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Overtaking Collisions of Ion Acoustic *N*-Shocks in a Collisionless Plasma with Pair-Ion and (α, q) Distribution Function for Electrons

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Abstract: In this work, the effects of plasma parameters on overtaking collisions of ion acoustic multi-shocks are studied in an unmagnetized collisionless plasma with positive and negative ions, and (α, q)-distributed electrons. To investigate such phenomena, the reductive perturbation technique is implemented to derive the Burgers equation. The *N*-shock wave solution is determined for this equation by directly implementing the exponential function. The result reveals that both the amplitudes and thicknesses of overtaking collisions of *N*-shock wave compressive and rarefactive electrostatic potential are significantly modified with the influences of viscosity coefficients of positive and negative ions. In addition, the density ratios also play an essential role to the formation of overtaking collisions of *N*-shocks. It is observed from all theoretical and parametric investigations that the outcomes may be very useful in understanding the dynamical behavior of overtaking collisions of multi-shocks in various environments, especially the D- and F-regions of the Earth's ionosphere and the future experimental investigations in Q-machine laboratory plasmas.

Keywords: unmagnetized plasma; pair-ion; Burgers equation; multi-shocks

1. Introduction

Rigorous experimental and theoretical investigations on the unmagnetized collisionless plasmas having pair-ion and electrons are reported by many researchers because of their significant applications in understanding several types of collective processors in diverse environments, especially in Q-machine plasmas [1–4], neutral beam source [5], semiconductor and material processing [6], and so on. To illustrate it, Ichiki et al. [7] and Weingarten et al. [8] have experimentally confirmed the production of negative ions with low temperature. Sato [2] has also confirmed that the negative ions with low temperature (~0.2 eV) are generated in potassium plasma in Q-machine. Goeler et al. [9] have detected a beam of pair ions (Cs^+, Cl^-) in the hot tungsten plate of a Q-machine. Besides, the negative ions exist in space and astrophysical objects [10,11], e.g., in the D-region altitudes, F-region of the Earth's ionosphere, etc. More notably, negative ions can coexist with electrons in the ionosphere of Earth at D-region altitudes as they are formed primarily by electron attachment to electronegative species [10]. The Cassini spacecraft has confirmed that the heavy negative ions are likely to have occurred in the upper region of Titan's atmosphere [10]. Wong et al. [12] have experimentally reported that ion acoustic (IA) waves with the presence of SF_6^- plasma species. In addition, Song et al. [13] have already described the IA wave propagation in a Q-machine plasma having pair ions (K^+, SF_6^-)

and electrons. They have investigated that the phase velocity of IA "fast" mode is increased with the increase in the negative to positive ions density ratio. It is noted that negative ion plasma system is mainly formed naturally, and it is composed of electrons, positive ions, and negative ions in space and astrophysical environments [14–16]. For instance, negative ions are produced in the lower ionosphere (D region) by the attaching of electrons to atoms or molecules with the presence of relatively high pressure [17]. The decrease in electron density allows the night-time reflection of radio waves from the E layer. Andersen et al. [18] have also experimentally observed that Landau damping prevented the formation of a shock, and only a "spreading" of the pulse with equal electron and negative ion temperatures, that is, $T_e = T_{ni}$. However, they have found that the Landau damping is reduced and the shock structure is formed by increasing electron temperature and cooling the ions via ion-neutral collisions. With the increase in wave phase velocity, the results can be sufficiently reduced by the wave particle resonance due to the involvement of of negative ions when $T_e > T_{ni}$ [12]. Further, the presence of energetic electrons as ubiquitous in a variety of environments and measurements of their distribution functions revealed them to be highly nonthermal or sub-thermal or superthermal [19–25]. Due to the energetic electrons, the electrons have long range interactions with other plasma species because the plasma species are actually occurred in different phase space [26]. Already, many researchers have examined the IA structures in the plasma system by assuming the plasma species in thermal equilibrium, which is followed the Maxwell–Boltzmann distribution. However, such an ideal thermal equilibrium assumption is no longer valid when some external agents, such as force field present in natural space environments, wave–particle interaction, turbulence, etc., are disturbed in the thermal equilibrium of the plasma systems. The fast ions and electrons mode in plasma environments is also suggested that the particles have a deviated velocity distributions far from Maxwellian distribution. Moreover, the systems with the presence of long-range interactions as well as long-time memory are intractable within the Boltzmann–Gibbs statistics. At these stages, the non-Maxwellian velocity distribution functions are an arena for obtaining the electron density function when the electron energy is higher/less than the isothermal energies or the electron energy becomes nonthermal. For instance, the $q-$ velocity distribution function [27] is very suitable to describe the particles system consisting of a very huge number of both low and high speed because of the long rang nature of Colombian interactions with the presence of sub-thermal and superthermal particles in the plasma environments. Besides, one can use the α (Cairns)-velocity distribution [28] for electrons because the lighter particles (electrons) have a great potential to be trapped in the potential wall due to its high energy rather than other plasma species (positive or negative ions). Elwakil et al. [29] have already described IA modulation instability characteristics in negative ion plasma having nonthermal electrons. They have also shown that the instability conditions affected by nonthermal electron parameters in D and F regions in Earth's ionosphere. It is therefore suggested to extend the hydrodynamic fluid model by considering (α, q)-distributed electrons [25] and pair-ions because the (α, q) velocity distribution function is not only used for superthermal, subthermal, and nonthermal, but also Maxwellian distributed electrons.

On the other hand, the most credible mechanism is the production of shock waves phenomena because they propagate superficial. Basically, the IA shock wave is generated in plasmas due to the comparative speed between the rarefaction wave and ambient plasma exceeds IA acoustic speed, where the concentration of the rarefaction wave and ambient medium are almost analogous. Owing to the significance of IA shock wave for understanding the physical issues in many environments, such wave phenomena have been studied by many researchers by considering different types of plasma environments [22,23,30,31]. In [32,33], the authors have confirmed the production of shock waves in the Q-machine with negative ions. Adak et al. [34,35] have reported the nonlinear IA shock in a pair-ion (C_{60}^{+}, C_{60}^{-}) plasma by deriving the Korteweg–de Vries equation Burgers equation (KdVBE) with the presence of weakly dissipative media. Further, the N-wave phenomena may interact with each other in two different ways. One of which is the overtaking phenomenon, where N-wave moves along the same direction and the other one is the face-to-face collision, where the angle between two propagation directions among N-wave is π. Very recently, Hafez [36] has been reported the face-to-face collisions of an IA multi-soliton described by two-sided KdVEs in a dense plasma. El-Shamya and Mahmoud [37] have reported the overtaking collisions of electrostatic N-soliton described by KdVE in electron—hole quantum plasmas. In addition, Hossain et al. [38] have theoretically described the nonlinear IA single shock waves via KdVBE in negative ion plasmas with q-distributed electrons by stretching the kinematic viscosity coefficient parameter of negative and positive ions as $\eta_+ = \epsilon^{1/2}\eta_{+0}$ and $\eta_- = \epsilon^{1/2}\eta_{-0}$, respectively. Very recently, Alam and Talukder [30] have investigated the face to face collision phenomena between two IA single-shock waves via the stationary solutions of two-sided KdVBEs in a plasma system having pair-ion and isothermal electrons. They have also considered the stretching of kinematic viscosity coefficients for negative and positive ions as $\eta_{1i} = \epsilon^{1/2}\eta_1$ and $\eta_{2i} = \epsilon^{1/2}\eta_2$, respectively. Mamun [39] has clearly shown that the stretching of kinematic viscosity coefficient parameters is not usually valid. The valid stretching coordinates supported only the Burgers equation, which divulges the formation of shock structures in the plasmas. However, the research work on nonlinear overtaking collisions of IA multi-shock wave phenomena is not previously reported by using the N-shock wave analytical solution of Burgers equation in a plasma by composing of pair-ion and (α, q)-distributed electrons. It is therefore still required to describe the influences of physical parameters on the salient features of small but finite amplitude electrostatic overtaking collisions of N-shock wave excitations through the valid stretching coordinates in the negative ions plasmas with (α, q)-distribution function for electrons.

Thus, the research work aims to explore the effects of the density ratio of negative to positive ions, the density ratio of electrons to positive ions, the viscosity coefficient of positive ions, the viscosity coefficient of negative ions, the population of nonthermal electrons and strength of non-extensive electrons on the phase velocity, the electrostatic overtaking collisions of N-shocks, and their corresponding electric field in a plasma having pair-ion and (α, q)-distributed electrons by determining the N-shock wave solution of the Burgers-like equation. The paper is arranged as follows. The hydrodynamic fluid model equations under some plasma assumptions are given in Section 2. Formation of a Burgers-like equation with an N-shock wave stationary solution is described in Section 3. The parametric analysis of the silent feature of small but finite overtaking collisions of N-shocks is presented in Section 4. In Section 5, some concluding remarks are given.

2. Theoretical Model Equations under Plasma Assumption

An unmagnetized collisionless plasma by composing of inertial pair-ion (e.g., positive ion (K^+) with mass (m_{pi}) and temperature (T_{pi}), and negative ion (SF_6^-) with mass (m_{ni}) and temperature (T_{ni})) and inertial-less (α, q) distributed electrons, where α and q are treated as the population of non-thermal and the strength of non-extensivity electrons, respectively, is considered. Under the above plasma assumption, one obtains the charge neutrality condition as $1 = N_{r1} + N_{r2}(N_{r1} = N_{ni0}/N_{pi0}, N_{r2} = N_{e0}/N_{pi0})$, where N_{pi0}, N_{ni0}, and N_{e0} are the unperturbed densities of positive ions, negative ions, and electrons, respectively. In addition, the density function for (α, q)-distributed electrons is defined [25] as

$$\rho_e = \rho_{e0}\left[1 + (q-1)\frac{e\phi}{k_B T_e}\right]^{\frac{(q+1)}{2(q-1)}} \times \left[1 + B_1\left(\frac{e\phi}{k_B T_e}\right) + B_2\left(\frac{e\phi}{k_B T_e}\right)^2\right], \quad (1)$$

where $B_1 = -16q\alpha/\left(3 - 14q + 15q^2 + 12\alpha\right)$, $B_2 = -B_1\left(2q-1\right)$, T_e is the electron temperature, k_B is Boltzmann constant, and ϕ is the electrostatic potential function. It is noted that the above expression is recovered by (i) the Maxwell–Boltzmann distributed concentration of electrons when $q \to 1$ and $\alpha = 0$, (ii) the cairn distributed concentration of electrons when $q \to 1$ and $\alpha \neq 0$, and (iii) the non-extensive distributed concentration of electrons when $\alpha = 0$, respectively. Thus, the following normalized hydrodynamic continuity and momentum equations are obtained by implementing the mass and momentum conservation low along with the above plasma assumptions for positive and negative ions:

$$\partial_t(N_{pi}) + \partial_z(N_{pi}U_{pi}) = 0, \quad (2)$$

$$\partial_t(N_{ni}) + \partial_z(N_{ni}U_{ni}) = 0, \tag{3}$$

$$\partial_t(U_{pi}) + U_{pi}\partial_z(U_{pi}) + \partial_z\Phi + \mu_{pi}\partial_{zz}(U_{pi}) = 0, \tag{4}$$

$$\partial_t(U_{ni}) + U_{ni}\partial_z(U_{ni}) - M_r\partial_z\Phi + \frac{\delta_{ei}}{N_{ni}}\partial_z(N_{ni}) + \mu_{ni}\partial_{zz}(U_{ni}) = 0. \tag{5}$$

The aforementioned plasma is bounded through the following normalized equation, which is obtained by considering the Maxwell equations along with the normalized electric field $E = -\nabla\Phi$,

$$\partial_{zz}\Phi = -N_{r2}\left\{[1+(q-1)\Phi]^{\frac{(q+1)}{2(q-1)}} \times \left[1 + B_1\Phi + B_2\Phi^2\right]\right\} + N_{pi} - N_{ni}, \tag{6}$$

where

$$M_r = \frac{m_{pi}}{m_{ni}}, \delta_{ei} = \frac{T_e}{(1-N_{r1})\,T_{ni}}. \tag{7}$$

Here, N_{pi} (N_{ni}) and N_e are the densities of the positive (negative) ions and electrons normalized by N_{pi0}, respectively. U_{pi} (U_{ni}) is the fluid positive (negative) ions velocity normalized by the positive ion acoustic speed $C_{ips} = \sqrt{k_B T_e/m_{pi}}\sqrt{(N_{r1}+N_{r2})\,/N_{r2}(1-N_{r1})}$ based on the neutrality condition. Φ is the normalized electrostatic potential by introducing $\Phi \to e\phi/k_B T_e$. z and t are normalized by electron Debye length ($\lambda_{De} = \sqrt{k_B T_e/4\pi N_{e0}e^2}$) and positive ion plasma frequency $\omega_{pi}^{-1} = \lambda_{De}/C_{ips}$, respectively. μ_{pi} (μ_{ni}) is the normalized viscosity coefficient of positive (negative) ions that normalized by introducing $\omega_{pi}^{-1}/m_{pi}N_{pi}C_{ips}^2$ ($\omega_{pi}^{-1}/m_{ni}N_{ni}C_{ips}^2$). Without loss of generality, it is assumed that $T_{pi} \approx 0$ because T_e is modified by $T_e/\,(1-N_{r1})$ [4]. In addition, the ion acoustic mode is generated by the positive ion mode because the mass number (146) of SF_6^- is greater than the mass number (39) of (K^+). With the presence of negative ions [4], $T_e \approx T_e/\,(1-N_{r1})$ is mainly responsible for the growth of the phase velocity, which is indicated that the Landau damping of IA waves losses. It is actually equivalent to the increase in T_e/T_{pi} due to electron heating via electron cyclotron resonance [40]. In such situations, the ion Landau damping for IA mode can be ignored [41].

3. Mathematical Formulation

3.1. Formation of a Nonlinear Evolution Equation

It is well established that the nonlinear IA wave mode cannot be easily described by solving Equations (2)–(6) directly. At this stage, one can use the tedious mathematical technique for deriving the nonlinear evolution equations (NLEEs) to study the basic features of physical phenomena in diverse environment. To derive a NLEE form the aforementioned model, the reduction perturbation technique [22,39] is allowed to consider the new coordinates instead of the scaling of variables as

$$X = \epsilon(z - V_p t), T = \epsilon^2 t, \tag{8}$$

and the expansions for physical quantities around the equilibrium values as

$$\begin{bmatrix} N_{pi} \\ N_{ni} \\ U_{pi} \\ U_{ni} \\ \Phi \end{bmatrix} = \begin{bmatrix} 1 \\ N_{r1} \\ 0 \\ 0 \\ 0 \end{bmatrix} + \sum_i^\infty \epsilon^i \begin{bmatrix} N_{pi}^{(i)} \\ N_{ni}^{(i)} \\ U_{pi}^{(i)} \\ U_{ni}^{(i)} \\ \Phi^{(i)} \end{bmatrix}, \tag{9}$$

where V_p and ϵ are the linear phase speed of the IA mode and a proper fraction number recovering the weakness of dissipation, respectively. By employing Equations (8) and (9) systematically into Equations (2)–(6), one can derive the set of evolution equations based on the different order of ϵ.

For $O(\epsilon)$ equations (ignored for simplicity), one obtains the following,

$$N_{pi}^{(1)} = \frac{1}{V_p^2}\Phi^{(1)}, U_{pi}^{(1)} = \frac{1}{V_p}\Phi^{(1)}, N_{ni}^{(1)} = -\frac{N_{r1}M_r}{V_p^2-\delta_{ei}}\Phi^{(1)}, U_{ni}^{(1)} = -\frac{V_pM_r}{V_p^2-\delta_{ei}}\Phi^{(1)}, \tag{10}$$

and

$$-N_{r2}K_1\Phi^{(1)} + N_{pi}^{(1)} - N_{ni}^{(1)} = 0, \tag{11}$$

where $K_1 = \Omega\left[(q+1)/2 + B_1\right]\left(1+\rho_{ep}\sigma_{ep}\right)$. By simplifying Equations (10) and (11), the linear phase velocity is obtained as

$$V_p = \sqrt{\frac{(1+N_{r1}M_r+\delta_{ei}N_{r2}K_1) \pm \sqrt{(1+N_{r1}M_r+\delta_{ei}N_{r2}K_1)^2 - 4N_{r2}\delta_{ei}}}{2N_{r2}K_1\delta_{ei}}}. \tag{12}$$

In Equation (12) it is clearly indicated that V_p is strongly dependent on N_{r1}, M_r, δ_{ei}, N_{r2}, α, and q, but not on μ_{pi} and μ_{ni}. Moreover, it is validated only if $(1+N_{r1}M_r+\delta_{ei}N_{r2}K_1)^2 - 4N_{r2}\delta_{ei} \geq 0$.

Now, one can easily derive the following equations with the help of Equation (10) by taking $O(\epsilon^2)$ equations:

$$\frac{1}{V_p^2}\frac{\partial\Phi^{(1)}}{\partial T} - V_p\frac{\partial N_{pi}^{(2)}}{\partial X} + \frac{\partial U_{pi}^{(2)}}{\partial X} + \frac{2}{V_p^3}\Phi^{(1)}\frac{\partial\Phi^{(1)}}{\partial X} = 0, \tag{13}$$

$$\frac{1}{V_p}\frac{\partial\Phi^{(1)}}{\partial T} - V_p\frac{\partial U_{pi}^{(2)}}{\partial X} + \frac{\partial\Phi^{(2)}}{\partial X} + \frac{1}{V_p^2}\Phi^{(1)}\frac{\partial\Phi^{(1)}}{\partial X} + \frac{\mu_{pi}}{V_p}\frac{\partial^2\Phi^{(1)}}{\partial X^2} = 0, \tag{14}$$

$$-\frac{N_{r1}M_r}{V_p^2-\delta_{ei}}\frac{\partial\Phi^{(1)}}{\partial T} - V_p\frac{\partial N_{ni}^{(2)}}{\partial X} + N_{r1}\frac{\partial U_{ni}^{(2)}}{\partial X} + \frac{2N_{r1}V_pM_r^2}{\left(V_p^2-\delta_{ei}\right)^2}\Phi^{(1)}\frac{\partial\Phi^{(1)}}{\partial X} = 0, \tag{15}$$

$$\begin{aligned} &-\frac{V_pM_r}{V_p^2-\delta_{ei}}\frac{\partial\Phi^{(1)}}{\partial T} - V_p\frac{\partial U_{ni}^{(2)}}{\partial X} + \frac{\delta_{ei}}{N_{r1}}\frac{\partial N_{ni}^{(2)}}{\partial X} - M_r\frac{\partial\Phi^{(2)}}{\partial X} + \frac{V_p^2M_r^2}{\left(V_p^2-\delta_{ei}\right)^2}\Phi^{(1)}\frac{\partial\Phi^{(1)}}{\partial X} \\ &-\frac{\delta_{ei}M_r^2}{\left(V_p^2-\delta_{ei}\right)^2}\Phi^{(1)}\frac{\partial\Phi^{(1)}}{\partial X} - \frac{\mu_{ni}V_pM_r}{V_p^2-\delta_{ei}}\frac{\partial^2\Phi^{(1)}}{\partial X^2} = 0, \end{aligned} \tag{16}$$

$$0 = -N_{r2}K_1\Phi^{(2)} + N_{r2}K_2\left[\Phi^{(1)}\right]^2 + N_{pi}^{(2)} - N_{ni}^{(2)}, \tag{17}$$

where $K_2 = \Omega\left[B_2 + B_1(q+1)/2 + (q+1)(3-q)/8\right]\left(1+\rho_{ep}\sigma_{ep}^2\right)$.

After cumbersome calculations by taking Equations (13)–(17), the following NLEE is obtained,

$$A\frac{\partial\Phi}{\partial T} + B\Phi\frac{\partial\Phi}{\partial X} + C\frac{\partial^2\Phi}{\partial X^2} = 0, \tag{18}$$

where

$$
\begin{aligned}
A &= \frac{2}{V_p^3} + \frac{2N_{r1}V_pM_r}{\left(V_p^2-\delta_{ei}\right)^2},\\
B &= \frac{3}{V_p^4} - \frac{3N_{r1}V_p^2M_r^2}{\left(V_p^2-\delta_{ei}\right)^3} + \frac{\delta_{ei}N_{r1}M_r^2}{\left(V_p^2-\delta_{ei}\right)^3} - 2N_{r2}K_2,\\
C &= \frac{\mu_{pi}}{V_p^3} + \frac{\mu_{ni}V_pM_r}{\left(V_p^2-\delta_{ei}\right)^2},
\end{aligned}
\tag{19}
$$

and $\Phi^{(1)} \sim \Phi$ (for simplicity). It is noted that Equation (18) divulges only the shock wave structures in plasmas because it is a Burgers-like equation.

3.2. N-Shock Wave Solutions

Before determining the useful N-shock wave solution of Equation (18), one first needs to transform Equation (18) to the well-established standard type Burgers equation by using suitable variables transformation. To do so, one can easily convert Equation (18) to the following standard type Burgers equation by considering the transforms $X = Y(C/A)^{1/2}$ and $\Phi = U(B/A)^{-1}(C/A)^{1/2}$ into account,

$$\frac{\partial U}{\partial T} + U\frac{\partial U}{\partial Y} + \frac{\partial^2 U}{\partial Y^2} = 0, \tag{20}$$

where the nonlinear and dissipative coefficients are unity. Equation (20) is also converted to a heat type equation by linearizing with the well-known Cole–Hopf transformation [42] as $U \rightarrow 2\frac{\partial}{\partial Y}(\ln U(Y,T))$. The heat type equation provides a simple solution as $V_i(Y,T) = \exp(k_iY - \omega_iT)$, where $i = 1,2,3,\ldots N$, k_i is the wave number, and ω_i is the frequency and their linear superposition as $V(Y,T) = K + \sum_i^N[\exp(k_iY - \omega_iT)]$ (K is real constant) is also solution. In addition, one can easily derive the multi-shocks of Equation (20) by considering $U = (\frac{\partial V}{\partial Y}/V)$. Based on the above assumptions, a few authors [42–44] have proposed the multi-shocks solutions of only the standard Burgers equation having nonlinear and dissipative coefficients unity, like $U_T + UU_x = U_{xx}$. Besides, the nonlinear and dissipative coefficients of the Burgers equation are not in unity, but strongly dependent on the physical parameters. Therefore, we sill require the N-shock wave solutions of Equation (18) to understand the nature of multi-shocks in the aforementioned plasmas with the influences of plasma parameters. It is noted that Equation (18) can also be reduced to a heat type equation by introducing $\Phi(X,T) \rightarrow 2((B/A)^{-1}(C/A)^{1/2}\frac{\partial}{\partial X}(\ln(\Phi((C/A)^{-1/2}X,T))$. Besides, one can easily derive the dissipation relation of Equation (18) as $\omega_i = k_i^2$ by inserting $(\exp(k_iX - \omega_iT))$, $(i = 1,2,\ldots.N)$ with the above transform directly in Equation (18). Therefore, the useful N-shock wave solution of Equation (18) is defined as

$$\Phi(X,T) = \frac{A^{3/2}C^{1/2}}{B}\left\{\frac{2\sum_{i=1}^N\left[k_i\exp\left(\frac{k_iX}{\sqrt{(C/A)}} - k_i^2T\right)\right]}{1+\sum_{i=1}^N\left[\exp\left(\frac{k_iX}{\sqrt{C/A}} - k_i^2T\right)\right]}\right\}, \tag{21}$$

where N is a positive integer number, $A = \frac{2}{V_p^3} + \frac{2N_{r1}V_pM_r}{\left(V_p^2-\delta_{ei}\right)^2}$, $B = \frac{3}{V_p^4} - \frac{3N_{r1}V_p^2M_r^2}{\left(V_p^2-\delta_{ei}\right)^3} + \frac{\delta_{ei}N_{r1}M_r^2}{\left(V_p^2-\delta_{ei}\right)^3} - 2N_{r2}K_2$, and $C = \frac{\mu_{pi}}{V_p^3} + \frac{\mu_{ni}V_pM_r}{\left(V_p^2-\delta_{ei}\right)^2}$. It is obviously found from Equation (21) that the electrostatic N-shocks are only dependent on the plasma parameters, which indicates that Equation (21) is very useful for studying the nature of N-shocks in plasmas. Equation (21) is also checked by insetting in Equation (18) with the aid of computational software, like Maple 18, and found correct. One can easily study the nature of electrostatic shocks and their corresponding electric filed ($E = -\nabla\Phi$) by using the analytical stationary solution as in Equation (21) of the Burgers-like Equation, as in Equation (18).

4. Results and Discussion

To report the influences of N_{r1} (density ratio of negative to positive ions), N_{r2} (density ratio of electrons to positive ions), μ_{pi} (viscosity coefficient of positive ions), μ_{ni} (viscosity coefficient of negative ions), α (population of nonthermal electrons), and q (strength of non-extensive electrons) on the basic features of single and overtaking collision among multi-shock waves formation, an unmagnetized collosionless three-component plasma having positive as well as negative ions and (α, q)-distributed electrons is considered. The Burgers equation with the exact N-shocks solution as mentioned in Equation (21) is obtained to reveal such physical phenomena. In the presented analysis, the physical parameters are taken as $M_r = 3.74$, $T_{ni} = 0.05$, $T_e = 0.2$, $0 < N_{r1} < 1$, $0 < N_{r2} < 1$, $\mu_{pi} = 0.1 \sim 0.5$, and $\mu_{ni} = 0.001 \sim 0.4$ based on the work in [4,30] and $-1 < q < 1$ (for superthermality), $q > 1$ (subthermality), $q = 1$ (isothermality), and $0 < \alpha < 1$ based on the work in [25].

(i) Figure 1a,b shows, respectively, the variation of phase velocity with regards to N_{r2} and α for different values of N_{r1} and q. It is observed from Figure 1 that the linear phase velocity (V_p) losses (gains) energy with increases the density ratio of electrons to positive ions (the density ratio of negative to positive ions). Whereas, V_p loses (gains) energy with increasing q for the case of subthermality (superthermality). Moreover, V_p gains more energy with increases the population of nonthermal nonextensive electrons.

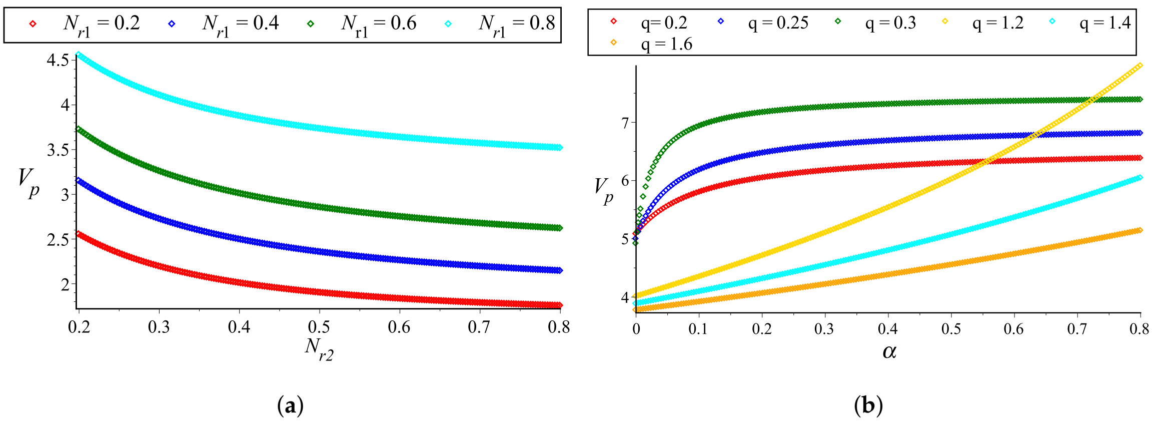
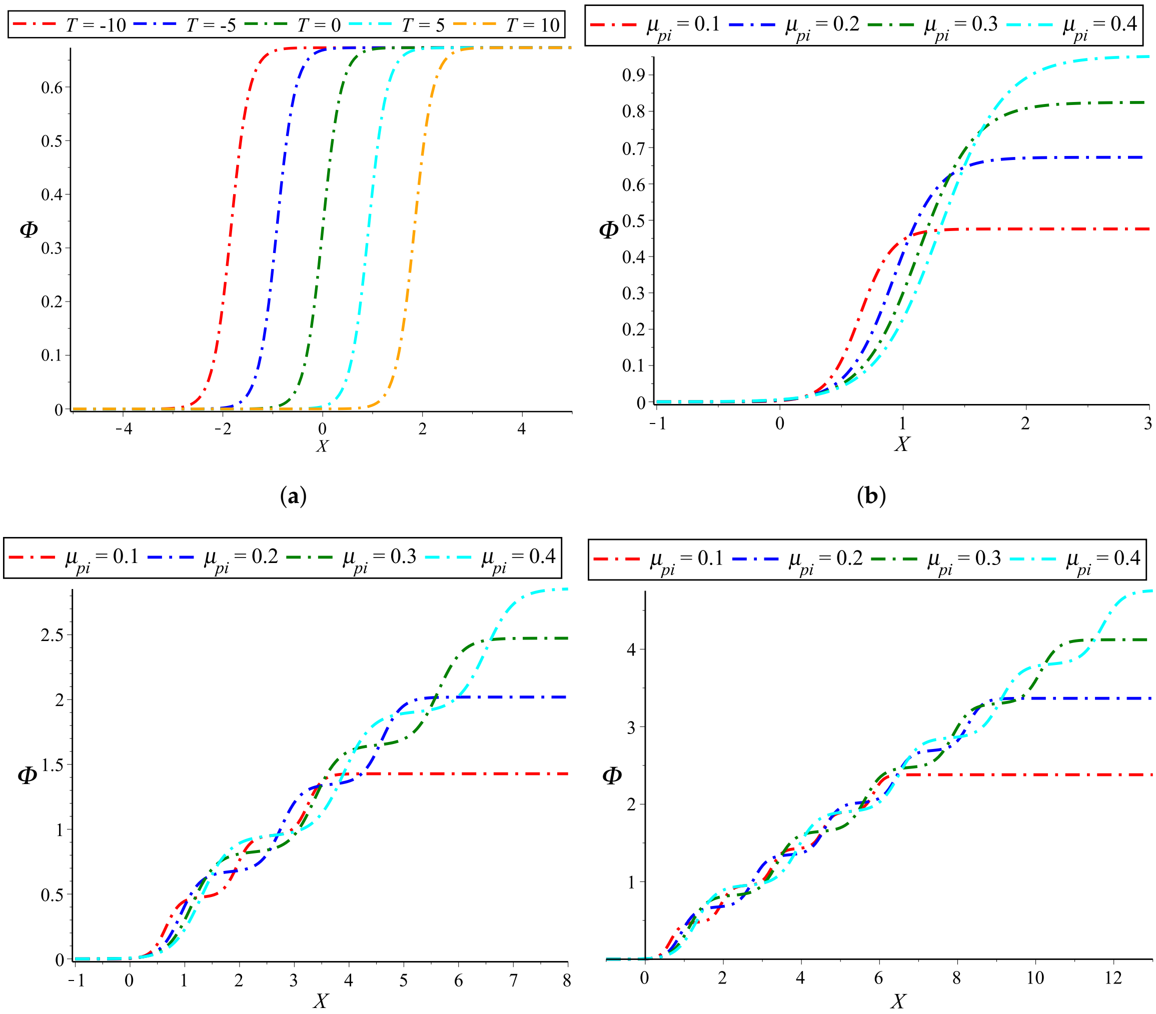


Figure 1. Variation of V_p with regards to (**a**) N_{r2} for different values of N_{r1} ($M_r = 3.75$, $T_{ni} = 0.05$, $T_e = 0.2$, $\alpha = 0.5$, and $q = 1.6$), and (**b**) α for different values of q ($M_r = 3.75$, $T_{ni} = 0.05$, $T_e = 0.2$, $N_{r1} = 0.5$, and $N_{r2} = 0.1$), respectively.

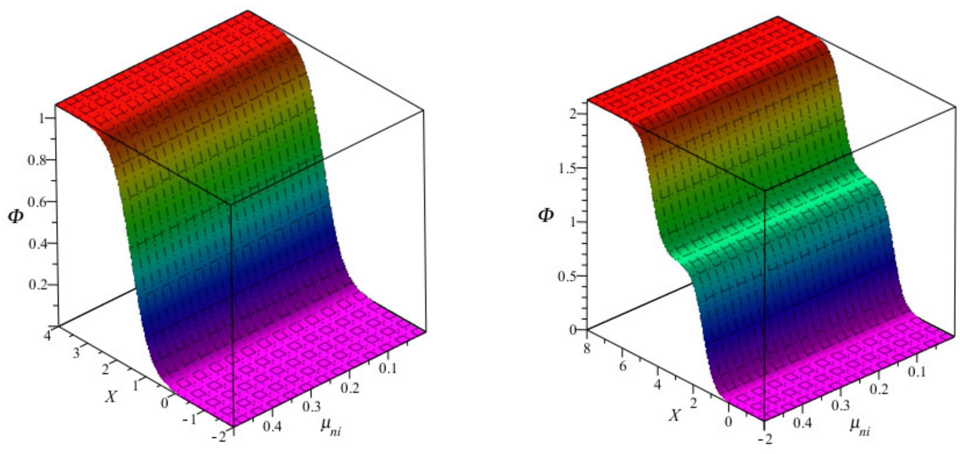
(ii) Figure 2b–d demonstrates the influence of μ_{pi} on the nonlinear IA single-, and overtaking collision of triple-, as well as five-shock wave excitations, while Figure 2a demonstrates the IA single-shock wave excitation for different values of time T by choosing the other parameters constant. Besides, the influence of μ_{ni} on the nonlinear IA single-, and overtaking collision of double-, triple-, as well as four-shock wave excitations by choosing the other parameters constant is demonstrated in Figure 3. It is evidently reveled from these figures that both μ_{pi} and μ_{ni} play a vital role in the formation of not only nonlinear electrostatic single-, but also overtaking collision of multi-shock waves in the negative ions plasmas. Because, the amplitude and thickness of monotonically shock wave remarkably increases with increases in μ_{pi}, but very sightly increases with increases in μ_{ni}. Besides, the amplitude and thickness of shock wave remains unchanged, but the position of shock wave changes with increases in time.



(**a**) (**b**)

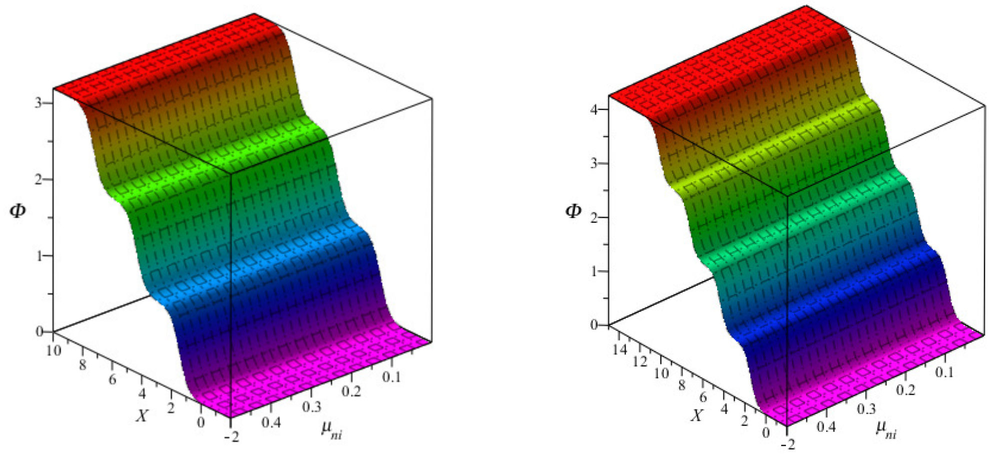
(**c**) (**d**)

Figure 2. (**a**) Electrostatic single-shock wave excitation due to the variation of T with $\mu_{pi} = 0.3$ and effect of μ_{pi} on the electrostatic (**b**) single-, (**c**) overtaking collision of triple-, and (**d**) overtaking collision of five-shock wave excitations with $T = 5$, respectively. The other parameters are chosen as $\alpha = 0.3$, $q = 0.35$, $M_r = 3.75$, $T_{ni} = 0.05$, $T_e = 0.2$, $N_{r1} = 0.5$, $N_{r2} = 0.02$, and $\mu_{ni} = 0.01$.



(**a**) (**b**)

Figure 3. *Cont.*



(**c**) (**d**)

Figure 3. Effect of μ_{ni} on the electrostatic (**a**) single-, (**b**) overtaking collision of double-, (**c**) overtaking collision of triple-, and (**d**) overtaking collision of four-shock wave excitations, respectively, by choosing $\alpha = 0.3$, $q = 0.35$, $M_r = 3.75$, $T_{ni} = 0.05$, $T_e = 0.2$, $N_{r1} = 0.5$, $N_{r2} = 0.02$, $\mu_{pi} = 0.5$, and $T = 5$.

(iii) Figure 4 displays the influence of N_{r2} (Figure 4a,b) and N_{r1} (Figure 4c,d) on the nonlinear electrostatic overtaking collision of double- and triple-shock wave excitations by assuming other parameters are constant. It is found from Figure 4 that both density ratios remarkably play distinct roles to each other in the formation of overtaking collision of N-shocks in the considered plasmas. This is because the monotonically shock wave also occurred, in which the amplitude and thickness of shocks decreases monotonically with increases in N_{r2}. However, the amplitude and thickness of shocks increases with increases in N_{r1}.

(iv) Figures 5 and 6 show the influence of α and q, respectively, on the nonlinear IA single-, and overtaking collision of double-, triple-, as well as four-shock wave excitations by choosing other parameters constant. It is clearly shown form these figures that the amplitude and thickness of shock wave excitations decrease (increases) with increasing in q and α. It is also found that the compressive and rarefactive electrostatic of not only single- but also overtaking collision of multi-shock wave excitations are supported in the considered plasmas.

(v) Figure 7 shows the rarefactive electrostatic single- and overtaking collision of multi-shock wave and behavior of its corresponding normalized electric field by taking $N = 1, 2, 3, 4$ and different values of α with $q = 1$, $M_r = 3.75$, $T_{ni} = 0.05$, $T_e = 0.2$, $N_{r1} = 0.9$, $N_{r2} = 0.5$, $\mu_{ni} = 0.05$, and $\mu_{pi} = 0.35$, whereas Figure 8 shows the rarefactive electrostatic overtaking collision of multi-shocks and behavior of its corresponding normalized electric field by taking $N = 2, 3$, different values of T and isothermal electrons, i.e., $\alpha = 0$, $q = 1$ with $M_r = 3.75$, $T_{ni} = 0.05$, $T_e = 0.2$, $N_{r1} = 0.9$, $N_{r2} = 0.5$, $\mu_{ni} = 0.05$, and $\mu_{pi} = 0.35$. It is evidently found from these figures that the aforementioned plasma supports the rarefactive shock wave phenomena with the presence of isothermal electrons ($\alpha = 0$, $q = 1$), which is in good agreement with the experimental investigations [4]. The rarefactive shock wave phenomena also exists with the presence of only nonthermal populations electrons ($q = 1$). It is also observed that the electric field becomes not only single- but also overtaking collision of multi-hump-shaped by depending on the value of N in the shock wave solution and pulseable with increases in time, as is expected.

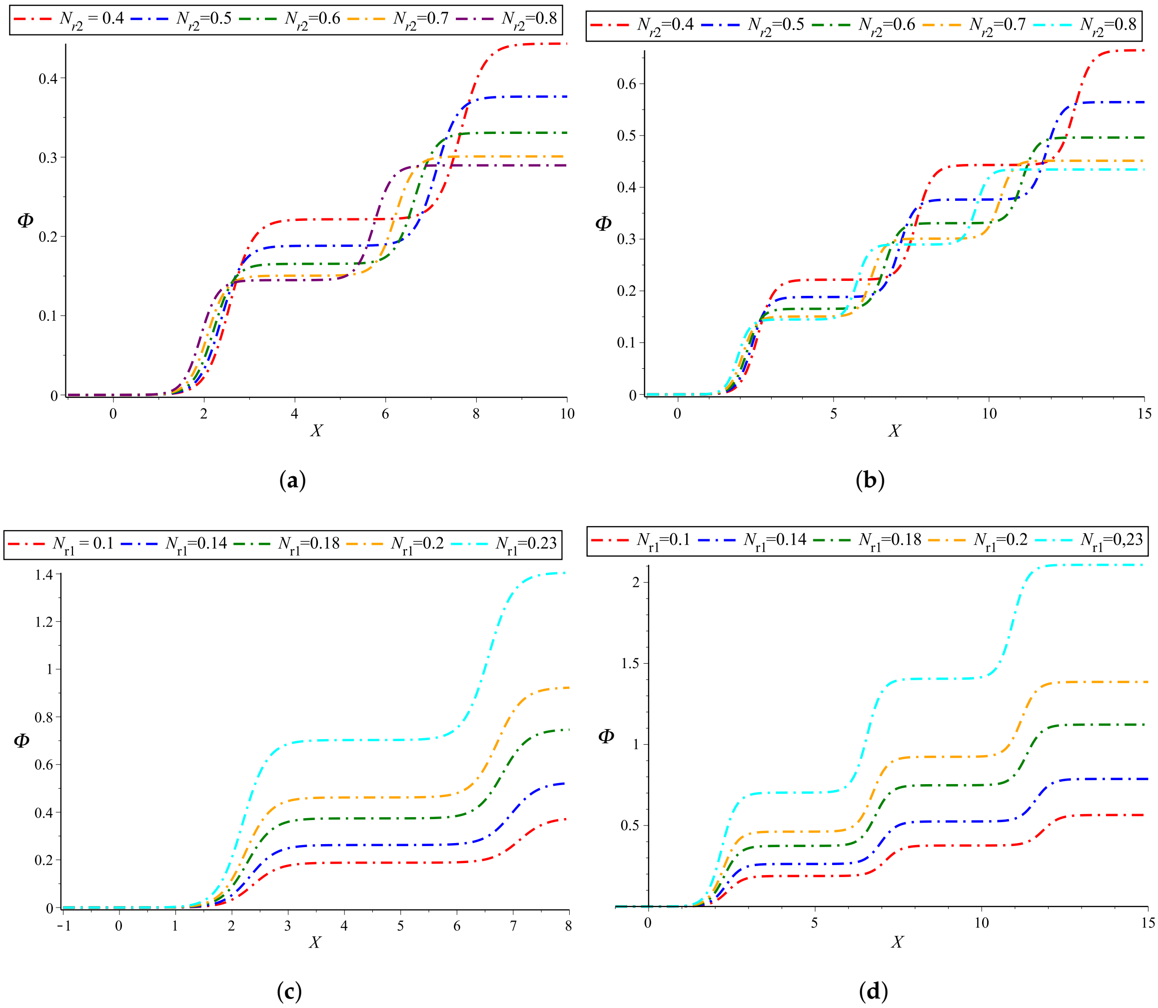


Figure 4. Effect of N_{r2} when $N_{r1} = 0.1$ on the electrostatic overtaking collision of (**a**) double-, and (**b**) triple-, and N_{r1} when $N_{r2} = 0.8$ on the electrostatic overtaking collision of (**c**) double-, and (**d**) triple-shock wave excitations, respectively, by choosing $\alpha = 0.3$, $q = 0.35$, $M_r = 3.75$, $T_{ni} = 0.05$, $T_e = 0.2$, $\mu_{pi} = 0.35$, $\mu_{ni} = 0.05$, and $T = 10$.

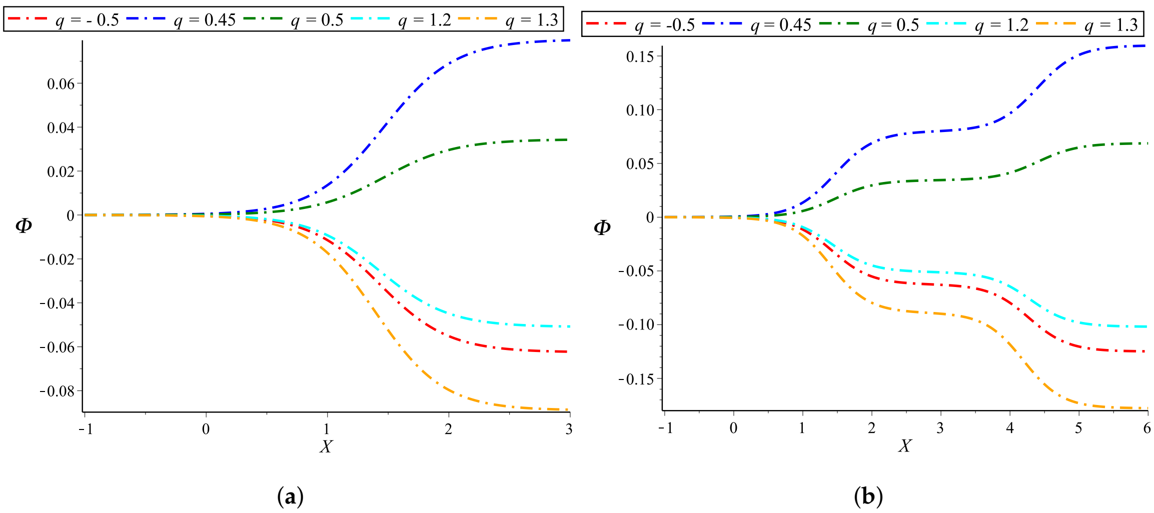


Figure 5. *Cont.*

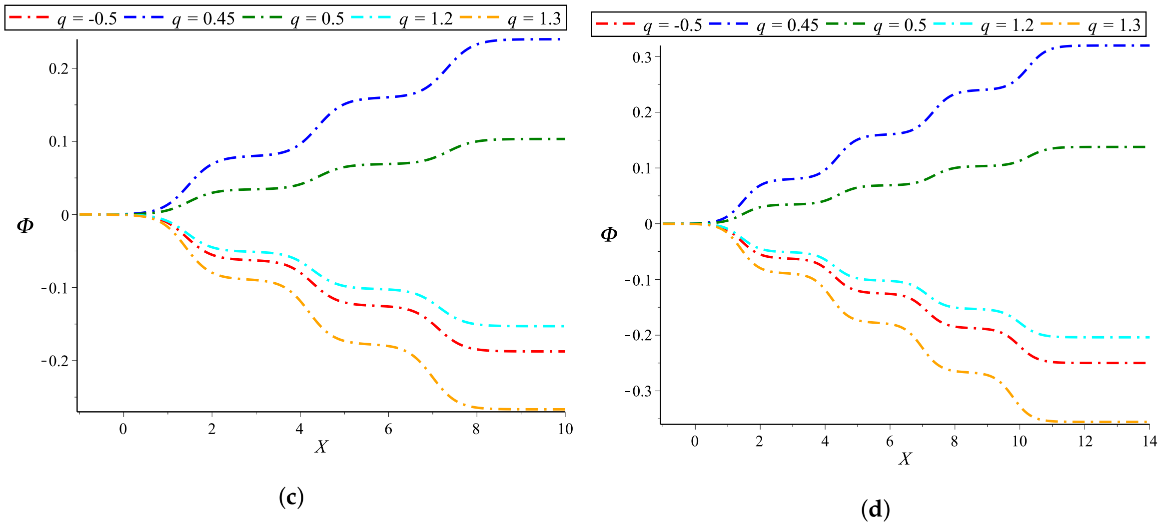


Figure 5. Effect of q on the electrostatic (**a**) single-, (**b**) overtaking collision of double-, (**c**) overtaking collision of triple-, and (**d**) overtaking collision of four-shock wave excitations, respectively, by choosing $\alpha = 0.3$, $M_r = 3.75$, $T_{ni} = 0.05$, $T_e = 0.2$, $N_{r1} = 0.5$, $N_{r2} = 0.02$, $\mu_{ni} = 0.01$, and $\mu_{pi} = 0.5$.

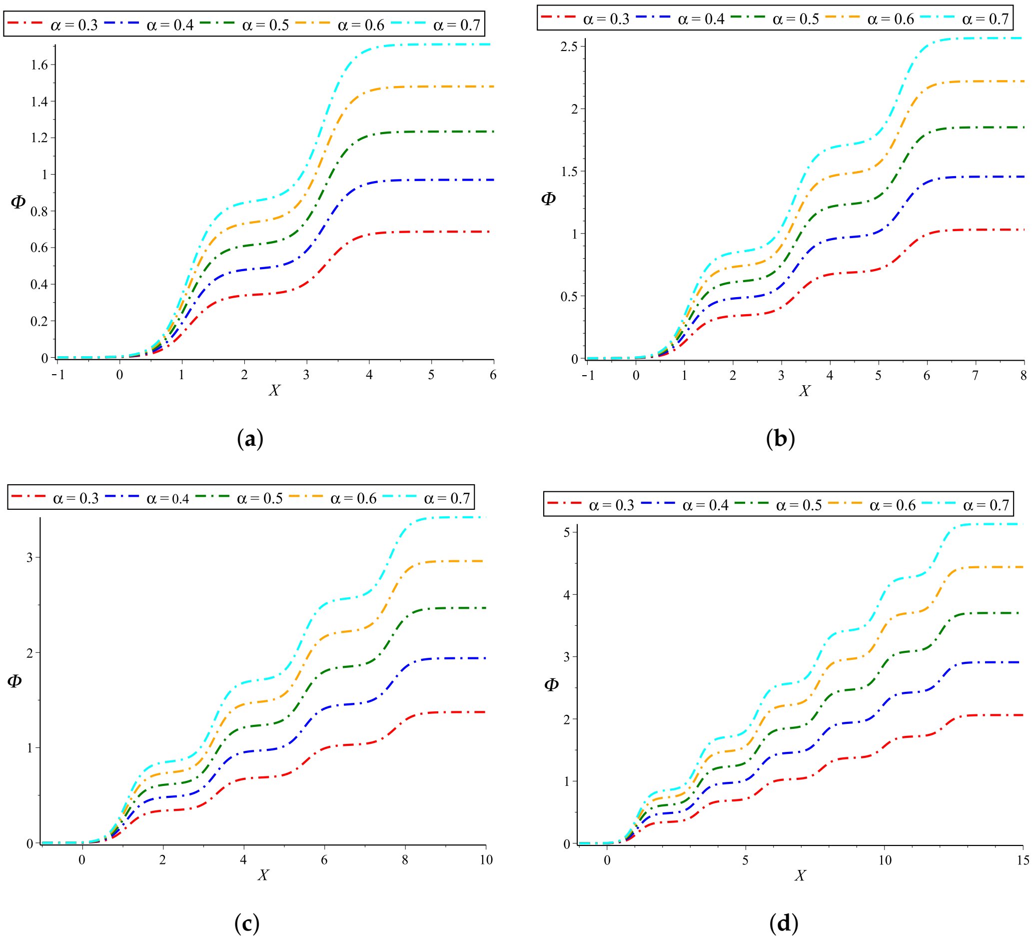
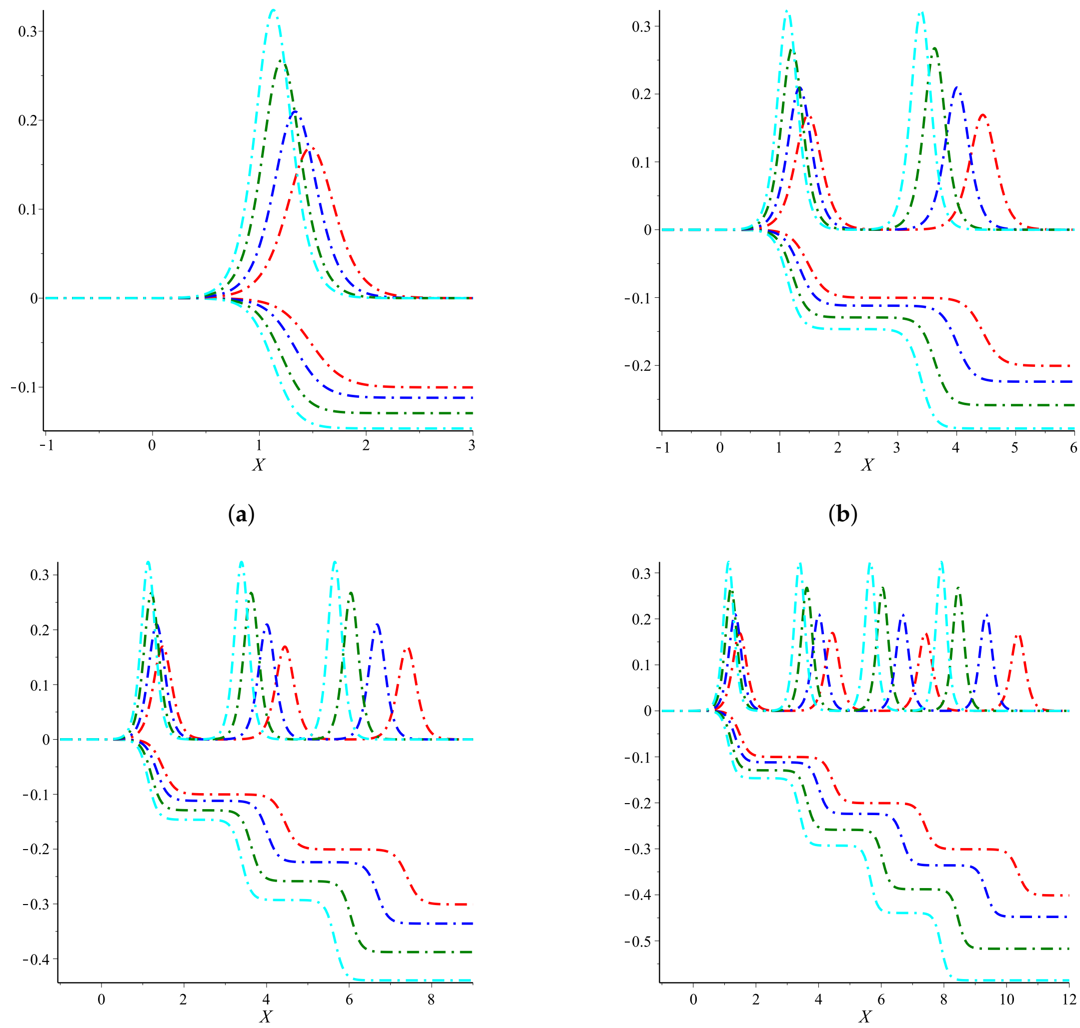


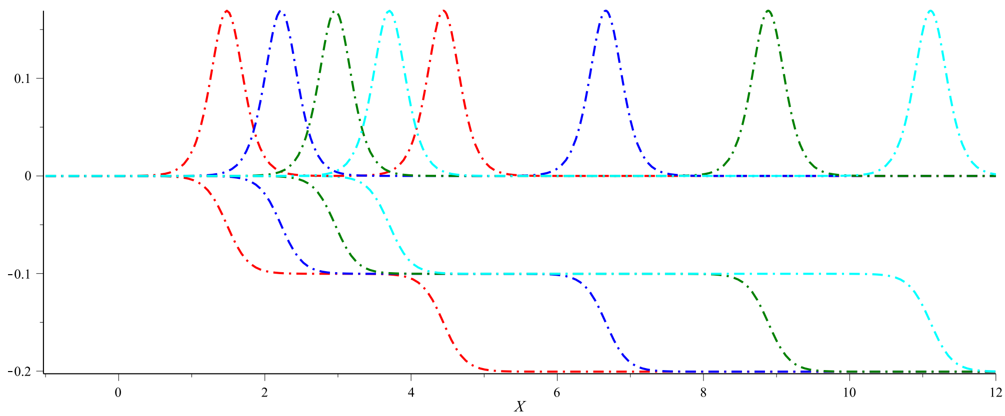
Figure 6. Effect of α on the electrostatic (**a**) single-, (**b**) overtaking collision of double-, (**c**) overtaking collision of triple-, and (**d**) overtaking collision of four–shock wave excitations, respectively, by choosing $p = 0.45$, $M_r = 3.75$, $T_{ni} = 0.05$, $T_e = 0.2$, $N_{r1} = 0.5$, $N_{r2} = 0.02$, $\mu_{ni} = 0.01$, and $\mu_{pi} = 0.5$.



(**a**) (**b**)

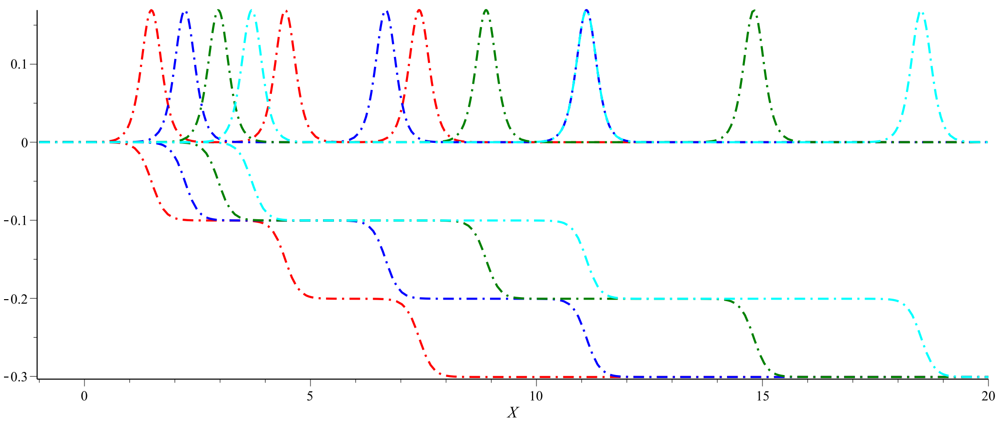
(**c**) (**d**)

Figure 7. Rarefactive overtaking collision of electrostatic shock wave (lower curves) and its corresponding electric field behavior (upper curves) for (**a**) $N = 1$, (**b**) $N = 2$, (**c**) $N = 3$, and (**d**) $N = 4$ with $\alpha = 0$ (red color), 0.1 (blue color), 0.2 (green color), 0.3 (cyan color), $M_r = 3.75$, $T_{ni} = 0.05$, $T_e = 0.2$, $N_{r1} = 0.9$, $N_{r2} = 0.5$, $\mu_{ni} = 0.05$, and $\mu_{pi} = 0.35$.



(**a**)

Figure 8. *Cont.*



(**b**)

Figure 8. Rarefactive electrostatic overtaking collision of shock wave (lower curves) and its corresponding electric field behaviors (upper curves) for (**a**) $N = 2$ and (**b**) $N = 3$ with $T = 10$ (red color), 15 (blue color), 20 (green color), 25 (cyan color), $\alpha = 0$ and $q = 1$, $M_r = 3.75$, $T_{ni} = 0.05$, $T_e = 0.2$, $N_{r1} = 0.9$, $N_{r2} = 0.5$, $\mu_{ni} = 0.05$, and $\mu_{pi} = 0.35$.

On the other hand, the considered plasma system coincides with the plasma system that is proposed in [30] when $\alpha = 0$ and $q = 1$. However, no comparison needs to be made with the single shock wave phenomena described by the Burgers equation that is presented in the theoretical investigation of the work in [30]. However, the phase velocity may be incorrectly defined, which leads to truncation errors in [30]. However, this research works reveals the novel electrostatic N-shocks potential function, which is obtained directly via the exponential functions. To best of our knowledge, the N-shocks potential defined in Equation (21) is not previously archived for Equation (18). Thus, the theoretical investigations presented in this work would be useful to describe the nonlinear dynamical behaviour of IA single- ($N = 1$), and overtaking collisions of multi-shocks ($N \geq 1$) in various environments as mentioned earlier, where the plasma system consists pair-ion and not only iosthermal, but also superthermal, subthermal, nonthermal, and nonextensive nonthermal electrons.

5. Concluding Remarks

A plasma system consisting of positive as well as negative ions and electrons is proposed to study the nature of not only single, but also overtaking collisions of multi-shocks electrostatic potential structures with the influences of plasma parameters, where electrons are assumed to follow not only iosthermality but also superthermality, subthermality, nonthermality, and nonthermality with the presence of nonextensitity. The (α, q)-velocity distribution function is considered because it is very effective in all cases of thermality conditions. By employing the well-established reductive perturbation approach, the Burger equation is obtained which divulges the shocks in the proposed plasmas. The new exact N-shock wave solution is archived by assuming directly the exponential function. It is found that μ_{pi}, μ_{ni}, and N_{r2} are significantly affected on the formation of monotonically single, and overtaking collisions of N-shocks by playing different roles. This is because the steepness and amplitude of shocks are sensitively increased (decreased) with the increase of μ_{pi} (N_{r2}), but very slightly increased with the increase of μ_{ni}. In addition, both compressive and rarefactive single as well as overtaking collisions of N-shocks exist with the influences of parameters. The density ratios and strength of non-extensive as well as the population of nonthermal electrons are also significantly contributed in the formation of overtaking collisions of N-shocks in the plasmas. Besides, the electric field is produced overtaking collisions of N-hump shape structures due to the overtaking collisions of N-shocks electrostatic potential. Thus, the outcomes presented in this work would be very useful to understand the dynamics of overtaking collisions of electrostatic N-shocks and their corresponding electric field behaviors in many space and astrophysical environments [14–17,29] (especially, the electrostatic overtaking collisions of multi-shock noise in the Earth's ionosphere) and future experimental studies [4,11,12]. It is noted that the basic features of small but finite amplitude single- and overtaking collisions of multi-shocks are validated only if the nonlinear coefficient $B \neq 0$ is used, because the electrostatic potential function as mentioned in Equation (21) becomes infinite when $B \to 0$. Further theoretical studies in the formation of overtaking collisions of N–shocks around the critical values by deriving a NLEE with higher order correction is still required. This will be our main focus for future studies.

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