Article

Exact Soliton Solutions for Nonlinear Perturbed Schrödinger Equations with Nonlinear Optical Media

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Abstract: The nonlinear perturbed Schrödinger equations (NPSEs) with nonlinear terms as Kerr law, power law, quadratic-cubic law, and dual-power law nonlinearity media play an important role in optical fibers. In this article we implement the rational solitary wave method to study the NPSEs when nonlinear terms take some different forms. Additionally, we use the q-deformed hyperbolic function and q-deformed trigonometric function methods to study the exact solutions to NPSEs. Different kind of soliton solutions are obtained such as bright, dark, and singular periodic solutions to the NPSEs.

Keywords: nonlinear perturbed Schrödinger equations; Kerr law nonlinearity; nonlinear power law; quadratic-cubic law; anti-cubic law; solitary solutions

1. Introduction

In recent years, the research of optical (bright, dark, and singular) solutions for nonlinear Schrödinger equations have played an important role in nonlinear optical media. The optical pulses in the optical solutions play a significant role in communication systems, optical fibers, and so on. Many authors have discussed the optical solutions for the nonlinear Schrödinger equation (NSE) when the nonlinear terms are Kerr law, power law, parabolic law, dual-power law, saturating law, exponential law, higher order polynomial law, and triple-power law (see [1–17]). There are several previous studies which are of great interest in the optical solitons (see [18–22]). Ma et al. [23,24] have discussed exact solutions to NSEs and Lump solutions to higher order rational dispersion relations. Additionally, Ma [25] has provided better approximations to real physical nonlinear waves, which could exist in even linear wave models. Recently, Arshed [3] studied the exact solution for perturbed Schrödinger equations by using the $Exp(-\varphi(\xi))$ expansion function method.

In this research, we use the rational solitary solution methods and q-deformed functions methods for the following nonlinear perturbed Schrödinger equation [2–4,7–9]:

$$iQ_t + \sigma Q_{xt} + \varepsilon Q_{xx} + \zeta Q H(|Q|^2) = i [\alpha Q_x + \eta [|Q|^{2k} Q]_x + \nu Q (|Q|^{2k})_x]. \quad (1)$$

where σ represents the linear evolution phenomena, ε is dispersion velocity, ζ is the nonlinear term coefficient, α is the temporal dispersion, η is the self-steepening perturbation term, ν is the coefficient of nonlinear dispersion, and k is the power of nonlinear terms. We will ansatz the traveling wave transformation as

$$Q(x, t) = \psi(\xi) e^{i(-s_0 x + \omega t + \rho)}, \quad \xi = x - vt. \quad (2)$$

where $\psi(\xi)$ is the amplitude real function of the wave solution, v is the soliton velocity, s_0 is the frequency of the soliton, ω is the wave number, and the phase constant is ρ . Equation (2) leads to write Equation (1) as

$$(\varepsilon - \sigma v) \psi'' - (w + \alpha s_0 - \sigma w s_0 + \varepsilon s_0^2) \psi - s_0 \eta \psi^{2k+1} + \zeta \psi H(\psi^2) = 0 \quad (3)$$

and

$$(v(1 - \sigma s_0) - \sigma w + 2\varepsilon s_0 + \alpha) - [(2k + 1)\eta + 2kv] \psi^{2k} = 0. \quad (4)$$

Equations (3) and (4) are independent; consequently the soliton velocity is given by

$$v = \frac{2\varepsilon s_0 + \alpha - \sigma w}{(\sigma s_0 - 1)}, \quad \sigma s_0 - 1 \neq 0 \quad (5)$$

and the constraint condition

$$(2k + 1)\eta + 2kv = 0. \quad (6)$$

2. Analysis of Rational Solitary Solutions Methods (RSSM)

There are many methods to solve nonlinear partial differential equations (NPDEs) such as transformed rational function (TRF) method [26] and the multiple exp-function (MEF) method [24,25]. The RSSM is one of the important applications to the TRF method and MEF methods for solving the NPDEs. The transformation in Equation (2) is applied to transfer the NPDEs to NODEs. We suppose the solutions of the NODEs according to the RSSM have the shape

$$W(\xi) = a_0 + \sum_{i=0}^N a_i g^i(\xi) + \sum_{j=1}^N b_j g^{j-1}(\xi) f(\xi), \quad \alpha_N \neq 0, \text{ or } b_N \neq 0, \quad (7)$$

where the conditions on the two functions $g(\xi)$ and $f(\xi)$ take shapes:

$$\begin{aligned} f'(\xi) &= -A g^2(\xi) + \frac{\delta A}{B} g(\xi) [1 - B g(\xi)], \\ g'(\xi) &= -A f(\xi) g(\xi), \\ f^2(\xi) &= g^2(\xi) + \frac{\delta}{A^2} [1 - B g(\xi)]^2, \end{aligned} \quad (8)$$

$A, B, a_i, i = 0, 1, 2, \dots$ and $b_i, i = 0, 1, 2, \dots$ are constants and can be evaluated later, while $\delta = \pm 1$. The nonlinear auxiliary conditions in Equation (8) can be solved as follows:

If $\delta = 1$, the hyperbolic solutions take the following form:

$$f(\xi) = \frac{1}{A \tanh(\xi) + B \operatorname{sech}(\xi)}, \quad g(\xi) = \frac{\operatorname{sech}(\xi)}{A \tanh(\xi) + B \operatorname{sech}(\xi)}. \quad (9)$$

or

$$f(\xi) = \frac{\coth(\xi)}{A + B \operatorname{csch}(\xi)}, \quad g(\xi) = \frac{\operatorname{csch}(\xi)}{A + B \operatorname{csch}(\xi)}. \quad (10)$$

If $\delta = -1$, the rational periodic solutions take the following form:

$$f(\xi) = \frac{1}{A \tan(\xi) + B \sec(\xi)}, \quad g(\xi) = \frac{\sec(\xi)}{A \tan(\xi) + B \sec(\xi)}, \quad (11)$$

or

$$f(\xi) = \frac{\cot(\xi)}{A + B \csc(\xi)}, \quad g(\xi) = \frac{\csc(\xi)}{A + B \csc(\xi)}. \quad (12)$$

The homogenous idea of the balance power is used to determine the end sum of the series in Equation (7). Insert Equation (7) with the auxiliary Equation (8) into the NODEs and deduce the system of algebraic equations from comparison between the coefficients of $f(\xi)$ and $g(\xi)$. Resolve the aforesaid system of equations to $A, B, a_i, i = 0, 1, 2, \dots$ and $b_i, i = 0, 1, 2, \dots$ and insert the results in the Equation (3) solution to get the solutions of the NPDEs.

Remark 1. The system in Equation (8) can be presented by a single of NODE $Bg''(\xi) = g(\xi) [Bg^2\{(\delta + 2)A^2 + \delta B^2\} - \delta(A^2 + 2B^2)g + \delta B]$ and it has the same solutions (9)–(12) when $A^2 = B^2$.

2.1. Rational Soliton Solution to Nonlinear Perturbed Schrödinger Equations NPSEs with Kerr Law $H(\psi) = \psi$

The nonlinearity Kerr law PNSEs takes the form:

$$iQ_t + \sigma Q_{xt} + \varepsilon Q_{xx} + \zeta Q|Q|^2 = i[\alpha Q_x + \eta[|Q|^{2k}Q]_x + \nu Q(|Q|^{2k})_x]. \quad (13)$$

Equations (3) and (4) lead to write PNSEs in Equation (3) as follows

$$(\varepsilon - \sigma v)\psi'' - (w + \alpha s_0 - \sigma w s_0 + \varepsilon s_0^2)\psi + (\zeta - s_0\eta)\psi^3 = 0. \quad (14)$$

when the soliton velocity is given by $v = \frac{2\varepsilon s_0 + \alpha - \sigma w}{(\sigma s_0 - 1)}$, the condition $\eta = \frac{-2v}{3}$, and put $k = 1$ in the nonlinear media to be integrable. Balancing ψ'' and ψ^3 , we get the Equation (14), which can be written in the form:

$$\psi(\xi) = a_0 + a_1 f(\xi) + b_1 g(\xi), \alpha_1 \neq 0, \text{ or } b_1 \neq 0 \quad (15)$$

where $g(\xi)$, $f(\xi)$ satisfies the conditions in Equation (8) and a_0, a_1, b_1 are arbitrary constants. The solution formula in Equation (15) is a solution to Equation (14) under the conditions of Equation (8) if and only if:

Family 1. $\delta = 1$

$$a_0 = 0, a_1 = A \sqrt{\frac{\varepsilon - \sigma v}{2\eta s_0 - 2\zeta}}, (2\eta s_0 - 2\zeta)(\varepsilon - \sigma v) > 0, \eta s_0 - \zeta \neq 0, \\ b_1 = \sqrt{\frac{(A^2 + B^2)(\varepsilon - \sigma v)}{2\eta s_0 - 2\zeta}}, w = \frac{2\alpha s_0 + 2\varepsilon s_0^2 + \varepsilon - \sigma v}{2(\sigma s_0 - 1)}. \quad (16)$$

In this family the solution of PNSEs in Equation (13) take the form:

$$Q_1(x, t) = e^{i(-s_0 x + wt + \rho)} \left[\sqrt{\frac{\varepsilon - \sigma v}{2\eta s_0 - 2\zeta}} \frac{A}{A \tanh(x - vt) + B \operatorname{sech}(x - vt)} + \sqrt{\frac{(A^2 + B^2)(\varepsilon - \sigma v)}{2\eta s_0 - 2\zeta}} \frac{\operatorname{sech}(x - vt)}{A \tanh(x - vt) + B \operatorname{sech}(x - vt)} \right] \quad (17)$$

or

$$Q_2(x, t) = e^{i(-s_0 x + wt + \rho)} \left[\sqrt{\frac{\varepsilon - \sigma v}{2\eta s_0 - 2\zeta}} \frac{A \coth(x - vt)}{A + B \operatorname{csch}(x - vt)} + \sqrt{\frac{(A^2 + B^2)(\varepsilon - \sigma v)}{2\eta s_0 - 2\zeta}} \frac{\operatorname{csch}(x - vt)}{A + B \operatorname{csch}(x - vt)} \right] \quad (18)$$

Family 2. $\delta = -1$

$$a_0 = 0, a_1 = A \sqrt{\frac{\varepsilon - \sigma v}{2\eta s_0 - 2\zeta}}, (\varepsilon - \sigma v)(2\eta s_0 - 2\zeta) > 0, \eta s_0 - \zeta \neq 0 \\ b_1 = \sqrt{\frac{(A^2 - B^2)(\varepsilon - \sigma v)}{2\eta s_0 - 2\zeta}}, A > B, w = \frac{2\alpha s_0 + 2\varepsilon s_0^2 - \varepsilon + \sigma v}{2(\sigma s_0 - 1)}. \quad (19)$$

In this family, the solution of PNSEs in Equation (13) take the form:

$$Q_3(x, t) = e^{i(-s_0 x + wt + \rho)} \left[\sqrt{\frac{\varepsilon - \sigma v}{2\eta s_0 - 2\zeta}} \frac{A}{A \tan(x - vt) + B \sec(x - vt)} + \sqrt{\frac{(A^2 - B^2)(\varepsilon - \sigma v)}{2\eta s_0 - 2\zeta}} \frac{\sec(x - vt)}{A \tan(x - vt) + B \sec(x - vt)} \right], \quad (20)$$

or

$$Q_4(x, t) = e^{i(-s_0 x + wt + \rho)} \left[\sqrt{\frac{\varepsilon - \sigma v}{2\eta s_0 - 2\zeta}} \frac{A \cot(x - vt)}{A + B \csc(x - vt)} + \sqrt{\frac{(A^2 - B^2)(\varepsilon - \sigma v)}{2\eta s_0 - 2\zeta}} \frac{\csc(x - vt)}{A + B \csc(x - vt)} \right]. \quad (21)$$

2.2. Rational Soliton Solution to NPSEs with Power Law $H(\psi) = \psi^k, 0 < k < 0$

The nonlinearity power law PNSEs take the form:

$$iQ_t + \sigma Q_{xt} + \varepsilon Q_{xx} + \zeta Q |Q|^{2k} = i [\alpha Q_x + \eta [|Q|^{2k} Q]_x + \nu Q (|Q|^{2k})_x]. \quad (22)$$

Equations (3) and (4) lead to write PNSEs in Equation (22) as follows:

$$(\varepsilon - \sigma v) \psi'' - (w + \alpha s_0 - \sigma w s_0 + \varepsilon s_0^2) \psi + (\zeta - s_0 \eta) \psi^{2k+1} = 0, \quad (23)$$

when the soliton velocity is given by $v = \frac{2\varepsilon s_0 + \alpha - \sigma w}{(\sigma s_0 - 1)}$ and $\eta = \frac{-2kv}{2k+1}$. From balance the highest power of ψ'' and ψ^{2k+1} , we get $N = \frac{1}{k}$ as rational number, and we take the transformation

$$\psi(\xi) = [\varphi(\xi)]^{1/2k}. \quad (24)$$

The transformation in Equation (24) leads to obtain

$$(\varepsilon - \sigma v)(1 - 2k) \varphi'^2 + 2k(\varepsilon - \sigma v) \varphi \varphi'' - 4k^2(w + \alpha s_0 - \sigma w s_0 + \varepsilon s_0^2) \varphi^2 + 4k^2(\zeta - s_0 \eta) \varphi^3 = 0. \quad (25)$$

From $\varphi \varphi''$ and φ^3 , the solution of Equation (21) can be written in the form:

$$\psi(\xi) = a_0 + a_1 f(\xi) + b_1 g(\xi) + a_2 f(\xi) g(\xi) + b_2 g^2(\xi), \quad \alpha_2 \neq 0, \text{ or } b_2 \neq 0, \quad (26)$$

where $g(\xi), f(\xi)$ satisfies the conditions in Equation (8), and a_0, a_1, a_2, b_1, b_2 are arbitrary constants. The solution formula in Equation (26) is a solution to Equation (25) under conditions (8) if and only if:

Family 3. $\delta = 1$

$$\begin{aligned} a_2 &= \frac{A(k+1)(v\sigma-\varepsilon)\sqrt{A^2+B^2}}{2k^2(\eta s_0-\zeta)}, \quad a_1 = a_0 = 0, \quad \eta s_0 - \zeta \neq 0 \\ b_1 &= \frac{B(k+1)(v\sigma-\varepsilon)}{2k^2(\eta s_0-\zeta)}, \quad w = \frac{v\sigma-\varepsilon+4k^2s_0\alpha+4k^2s_0^2\varepsilon}{4k^2(\sigma s_0-1)}, \\ b_2 &= -\frac{(A^2+B^2)(k+1)(v\sigma-\varepsilon)}{2k^2(\eta s_0-\zeta)}. \end{aligned} \quad (27)$$

In this family the solution of PNSEs in Equation (22) takes the form:

$$Q_5(x, t) = e^{i(-s_0x+wt+\rho)} \left[\frac{B(k+1)(v\sigma-\varepsilon)\operatorname{sech}(x-vt)}{2k^2(\eta s_0-\zeta)[A\tanh(x-vt)+B\operatorname{sech}(x-vt)]} + \frac{A(k+1)(v\sigma-\varepsilon)\sqrt{A^2+B^2}\operatorname{sech}(x-vt)}{2k^2(\eta s_0-\zeta)[A\tanh(x-vt)+B\operatorname{sech}(x-vt)]^2} - \frac{(A^2+B^2)(k+1)(v\sigma-\varepsilon)\operatorname{sech}^2(x-vt)}{2k^2(\eta s_0-\zeta)[A\tanh(x-vt)+B\operatorname{sech}(x-vt)]^2} \right]^{1/2k} \quad (28)$$

or

$$Q_6(x, t) = e^{i(-s_0x+wt+\rho)} \left[\frac{B(k+1)(v\sigma-\varepsilon)\coth(x-vt)}{2k^2(\eta s_0-\zeta)[A+B\operatorname{csch}(x-vt)]} + \frac{A(k+1)(v\sigma-\varepsilon)\sqrt{A^2+B^2}\coth(x-vt)\operatorname{csch}(x-vt)}{2k^2(\eta s_0-\zeta)[A+B\operatorname{csch}(x-vt)]^2} - \frac{(A^2+B^2)(k+1)(v\sigma-\varepsilon)\operatorname{csch}^2(x-vt)}{2k^2(\eta s_0-\zeta)[A+B\operatorname{csch}(x-vt)]^2} \right]^{1/2k} \quad (29)$$

Family 4. $\delta = -1$

$$\begin{aligned} a_2 &= \frac{A(k+1)(v\sigma-\varepsilon)\sqrt{A^2-B^2}}{2k^2(\eta s_0-\zeta)}, \quad a_1 = a_0 = 0, \quad A^2 - B^2 > 0, \quad \eta s_0 - \zeta \neq 0 \\ b_1 &= -\frac{B(k+1)(v\sigma-\varepsilon)}{2k^2(\eta s_0-\zeta)}, \quad w = \frac{-v\sigma+\varepsilon+4k^2s_0\alpha+4k^2s_0^2\varepsilon}{4k^2(\sigma s_0-1)}, \\ b_2 &= -\frac{(A^2-B^2)(k+1)(v\sigma-\varepsilon)}{2k^2(\eta s_0-\zeta)}. \end{aligned} \quad (30)$$

In this family, the solution of PNSEs in Equation (22) take the form:

$$Q_7(x, t) = e^{i(-s_0x+wt+\rho)} \left[-\frac{B(k+1)(v\sigma-\varepsilon) \sec(x-vt)}{2k^2(\eta s_0-\zeta)[A \tan(x-vt)+B \sec(x-vt)]} + \frac{A(k+1)(v\sigma-\varepsilon) \sqrt{A^2-B^2} \sec(x-vt)}{2k^2(\eta s_0-\zeta)[A \tan(x-vt)+B \sec(x-vt)]^2} - \frac{(A^2-B^2)(k+1)(v\sigma-\varepsilon) \sec^2(x-vt)}{2k^2(\eta s_0-\zeta)[A \tan(x-vt)+B \sec(x-vt)]^2} \right]^{1/2k}. \quad (31)$$

or

$$Q_8(x, t) = e^{i(-s_0x+wt+\rho)} \left[-\frac{B(k+1)(v\sigma-\varepsilon) \csc(x-vt)}{2k^2(\eta s_0-\zeta)[A+B \csc(x-vt)]} + \frac{A(k+1)(v\sigma-\varepsilon) \sqrt{A^2-B^2} \cot(x-vt) \csc(x-vt)}{2k^2(\eta s_0-\zeta)[A+B \csc(x-vt)]^2} - \frac{(A^2-B^2)(k+1)(v\sigma-\varepsilon) \csc^2(x-vt)}{2k^2(\eta s_0-\zeta)[A+B \csc(x-vt)]^2} \right]^{1/2k}. \quad (32)$$

There are many different other solutions, but we leave it as a kind of convenience.

2.3. Rational Soliton Solution to NPSEs with Quadratic-Cubic Law $H(Q) = E_1 \sqrt{Q} + E_2 Q$

The nonlinearity quadratic law PNSEs take the form:

$$iQ_t + \sigma Q_{xt} + \varepsilon Q_{xx} + \zeta[E_1|Q| + E_2|Q|^2]Q = i[\alpha Q_x + \eta[|Q|^{2k}Q]_x + \nu Q(|Q|^{2k})_x], \quad (33)$$

where E_1 and E_2 are constants. Equations (3) and (4) lead to write PNSEs in Equation (33) as follows:

$$(\varepsilon - \sigma v) \psi'' - (w + \alpha s_0 - \sigma w s_0 + \varepsilon s_0^2) \psi + E_1 \psi^2 + E_2 \psi^3 - s_0 \eta \psi^{2k+1} = 0. \quad (34)$$

When the soliton velocity is given by $v = \frac{2\varepsilon s_0 + \alpha - \sigma w}{(\sigma s_0 - 1)}$, the condition $\eta = \frac{-2kv}{2k+1}$. Balancing the power of ψ'' and ψ^3 in the series solution in Equation (7) with the restriction in Equation (8), we have $N = 1$ and Equation (28) is integrable when $k = 1$. Consequently, we set the solution in the following form:

$$\psi(\xi) = a_0 + a_1 f(\xi) + b_1 g(\xi) \quad \alpha_1 \neq 0, \text{ or } b_1 \neq 0, \quad (35)$$

where $g(\xi)$, $f(\xi)$ satisfies the conditions in Equation (8), and a_0, a_1, b_1 are arbitrary constants. The solution formula in Equation (35) is a solution to Equation (34) under conditions in Equation (8) if and only if:

Family 5. $\delta = 1$

$$\begin{aligned} a_0 &= \frac{-E_1}{3(E_2 - s_0 \eta)}, \quad a_1 = \frac{AE_1}{3(E_2 - s_0 \eta)}, \\ b_1 &= \frac{E_1 \sqrt{A^2 + B^2}}{3(E_2 - s_0 \eta)}, \quad \sigma = \frac{2E_1^2 + 9E_2 \varepsilon - 9s_0 \varepsilon \eta}{9v(E_2 - s_0 \eta)}, \quad E_2 - s_0 \eta \neq 0 \\ w &= \frac{v(-9\varepsilon s_0^3 \eta + 9\varepsilon s_0^2 E_2 - 9\alpha s_0^2 \eta + 9\alpha s_0 E_2 + 2E_1^2)}{9s_0 \varepsilon E_2 + 9vs_0 \eta - 9s_0^2 \varepsilon \eta - 9vE_2 + 2s_0 E_1^2}. \end{aligned} \quad (36)$$

In this family the solution of PNSEs in Equation (33) take the form:

$$Q_9(x, t) = e^{i(-s_0x+wt+\rho)} \left[\frac{-E_1}{3(E_2 - s_0 \eta)} + \frac{AE_1}{3(E_2 - s_0 \eta)[A \tanh(x-vt) + B \operatorname{sech}(x-vt)]} + \frac{E_1 \sqrt{A^2 + B^2} \operatorname{sech}(x-vt)}{3(E_2 - s_0 \eta)[A \tanh(x-vt) + B \operatorname{sech}(x-vt)]} \right]. \quad (37)$$

or

$$Q_{10}(x, t) = e^{i(-s_0x+wt+\rho)} \left[\frac{-E_1}{3(E_2 - s_0 \eta)} + \frac{AE_1 \coth(x-vt)}{3(E_2 - s_0 \eta)[A + B \operatorname{csch}(x-vt)]} + \frac{E_1 \sqrt{A^2 + B^2} \operatorname{csch}(x-vt)}{3(E_2 - s_0 \eta)[A + B \operatorname{csch}(x-vt)]} \right]. \quad (38)$$

Family 6. $\delta = -1$

$$\begin{aligned}
 b_1 &= -\frac{3Ba_0(2E_2a_0-2s_0\eta a_0+E_1)}{3E_2a_0+E_1-3s_0\eta a_0}, A = \frac{B\sqrt{2E_1^2-3s_0E_1\eta a_0+3E_1E_2a_0}}{\sqrt{2}[3E_2a_0+E_1-3s_0\eta a_0]}, \\
 w &= \frac{v(-s_0\eta a_0^2+E_1a_0+E_2a_0^2-\varepsilon s_0^2-\alpha s_0)}{2s_0E_2a_0^2-2s_0^2\eta a_0^2-s_0\varepsilon+v+s_0E_1a_0}, \sigma = -\frac{1}{v}(2E_2a_0^2-2s_0\eta a_0^2+E_1a_0-\varepsilon), \\
 a_1 &= 0.2E_1^2-3s_0E_1\eta a_0+3E_1E_2a_0 > 0
 \end{aligned} \quad (39)$$

In this family, the solution of PNSEs in Equation (33) takes the form:

$$Q_{11} = \left[a_0 - \frac{3a_0(2E_2a_0-2s_0\eta a_0+E_1)\sec(x-vt)}{[\sqrt{2E_1^2-3s_0E_1\eta a_0+3E_1E_2a_0}\tan(x-vt)+(3E_2a_0+E_1-3s_0\eta a_0)\sec(x-vt)]} \right] e^{i(-s_0x+wt+\rho)}. \quad (40)$$

or

$$Q_{12}(x,t) = \left[a_0 - \frac{3a_0(2E_2a_0-2s_0\eta a_0+E_1)\csc(x-vt)}{[\sqrt{2E_1^2-3s_0E_1\eta a_0+3E_1E_2a_0}+(3E_2a_0+E_1-3s_0\eta a_0)\csc(x-vt)]} \right] e^{i(-s_0x+wt+\rho)}. \quad (41)$$

There are many different other families, but we leave it as a kind of convenience to the readers.

2.4. Rational Soliton Solution to NPS's with Anti-Cubic Law $H(Q) = \frac{M_1}{Q^2} + M_2Q + M_3Q^2$

The nonlinearity anti cubic law PNSEs takes the form:

$$iQ_t + \sigma Q_{xt} + \varepsilon Q_{xx} + \zeta[M_1|Q|^{-4} + M_2|Q|^2 + M_3|Q|^4]Q = i[\alpha Q_x + \eta[|Q|^{2k}Q]_x + \nu Q(|Q|^{2k})_x], \quad (42)$$

where M_1, M_2 and M_3 are nonzero constants. Equations (3) and (4) lead to write PNSEs in Equation (34) as follows:

$$(\varepsilon - \sigma v)\psi^3\psi'' - (w + \alpha s_0 - \sigma w s_0 + \varepsilon s_0^2)\psi^4 - 4s_0\eta\psi^{2k+4} + M_1 + M_2\psi^6 + M_3\psi^8 = 0. \quad (43)$$

when the soliton velocity is given by $v = \frac{2\varepsilon s_0 + \alpha - \sigma w}{(\sigma s_0 - 1)}$, the condition $\eta = \frac{-2kv}{2k+1}$. From equating the power of terms $\psi^3\psi''$ and We suppose $N = 1/2$, we have ψ^8

$$\psi = [\Omega(\xi)]^{1/2}. \quad (44)$$

From Equations (44) and (43) we get:

$$(\varepsilon - \sigma v)[2\Omega\Omega'' - \Omega'^2] - 4(w + \alpha s_0 - \sigma w s_0 + \varepsilon s_0^2)\Omega^2 - 4s_0\eta\Omega^4 + 4(M_1 + M_2\Omega^3 + M_3\Omega^4) = 0. \quad (45)$$

Consequently, we set the solution of Equation (45) in the following form:

$$\Omega(\xi) = a_0 + a_1f(\xi) + b_1g(\xi) \quad \alpha_1 \neq 0, \text{ or } b_1 \neq 0, \quad (46)$$

where $g(\xi), f(\xi)$ satisfies the conditions in Equation (8), and a_0, a_1, b_1 are arbitrary constants. The solution formula in Equation (46) is a solution to Equation (45) under conditions of Equation (8) if and only if:

Family 7. $\delta = 1$

$$\begin{aligned}
M_1 &= -\frac{a_0^2(\sigma v - \varepsilon)}{4b_1^2} [2a_0b_1B + a_0^2(A^2 + B^2) + b_1^2], \quad a_1 = 0, \\
M_2 &= -\frac{(\sigma v - \varepsilon)}{b_1^2} [2a_0(A^2 + B^2) + Bb_1], \quad b_1 \neq 0 \\
M_3 &= \frac{1}{4b_1^2} [3(\sigma v - \varepsilon)(A^2 + B^2) + 4s_0\eta b_1^2], \\
w &= \frac{1}{4b_1^2(\sigma s_0 - 1)} [6a_0^2(\sigma v - \varepsilon)(A^2 + B^2) + (\sigma v - \varepsilon)(b_1^2 + 6a_0b_1B) + 4\varepsilon s_0^2b_1^2 + 4\alpha s_0b_1^2].
\end{aligned} \tag{47}$$

In this family, the solution of PNSEs in Equation (42) takes the form:

$$Q_{13}(x, t) = e^{i(-s_0x + wt + \rho)} \left[a_0 + \frac{b_1 \operatorname{sech}(x - vt)}{[A \tanh(x - vt) + B \sec h(x - vt)]} \right]^{1/2}. \tag{48}$$

or

$$Q_{14}(x, t) = e^{i(-s_0x + wt + \rho)} \left[a_0 + \frac{b_1 \operatorname{csch}(x - vt)}{[A + B \operatorname{csch}(x - vt)]} \right]^{1/2}. \tag{49}$$

Family 8. $\delta = -1$

$$\begin{aligned}
M_1 &= \frac{a_0^2(\sigma v - \varepsilon)}{4b_1^2} [2a_0b_1B - a_0^2(A^2 - B^2) + b_1^2], \quad a_1 = 0, \\
M_2 &= \frac{(\sigma v - \varepsilon)}{b_1^2} [-2a_0(A^2 - B^2) + Bb_1], \\
M_3 &= \frac{1}{4b_1^2} [3(\sigma v - \varepsilon)(A^2 - B^2) + 4s_0\eta b_1^2], \\
w &= \frac{1}{4b_1^2(\sigma s_0 - 1)} [6a_0^2(\sigma v - \varepsilon)(A^2 - B^2) - (\sigma v - \varepsilon)(b_1^2 + 6a_0b_1B) + 4\varepsilon s_0^2b_1^2 + 4\alpha s_0b_1^2].
\end{aligned} \tag{50}$$

In this family, the solution of PNSEs in Equation (42) takes the form:

$$Q_{15}(x, t) = e^{i(-s_0x + wt + \rho)} \left[a_0 + \frac{b_1 \sec(x - vt)}{[A \tan(x - vt) + B \sec(x - vt)]} \right]^{1/2}, \tag{51}$$

or

$$Q_{16}(x, t) = e^{i(-s_0x + wt + \rho)} \left[a_0 + \frac{b_1 \csc(x - vt)}{[A + B \csc(x - vt)]} \right]^{1/2} \tag{52}$$

There are many different other families, but we leave it as a kind convenience to the readers.

3. Q-Deformed Hyperbolic Functions Method to NPSEs

Many authors have introduced the definition of q-deformed hyperbolic functions (see, for example, [7,12–14]). The q-deformed hyperbolic functions are defined by $\sinh_q(\xi) = \frac{1}{2}(e^\xi - qe^{-\xi})$, and $\cosh_q(\xi) = \frac{1}{2}(e^\xi + qe^{-\xi})$. There are many applications of q-deformed hyperbolic and trigonometric functions to NPDEs (see [26]). In the q-deformed hyperbolic functions methods, we suppose the solution of NODEs in the following form:

Type 1.

$$\psi(\xi) = A_0 + \sum_{i=1}^n (\operatorname{sech}_q \xi)^{i-1} (A_i \operatorname{sech}_q \xi + B_i \tanh_q \xi), \tag{53}$$

where

$$(\operatorname{sech}_q \xi)' = -\operatorname{sech}_q \xi \tanh_q \xi, \quad (\tanh_q \xi)' = q \operatorname{sech}_q^2(\xi), \quad q \operatorname{sech}_q^2(\xi) + \tanh_q^2 \xi = 1. \tag{54}$$

Type 2.

$$\varphi(\xi) = A_0 + \sum_{i=1}^n (\operatorname{csch}_q \xi)^{i-1} (A_i \operatorname{csch}_q \xi + B_i \coth_q \xi), \tag{55}$$

where

$$(\operatorname{csch}_q \xi)' = -\operatorname{csch}_q \xi \coth_q \xi, (\coth_q \xi)' = q \operatorname{csch}_q^2(\xi), q \operatorname{csch}_q^2(\xi) + \coth_q^2 \xi = 1. \quad (56)$$

3.1. Kerr Law $H(\psi) = \psi$

From Type 1, we suppose the solution to NPSEs in Equation (14) in the form in Equation (53). The formula in Equation (53) is a solution to Equation (14) when $n = 1$, hence

$$\psi(\xi) = A_0 + A_1 \operatorname{sech}_q \xi + B_1 \tanh_q \xi \quad (57)$$

Substituting Equation (57) into Equation (14) and setting the coefficients of $(\operatorname{sech}_q \xi)^j (\tanh_q \xi)^i$, $j = 0, 1, i = 0, 1, 2, \dots$, to be zero, we acquire the system equations, which can be solved to get:

Set 1.

$$A_0 = 0, A_1 = \sqrt{\frac{2q(\sigma v - \varepsilon)}{\eta s_0 - \varsigma}}, B_1 = 0, w = \frac{\varepsilon s_0^2 + \alpha s_0 + \sigma v - \varepsilon}{\sigma s_0 - 1}, q > 0, (\sigma v - \varepsilon)(\eta s_0 - \varsigma) > 0. \quad (58)$$

In this set, the solution to NPSEs in Equation (13) is

$$Q_{17}(x, t) = \sqrt{\frac{2q(\sigma v - \varepsilon)}{\eta s_0 - \varsigma}} e^{i(-s_0 x + wt + \rho)} \operatorname{sech}_q(x - vt) \quad (59)$$

Set 2.

$$A_0 = 0, A_1 = \sqrt{\frac{q(\sigma v - \varepsilon)}{2(\eta s_0 - \varsigma)}}, B_1 = \sqrt{\frac{-(\sigma v - \varepsilon)}{2(\eta s_0 - \varsigma)}}, w = \frac{\sigma v - 2\varepsilon s_0^2 - 2\alpha s_0 - \varepsilon}{2(1 - \sigma s_0)}, q < 0, \frac{(\sigma v - \varepsilon)}{\eta s_0 - \varsigma} < 0. \quad (60)$$

In this set, the solution to NPSEs in Equation (13) is

$$Q_{18}(x, t) = e^{i(-s_0 x + wt + \rho)} \left[\sqrt{\frac{q(\sigma v - \varepsilon)}{2(\eta s_0 - \varsigma)}} \operatorname{sech}_q(x - vt) + \sqrt{\frac{-(\sigma v - \varepsilon)}{2(\eta s_0 - \varsigma)}} \tanh_q(x - vt) \right] \quad (61)$$

For the type 2. the solutions of the NPSEs in Equation (13) are given by:

$$Q_{19}(x, t) = e^{i(-s_0 x + wt + \rho)} \left[\sqrt{\frac{2q(\varepsilon - \sigma v)}{\eta s_0 - \varsigma}} \operatorname{csch}_q(x - vt) \right], \frac{q(\varepsilon - \sigma v)}{v s_0 - \eta} > 0, w = \frac{\varepsilon s_0^2 + \sigma v - \varepsilon + \alpha s_0}{\sigma s_0 - 1} \quad (62)$$

$$Q_{20}(x, t) = e^{i(-s_0 x + wt + \rho)} \left[\sqrt{\frac{2(\varepsilon - \sigma v)}{\eta s_0 - \varsigma}} \coth_q(x - vt) \right], \frac{(\varepsilon - \sigma v)}{\eta s_0 - \varsigma} > 0, w = \frac{\varepsilon s_0^2 + \alpha s_0 + 2\varepsilon - 2\sigma v}{\sigma s_0 - 1}, \quad (63)$$

$$Q_{21}(x, t) = e^{i(-s_0 x + wt + \rho)} \left[\sqrt{\frac{q(\varepsilon - \sigma v)}{2(\eta s_0 - \varsigma)}} \operatorname{csch}_q(x - vt) + \sqrt{\frac{\varepsilon - \sigma v}{2(\eta s_0 - \varsigma)}} \coth_q(x - vt) \right], \quad (64)$$

$$\frac{(\varepsilon - \sigma v)}{\eta s_0 - \varsigma} > 0, w = \frac{2\varepsilon s_0^2 + 2\alpha s_0 + \varepsilon - \sigma v}{2\sigma s_0 - 1},$$

3.2. Power Law $H(\psi) = \psi^k, 0 < k < 2$

From Type 1, we suppose the solution to NPSEs in Equation (25) in the form in Equation (53). The formula in Equation (53) is a solution to Equation (25) when $n = 2$, consequently we have:

$$\psi(\xi) = A_0 + A_1 \operatorname{sech}_q \xi + B_1 \tanh_q \xi + A_2 \operatorname{sech}_q^2 \xi + B_2 \operatorname{sech}_q \xi \tanh_q \xi. \quad (65)$$

Equation (65) is a solution to Equation (25) when

Set 1.

$$A_2 = \frac{q(k+1)(\sigma v - \varepsilon)}{2k^2(\eta s_0 - \varsigma)}, B_2 = \frac{\sqrt{-q}(k+1)(\sigma v - \varepsilon)}{2k^2(\eta s_0 - \varsigma)}, w = \frac{4\varepsilon s_0^2 k^2 + 4\alpha s_0 k^2 + \sigma v - \varepsilon}{4k^2(\sigma s_0 - 1)}, q < 0, \\ A_0 = A_1 = B_1 = 0. \quad (66)$$

In this set, the solution to NPSEs in Equation (22) is

$$Q_{22}(x, t) = e^{i(-s_0 x + wt + \rho)} \left[\frac{q(k+1)(\sigma v - \varepsilon)}{2k^2(\eta s_0 - \varsigma)} \operatorname{sech}_q^2 \xi + \frac{\sqrt{-q}(k+1)(\sigma v - \varepsilon)}{2k^2(\eta s_0 - \varsigma)} \operatorname{sech}_q \xi \tanh_q \xi \right]^{1/2k} \quad (67)$$

Set 2.

$$A_2 = \frac{q(k+1)(\sigma v - \varepsilon)}{2k^2(\eta s_0 - \varsigma)}, w = \frac{\varepsilon s_0^2 k^2 + \alpha s_0 k^2 + \sigma v - \varepsilon}{k^2(\sigma s_0 - 1)}, A_0 = A_1 = B_1 = B_2 = 0, \quad (68)$$

In this set, the solution to NPSEs in Equation (22) is

$$Q_{23}(x, t) = e^{i(-s_0 x + wt + \rho)} \left[\frac{q(k+1)(\sigma v - \varepsilon)}{2k^2(\eta s_0 - \varsigma)} \operatorname{sech}_q^2 \xi \right]^{1/2k} \quad (69)$$

For the type 2. The solutions of the NPSEs in Equation (22) are given by:

$$Q_{22}(x, t) = e^{i(-s_0 x + wt + \rho)} \left[-\frac{q(k+1)(\sigma v - \varepsilon)}{2k^2(\eta s_0 - \varsigma)} \operatorname{csch}_q^2(\xi) + \frac{\sqrt{q}(k+1)(\sigma v - \varepsilon)}{2k^2(\eta s_0 - \varsigma)} \operatorname{csch}_q \xi \coth_q \xi \right]^{1/2k} \quad (70)$$

where $w = \frac{4\varepsilon s_0^2 k^2 + 4\alpha s_0 k^2 + \sigma v - \varepsilon}{4k^2(\sigma s_0 - 1)}$ and $q > 0$

$$Q_{23}(x, t) = e^{i(-s_0 x + wt + \rho)} \left[\frac{-q(k+1)(\sigma v - \varepsilon)}{k^2(\eta s_0 - \varsigma)} \operatorname{csch}_q^2 \xi \right]^{1/2k}. \quad (71)$$

where $w = \frac{\varepsilon s_0^2 k^2 + \alpha s_0 k^2 + \sigma v - \varepsilon}{k^2(\sigma s_0 - 1)}$.

3.3. Quadratic-Cubic Law $H(Q) = E_1 \sqrt{Q} + E_2 Q$

From Type 1, we suppose the solution to NPSEs in Equation (34) in the form in Equation (53). The formula in Equation (53) is a solution to Equation (34) when $n = 1$, and Equation (34) is integrable when $k = 1$. Suppose a solution of Equation (34) in the following form:

$$\psi(\xi) = A_0 + A_1 \operatorname{sech}_q \xi + B_1 \tanh_q \xi. \quad (72)$$

Equation (72) is a solution of Equation (34) when

Set 1.

$$A_0 = \frac{-6(\sigma v - \varepsilon)}{E_1}, A_1 = 0, B_1 = \frac{6(\sigma v - \varepsilon)}{E_1}, w = \frac{\varepsilon s_0^2 + \alpha s_0 + 4\sigma v - 4\varepsilon}{\sigma s_0 - 1}, E_2 = \frac{18s_0 \eta (\sigma v - \varepsilon) + E_1^2}{18(\sigma v - \varepsilon)}. \quad (73)$$

In this set, the solution to NPSEs in Equation (33) is

$$Q_{24}(x, t) = \left[\frac{-6(\sigma v - \varepsilon)}{E_1} + \frac{6(\sigma v - \varepsilon)}{E_1} \tanh_q(x - vt) \right] e^{i(-s_0 x + wt + \rho)}. \quad (74)$$

Set 2.

$$A_0 = \frac{3(\sigma v - \varepsilon)}{E_1}, A_1 = \frac{3\sqrt{2q}(\sigma v - \varepsilon)}{E_1}, B_1 = 0, w = \frac{\varepsilon s_0^2 + \alpha s_0 - 2\sigma v + 2\varepsilon}{\sigma s_0 - 1}, E_2 = -\frac{-9s_0 \eta (\sigma v - \varepsilon) + E_1^2}{9(\sigma v - \varepsilon)}, q > 0. \quad (75)$$

In this set, the solution to NPSEs in Equation (33) is

$$Q_{25}(x, t) = \left[\frac{3(\sigma v - \varepsilon)}{E_1} + \frac{3\sqrt{2q}(\sigma v - \varepsilon)}{E_1} \operatorname{sech}_q(x - vt) \right] e^{i(-s_0 x + wt + \rho)}. \quad (76)$$

Set 3.

$$A_0 = \frac{-3(\sigma v - \varepsilon)}{2E_1}, A_1 = \frac{3(\sigma v - \varepsilon)\sqrt{-q}}{2E_1}, B_1 = \frac{3(\sigma v - \varepsilon)}{2E_1}, \\ w = \frac{\varepsilon s_0^2 + \alpha s_0 + \sigma v - \varepsilon}{\sigma s_0 - 1}, E_2 = \frac{9s_0\eta(\sigma v - \varepsilon) + 2E_1^2}{9(\sigma v - \varepsilon)}, q < 0 \quad (77)$$

In this set, the solution to NPSEs in Equation (33) is

$$Q_{26}(x, t) = \left[\frac{-3(\sigma v - \varepsilon)}{2E_1} + \frac{3(\sigma v - \varepsilon)\sqrt{-q}}{2E_1} \operatorname{sech}_q(x - vt) + \frac{3(\sigma v - \varepsilon)}{2E_1} \tanh_q(x - vt) \right] e^{i(-s_0 x + wt + \rho)}. \quad (78)$$

For the type 2. The solutions of the NPSEs in Equation (33) are given by

$$Q_{27}(x, t) = \left[-\frac{6(\sigma v - \varepsilon)}{E_1} - \frac{6(\sigma v - \varepsilon)}{E_1} \coth_q(x - vt) \right] e^{i(s_0 x + wt + \rho)}, \quad (79)$$

where $w = \frac{\varepsilon s_0^2 + \alpha s_0 + 4(\sigma v - \varepsilon)}{(\sigma s_0 - 1)}$ and $E_2 = \frac{18s_0\eta(\sigma v - \varepsilon) + E_1^2}{18(\sigma v - \varepsilon)}$.

$$Q_{28}(x, t) = \left[\frac{3(\sigma v - \varepsilon)}{E_1} + \frac{3\sqrt{-2q}(\sigma v - \varepsilon)}{E_1} \operatorname{csch}_q(x - vt) \right] e^{i(s_0 x + wt + \rho)}. \quad (80)$$

where $w = \frac{\varepsilon s_0^2 + \alpha s_0 - 2\sigma v + 2\varepsilon}{\sigma s_0 - 1}$ and $E_2 = -\frac{9s_0\eta(\sigma v - \varepsilon) + E_1^2}{9(\sigma v - \varepsilon)}, q < 0$.

$$Q_{29}(x, t) = \left[\frac{-3(\sigma v - \varepsilon)}{2E_1} + \frac{3(\sigma v - \varepsilon)\sqrt{q}}{2E_1} \operatorname{csch}_q(x - vt) - \frac{3(\sigma v - \varepsilon)}{2E_1} \coth_q(x - vt) \right] e^{i(-s_0 x + wt + \rho)}. \quad (81)$$

where $w = \frac{\varepsilon s_0^2 + \alpha s_0 + (\sigma v - \varepsilon)}{(\sigma s_0 - 1)}$ and $E_2 = \frac{9s_0\eta(\sigma v - \varepsilon) + 2E_1^2}{9(\sigma v - \varepsilon)}, q > 0$.

3.4. Anti-Cubic Law $H(Q) = M_1 Q^{-2} + M_2 Q + M_3 Q^2$

From Type 1, we suppose the solution to NPSEs in Equation (45) in the form in Equation (53). The formula in Equation (53) is a solution to Equation (45) when $n = 1$. Suppose a solution of Equation (45) in the following form:

$$\Omega(\xi) = A_0 + A_1 \operatorname{sech}_q \xi + B_1 \tanh_q \xi, \quad (82)$$

Equation (82) is a solution of Equation (45) when

Set 1.

$$A_1 = 0, B_1 = \sqrt{\frac{3(\sigma v - \varepsilon)}{4(M_3 - \eta s_0)}}, \frac{(\sigma v - \varepsilon)}{(M_3 - \eta s_0)} > 0, w = \frac{4A_0^2 (M_3 - \eta s_0) + 2\varepsilon s_0^2 + 2\alpha s_0 + \varepsilon - \sigma v}{2(\sigma s_0 - 1)}, \\ M_2 = -\frac{8A_0}{3}(M_3 - \eta s_0), M_1 = -\frac{1}{48(M_3 - \eta s_0)}[(4M_3 A_0^2 - 3\sigma v) + (3\varepsilon - 4\eta s_0 A_0^2)]^2. \quad (83)$$

In this set, the solution to NPSEs in Equation (45) is

$$Q_{30}(x, t) = \left[A_0 + \sqrt{\frac{3(\sigma v - \varepsilon)}{4(M_3 - \eta s_0)}} \tanh_q(x - vt) \right]^{1/2} e^{i(-s_0 x + wt + \rho)}. \quad (84)$$

Set 2.

$$B_1 = 0, A_1 = \sqrt{\frac{3q(\sigma v - \varepsilon)}{4(M_3 - \eta s_0)}}, \frac{q(\sigma v - \varepsilon)}{(M_3 - \eta s_0)} > 0, w = \frac{8A_0^2 (M_3 - \eta s_0) + 4\varepsilon s_0^2 + 4\alpha s_0 + \sigma v - \varepsilon}{4(\sigma s_0 - 1)}, \quad (85)$$

$$M_2 = -\frac{8}{3}A_0(M_3 - \eta s_0), M_1 = -\frac{A_0^2}{12}[4A_0^2(M_3 - \eta s_0) + 3(\sigma v - \varepsilon)].$$

In this set, the solution to NPSEs in Equation (42) is

$$Q_{31}(x, t) = \left[A_0 + \sqrt{\frac{3q(\sigma v - \varepsilon)}{4(M_3 - \eta s_0)}} \operatorname{sech}_q(x - vt) \right]^{1/2} e^{i(-s_0 x + wt + \rho)}. \quad (86)$$

Set 3.

$$A_1 = \frac{1}{4} \sqrt{\frac{-3q(\sigma v - \varepsilon)}{(M_3 - \eta s_0)}}, B_1 = \frac{1}{4} \sqrt{\frac{3(\sigma v - \varepsilon)}{(M_3 - \eta s_0)}}, \frac{(\sigma v - \varepsilon)}{(M_3 - \eta s_0)} > 0, q < 0, M_2 = -\frac{8A_0}{3}(M_3 - \eta s_0), \quad (87)$$

$$w = \frac{16A_0^2(M_3 - \eta s_0) + 8\varepsilon s_0^2 + 8\alpha s_0 + \varepsilon - \sigma v}{8(\sigma s_0 - 1)}, M_1 = -\frac{1}{768(M_3 - \eta s_0)}[(16M_3A_0^2 + 3\varepsilon) - (16\eta s_0A_0^2 + 3\sigma v)]^2.$$

In this set, the solution to NPSEs in Equation (42) is

$$Q_{32}(x, t) = \left[A_0 + \frac{1}{4} \sqrt{\frac{-3q(\sigma v - \varepsilon)}{(M_3 - \eta s_0)}} \operatorname{sech}_q(x - vt) + \frac{1}{4} \sqrt{\frac{3(\sigma v - \varepsilon)}{(M_3 - \eta s_0)}} \tanh_q(x - vt) \right]^{1/2} e^{i(-s_0 x + wt + \rho)}. \quad (88)$$

For the type 2. The solutions of the NPSEs in Equation (42) are given by

$$Q_{33}(x, t) = \left[A_0 + \sqrt{\frac{3(\sigma v - \varepsilon)}{4(M_3 - \eta s_0)}} \coth_q(x - vt) \right]^{1/2} e^{i(-s_0 x + wt + \rho)}, \frac{3(\sigma v - \varepsilon)}{4(M_3 - \eta s_0)} > 0, \quad (89)$$

where $M_1 = -\frac{1}{48(M_3 - \eta s_0)}[(4M_3A_0^2 - 3\sigma v) + (3\varepsilon - 4\eta s_0A_0^2)]^2$, $M_2 = -\frac{8}{3}(M_3 - \eta s_0)$, and $w = \frac{4A_0^2(M_3 - \eta s_0) + 2\varepsilon s_0^2 + 2\alpha s_0 + \varepsilon - \sigma v}{2(\sigma s_0 - 1)}$.

$$Q_{34}(x, t) = \left[A_0 + \sqrt{\frac{3q(\sigma v - \varepsilon)}{4(M_3 - \eta s_0)}} \operatorname{csch}_q(x - vt) \right]^{1/2} e^{i(-s_0 x + wt + \rho)}, \frac{3q(\sigma v - \varepsilon)}{4(M_3 - \eta s_0)} > 0, \quad (90)$$

where $M_1 = -\frac{A_0^2}{12}[4A_0^2(M_3 - \eta s_0) + 3(\sigma v - \varepsilon)]$, $M_2 = -\frac{8}{3}A_0(M_3 - \eta s_0)$, and $w = \frac{8A_0^2(M_3 - \eta s_0) + 4\varepsilon s_0^2 + 4\alpha s_0 + \sigma v - \varepsilon}{4(\sigma s_0 - 1)}$.

$$Q_{35}(x, t) = \left[A_0 + \frac{1}{4} \sqrt{\frac{3q(\sigma v - \varepsilon)}{(M_3 - \eta s_0)}} \operatorname{csch}_q(x - vt) + \frac{1}{4} \sqrt{\frac{3(\sigma v - \varepsilon)}{(M_3 - \eta s_0)}} \coth_q(x - vt) \right]^{1/2} e^{i(-s_0 x + wt + \rho)}. \quad (91)$$

where $\frac{3(\sigma v - \varepsilon)}{(M_3 - \eta s_0)} > 0$, $M_1 = -\frac{1}{768(M_3 - \eta s_0)}[(16M_3A_0^2 + 3\varepsilon) - (16\eta s_0A_0^2 + 3\sigma v)]^2$ and $q > 0$, $M_2 = -\frac{8A_0}{3}(M_3 - \eta s_0)$ and $w = \frac{16A_0^2(M_3 - \eta s_0) + 8\varepsilon s_0^2 + 8\alpha s_0 + \varepsilon - \sigma v}{8(\sigma s_0 - 1)}$.

4. Q-Deformed Trigonometric Functions Method

Many authors have defined the q-deformed trigonometric functions in [7,12–14,26] as $\sin_q(\xi) = \frac{1}{2i}(e^{i\xi} - qe^{-i\xi})$, and $\cos_q(\xi) = \frac{1}{2}(e^{i\xi} + qe^{-i\xi})$. In the q-deformed trigonometric functions methods, we suppose the solution of Equation (14) in the following form:

Type 3.

$$\psi(\xi) = A_0 + \sum_{i=0}^n (\sec_q \xi)^{i-1} (A_i \sec_q \xi + B_i \tan_q \xi), \quad (92)$$

where

$$(\sec_q \xi)' = \sec_q \xi \tan_q \xi, \quad (\tan_q \xi)' = q \sec_q^2(\xi), \quad q \sec_q^2(\xi) - \tan_q^2 \xi = 1 \quad (93)$$

Type 4.

$$\psi(\xi) = A_0 + \sum_{i=0}^n (\csc_q \xi)^{i-1} (A_i \csc_q \xi + B_i \cot_q \xi), \quad (94)$$

where

$$(\csc_q \xi)' = -\csc_q \xi \cot_q \xi, \quad (\cot_q \xi)' = -q \csc_q^2(\xi), \quad q \csc_q^2(\xi) - \cot_q^2 \xi = 1 \quad (95)$$

4.1. Kerr Law $H(\psi) = \psi$

From Type 3, we suppose the solution to NPSEs in Equation (14) in the form in Equation (92). The formula in Equation (92) is a solution to Equation (14) when $n = 1$, hence

$$\psi(\xi) = A_0 + A_1 \sec_q \xi + B_1 \tan_q \xi. \quad (96)$$

Substituting Equation (96) into Equation (14) and setting the coefficients of $(\sec_q \xi)^j (\tan_q \xi)^i$, $j = 0, 1, i = 0, 1, 2, \dots$, to be zero, we acquire the system equations which can be solved to have:

Set 1.

$$A_0 = 0, \quad A_1 = \sqrt{\frac{2q(\varepsilon - \sigma v)}{\eta s_0 - \varsigma}}, \quad B_1 = 0, \quad w = \frac{\varepsilon s_0^2 + \alpha s_0 - \sigma v + \varepsilon}{\sigma s_0 - 1}, \quad q > 0, \quad \frac{\varepsilon - \sigma v}{\eta s_0 - \varsigma} > 0. \quad (97)$$

In this set, the solution to NPSEs in Equation (13) is

$$Q_{36}(x, t) = \sqrt{\frac{2q(\varepsilon - \sigma v)}{\eta s_0 - \varsigma}} e^{i(-s_0 x + wt + \rho)} \sec_q(x - vt) \quad (98)$$

Set 2.

$$A_0 = 0, \quad A_1 = 0, \quad B_1 = \sqrt{\frac{2(\varepsilon - \sigma v)}{\eta s_0 - \varsigma}}, \quad w = \frac{\varepsilon s_0^2 + \alpha s_0 + 2\sigma v - 2\varepsilon}{\sigma s_0 - 1}, \quad \frac{\varepsilon - \sigma v}{\eta s_0 - \varsigma} > 0. \quad (99)$$

In this set, the solution to NPSEs in Equation (13) is

$$Q_{37}(x, t) = \sqrt{\frac{2(\varepsilon - \sigma v)}{\eta s_0 - \varsigma}} e^{i(-s_0 x + wt + \rho)} \tan_q(x - vt) \quad (100)$$

Set 3.

$$A_0 = 0, \quad A_1 = \sqrt{\frac{q(\varepsilon - \sigma v)}{2(\eta s_0 - \varsigma)}}, \quad B_1 = \sqrt{\frac{(\varepsilon - \sigma v)}{2(\eta s_0 - \varsigma)}}, \quad w = \frac{\sigma v + 2\varepsilon s_0^2 + 2\alpha s_0 - \varepsilon}{2(\sigma s_0 - 1)}, \quad q > 0, \quad \frac{(\varepsilon - \sigma v)}{(\eta s_0 - \varsigma)} > 0. \quad (101)$$

In this set, the solution to NPSEs in Equation (13) is

$$Q_{38}(x, t) = e^{i(-s_0x+wt+\rho)} \left[\sqrt{\frac{q(\varepsilon - \sigma v)}{2(\eta s_0 - \varsigma)}} \sec_q(x - vt) + \sqrt{\frac{(\varepsilon - \sigma v)}{2(\eta s_0 - \varsigma)}} \tan_q(x - vt) \right] \quad (102)$$

For the type 4. The solutions of the NPSEs in Equation (13) are given by:

$$Q_{39}(x, t) = \sqrt{\frac{2q(\varepsilon - \sigma v)}{\eta s_0 - \varsigma}} e^{i(-s_0x+wt+\rho)} \csc_q(x - vt) \quad (103)$$

where $w = \frac{\varepsilon s_0^2 + \alpha s_0 - \sigma v + \varepsilon}{\sigma s_0 - 1}$, $q > 0$, and $\frac{\varepsilon - \sigma v}{\eta s_0 - \varsigma} > 0..$

$$Q_{40}(x, t) = \sqrt{\frac{2(\varepsilon - \sigma v)}{\eta s_0 - \varsigma}} e^{i(-s_0x+wt+\rho)} \cot_q(x - vt) \quad (104)$$

where $w = \frac{\varepsilon s_0^2 + \alpha s_0 + 2\sigma v - 2\varepsilon}{\sigma s_0 - 1}$, $q > 0$, and $\frac{\varepsilon - \sigma v}{\eta s_0 - \varsigma} > 0..$

$$Q_{41}(x, t) = e^{i(-s_0x+wt+\rho)} \left[\sqrt{\frac{q(\varepsilon - \sigma v)}{2(\eta s_0 - \varsigma)}} \csc_q(x - vt) + \sqrt{\frac{(\varepsilon - \sigma v)}{2(\eta s_0 - \varsigma)}} \cot_q(x - vt) \right], \quad (105)$$

where $w = \frac{\sigma v + 2\varepsilon s_0^2 + 2\alpha s_0 - \varepsilon}{2(\sigma s_0 - 1)}$ and $\frac{q(\varepsilon - \sigma v)}{(\eta s_0 - \varsigma)} > 0$.

4.2. Power Law $H(\psi) = \psi^k, 0 < k < 2$

From Type 3, we suppose the solution to NPSEs in Equation (25) in the form in Equation (92). The formula in Equation (92) is a solution to Equation (25) when $n = 2$; consequently we have:

$$\psi(\xi) = A_0 + A_1 \sec_q \xi + B_1 \tan_q \xi + A_2 \sec_q^2 \xi + B_2 \sec_q \xi \tan_q \xi \quad (106)$$

Equation (106) is a solution to Equation (25) when

Set 1.

$$A_2 = -\frac{q(k+1)(\sigma v - \varepsilon)}{2k^2(\eta s_0 - \varsigma)}, \quad B_2 = \frac{\sqrt{q}(k+1)(\sigma v - \varepsilon)}{2k^2(\eta s_0 - \varsigma)}, \quad w = \frac{4\varepsilon s_0^2 k^2 + 4\alpha s_0 k^2 - \sigma v + \varepsilon}{4k^2(\sigma s_0 - 1)}, \quad q > 0, \\ A_0 = A_1 = B_1 = 0, \quad (107)$$

In this set, the solution to NPSEs in Equation (22) is

$$Q_{42}(x, t) = e^{i(-s_0x+wt+\rho)} \left[-\frac{q(k+1)(\sigma v - \varepsilon)}{2k^2(\eta s_0 - \varsigma)} \sec_q^2(x - vt) + \frac{\sqrt{q}(k+1)(\sigma v - \varepsilon)}{2k^2(\eta s_0 - \varsigma)} \sec_q(x - vt) \tan_q(x - vt) \right]^{1/2k}. \quad (108)$$

Set 2.

$$A_2 = \frac{q(k+1)(\sigma v - \varepsilon)}{2k^2(\eta s_0 - \varsigma)}, \quad w = \frac{\varepsilon s_0^2 k^2 + \alpha s_0 k^2 - \sigma v + \varepsilon}{k^2(\sigma s_0 - 1)}, \quad A_0 = A_1 = B_1 = B_2 = 0, \quad (109)$$

In this set, the solution to NPSEs in Equation (22) is

$$Q_{43}(x, t) = e^{i(-s_0x+wt+\rho)} \left[\frac{q(k+1)(\sigma v - \varepsilon)}{2k^2(\eta s_0 - \varsigma)} \sec_q^2(x - vt) \right]^{\frac{1}{2k}}. \quad (110)$$

For the type 4. The solutions of the NPSEs in Equation (22) are given by:

$$Q_{43}(x, t) = e^{i(-s_0x+wt+\rho)} \left[-\frac{q(k+1)(\sigma v - \varepsilon)}{2k^2(\eta s_0 - \varsigma)} \csc^2 q(x - vt) + \frac{\sqrt{q}(k+1)(\sigma v - \varepsilon)}{2k^2(\eta s_0 - \varsigma)} \csc q(x - vt) \cot q(x - vt) \right]^{\frac{1}{2k}}. \quad (111)$$

where $w = \frac{4\varepsilon s_0^2 k^2 + 4\alpha s_0 k^2 - \sigma v + \varepsilon}{4k^2(\sigma s_0 - 1)}$ and $q > 0$

$$Q_{44}(x, t) = e^{i(-s_0x+wt+\rho)} \left[\frac{-q(k+1)(\sigma v - \varepsilon)}{k^2(\eta s_0 - \varsigma)} \csc^2 q(x - vt) \right]^{\frac{1}{2k}}. \quad (112)$$

where $w = \frac{\varepsilon s_0^2 k^2 + \alpha s_0 k^2 + \sigma v - \varepsilon}{k^2(\sigma s_0 - 1)}$

4.3. Quadratic-Cubic Law $H(Q) = E_1 \sqrt{Q} + E_2 Q$

From Type 3, we suppose the solution to NPSEs in Equation (34) in the form in Equation (92). The formula in Equation (92) is a solution to Equation (34) when $n = 1$, and Equation (34) is integral when $k = 1$. Suppose a solution of Equation (34) in the following form:

$$\psi(\xi) = A_0 + A_1 \sec_q \xi + B_1 \tan_q \xi. \quad (113)$$

Equation (113) is a solution of Equation (34) when

Set 1.

$$A_0 = -\frac{3(\sigma v - \varepsilon)}{E_1}, A_1 = \frac{3\sqrt{2q}(\sigma v - \varepsilon)}{E_1}, B_1 = 0, w = \frac{\varepsilon s_0^2 + \alpha s_0 + 2\sigma v - 2\varepsilon}{\sigma s_0 - 1}, E_2 = \frac{9s_0\eta(\sigma v - \varepsilon) + E_1^2}{9(\sigma v - \varepsilon)}, q > 0. \quad (114)$$

In this set, the solution to NPSEs in Equation (33) is

$$Q_{45}(x, t) = \left[-\frac{3(\sigma v - \varepsilon)}{E_1} + \frac{3\sqrt{2q}(\sigma v - \varepsilon)}{E_1} \sec_q(x - vt) \right] e^{i(-s_0x+wt+\rho)}. \quad (115)$$

For the type 4. The solutions of the NPSEs in Equation (33) are given by:

$$Q_{46}(x, t) = \left[-\frac{3(\sigma v - \varepsilon)}{E_1} + \frac{3\sqrt{2q}(\sigma v - \varepsilon)}{E_1} \csc_q(x - vt) \right] e^{i(-s_0x+wt+\rho)}. \quad (116)$$

where $w = \frac{\varepsilon s_0^2 + \alpha s_0 + 2\sigma v - 2\varepsilon}{\sigma s_0 - 1}$ and $E_2 = \frac{9s_0\eta(\sigma v - \varepsilon) + E_1^2}{9(\sigma v - \varepsilon)}, q > 0$.

4.4. Anti-Cubic Law $H(Q) = M_1 Q^{-2} + M_2 Q + M_3 Q^2$

From Type 3, we suppose the solution to NPSEs in Equation (45) in the form in Equation (92). The formula in Equation (92) is a solution to Equation (45) when $n = 1$. Suppose a solution of Equation (45) in the following form:

$$\Omega(\xi) = A_0 + A_1 \sec_q \xi + B_1 \tan_q \xi \quad (117)$$

Equation (117) is a solution of Equation (45) when

Set 1.

$$A_1 = 0, B_1 = \sqrt{\frac{3(\sigma v - \varepsilon)}{4(M_3 - \eta s_0)}}, \frac{(\sigma v - \varepsilon)}{(M_3 - \eta s_0)} > 0, w = \frac{4A_0^2(M_3 - \eta s_0) + 2\varepsilon s_0^2 + 2\alpha s_0 - \varepsilon + \sigma v}{2(\sigma s_0 - 1)}, \quad (118)$$

$$M_2 = -\frac{8}{3}A_0(M_3 - \eta s_0), M_1 = -\frac{1}{48(M_3 - \eta s_0)}[(4M_3A_0^2 + 3\sigma v) - (3\varepsilon + 4\eta s_0A_0^2)]^2$$

In this set, the solution to NPSEs in Equation (42) is

$$Q_{47}(x, t) = \left[A_0 + \sqrt{\frac{3q(\sigma v - \varepsilon)}{4(M_3 - \eta s_0)}} \tan_q(x - vt) \right]^{1/2} e^{i(-s_0 x + wt + \rho)}. \quad (119)$$

Set 2.

$$B_1 = 0, \quad A_1 = \frac{1}{2} \sqrt{\frac{3q(\sigma v - \varepsilon)}{(M_3 - \eta s_0)}}, \quad \frac{q(\sigma v - \varepsilon)}{(M_3 - \eta s_0)} > 0, \quad w = \frac{8A_0^2 (M_3 - \eta s_0) + 4\varepsilon s_0^2 + 4\alpha s_0 - \sigma v + \varepsilon}{4(\sigma s_0 - 1)}, \quad (120)$$

$$M_2 = -\frac{8}{3}A_0(M_3 - \eta s_0), \quad M_1 = -\frac{A_0^2}{12}[4A_0^2(M_3 - \eta s_0) - 3(\sigma v - \varepsilon)].$$

In this set, the solution to NPSEs in Equation (42) is

$$Q_{48}(x, t) = \left[A_0 + \frac{1}{2} \sqrt{\frac{3q(\sigma v - \varepsilon)}{(M_3 - \eta s_0)}} \sec_q(x - vt) \right]^{1/2} e^{i(-s_0 x + wt + \rho)}. \quad (121)$$

Set 3.

$$A_1 = \frac{1}{4} \sqrt{\frac{3q(\sigma v - \varepsilon)}{(M_3 - \eta s_0)}}, \quad B_1 = \frac{1}{4} \sqrt{\frac{3(\sigma v - \varepsilon)}{(M_3 - \eta s_0)}}, \quad \frac{(\sigma v - \varepsilon)}{(M_3 - \eta s_0)} > 0, \quad q > 0, \quad M_2 = -\frac{8A_0}{3}(M_3 - \eta s_0), \quad (122)$$

$$w = \frac{16A_0^2 (M_3 - \eta s_0) + 8\varepsilon s_0^2 + 8\alpha s_0 - \varepsilon + \sigma v}{8(\sigma s_0 - 1)}, \quad M_1 = -\frac{1}{768(M_3 - \eta s_0)}[(16M_3A_0^2 - 3\varepsilon) - (16\eta s_0A_0^2 - 3\sigma v)]^2$$

In this set, the solution to NPSEs in Equation (42) is

$$Q_{49}(x, t) = \left[A_0 + \frac{1}{4} \sqrt{\frac{3q(\sigma v - \varepsilon)}{(M_3 - \eta s_0)}} \sec_q(x - vt) + \frac{1}{4} \sqrt{\frac{3(\sigma v - \varepsilon)}{(M_3 - \eta s_0)}} \tan_q(x - vt) \right]^{1/2} e^{i(-s_0 x + wt + \rho)}. \quad (123)$$

For the type 4. The solutions of the NPSEs in Equation (42) are given by

$$Q_{50}(x, t) = \left[A_0 + \frac{1}{2} \sqrt{\frac{3(\sigma v - \varepsilon)}{(M_3 - \eta s_0)}} \cot_q(x - vt) \right]^{1/2} e^{i(-s_0 x + wt + \rho)}, \quad \frac{(\sigma v - \varepsilon)}{(M_3 - \eta s_0)} > 0, \quad (124)$$

where $M_1 = -\frac{1}{48(M_3 - \eta s_0)}[(4M_3A_0^2 + 3\sigma v) - (3\varepsilon + 4\eta s_0A_0^2)]^2$, $M_2 = -\frac{8}{3}A_0(M_3 - \eta s_0)$ and $w = \frac{4A_0^2 (M_3 - \eta s_0) + 2\varepsilon s_0^2 + 2\alpha s_0 - \varepsilon + \sigma v}{2(\sigma s_0 - 1)}$

$$Q_{51}(x, t) = \left[A_0 + \frac{1}{2} \sqrt{\frac{3q(\sigma v - \varepsilon)}{(M_3 - \eta s_0)}} \csc_q(x - vt) \right]^{1/2} e^{i(-s_0 x + wt + \rho)}, \quad \frac{q(\sigma v - \varepsilon)}{(M_3 - \eta s_0)} > 0, \quad (125)$$

where $M_2 = -\frac{8}{3}A_0(M_3 - \eta s_0)$, $w = \frac{8A_0^2 (M_3 - \eta s_0) + 4\varepsilon s_0^2 + 4\alpha s_0 - \sigma v + \varepsilon}{4(\sigma s_0 - 1)}$, and $M_1 = -\frac{A_0^2}{12}[4A_0^2(M_3 - \eta s_0) - 3(\sigma v - \varepsilon)]$.

$$Q_{52}(x, t) = \left[A_0 + \frac{1}{4} \sqrt{\frac{3q(\sigma v - \varepsilon)}{(M_3 - \eta s_0)}} \csc_q(x - vt) + \frac{1}{4} \sqrt{\frac{3(\sigma v - \varepsilon)}{(M_3 - \eta s_0)}} \cot_q(x - vt) \right]^{1/2} e^{i(-s_0 x + wt + \rho)}. \quad (126)$$

where $M_1 = -\frac{1}{768(M_3 - \eta s_0)}[(16M_3A_0^2 - 3\varepsilon) - (16\eta s_0A_0^2 - 3\sigma v)]^2$ and $w = \frac{16A_0^2 (M_3 - \eta s_0) + 8\varepsilon s_0^2 + 8\alpha s_0 - \varepsilon + \sigma v}{8(\sigma s_0 - 1)}$, $q > 0$

5. Behavior of Soliton Solutions

In this section we plot the two- and one-dimensional solutions when the parameters take some suitable values to determine the type of the optical solutions. We show the effect of the parameter q in the q -deformed functions in optical solutions. We plot the absolute value of the complex exact solution to discuss the amplitude of the soliton solutions. In Figure 1, we plot the traveling wave solution (17) when the parameters take the values $s_0 = 0.01, \varepsilon = 2, \alpha = 5, \sigma = 0.5, \eta = 1, \zeta = -100, A = 3$ and $B = 2$. We get a singularity at $x = -1.4$ when the velocity of solion is equal to 2 and the wave number is 4; consequently the soliton solution in Equation (17) is the singular soliton solution while in the Figure 2, the soliton solutions in Equation (18) is periodic singular solution. Figure 3 presents the exact solution of Equation (59) when $s_0 = 2, \varepsilon = 0.2, \alpha = -2, \sigma = 0.1, \eta = 2, \zeta = 1, q = 11$ is the bright soliton equations while the exact Equation (80) in the Figure 4 is dark soliton equations. Figure 5 illustrates the periodic soliton Equation (119) when $s_0 = 2, \varepsilon = 0.2, \alpha = -2, \sigma = 0.1, \eta = 2, \zeta = 1, q = 11$.

Figure 1. (a) The Absolute value of exact solution Q_1 when $s_0 = 2, \varepsilon = 0.2, \alpha = -2, \sigma = 0.1, \eta = 2, \zeta = 1, A = 3$ and $B = 2$ and (b) is the projection Q_1 in 2D.

Figure 2. (a) The Absolute value of exact solution Q_2 when $s_0 = 2, \varepsilon = 0.2, \alpha = -2, \sigma = 0.1, \eta = 2, \zeta = 1, A = 3$ and $B = 2$ and (b) is the projection Q_2 in 2D.

Figure 3. (a) The Absolute value of exact solution Q_{17} when $s_0 = 2, \varepsilon = 0.2, \alpha = -2, \sigma = 0.1, \eta = 2, \zeta = 1, q = 11$ and (b) is the projection Q_{17} in 2D.

Figure 4. (a) The Absolute value of exact solution Q_{28} when $s_0 = 2, \varepsilon = 0.2, \alpha = -2, \sigma = 0.1, \eta = 2, \zeta = 1, q = 11$ and (b) is the projection of Q_{28} in 2D.

Figure 5. (a) The Absolute value of exact solution Q_{47} when $s_0 = 2, \varepsilon = 0.2, \alpha = -2, \sigma = 0.1, \eta = 2, \zeta = 1, q = 11$ and (b) is the projection of Q_{47} in 2D.

6. Conclusions

In this paper, we applied some interesting algebraic methods to find a variety of explicit solutions to nonlinear perturbed Schrödinger equations when the nonlinear parts take different terms: nonlinear Kerr law, power law, quadratic-cubic law, and anti-cubic law. In this article, we construct the rational solitary solutions, q-deformed hyperbolic solutions, and q-deformed trigonometric solutions for one of the important equations in optical media which has many applications in optics and communications. Through our study that we conducted on the non-linear Schrödinger equation, it is clear that the proposed methods are effective methods in finding the soliton solutions. The drawings studied during this work illustrate the different optical solutions that have many applications in optics.

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