10] developed a non-parametric GWMA sign chart to improve detection ability in small process mean shifts. Lu [11] proposed a mixed GWMA-Cumulative Sum (GWMA-CUSUM) chart and its reverse CUSUM-GWMA chart to enhance detection sensitivity over existing counterparts. Other studies of GWMA-type charts include Sheu and Yang [12], Sheu and Hsieh [13], Huang et al. [14], Chakraborty et al. [15], and Aslam et al. [16].

More recently, significant attempts have been made to develop auxiliary information-based (AIB) charts to enhance the detection ability when auxiliary information is lacking. The plotting statistics of AIB charts are mainly based on parameter estimators estimated from the study variable (quality characteristic) along with one or more correlated auxiliary variables. AIB mean estimators such as classical ratio, product, and regression estimators are more accurate than those estimated only using the study variable. The advantages of accurate estimators have encouraged many authors to investigate the designs of AIB charts. Riaz [17] first proposed the AIB-Shewhart chart based on regression estimators and showed that it is more sensitive than the classic Shewhart mean chart for monitoring the process mean. Ahmad et al. [18] and Riaz [19] proposed AIB-Shewhart charts using ratio-type and location estimators, respectively. For monitoring small process mean shifts, Abbas et al. [20] proposed the AIB-EWMA chart and showed that it outperforms both the classical EWMA chart and the AIB-Shewhart chart. Haq and Abidin [21] further enhanced the sensitivity of the AIB-EWMA chart by developing the AIB-GWMA chart for effectively monitoring the process mean and indicated that existing AIB-Shewhart and AIB-EWMA charts are special cases of the proposed chart.

The EWMA-type and GWMA-type charts mentioned above use the sample mean to monitor process mean shifts. A practical issue is that an out-of-control signal detection is caused by a change in the process standard deviation instead of process mean shifts. To overcome this problem, Zhang et al. [22] proposed using EWMA-t charts to monitor the process mean, finding them to be robust against changes in the process standard deviation compared with EWMA mean charts despite being less sensitive. Celano et al. [23] utilized an EWMA-t chart to monitor the process mean in the short run and showed that it can quickly detect an out-of-control signal. Thereafter, Celano et al. [24] investigated the performance of Shewhart-t, EWMA-t, and CUSUM-t charts for short production runs with an unknown shift size of the process mean. Recently, Haq et al. [25] extended the work of Zhang et al. [22] and Abbas et al. [20] to propose an AIB-EWMA-t chart for monitoring the process mean when the process standard deviation is unstable. They showed that the AIB-EWMA-t chart not only outperforms the existing EWMA-t chart but also can replace the AIB-EWMA mean chart when the process standard deviation has changed.

This study aims to enhance the sensitivity of two types of EWMA-t charts, namely the GWMA-t chart and AIB-GWMA-t chart, to efficiently monitor small process mean shifts when the process standard deviation is unstable. Existing EWMA-t and AIB-EWMA-t charts are special cases of the AIB-GWMA-t chart. The potential limits of the proposed AIB-GWMA-t chart such as single auxiliary variable is considered and correlated to the study variable. Also we assume that two variables follow a bivariate normal distribution with known population means, variances, and correlation coefficient. The Monte Carlo numerical simulations are evaluated using the run length profiles, specifically the average run length (ARL). A comprehensive comparative study shows that the AIB-GWMA-t chart not only improves the detection ability of the GWMA-t chart, but also surpasses that of the competitive AIB-EWMA-t chart.

The organization of the rest of the paper is as follows. Section 2 reviews the EWMA-t and AIB-EWMA-t charts. Section 3 proposes the GWMA-t and AIB-GWMA-t charts. Section 4 evaluates the performance of the GWMA-t and AIB-GWMA-t charts in terms of their *ARL* values. Section 5 presents a numerical simulation to explain the implementation of both existing and proposed charts. Section 6 concludes.

2. Exponentially Weighted Moving Average (EWMA)-t and Auxiliary Information-Based (AIB)-EWMA-t Control Charts

Zhang et al. [22] investigated whether the EWMA-t chart is more robust than the EWMA mean chart when the process standard deviation changes. Recently, Haq et al. [25] showed that the AIB-EWMA-t chart is better than the EWMA-t chart in detecting process mean shifts when the process standard deviation is unstable. In this section, we briefly introduce the design structures of the well-known EWMA-t and AIB-EWMA-t charts.

2.1. EWMA-t Control Charts

Let X be the quality characteristic of interest and assume that X is a normally distributed random variable with mean $\mu_X + \delta\sigma_X$ and variance σ_X^2, that is, $X \sim N(\mu_X + \delta\sigma_X, \sigma_X^2)$. The process is in the control state when $\delta = 0$; otherwise, the process has shifted. Let X_{ij}, $i = 1, 2, \ldots$, $j = 1, 2, \ldots, n$, be a random sample of size n drawn from the process at time i. Let $\overline{X}_i$ and $S_{X,i}^2$ be the sample mean and sample variance of the ith subgroup, respectively, where $\overline{X}_i = \sum_{j=1}^{n} X_{ij}/n$ and $S_{X,i}^2 = \sum_{j=1}^{n} \left(X_{ij} - \overline{X}_i\right)^2/(n-1)$. $\overline{X}_i$ follows a normal random variable with mean $\mu_X + \delta\sigma_X$ and variance σ_X^2/n, whereas $(n-1)S_{X,i}^2/\sigma_X^2$ is a chi-squared random variable with $n-1$ degrees of freedom. $\overline{X}_i$ and $S_{X,i}^2$ are mutually independent random variables for an in-control process. Suppose that the process is in control $(\delta = 0)$; then, the statistic T_i is defined by:

$$T_i = \frac{\sqrt{n}(\overline{X}_i - \mu_X)}{S_{X,i}}, \tag{1}$$

is a Student's t-distribution with $n-1$ degrees of freedom and $S_{X,i}$ is the sample standard deviation. Therefore, the EWMA-t statistic Z_i at time i can be defined as:

$$Z_i = \lambda T_i + (1-\lambda)Z_{i-1} \tag{2}$$

where T_i is a t distributed random variable at time i and λ is the smoothing constant satisfying $0 < \lambda \leq 1$. The initial value of Z_i is usually set to zero (i.e., $Z_0 = 0$). The upper control limit (UCL), central line (CL), and the lower control limit (LCL) for the well-known EWMA-t control chart based on Z_i are:

$$\begin{cases} UCL = 0 + L_e\sqrt{\frac{\lambda}{2-\lambda} \cdot \frac{n-1}{n-3}} \\ CL = 0 \\ LCL = 0 - L_e\sqrt{\frac{\lambda}{2-\lambda} \cdot \frac{n-1}{n-3}} \end{cases} \tag{3}$$

where L_e is the control limit constant chosen to match the desired in-control ARL. When statistics Z_i exceeds the UCL or LCL, the EWMA-t control chart initiates an out-of-control signal; otherwise, the process remains in-control. For more details on the EWMA-t chart, see Zhang et al. [22] and Haq et al. [25].

2.2. AIB-EWMA-t Control Charts

Assume a corresponding auxiliary variable Y accompanies the quality characteristic X of interest. Let (X, Y) follow a bivariate normally distributed process with means (μ_X, μ_Y) and variances (σ_X^2, σ_Y^2); ρ is the correlation between X and Y, that is, $(X, Y) \sim N_2(\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho)$. Suppose (X_{ij}, Y_{ij}), $j = 1, 2, \ldots, n$, is a random sample of size n taken from the process at time i, for $i = 1, 2, \ldots$. Let $\overline{X}_i = \sum_{j=1}^{n} X_{ij}/n$ and $\overline{Y}_i = \sum_{j=1}^{n} Y_{ij}/n$ be the sample mean based on $(X_{i1}, X_{i2}, \ldots, X_{in})$ and $(Y_{i1}, Y_{i2}, \ldots, Y_{in})$, respectively. Then, following Abbas et al. [20], the regression estimator of the process mean μ_X is given by:

$$X_i^* = \overline{X}_i + \rho\left(\frac{\sigma_X}{\sigma_Y}\right)(\mu_Y - \overline{Y}_i) \tag{4}$$

The mean and variance of X_i^* are:

$$E(X_i^*) = \mu_X \tag{5}$$

$$Var(X_i^*) = \frac{\sigma_X^2}{n}(1-\rho^2) \tag{6}$$

Assume that the underlying process remains in an in-control state; then, the statistic T_i^* is defined by:

$$T_i^* = \frac{\sqrt{n}(X_i^* - \mu_X)}{S_{X,i}\sqrt{1-\rho^2}} \tag{7}$$

As this is also a Student's t-distribution with $n-1$ degrees of freedom, the AIB-EWMA-t statistic Z_i^* at time i can be defined as:

$$Z_i^* = \lambda T_i^* + (1-\lambda)Z_{i-1}^* \tag{8}$$

where λ is the smoothing constant satisfying $0 < \lambda \leq 1$. The initial value of Z_i^* is usually set to zero (i.e., $Z_i^* = 0$). The asymptotic control limits for the well-known AIB-EWMA-t control chart based on Z_i^* are:

$$\begin{cases} UCL = 0 + L_{ae}\sqrt{\frac{\lambda}{2-\lambda} \cdot \frac{n-1}{n-3}} \\ CL = 0 \\ LCL = 0 - L_{ae}\sqrt{\frac{\lambda}{2-\lambda} \cdot \frac{n-1}{n-3}} \end{cases} \tag{9}$$

where L_{ae} is the control limit constant which is similar to that of the EWMA-t control chart. The AIB-EWMA-t control chart initiates an out-of-control signal whenever statistic $Z_i^* > UCL$ or $Z_i^* < LCL$; otherwise, the process remains in-control. For more details on the AIB-EWMA-t chart, see Haq et al. [25].

3. Proposed Control Charts

Sheu and Lin [7] first extended the EWMA chart to the GWMA chart by adding design parameter q and adjustment parameter α. Owing to this novel feature, the GWMA chart is capable of monitoring small process changes. The GWMA statistic H_i at time i can be represented as:

$$H_i = \sum_{j=1}^{i} (q^{(j-1)^\alpha} - q^{j^\alpha})X_{i-j+1} + q^{i^\alpha}H_0 \tag{10}$$

where X_i is the quality characteristic mentioned as above and H_0 is usually set to zero.

3.1. Generally Weighted Moving Average (GWMA-t) Control Charts

This study further extends the EWMA-t chart by adding design and adjustment parameters to enhance detection capability. The statistic T_i in Equation (1) is used instead of the statistic X_i in Equation (10); hence, the novel statistic for the GWMA-t chart is G_i, defined as follows:

$$G_i = \sum_{j=1}^{i} (q^{(j-1)^\alpha} - q^{j^\alpha})T_{i-j+1} + q^{i^\alpha}G_0 \tag{11}$$

where the initial value of G_i is set to zero (i.e., $G_0 = 0$). For the in-control case, we express the mean and variance of G_i as:

$$E(G_i) = 0 \tag{12}$$

$$Var(G_i) = Q \cdot \frac{n-1}{n-3} \tag{13}$$

where $Q = \lim_{i\to\infty}\left(\sum_{j=1}^{i} (q^{(j-1)^\alpha} - q^{j^\alpha})^2\right)$ is an asymptotic value. We use the asymptotic control limits instead of the time-varying control limits in this study to simplify the control chart. Assuming that L_g denotes the width of the control limit, the GWMA-t chart is:

$$\begin{cases} UCL = 0 + L_g \sqrt{Q \cdot \frac{n-1}{n-3}} \\ CL = 0 \\ LCL = 0 - L_g \sqrt{Q \cdot \frac{n-1}{n-3}} \end{cases} \quad (14)$$

When the statistic G_i remains inside the control limits (LCL, UCL), the process stays in control. However, if $G_i > UCL$ or $G_i < LCL$, the process mean shifts. We can determine the value of L_g using a numerical simulation to achieve the desired in-control ARL. Since the design parameter q is a constant, we can use parameter α to adjust the kurtosis of the weighting function slightly. In particular, the EWMA-t chart is a special case of the GWMA-t chart when $\alpha = 1$ and $q = 1 - \lambda$.

3.2. AIB-GWMA-t Control Charts

This study also extends the AIB-EWMA-t chart by adding design and adjustment parameters to enhance detection capability. Now, we use the statistic T_i^* in Equation (7) to replace the statistic X_i in Equation (10); hence, the statistic for the AIB-GWMA-t chart is G_i^*, which we define as follows:

$$G_i^* = \sum_{j=1}^{i} (q^{(j-1)^\alpha} - q^{j^\alpha}) T_{i-j+1}^* + q^{i^\alpha} G_0^* \quad (15)$$

where the initial value of G_i^* is equal to zero (i.e., $G_i^* = 0$). For the in-control case, the mean and variance of G_i^* are the same as those of G_i in Equations (12) and (13). The chart based on G_i^* is called the AIB-GWMA-t chart. The constant control limits of the AIB-GWMA-t chart are:

$$\begin{cases} UCL = 0 + L_{ag} \sqrt{Q \cdot \frac{n-1}{n-3}} \\ CL = 0 \\ LCL = 0 - L_{ag} \sqrt{Q \cdot \frac{n-1}{n-3}} \end{cases} \quad (16)$$

where $Q = \lim_{i \to \infty} \left(\sum_{j=1}^{i} (q^{(j-1)^\alpha} - q^{j^\alpha})^2 \right)$ is an asymptotic value. L_{ag} is the control limit constant, as in L_{ae} in Equation (9) or L_g in Equation (14), which is used to determine the desired in-control ARL. When the statistic G_i^* moves beyond the control limit, it indicates a process mean shift.

Figure 1 depicts the relationship among the AIB-GWMA-t chart, AIB-EWMA-t chart, GWMA-t chart, and EWMA-t chart.

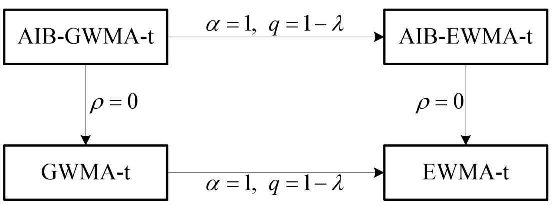


Figure 1. The structure of the AIB-GWMA-t chart and its special cases.

In Figure 1, the AIB-EWMA-t and EWMA-t charts are special cases of the AIB-GWMA-t and GWMA-t charts, respectively when adjustment parameter $\alpha = 1$ and design parameter $q = 1 - \lambda$. Moreover, the AIB-GWMA-t and AIB-EWMA-t charts reduce to the GWMA-t and EWMA-t charts, respectively when correlation coefficient $\rho = 0$.

4. Performance Measurement and Comparison

4.1. In-Control ARL Profiles

The ARL (Average Run Length) is a popular indicator for evaluating the performance of control charts. When the process operates in an in-control state, the in-control ARL, termed ARL_0, of a AIB-GWMA-t chart is a function of (q, α, L_{ag}) and this is expected to be sufficiently large to avoid false alarms. To compute the ARL values, the Monte Carlo simulation was recommended by Sheu and Lin [7] instead of the Markov chain or integral equation approaches since the GWMA statistic is complicated. An algorithm in R has been developed in Appendix A to calculate the ARL values, which are 50,000 run lengths on average. Without loss of generality, the random samples (X_{ij}, Y_{ij}), $j = 1, 2, \ldots, n$, for $i = 1, 2, \ldots$, are drawn from a bivariate normal distribution, that is, $(X, Y) \sim N_2(\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho)$. For $\mu_X = \mu_Y = 0$, $\sigma_X = \sigma_Y = 1$, and the known correlation coefficient ρ, we find that the underlying process is in-control. The control limit coefficient L_{ag} under various combinations of n, q, and α can be adjusted to match the desired ARL_0.

Table 1 presents the L_{ag} values for the AIB-GWMA-t charts. The various parameter combinations of $q \in \{0.1, 0.2, \ldots, 0.9\}$ and $\alpha \in \{0.1, 0.2, \ldots, 1.0\}$, correlation coefficient value of $\rho = 0$, and sample size $n \in \{5, 10\}$ are considered to achieve an in-control ARL at approximately 500. Note that sample sizes $n = 5$ and 10 are selected for brief discussion to compare with that of Haq et al. [25] and explore the effect of different sample sizes on L_{ag}. When $\rho = 0$, the results in Table 1 correspond to the proposed GWMA-t charts.

Table 1. Values for the auxiliary information-based generally weighted moving average (AIB-GWMA-t) charts when $\rho = 0$, $ARL_0 \approx 500$.

n	q	α									
		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
5	0.1	5.079	5.078	5.076	5.073	5.069	5.067	5.064	5.061	5.060	5.058
	0.2	5.077	5.071	5.060	5.053	5.037	5.029	5.021	5.007	5.001	4.993
	0.3	5.076	5.060	5.045	5.017	4.999	4.975	4.949	4.925	4.901	4.881
	0.4	5.075	5.051	5.016	4.976	4.939	4.891	4.850	4.809	4.766	4.726
	0.5	5.070	5.038	4.989	4.924	4.861	4.791	4.718	4.652	4.585	4.521
	0.6	5.067	5.025	4.953	4.865	4.762	4.655	4.548	4.449	4.355	4.264
	0.7	5.063	5.008	4.913	4.782	4.629	4.474	4.319	4.180	4.053	3.943
	0.8	5.062	4.990	4.854	4.667	4.444	4.217	4.005	3.824	3.671	3.550
	0.9	5.060	4.969	4.787	4.497	4.154	3.815	3.526	3.302	3.145	3.047
	q	α									
		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
10	0.1	3.792	3.791	3.790	3.788	3.788	3.787	3.786	3.785	3.785	3.784
	0.2	3.791	3.787	3.784	3.780	3.778	3.774	3.770	3.765	3.757	3.754
	0.3	3.790	3.783	3.774	3.767	3.756	3.744	3.734	3.722	3.710	3.704
	0.4	3.790	3.777	3.768	3.746	3.728	3.705	3.689	3.666	3.647	3.629
	0.5	3.788	3.775	3.751	3.724	3.692	3.658	3.625	3.595	3.566	3.540
	0.6	3.786	3.766	3.733	3.692	3.643	3.591	3.548	3.503	3.465	3.431
	0.7	3.784	3.758	3.711	3.643	3.578	3.511	3.444	3.385	3.336	3.296
	0.8	3.783	3.708	3.681	3.593	3.491	3.391	3.301	3.226	3.166	3.125
	0.9	3.782	3.738	3.645	3.502	3.339	3.181	3.059	2.968	2.910	2.874

Table 1 shows that when q and α are fixed, a large sample size corresponds to a small control limit coefficient and vice versa. The L_{ag} values decrease as design parameter q or adjustment parameter α increases. In particular, the difference among the L_{ag} values is large for larger q or α values, whereas the difference among the L_{ag} values become small for smaller q or α values. For example, when $n = 5$, the values of L_{ag} for $q = 0.1$ and 0.9 at $\alpha = 0.9$ are 5.060 and 3.145, respectively; however, those for $q = 0.1$ and 0.9 at $\alpha = 0.1$ are 5.079 and 5.060, respectively. Moreover, when $\alpha = 1.0$, the L_{ag} value of the

AIB-GWMA-t chart at $q = 0.9$ is 3.047, which is close to 3.046 at $\lambda = 0.1$ for the existing AIB-EWMA-t chart, as shown by Haq et al. [25].

4.2. Performance Comparison

The out-of-control ARL, denoted by ARL_1, is expected to be sufficiently small to detect shifts early when the process is out of control. It is customary to set an ARL_0 value and then compute the ARL_1 values. The smaller the ARL_1 value, the better is the statistical performance. To compare the performance of the AIB-GWMA-t charts, this study keeps ARL_0 close to 500 and the $ARL_1 s$ are evaluated for specific process mean shifts. To ensure the discussion remains brief, design parameter $q = \{0.5, 0.7, 0.9, 0.95\}$, adjustment parameter $\alpha = \{0.5, 0.7, 0.9, 1.0\}$, correlation coefficient $\rho = \{0.00, 0.25, 0.50, 0.75, 0.95\}$, and process mean shifts $\delta = \{0.1, 0.2, 0.4, 0.6, 1.0, 2.0\}$ are used to compute the ARL values of the AIB-GWMA-t charts with $n = 5$ and $n = 10$ (see Tables 2 and 3, respectively). In particular, $\alpha = 1.0$ and $q = 1 - \lambda$ reduce to the AIB-EWMA-t chart proposed by Haq et al. [25]. Furthermore, when $\rho = 0$, the AIB-GWMA-t and AIB-EWMA-t charts reduce to the GWMA-t and EWMA-t charts, respectively.

The boldface values in Tables 2 and 3 indicate smaller ARL_1 values than those of the AIB-EWMA-t and EWMA-t charts under various process mean shifts with the specific design parameter q and correlation coefficient ρ. The main findings from Tables 2 and 3 are as follows:

(1) For fixed q and α, the values of L_{ag} are close and unaffected by the choices of ρ, a finding consistent with that of Haq and Abidin [21] and Haq et al. [25].
(2) A larger sample size n results in a smaller ARL_1 value at fixed parameter combinations of (q, α, ρ).
(3) For fixed $q\alpha$, and ρ, the ARL_1 values decrease as the value of δ increases. Moreover, for fixed α, δ, and ρ, the ARL_1 values tend to decrease as the value of q increases. Similarly, the ARL_1 value is a decreasing function of ρ when q, α, and δ are fixed.
(4) The AIB-GWMA-t and AIB-EWMA-t charts uniformly perform better than the GWMA-t and EWMA-t charts, respectively. This result reveals that the use of auxiliary information enhances the performance of the GWMA-t and EWMA-t charts, especially for large values of ρ.
(5) To detect small process mean shifts, the AIB-GWMA-t chart with large q and α performs better than the AIB-EWMA-t chart with $0.25 \leq \rho \leq 0.75$. However, the AIB-GWMA-t chart performs comparably to the AIB-GWMA-t chart at $\rho = 0.95$.

Facing an auxiliary variable with different correlations, suitable parameter combinations of (q, α) facilitate the use of our proposed charts to monitor small process mean shifts. Figure 2 depicts the ARL_1 curves of the AIB-GWMA-t charts at different correlation coefficients with specific parameter combinations. It shows that when we apply the proposed GWMA-t or AIB-GWMA-t charts to monitor small process mean shifts $(\delta = 0.1)$, large values of q and α under $\rho \leq 0.75$ are recommended for practical applications. Moreover, the performance of the AIB-GWMA-t chart is comparable to the AIB-EWMA-t chart at $q = 0.95$ regardless of the α value for $\rho = 0.95$.

Table 2. Values for the AIB-GWMA-t charts when $n = 5$.

$q = 0.5$	$\rho = 0.00$				$\rho = 0.25$				$\rho = 0.50$				$\rho = 0.75$				$\rho = 0.95$			
α	0.5	0.7	0.9	1.0	0.5	0.7	0.9	1.0	0.5	0.7	0.9	1.0	0.5	0.7	0.9	1.0	0.5	0.7	0.9	1.0
$\delta \backslash L_{ag}$	4.861	4.718	4.585	4.521	4.852	4.713	4.584	4.521	4.861	4.724	4.590	4.526	4.854	4.715	4.580	4.516	4.861	4.718	4.585	4.521
0.00	499.99	500.10	500.14	500.10	499.76	499.76	500.26	500.03	500.25	500.17	499.77	499.90	500.02	499.89	499.93	499.85	500.95	499.45	500.22	499.56
0.10	427.92	426.79	425.88	423.93	421.41	420.66	**421.33**	421.76	404.82	406.46	406.53	405.86	355.45	354.09	352.08	351.54	163.20	160.85	160.17	159.38
0.20	290.35	286.99	286.66	285.44	279.40	278.21	278.90	278.42	251.58	250.63	249.57	248.82	177.55	176.01	174.51	173.79	39.83	37.61	36.92	36.59
0.40	112.49	110.01	109.08	108.31	103.94	102.08	102.07	101.59	83.63	81.60	80.88	80.45	45.47	43.25	42.46	42.23	7.51	6.57	6.09	5.92
0.60	46.53	44.25	43.55	43.22	42.80	40.51	39.99	39.78	32.90	30.84	30.06	29.78	16.84	15.19	14.48	14.27	3.33	3.00	2.79	2.71
1.00	13.33	11.80	11.14	10.92	12.27	10.86	10.23	10.05	9.51	8.35	7.79	7.61	5.27	4.63	4.25	4.13	1.53	1.50	**1.48**	1.48
2.00	3.11	2.82	2.63	2.55	2.92	2.67	2.50	2.44	2.43	2.27	2.15	2.10	1.64	1.60	1.57	1.56	**1.01**	**1.01**	**1.01**	1.01
$q = 0.7$	$\rho = 0.00$				$\rho = 0.25$				$\rho = 0.50$				$\rho = 0.75$				$\rho = 0.95$			
α	0.5	0.7	0.9	1.0	0.5	0.7	0.9	1.0	0.5	0.7	0.9	1.0	0.5	0.7	0.9	1.0	0.5	0.7	0.9	1.0
$\delta \backslash L_{ag}$	4.629	4.319	4.053	3.943	4.619	4.315	4.048	3.943	4.631	4.321	4.048	3.939	4.619	4.313	4.045	3.936	4.624	4.315	4.048	3.941
0.00	500.13	500.27	500.33	500.16	499.96	499.66	499.83	499.73	500.05	499.81	500.13	499.54	500.21	500.13	500.62	500.05	500.31	499.99	499.93	500.05
0.10	382.43	376.07	373.81	373.29	374.91	371.57	**367.30**	368.48	354.13	349.12	344.27	343.39	285.94	279.80	276.92	275.77	100.18	90.96	87.98	87.84
0.20	212.28	203.76	201.07	200.70	201.70	193.99	**190.96**	191.16	175.02	165.51	161.61	161.40	110.43	101.66	98.75	98.44	23.65	18.63	16.66	16.30
0.40	64.52	55.77	53.54	53.31	59.92	51.62	49.42	49.31	47.58	39.99	37.47	37.13	26.62	21.19	19.10	18.69	5.98	4.87	4.23	4.02
0.60	27.18	21.62	19.53	19.14	25.29	20.06	17.99	17.67	20.11	15.69	13.78	13.35	11.48	8.97	7.71	7.36	3.02	2.68	2.47	2.39
1.00	9.51	7.49	6.43	6.11	8.90	7.06	6.06	5.75	7.28	5.83	5.00	4.73	4.45	3.76	3.34	3.19	1.52	**1.50**	**1.50**	1.51
2.00	2.85	2.55	2.36	2.30	2.70	2.44	2.28	2.22	2.30	2.13	2.02	1.99	1.62	1.59	**1.58**	1.58	**1.01**	**1.01**	**1.01**	1.01
$q = 0.9$	$\rho = 0.00$				$\rho = 0.25$				$\rho = 0.50$				$\rho = 0.75$				$\rho = 0.95$			
α	0.5	0.7	0.9	1.0	0.5	0.7	0.9	1.0	0.5	0.7	0.9	1.0	0.5	0.7	0.9	1.0	0.5	0.7	0.9	1.0
$\delta \backslash L_{ag}$	4.154	3.526	3.146	3.047	4.147	3.520	3.141	3.044	4.150	3.522	3.142	3.042	4.145	3.521	3.142	3.042	4.150	3.522	3.143	3.045
0.00	500.46	500.38	500.24	500.15	500.12	499.68	500.11	500.18	500.23	500.34	500.63	500.43	499.96	500.19	500.08	499.83	500.08	500.00	500.17	499.83
0.10	244.10	**200.66**	**195.69**	202.54	236.52	**191.66**	**187.69**	194.02	210.36	**166.25**	**160.91**	166.75	150.93	111.04	**104.44**	108.00	49.88	32.76	26.94	26.30
0.20	103.32	70.65	**63.21**	64.58	97.97	66.77	**59.24**	60.59	82.72	55.59	**48.32**	48.70	54.38	35.84	29.74	29.19	15.97	11.30	9.25	8.70
0.40	34.90	23.09	18.59	17.71	32.98	21.86	17.58	16.76	27.45	18.44	14.80	13.99	17.53	12.28	10.01	9.39	5.24	4.37	3.96	3.84
0.60	17.86	12.49	10.13	9.51	16.93	11.86	9.65	9.08	14.03	10.10	8.33	7.83	9.04	6.92	5.89	5.60	2.89	2.67	**2.61**	2.62
1.00	7.75	6.06	5.25	5.02	7.35	5.79	5.05	4.84	6.18	5.00	4.44	4.28	4.07	3.55	3.31	3.26	**1.54**	**1.59**	**1.70**	1.76
2.00	2.74	2.56	**2.52**	2.53	2.61	2.47	**2.44**	2.46	2.26	**2.19**	**2.21**	2.24	**1.64**	**1.68**	**1.77**	1.82	**1.01**	**1.02**	**1.05**	1.08
$q = 0.95$	$\rho = 0.00$				$\rho = 0.25$				$\rho = 0.50$				$\rho = 0.75$				$\rho = 0.95$			
α	0.5	0.7	0.9	1.0	0.5	0.7	0.9	1.0	0.5	0.7	0.9	1.0	0.5	0.7	0.9	1.0	0.5	0.7	0.9	1.0
$\delta \backslash L_{ag}$	3.893	3.114	2.750	2.682	3.888	3.109	2.747	2.684	3.893	3.112	2.745	2.681	3.886	3.104	2.744	2.682	3.893	3.110	2.750	2.685
0.00	500.18	499.75	500.21	500.46	499.56	499.71	499.93	500.69	499.56	500.45	500.63	500.16	500.01	499.81	499.93	500.77	500.40	500.57	500.25	500.06
0.10	199.63	140.33	**130.70**	136.63	192.76	134.41	**124.31**	130.51	171.81	116.71	**105.30**	109.43	123.51	80.80	**69.88**	71.12	43.87	28.66	23.25	22.06
0.20	85.93	55.11	45.90	45.06	81.97	52.56	43.46	42.80	70.37	45.10	36.89	35.90	47.50	30.95	25.10	23.91	15.09	11.13	9.55	9.10
0.40	31.44	21.16	17.21	16.21	29.77	20.15	16.48	15.58	25.19	17.33	14.26	13.48	16.46	11.97	10.21	9.73	5.19	4.55	4.39	4.37
0.60	16.75	12.18	10.33	9.80	15.89	11.62	9.92	9.44	13.34	10.03	8.68	8.31	8.76	7.03	6.36	6.18	**2.90**	**2.82**	**2.94**	3.02
1.00	7.55	6.22	5.72	5.59	7.19	5.96	5.51	5.41	6.08	5.17	4.89	4.83	4.06	**3.71**	**3.70**	3.73	**1.56**	**1.69**	**1.89**	2.00
2.00	**2.75**	**2.71**	**2.83**	2.91	**2.63**	**2.61**	**2.74**	2.84	**2.28**	**2.32**	**2.48**	2.59	**1.66**	**1.77**	**1.98**	2.09	**1.01**	**1.04**	**1.14**	1.24

Table 3. Values for the AIB-GWMA-t charts when $n = 10$.

$q = 0.5$	$\rho = 0.00$				$\rho = 0.25$				$\rho = 0.50$				$\rho = 0.75$				$\rho = 0.95$			
α	0.5	0.7	0.9	1.0	0.5	0.7	0.9	1.0	0.5	0.7	0.9	1.0	0.5	0.7	0.9	1.0	0.5	0.7	0.9	1.0
$\delta \backslash L_{ag}$	3.692	3.625	3.566	3.540	3.688	3.624	3.564	3.539	3.688	3.623	3.565	3.538	3.685	3.621	3.563	3.538	3.688	3.621	3.563	3.538
0.00	500.43	499.55	500.18	499.76	500.32	500.29	499.80	500.29	499.91	499.58	500.33	499.70	500.05	499.90	499.91	500.32	500.34	499.93	499.96	500.09
0.10	**267.97**	**266.89**	**268.56**	269.64	**257.46**	**257.98**	**258.65**	259.83	**226.70**	**226.83**	**229.40**	230.53	**155.00**	**155.70**	**157.67**	159.09	33.21	**31.40**	**31.54**	31.91
0.20	94.45	**93.47**	**95.03**	96.05	88.03	**87.56**	**88.53**	89.77	70.22	**68.89**	**69.94**	70.61	37.87	**36.20**	**36.45**	36.96	6.60	5.85	5.52	5.44
0.40	19.47	17.84	17.58	17.67	17.95	16.38	16.06	16.16	13.79	12.41	12.07	12.08	7.46	6.63	6.28	6.20	1.79	1.75	1.73	1.73
0.60	7.64	6.76	6.39	6.31	7.09	6.29	5.93	5.86	5.58	4.98	4.68	4.59	3.26	3.00	2.84	2.79	**1.11**	**1.12**	**1.14**	1.14
1.00	2.73	2.55	2.44	2.40	2.57	2.42	2.32	2.29	2.12	2.04	1.98	1.97	**1.42**	**1.42**	**1.43**	1.44	**1.00**	**1.00**	**1.00**	1.00
2.00	**1.08**	**1.09**	**1.10**	1.11	**1.06**	**1.07**	**1.08**	1.09	**1.02**	**1.02**	**1.03**	1.03	**1.00**	**1.00**	**1.00**	1.00	**1.00**	**1.00**	**1.00**	1.00
$q = 0.7$	$\rho = 0.00$				$\rho = 0.25$				$\rho = 0.50$				$\rho = 0.75$				$\rho = 0.95$			
α	0.5	0.7	0.9	1.0	0.5	0.7	0.9	1.0	0.5	0.7	0.9	1.0	0.5	0.7	0.9	1.0	0.5	0.7	0.9	1.0
$\delta \backslash L_{ag}$	3.578	3.444	3.336	3.296	3.577	3.443	3.336	3.297	3.577	3.443	3.335	3.294	3.574	3.439	3.331	3.291	3.576	3.438	3.331	3.291
0.00	499.91	500.22	500.45	499.63	500.32	500.38	500.05	500.25	500.36	500.12	499.88	500.39	500.21	499.81	499.61	499.65	499.82	500.03	500.12	499.96
0.10	**171.35**	**170.28**	**178.13**	182.98	**164.04**	**161.84**	**169.32**	174.93	**138.46**	**136.56**	**142.45**	146.74	**87.55**	**83.15**	**86.67**	89.91	20.41	17.09	**16.38**	16.58
0.20	51.56	**46.71**	**48.01**	49.67	48.50	**43.47**	**44.63**	46.35	38.89	**34.15**	**34.47**	35.65	22.81	19.20	**18.55**	18.85	5.57	4.81	4.41	4.29
0.40	13.28	11.02	10.20	10.15	12.45	10.36	9.57	9.49	10.12	8.41	7.69	7.57	6.16	5.29	4.82	4.69	**1.80**	**1.80**	**1.83**	1.84
0.60	6.24	5.36	4.89	4.75	5.90	5.08	4.64	4.52	4.84	4.26	3.93	3.81	3.06	2.83	2.70	2.66	**1.13**	**1.17**	**1.22**	1.26
1.00	2.62	2.49	2.41	2.38	2.49	2.38	2.32	2.30	2.10	2.05	2.04	2.04	**1.45**	**1.49**	**1.55**	1.59	**1.00**	**1.00**	**1.00**	1.00
2.00	**1.10**	**1.13**	**1.18**	1.22	**1.08**	**1.11**	**1.15**	1.18	**1.03**	**1.04**	**1.07**	1.08	**1.00**	**1.00**	**1.00**	1.00	**1.00**	**1.00**	**1.00**	1.00
$q = 0.9$	$\rho = 0.00$				$\rho = 0.25$				$\rho = 0.50$				$\rho = 0.75$				$\rho = 0.95$			
α	0.5	0.7	0.9	1.0	0.5	0.7	0.9	1.0	0.5	0.7	0.9	1.0	0.5	0.7	0.9	1.0	0.5	0.7	0.9	1.0
$\delta \backslash L_{ag}$	3.339	3.059	2.910	2.874	3.339	3.061	2.911	2.875	3.339	3.065	2.911	2.876	3.339	3.061	2.914	2.876	3.336	3.057	2.911	2.878
0.00	500.14	499.71	500.89	499.97	499.98	500.19	500.55	499.67	500.04	500.20	499.96	499.56	500.04	499.98	500.36	499.53	499.75	500.21	500.20	500.67
0.10	89.90	**73.59**	**76.23**	81.53	86.01	**70.34**	**72.33**	77.25	49.87	**59.59**	**59.83**	63.46	49.87	39.12	**37.56**	38.82	15.72	12.81	11.51	11.16
0.20	32.52	25.56	**23.49**	23.55	31.00	24.38	**22.30**	22.31	17.19	20.76	18.76	18.53	17.19	13.96	12.53	12.13	5.30	4.92	4.73	4.65
0.40	11.07	9.33	8.49	8.20	10.50	8.93	8.14	7.87	5.80	7.66	7.04	6.82	5.80	5.31	5.06	4.96	**1.90**	**2.08**	**2.26**	2.34
0.60	5.86	5.36	5.10	4.99	5.57	5.14	4.91	4.82	**3.12**	4.45	4.32	4.26	**3.12**	**3.14**	**3.21**	3.22	**1.19**	**1.36**	**1.61**	1.73
1.00	**2.70**	**2.78**	**2.89**	2.93	**2.58**	**2.68**	**2.80**	2.84	**1.55**	**2.35**	**2.50**	2.56	**1.55**	**1.75**	**1.95**	2.03	**1.00**	**1.00**	**1.04**	1.08
2.00	**1.15**	**1.31**	**1.56**	1.68	**1.12**	**1.27**	**1.51**	1.64	**1.00**	**1.14**	**1.35**	1.48	**1.00**	**1.01**	**1.06**	1.13	**1.00**	**1.00**	**1.00**	1.00
$q = 0.95$	$\rho = 0.00$				$\rho = 0.25$				$\rho = 0.50$				$\rho = 0.75$				$\rho = 0.95$			
α	0.5	0.7	0.9	1.0	0.5	0.7	0.9	1.0	0.5	0.7	0.9	1.0	0.5	0.7	0.9	1.0	0.5	0.7	0.9	1.0
$\delta \backslash L_{ag}$	3.181	2.804	2.654	2.634	3.182	2.802	2.654	2.633	3.187	2.806	2.653	2.634	3.185	2.803	2.657	2.638	3.181	2.803	2.655	2.635
0.00	499.85	499.90	500.00	499.92	499.81	500.09	499.76	499.61	499.88	500.30	499.66	499.61	500.13	500.62	500.63	500.35	500.15	500.70	500.46	499.83
0.10	80.22	63.94	**60.95**	62.94	77.00	61.22	**58.25**	59.83	66.89	52.96	**49.34**	50.21	46.18	36.65	33.55	33.21	15.42	13.32	12.32	11.90
0.20	30.78	25.07	22.68	22.07	29.42	23.94	21.74	21.10	25.12	20.66	18.76	18.15	16.82	14.38	13.27	12.80	**5.39**	5.36	5.45	5.44
0.40	10.99	9.90	9.38	9.12	10.47	9.47	9.02	8.79	8.87	8.19	7.90	7.74	5.89	5.77	5.82	5.79	**1.97**	**2.31**	**2.66**	2.80
0.60	5.94	5.82	5.86	5.82	5.66	5.58	5.65	5.63	4.80	4.86	**4.99**	5.02	**3.21**	**3.47**	**3.76**	3.85	**1.22**	**1.52**	**1.88**	2.00
1.00	**2.78**	**3.08**	**3.40**	3.51	**2.66**	**2.96**	**3.29**	3.40	**2.28**	**2.60**	**2.94**	3.07	**1.60**	**1.93**	**2.28**	2.43	**1.00**	**1.02**	**1.18**	1.36
2.00	**1.18**	**1.46**	**1.82**	1.95	**1.15**	**1.41**	**1.78**	1.92	**1.06**	**1.26**	**1.64**	1.80	**1.00**	**1.03**	**1.25**	1.45	**1.00**	**1.00**	**1.00**	1.00

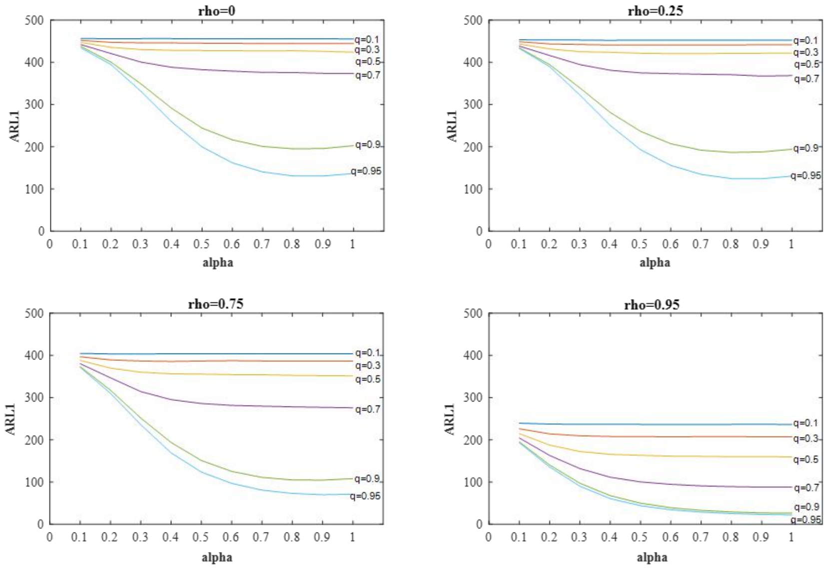


Figure 2. Curves of the AIB-GWMA-t charts with various correlation coefficients at small process mean shift $\delta = 0.1$.

The near optimal parameters of the proposed AIB-GWMA-t charts at various correlation coefficients are next investigated. Considering $ARL_0 \approx 500$ and correlation coefficient ρ = {0.00, 0.25, 0.50, 0.75, 0.95}, Table 4 proposes the near optimal parameters of the proposed charts under a small shift $\delta = 0.1$, median shift $\delta = 0.6$, and large shift $\delta = 2.0$ to reach the minimum ARL_1.

Table 4. The near optimal parameters under various process mean shifts at $ARL_0 \approx 500$.

n	ρ	δ	The Near Optimal Design				
			Schemes	q^*	α^*	L^*_{ag}	ARL_1
5	0.00	0.1	GWMA-t	0.95	0.9	2.750	130.698
		0.6	EWMA-t	0.9	1.0	3.047	9.506
		2.0	EWMA-t	0.7	1.0	3.949	2.296
	0.25	0.1	AIB-GWMA-t	0.95	0.9	2.747	124.315
		0.6	AIB-EWMA-t	0.9	1.0	3.044	9.081
		2.0	AIB-GWMA-t	0.7	1.0	3.943	2.221
	0.50	0.1	AIB-GWMA-t	0.95	0.9	2.745	105.297
		0.6	AIB-EWMA-t	0.9	1.0	3.042	7.828
		2.0	AIB-EWMA-t	0.7	1.0	3.939	1.989
	0.75	0.1	AIB-GWMA-t	0.95	0.9	2.744	69.875
		0.6	AIB-EWMA-t	0.9	1.0	3.042	5.599
		2.0	AIB-EWMA-t	0.5	1.0	4.516	1.557
	0.95	0.1	AIB-EWMA-t	0.95	1.0	2.685	22.064
		0.6	AIB-EWMA-t	0.7	1.0	3.941	2.395
		2.0	AIB-GWMA-t	0.5	0.4	4.925	1.005
10	0.00	0.1	GWMA-t	0.97	0.9	2.448	58.390
		0.6	EWMA-t	0.66	1.0	3.352	4.925
		2.0	GWMA-t	0.4	0.4	3.748	1.076
	0.25	0.1	AIB-GWMA-t	0.98	0.9	2.269	56.196
		0.6	AIB-EWMA-t	0.66	1.0	3.355	4.670
		2.0	AIB-GWMA-t	0.3	0.4	3.765	1.056
	0.50	0.1	AIB-GWMA-t	0.97	0.9	2.446	48.085
		0.6	AIB-EWMA-t	0.7	1.0	3.294	3.815
		2.0	AIB-GWMA-t	0.3	0.3	3.774	1.017
	0.75	0.1	AIB-EWMA-t	0.96	1.0	2.555	32.876
		0.6	AIB-EWMA-t	0.68	1.0	3.321	2.651
		2.0	AIB-GWMA-t	0.3	0.3	3.770	1.000
	0.95	0.1	AIB-EWMA-t	0.91	1.0	2.841	11.182
		0.6	AIB-GWMA-t	0.54	0.5	3.671	1.112
		2.0	AIB-GWMA-t	0.2	0.3	3.781	1.000

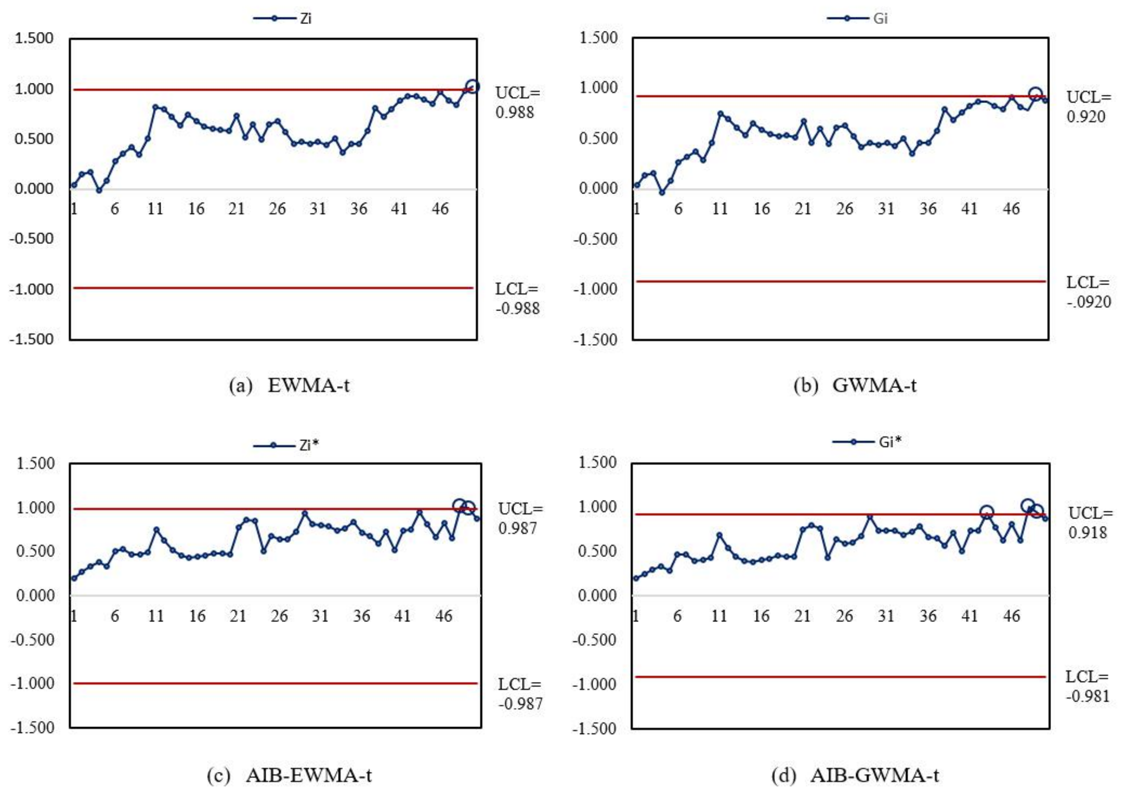
EWMA-t (Exponentially Weighted Moving Average-t); GWMA-t (Generally Weighted Moving Average-t); AIB-EWMA-t (Auxiliary Information-Based EWMA-t); AIB-GWMA-t (Auxiliary Information-Based GWMA-t).

As shown in Table 4, when $\rho = 0$ and $\delta = 0.1$, in the GWMA-t chart with the near optimal parameters $q^* = 0.95$, $\alpha^* = 0.9$, and $L^*_{ag} = 2.750$, for $n = 5$, the minimum ARL_1 is 130.698. Similarly, for the near optimal parameters $q^* = 0.97$, $\alpha^* = 0.9$, and $L^*_{ag} = 2.448$, for $n = 10$, the minimum ARL_1 is 58.390. A significant result is that the GWMA-t chart outperforms the EWMA-t chart at detecting small process mean shifts. When the auxiliary variable is related to the quality characteristic between $\rho = 0.25$ and $\rho = 0.75$, the AIB-GWMA-t chart with $q = 0.95$ and $\alpha = 0.9$ has the minimum ARL_1 at $\delta = 0.1$. However, the AIB-EWMA-t chart with large q performs better for small process mean shifts at $\rho = 0.95$.

5. Illustrative Example

A simulated dataset is used to demonstrate the implementation of the existing EWMA-t and AIB-EWMA-t charts as well as the proposed GWMA-t and AIB-GWMA-t charts when detecting process mean shifts. For this purpose, 50 bivariate samples, each of size $n = 5$, are generated from a bivariate normal distribution with $\mu_X = \mu_Y = 0$, $\sigma_X = \sigma_Y = 1$, and $\rho = 0.75$. The first 20 samples are referred to as in-control. Moreover, assuming the last 30 samples suffered from some assignable causes, the process mean shifted from μ_X to $\mu_X + \delta\sigma_X$, where $\delta = 0.2$, which is referred to as out-of-control. Table 5 lists these 50 simulated samples when $\rho = 0.75$.

To investigate the detection ability of these existing and proposed charts, their in-control ARL values are both set to 500. From Table 2, the parameter combinations $(q, \alpha, L_{e(g)})$ for the EWMA-t and GWMA-t charts are (0.9, 1.0, 3.047) and (0.9, 0.9, 3.146), respectively. When the auxiliary variable is available and correlated with the study variable at $\rho = 0.75$, the parameter combinations $(q, \alpha, L_{ae(ag)})$ for the AIB-EWMA-t and AIB-GWMA-t charts are (0.9, 1.0, 3.042) and (0.9, 0.9, 3.142), respectively. Moreover, Z_i, G_i, Z^*_i, and G^*_i represent the EWMA-t, GWMA-t, AIB-EWMA-t, and AIB-GWMA-t statistics, respectively. Table 6 lists the related upper control limits of these charts (see also Figure 3).



(a) EWMA-t (b) GWMA-t (c) AIB-EWMA-t (d) AIB-GWMA-t

Figure 3. The statistics and control limits of the AIB-GWMA-t chart and its reduced types at $\rho = 0.75$ and $\delta = 0.2$.

Table 5. Simulation bivariate dataset at process mean shift $\delta = 0.0$ in first 20 samples and $\delta = 0.2$ in the last 30 samples when $\rho = 0.75$.

No.	X_i	Y_i	No.	X_i	Y_i	No.	X_i	Y_i	No.	X_i	Y_i	No.	X_i	Y_i
1	2.340	1.092	11	−0.159	−1.007	21	0.037	−1.230	31	−0.366	−0.658	41	2.022	−0.519
	−0.260	−0.118		0.053	−0.597		0.395	−2.477		−0.200	−1.288		0.481	−0.168
	0.652	0.868		0.206	−0.093		0.310	−0.985		1.370	−1.397		0.432	−0.596
	0.435	−0.131		0.679	0.812		0.496	−0.836		−1.866	0.264		2.196	1.245
	1.700	0.010		0.408	−0.576		−0.940	−1.936		0.744	−0.579		−0.277	−1.524
2	2.440	0.179	12	0.499	0.607	22	1.383	−0.550	32	1.419	1.735	42	−0.728	−1.044
	−0.563	−1.099		−0.345	0.030		−1.220	−2.884		2.367	2.345		−0.536	−1.499
	1.586	1.341		0.309	0.111		1.082	0.349		−0.791	−0.602		−0.230	−0.374
	−0.088	1.220		−0.082	−0.034		1.281	0.917		−0.148	−1.442		0.287	−0.539
	0.627	1.814		−0.556	−0.237		−0.043	0.309		0.677	1.476		−0.307	−0.924
3	−0.545	1.675	13	0.053	0.215	23	−0.055	−0.271	33	0.429	0.049	43	0.479	0.597
	0.320	0.446		0.475	1.013		1.182	1.386		−1.341	−0.419		0.945	−0.120
	1.334	0.737		0.260	−1.199		0.074	1.125		−0.042	0.845		−0.040	0.059
	0.085	−1.383		0.600	0.563		−0.131	−0.812		0.178	−1.022		0.245	−0.921
	1.057	0.584		−0.081	2.508		0.988	0.984		1.414	0.533		0.691	0.928
4	−1.211	−0.250	14	−0.429	−0.321	24	0.229	0.811	34	0.196	0.168	44	−1.287	0.802
	2.311	1.635		−0.267	1.354		−1.408	0.201		0.431	−0.483		0.062	−0.003
	−0.585	−0.237		−1.340	−1.035		0.477	0.778		2.100	1.264		0.001	−1.385
	1.086	−0.031		1.375	0.738		−0.754	−0.063		1.131	1.347		−0.667	−2.131
	0.481	−1.361		−0.342	−2.611		−1.081	1.151		−1.040	−1.207		0.159	0.255
5	−0.029	1.020	15	0.791	−0.249	25	−0.659	−0.574	35	2.419	−0.344	45	0.574	0.835
	1.034	2.560		−1.589	−2.071		1.350	1.658		−0.005	−0.509		−0.238	0.525
	−1.428	−0.768		0.525	−0.146		0.417	−0.669		0.103	0.693		−0.422	0.074
	0.125	−1.362		−0.973	−1.290		1.378	−0.202		0.668	0.865		0.321	−0.817
	0.444	−0.944		−0.599	−0.767		0.175	−2.290		1.024	2.418		−0.546	0.065
6	2.452	0.919	16	−0.013	0.876	26	1.322	1.830	36	−1.768	−0.402	46	2.337	0.686
	0.493	−1.540		0.310	−0.413		0.830	0.310		−0.654	0.202		0.293	0.961
	−0.493	0.372		−1.075	−1.048		−0.406	−1.277		1.060	0.726		0.915	0.994
	0.631	0.102		1.715	1.670		0.117	0.457		0.035	−1.372		1.799	1.041
	0.977	−0.404		1.004	0.524		−0.564	0.410		−0.207	−0.748		0.697	1.118
7	0.751	−0.662	17	2.045	1.089	27	−1.482	−1.008	37	0.037	−0.843	47	−0.965	0.239
	0.476	−0.239		−1.118	−1.660		1.146	−0.650		1.616	0.210		−0.688	−0.737
	3.065	0.002		−0.834	0.093		−1.352	−0.646		0.588	−0.588		−0.199	−1.286
	−1.397	−0.744		0.068	−0.926		0.576	−0.001		−2.733	−1.173		0.903	2.380
	−0.593	1.603		0.344	−0.293		1.362	−0.715		1.003	0.786		0.373	0.701
8	−0.443	−1.017	18	0.083	0.581	28	0.887	1.137	38	−0.669	−0.845	48	1.430	0.246
	0.248	0.165		−2.046	−2.705		0.151	−0.572		0.866	0.254		0.494	−0.485
	0.043	−0.375		0.484	−1.689		0.131	−0.612		0.119	0.637		1.470	1.104
	−1.143	−1.297		0.273	−0.041		−0.505	−0.157		−0.567	−1.353		1.110	−0.264
	0.185	0.461		0.793	−0.155		1.076	0.178		−0.907	−0.482		2.529	1.371
9	−0.246	−0.560	19	−0.214	0.783	29	0.257	0.116	39	0.230	0.348	49	0.171	1.054
	−0.149	−0.102		−0.316	−0.840		2.920	1.049		1.926	−0.486		0.901	0.820
	0.805	0.085		0.851	0.944		0.390	−0.924		1.521	0.488		2.287	−1.175
	−0.735	0.261		−0.530	−0.497		0.824	−0.209		−0.478	−0.570		−0.139	−0.011
	0.430	−0.649		1.325	0.408		1.286	−0.797		0.729	0.922		−0.586	1.066
10	−0.535	−1.340	20	−0.564	−0.908	30	0.107	0.226	40	−1.405	−0.157	50	0.220	−0.684
	1.484	0.542		−0.197	−0.050		−0.239	−0.072		−0.345	−0.296		0.023	−0.388
	−0.932	−0.799		−1.455	0.047		−3.243	−2.441		−0.709	−0.750		1.429	0.920
	0.115	0.067		1.360	0.411		−0.317	0.450		−0.530	−1.302		−0.074	0.602
	0.373	0.214		2.273	1.148		2.015	0.848		0.654	1.663		−0.697	0.383

Figure 3 shows that the process remains in control during the first 20 samples. However, when an assignable cause produces small shifts in the process mean ($\delta = 0.2$), the EWMA-t chart triggers an out-of-control signal in the 50th sample, whereas the GWMA-t chart triggers an out-of-control signal after the 49th sample. When an auxiliary variable exists, the first out-of-control signal is detected with the AIB-EWMA-t chart in sample 48, while 29 samples are required for the AIB-GWMA-t chart. The simulation results suggest that the AIB-GWMA-t chart is more sensitive for detecting small process mean shifts than the AIB-EWMA-t chart and its counterpart, the GWMA-t chart.

Table 6. Simulation dataset of the EWMA-t and GWMA-t charts when $\rho = 0.0$ and AIB-EWMA-t and AIB-GWMA-t charts when $\rho = 0.75$ at process mean shift $\delta = 0.2$.

			EWMA-t		**GWMA-t**		**AIB-EWMA-t**		**AIB-GWMA-t**	
No.	X_i^*	T_i^*	Z_i	***UCL***	G_i	***UCL***	Z_i^*	***UCL***	G_i^*	***UCL***
1	0.801	1.993	0.041	0.988	0.041	0.920	0.199	0.987	0.199	0.918
2	0.455	0.960	0.144	0.988	0.140	0.920	0.252	0.987	0.275	0.918
3	0.244	0.834	0.173	0.988	0.155	0.920	0.295	0.987	0.331	0.918
4	0.441	0.821	–0.018	0.988	–0.041	0.920	0.333	0.987	0.380	0.918
5	–0.021	–0.061	0.086	0.988	0.082	0.920	0.281	0.987	0.336	0.918
6	0.867	2.098	0.276	0.988	0.265	0.920	0.461	0.987	0.512	0.918
7	0.464	0.709	0.351	0.988	0.317	0.920	0.462	0.987	0.532	0.918
8	–0.016	–0.070	0.418	0.988	0.371	0.920	0.397	0.987	0.472	0.918
9	0.117	0.503	0.344	0.988	0.286	0.920	0.407	0.987	0.475	0.918
10	0.233	0.646	0.509	0.988	0.457	0.920	0.427	0.987	0.492	0.918
11	0.383	3.070	0.822	0.988	0.754	0.920	0.686	0.987	0.750	0.918
12	–0.083	–0.486	0.798	0.988	0.692	0.920	0.535	0.987	0.626	0.918
13	–0.048	–0.441	0.719	0.988	0.608	0.920	0.439	0.987	0.519	0.918
14	–0.013	–0.035	0.636	0.988	0.534	0.920	0.400	0.987	0.464	0.918
15	0.083	0.213	0.745	0.988	0.657	0.920	0.387	0.987	0.439	0.918
16	0.227	0.557	0.682	0.988	0.588	0.920	0.408	0.987	0.451	0.918
17	0.271	0.562	0.627	0.988	0.542	0.920	0.422	0.987	0.462	0.918
18	0.318	0.728	0.599	0.988	0.525	0.920	0.451	0.987	0.488	0.918
19	0.143	0.454	0.593	0.988	0.529	0.920	0.447	0.987	0.485	0.918
20	0.219	0.375	0.576	0.988	0.517	0.920	0.438	0.987	0.474	0.918
21	0.806	3.563	0.728	0.988	0.676	0.920	0.750	0.987	0.783	0.918
22	0.682	1.579	0.519	0.988	0.454	0.920	0.796	0.987	0.862	0.918
23	0.170	0.705	0.643	0.988	0.602	0.920	0.765	0.987	0.847	0.918
24	–0.795	–2.493	0.493	0.988	0.444	0.920	0.430	0.987	0.513	0.918
25	0.740	2.227	0.644	0.988	0.613	0.920	0.641	0.987	0.684	0.918
26	0.087	0.279	0.677	0.988	0.635	0.920	0.592	0.987	0.644	0.918
27	0.352	0.664	0.566	0.988	0.520	0.920	0.600	0.987	0.646	0.918
28	0.351	1.417	0.448	0.988	0.415	0.920	0.680	0.987	0.723	0.918
29	1.212	2.908	0.474	0.988	0.459	0.920	0.892	0.987	**0.941**	0.918
30	–0.237	–0.324	0.450	0.988	0.438	0.920	0.741	0.987	0.815	0.918
31	0.302	0.634	0.469	0.988	0.462	0.920	0.734	0.987	0.797	0.918
32	0.354	0.731	0.438	0.988	0.431	0.920	0.733	0.987	0.790	0.918
33	0.129	0.336	0.509	0.988	0.507	0.920	0.691	0.987	0.745	0.918
34	0.455	1.010	0.363	0.988	0.354	0.920	0.726	0.987	0.771	0.918
35	0.529	1.400	0.453	0.988	0.461	0.920	0.790	0.987	0.834	0.918
36	–0.147	–0.369	0.455	0.988	0.455	0.920	0.665	0.987	0.714	0.918
37	0.263	0.402	0.578	0.988	0.577	0.920	0.649	0.987	0.683	0.918
38	–0.053	–0.189	0.805	0.988	0.791	0.920	0.570	0.987	0.596	0.918
39	0.715	1.907	0.725	0.988	0.681	0.920	0.715	0.987	0.727	0.918
40	–0.383	–1.328	0.801	0.988	0.759	0.920	0.499	0.987	0.521	0.918
41	1.127	2.686	0.878	0.988	0.828	0.920	0.739	0.987	0.738	0.918
42	0.135	0.909	0.932	0.988	0.871	0.920	0.737	0.987	0.755	0.918
43	0.410	2.766	0.931	0.988	0.863	0.920	0.933	0.987	**0.956**	0.918
44	–0.100	–0.418	0.895	0.988	0.826	0.920	0.771	0.987	0.819	0.918
45	–0.130	–0.691	0.851	0.988	0.788	0.920	0.631	0.987	0.668	0.918
46	0.728	2.244	0.967	0.988	0.912	0.920	0.809	0.987	0.825	0.918
47	–0.245	–0.828	0.882	0.988	0.818	0.920	0.631	0.987	0.660	0.918
48	1.209	4.225	0.840	0.988	0.785	0.920	**1.005**	0.987	**1.016**	0.918
49	0.351	0.808	0.979	0.988	**0.934**	0.920	0.948	0.987	**0.995**	0.918
50	0.097	0.321	**1.022**	0.988	0.877	0.920	0.877	0.987	0.877	0.918

6. Conclusions

EWMA-t and AIB-EWMA-t charts are effective for monitoring changes in the process mean when the process standard deviation is unstable or poorly estimated. To enhance the detection ability, this study combines the features of the GWMA chart to propose GWMA-t and AIB-GWMA-t charts to monitor small process mean shifts. The EWMA-t, GWMA-t, and AIB-EWMA-t charts are special cases of the AIB-GWMA-t chart. Performance comparisons of the proposed AIB-GWMA-t charts are evaluated using the *ARL* indicator. The numerical simulations indicate that the AIB-GWMA-t chart performs substantially better than its reduced cases such as the AIB-EWMA-t, GWMA-t, and EWMA-t charts in detecting small shifts in the process mean. Moreover, the simulation also recommends that when the auxiliary variable is related to the quality characteristic between $\rho = 0.25$ and $\rho = 0.75$, the AIB-GWMA-t chart with large values ($q = 0.95$ and $\alpha = 0.9$) is a suitable alternative. Even with a high correlation coefficient $\rho = 0.95$, the AIB-GWMA-t chart with large q and α performs comparable to the AIB-EWMA-t chart. Finally, an example is provided to illustrate the implementation of our proposed AIB-GWMA-t chart as well as its reduced charts.

Author Contributions: Conceptualization, J.-H.C.; methodology, S.-L.L.; software, J.-H.C.; validation, J.-H.C. and S.-L.L.; formal analysis, S.-L.L.; investigation, J.-H.C. and S.-L.L.; writing—original draft preparation, S.-L.L.; writing—review and editing, J.-H.C. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A

The algorithmic description of the AIB-GWMA-t chart.

Input:
Set sample size: $n = 5, 10$; correlation coefficients: $\rho = 0.00, 0.25, 0.50, 0.75, 0.95$
Set parameters: $q = 0.5, 0.7, 0.9, 0.95$; $\alpha = 0.5, 0.7, 0.9, 1.0$
Set mean shifts: $\delta = 0.1, 0.2, 0.4, 0.6, 1.0, 2.0$
Output:
Out-of-control ARL_1
1. Set the desired in-control $ARL_0 \approx 500$
2. Generate pseudo bivariate normal random numbers (X_{in}, Y_{in})
3. Calculate the statistic for the AIB-GWMA-t chart is G_i^* by Equations (7) and (15)
4. Given appropriate value of L_{ag}, the LCL and UCL can be calculated by Equation (16)
5. Count the Run Length when G_i^* exceeds LCL or UCL
6. Execute 50,000 iterations of the Steps 2-5, the ARL corresponding to the specific shift size (δ) and (n, q, α, L_{ag}) combination is calculated.
7. Using the "Bi-Section" researching method, L_{ag}^* corresponding to the desired $ARL_0 \approx 500$ is obtained through repeating Steps 2–6 under in-control $(\delta = 0)$
8. Repeat Steps 2-6 to compute ARL_1 under specific shift size (δ) and (n, q, α, L_{ag}^*) combination

References

1. Roberts, S.W. Control chart tests based on geometric moving averages. *Technometrics* **1959**, *1*, 239–250. [CrossRef]
2. Crowder, S.V. Design of exponentially weighted moving average schemes. *J. Qual. Technol.* **1989**, *21*, 155–162. [CrossRef]
3. Ng, C.H.; Case, K.E. Development and evaluation of control charts using exponentially weighted moving averages. *J. Qual. Technol.* **1989**, *21*, 242–250. [CrossRef]
4. Lucas, J.M.; Saccucci, M.S. Exponentially weighted moving average control schemes: Properties and enhancements. *Technometrics* **1990**, *32*, 1–12. [CrossRef]
5. Steiner, S.H. EWMA control charts with time-varying control limits and fast initial response. *J. Qual. Technol.* **1999**, *31*, 75–86. [CrossRef]

6. Capizzi, G.; Masarotto, G. An adaptive exponentially weighted moving average control chart. *Technometrics* **2003**, *45*, 199–207. [CrossRef]
7. Sheu, S.H.; Lin, T.C. The generally weighted moving average control chart for detecting small shifts in the process mean. *Qual. Eng.* **2003**, *16*, 209–231. [CrossRef]
8. Sheu, S.H.; Chiu, W.C. Poisson GWMA control chart. *Commun. Stat. Simul. Comput.* **2007**, *36*, 1099–1114. [CrossRef]
9. Sheu, S.H.; Lu, S.L. Monitoring the mean of autocorrelated observations with one generally weighted moving average control chart. *J. Stat. Comput. Simul.* **2009**, *79*, 1393–1406. [CrossRef]
10. Lu, S.L. An extended nonparametric exponentially weighted moving average sign control chart. *Qual. Reliab. Eng. Int.* **2015**, *31*, 3–13. [CrossRef]
11. Lu, S.L. Novel design of composite generally weighted moving average and cumulative sum charts. *Qual. Reliab. Eng. Int.* **2017**, *33*, 2397–2408. [CrossRef]
12. Sheu, S.H.; Yang, L. The generally weighted moving average control chart for monitoring the process median. *Qual. Eng.* **2006**, *3*, 333–344. [CrossRef]
13. Sheu, S.H.; Hsieh, Y.T. The extended GWMA control chart. *J. Appl. Stat.* **2009**, *36*, 135–147. [CrossRef]
14. Huang, C.J.; Tai, S.H.; Lu, S.L. Measuring the performance improvement of a double generally weighted moving average control chart. *Expert Syst. Appl.* **2014**, *41*, 3313–3322. [CrossRef]
15. Chakraborty, N.; Chakraborti, S.; Human, S.W.; Balakrishnan, N. A generally weighted moving average signed-rank control chart. *Qual. Reliab. Eng. Int.* **2016**, *32*, 2835–2845. [CrossRef]
16. Aslam, M.; Al-marshadi, A.H.; Jun, C.H. Monitoring process mean using generally weighted moving average chart for exponentially distributed characteristics. *Commun. Stat. Simul. Comput.* **2017**, *46*, 3712–3722. [CrossRef]
17. Riaz, M. Monitoring process mean level using auxiliary information. *Stat. Neerl.* **2008**, *62*, 458–481. [CrossRef]
18. Ahmad, S.; Riaz, M.; Abbasi, S.A.; Lin, Z. On efficient median control charting. *J. Chin. Inst. Eng.* **2014**, *37*, 358–375. [CrossRef]
19. Riaz, M. Control charting and survey sampling techniques in process monitoring. *J. Chin. Inst. Eng.* **2015**, *38*, 342–354. [CrossRef]
20. Abbas, N.; Riaz, M.; Does, R.J.M.M. An EWMA-type control chart for monitoring the process mean using auxiliary information. *Commun. Stat. Theory Methods.* **2014**, *43*, 3485–3498. [CrossRef]
21. Haq, A.; Abidin, Z.U. An enhanced GWMA chart for process mean. *Commun. Stat. Simul. Comput.* **2018**. [CrossRef]
22. Zhang, L.; Chen, G.; Castagliola, P. On t and EWMA t charts for monitoring changes in the process mean. *Qual. Reliab. Eng. Int.* **2009**, *25*, 933–945. [CrossRef]
23. Celano, G.; Castagliola, P.; Trovato, E.; Fichera, S. Shewhart and EWMA t control charts for short production runs. *Qual. Reliab. Eng. Int.* **2011**, *27*, 313–326. [CrossRef]
24. Celano, G.; Castagliola, P.; Fichera, S.; Nenes, G. Performance of t control charts in short runs with unknown shift sizes. *Comput. Ind. Eng.* **2013**, *64*, 56–68. [CrossRef]
25. Haq, A.; Abidin, Z.U.; Khoo, M.B.C. An enhanced EWMA-t control chart for monitoring the process mean. *Commun. Stat. Theory Methods.* **2019**, *48*, 1333–1350. [CrossRef]