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Linear Vibration Analysis of Shells Using a Seven-Parameter Spectral/*hp* Finite Element Model

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Abstract: In this paper, a seven-parameter spectral/*hp* finite element model to obtain natural frequencies in shell type structures is presented. This model accounts for constant and variable thickness of shell structures. The finite element model is based on a Higher-order Shear Deformation Theory, and the equations of motion are obtained by means of Hamilton's principle. Analysis is performed for isotropic linear elastic shells. A validation of the formulation is made by comparing the present results with those reported in the literature and with simulations in the commercial code ANSYS. Finally, results for shell like structures with variable thickness are presented, and their behavior for different ratios r/h and L/r is studied.

Keywords: 7-parameter shell; finite element model; natural frequencies; numerical results

1. Introduction

During the last several decades, studies of refined shell finite elements have taken importance due to their use in modeling variety of structures, mainly to study complex materials (e.g., laminated and functionally graded materials) and their applications in distinct areas of engineering. Consequently, many researchers have shown a strong interest in new and robust shell formulations, which require no artificial fixes such as reduced integration and stabilization. Through these developments, the performance of shell elements have been enhanced significantly. Continuous improvement of underlying mathematical models and computer implementations have been an area of significant research. Due to the amount of works in literature about shells, a detailed review is not presented here; the review of literature is limited to the present focus, namely, free vibration studies of shells.

Through the years, different theories of shells have been proposed and the most used can be classified as follows: Classical Theories (CT), First-order Shear Deformation Theories (FSDT), and High-order Shear Deformation Theories (HSDT). The latest theories have been developed with the intention to avoid locking (of various kinds) present in the numerical solution of some shell theories, to enhance the performance of the elements, and to eliminate limitations (e.g., the thickness inextensibility) of these theories. The classical theories are based on the Kirchhoff–Love's hypothesis in which lines normal to the undeformed mid-surface remain straight and normal to the surface after deformation [1]; the kinematics of these theories is defined using three displacements in each direction of a point located on the mid-surface [2]. The use of formulations based on CT is limited to thin shells, mainly because such assumption underestimates the effects of normal and shear strains in the thickness direction [3]. In addition, often one neglects rotatory inertia in CT and thus overestimates natural frequencies of moderately thick shells [2].

The FSDT is developed [4] to overcome the limitations of CT [2]; nevertheless, FSDT still presents a few limitations and exhibits shear locking (in addition to thickness and membrane locking present in the CT). Here, the term locking refers to the stiffening effects created by the use of low-order C^0 interpolation functions or by the kinematic assumptions. Generally, the shell kinematic is described with five parameters, three displacements in each coordinate direction at the mid-surface, and two rotations about the mid-surface coordinates (one can include the drilling degree of freedom or rotation about the transverse normal, as needed). In addition, this theory often is known as theory of five parameters. Limitation of this theory lies in its application to thin shells, need for shear correction factors, and transverse inextensibility. The shear correction factors are difficult to determine since they depend on the shell geometry, boundary conditions, and loads applied [3,5]. Results obtained are accurate while the shell does not show significant deformation through its thickness [2]. This is due to the fact that the theory is based on zero normal strain [6]. The good accuracy obtained with FSDT, with respect to CT, is a result of taking into account the effects of transverse shear strain and rotatory inertia [7]. The numerical implementation of FSDT exhibits locking phenomena (e.g., shear locking [8,9]), which is caused by the zero shear strains [10]. Some remedies to avoid locking phenomenon are the use of the method of assumed natural strain (ANS) [8], reduced integration techniques [9], and mixed formulations [11].

In HSDT, the displacement expansions considered are of high-order, and they can be applied to thick shells. These theories are developed using either polynomial or nonpolynomial expansions through thickness coordinate. The HSDT were introduced to avoid the use of shear correction factors needed in the FSDT [3]. Papers with high-order formulations can be found in the works of Sansour [12], Bischoff & Ramm [8], Hahn & Kikuchi [13], Payette & Reddy [14], and Amabili [2]. Among the finite element formulations based on these theories is the seven-parameter formulation with the drilling degree of freedom, the seventh parameter being the thickness stretch (see Arciniega & Reddy [11]). In addition, seven-parameter formulations were developed with the intention to eliminate the phenomenon of Poisson locking presented in six-parameter formulations (i.e., three displacements and three rotations). The Poisson locking arises when a constant state of normal strain through the thickness is assumed, and this phenomenon is due to the mathematical model and not to finite element model discretization [15]; in other words, a mesh refinement does not alleviate the locking.

There are numerous works related to shells in the literature. Finite element formulations to perform different types of analyses (e.g., static, dynamic and vibration or wave propagation analysis) can also be found. These finite element models can be classified as follows [14]: facet-shell elements, 2D elements based on shell theories, 3D elasticity elements, and continuum or degenerated elements. Often, these formulations are restricted to model elements with particular geometries, such as shallow shells, doubly curved shells, circular cylindrical shells, and so on. In addition, in general, formulations for dynamic analysis, free vibration, or transient response are developed for specific cases of materials (e.g., laminated shells and functionally graded materials). Examples related to these formulations can be found in the works of Cugnoni et al. [16], Park et al. [17], Jung & Han [18], Sahoo et al. [19], Hirwani et al. [20], Biswal & Mohanty [21], Moita et al. [22], among others. A review for plates and shells fabricated with FGM can be found in the work of Thai & Kim [3].

Kant et al. [23] presented a dynamic analysis model by means of degenerated shell elements, where the discretization of the 3D elasticity equations is made in terms of mid-surface nodal variables. In this formulation, five parameters were used to define the shell kinematic; however, they do not present any analysis to determine natural frequencies. Bardell et al. [24] reported a free vibration analysis of isotropic, open conical panels using the hp version of the finite element method (i.e., a refinement in the mesh size h and the order p of the polynomial interpolation functions) based on Kirchhoff–Love thin shell equations, and they assumed cones of uniform thickness. They presented natural frequencies and mode shapes, and compared their results with those obtained using a commercial finite element code and experimental results. Lee & Han [25] determined natural frequencies for plates and shells by means of a nine-node degenerated element with six degrees

of freedom, which was developed by them based on assumptions of the FSDT, and using the ANS approach to mitigate the locking phenomena. In the work of Lee & Han [26], a similar formulation to the one presented by these authors in 2001 [25] is used, but this time it is applied to laminated composite plates and shells. Recently, Bahrami et al. [27] reported a vibration analysis of thin shallow shells with the spectral interpolation functions and subjected to impact loads. Tornabene et al. [28] presented formulations based on differential and integral quadrature methods for the free vibration analysis of composite plates and shells; they considered nonuniform thickness. Żak & Krawczuk [29] presented a finite element with six degrees of freedom and spectral interpolation functions for dynamic analysis of isotropic shell-like structures; the transverse deformation was considered and the structures studied were of uniform thickness.

Additionally, many works related to the calculation of natural frequencies for plates and shells can be found in the literature; these use different numerical approaches, than the finite element method, and considered structures; see Ganesan & Sivadas [30] and Sivadas & Ganesan [31]. Both papers reported a semi-analytic finite element analysis of free vibration for thin circular isotropic cylindrical shells with variable thickness based on Kirchhoff–Love’s shell theory, but Ganesan & Sivadas [30] reported analysis of cantilevered cylindrical shells. Kang & Leissa [32] presented a three-dimensional vibration analysis to determine natural frequencies of thick spherical shell panels with variable thickness using the Ritz method. El-Kaabazi & Kennedy [33] determined natural frequencies and vibration modes of cylindrical shells with variable thickness based on the assumptions of Donell, Timoshenko, and Flüge shell theories; the natural frequencies were found using the Wittrick–Williams algorithm. Dai et al. [34] presented an exact series solution for the vibration analysis of circular cylindrical shells by means of the Flüge shell theory. Baccocchi et al. [35] performed free vibration analysis of variable thickness plates and shells using the Generalized Differential Quadrature method.

The aim of this work is to implement a seven-parameter finite element formulation (with thickness stretch as the seventh degree of freedom) to determine natural frequencies and mode shapes of arbitrary isotropic shell structures with uniform and nonuniform thicknesses. In this formulation, a continuum element with the hp spectral interpolation functions (see the text of Karniadakis & Sherwin [36]) is used. The description of shell kinematics with seven parameters, higher-order spectral interpolation of all variables, and the use of three-dimensional constitutive equations avoids all types of locking. By means of hp spectral interpolation functions, the errors in the numerical interpolation are reduced, achieving great accuracy and avoiding the Runge’s effect (more details about this effect can be found in the works of Reddy [15] and Pozrikidis [37]). The seven-parameter formulation was presented by Payette & Reddy [14] for static nonlinear analyses of isotropic, FGM, and laminated composite shells, and Gutierrez Rivera & Reddy [38] extended the work to thermo-mechanical nonlinear analysis of FGM plates and shells. Numerical results obtained were compared with those available in the literature for constant thickness formulations and with those obtained by means of the commercial code ANSYS.

2. Theoretical Formulation

In this work, the 3D geometry is approximated using a continuum shell element; in other words, the shell element Φ presented in Figure 1 is modeled only by means of its mid-surface Ω (shaded area). The shell element shown in figure has constant thickness and radii, R_1 and R_2 ; however, in reality, thickness can vary. In Figure 2, a discretization of shell’s mid-surface using spectral nodes can be observed; in this case, an element with polynomial order $p = 4$ is shown. In both figures, the reference system is Cartesian (X_1, X_2, X_3) , and the local system is curvilinear (ξ_1, ξ_2, ξ_3) and it is attached to the mid-surface.

Figure 1. Shell element and its mid-surface.

Figure 2. Finite element discretization of shell mid-surface.

The position vector $\underline{\mathbf{X}}^k$, shown in Figure 2, describes the mid-surface coordinates of the k th spectral node. The spectral nodes, which are not equally spaced, are distributed in positions that correspond to the roots of orthogonal polynomial families, in this case, the Legendre polynomials (more detailed information may be consulted in the work of Payette & Reddy [14]). Additionally, it is necessary to know the unit normal vector at each node, which is denoted by $\underline{\mathbf{n}}^k$. Both vectors are in terms of the reference system. In addition, the thickness at each node is denoted with h^k . All these geometric parameters define the spectral element and correspond to the code's input data. In Section 2.1, the approximation of these parameters will be described.

2.1. Geometric Parameters

The position vector of a point located on the element's mid-surface is approximated as follows:

$$\underline{\mathbf{X}} = \sum_{k=1}^n \psi_k(\tilde{\zeta}^1, \tilde{\zeta}^2) \underline{\mathbf{X}}^k, \quad (1)$$

where $\underline{\mathbf{X}}^k$ is the position vector of the k th spectral node, $\psi_k(\tilde{\zeta}^1, \tilde{\zeta}^2)$ is the k th interpolation function of order p (i.e., isoparametric formulation is used). The total number of nodes n per element is given by $n = (p + 1)(p + 1)$.

In this formulation, the normal vector of a point located on the element's mid-surface is approximated as

$$\underline{\mathbf{n}} = \sum_{k=1}^n \psi_k(\tilde{\zeta}^1, \tilde{\zeta}^2) \underline{\mathbf{n}}^k, \quad (2)$$

where $\underline{\mathbf{n}}^k$ is the unit normal vector of the k th spectral node.

Now, using the position vector in Equation (1) and scaling the normal vector in Equation (2) by a factor of $h/2$ (half of the thickness), the position of a typical point in the element Φ is given as

$$\mathbf{X} = \sum_{k=1}^n \psi_k(\xi_1, \xi_2) \left(\mathbf{X}^k + \xi^3 \frac{h}{2} \hat{\mathbf{n}}^k \right). \quad (3)$$

In addition to the above approximations, the covariant basis vectors for each point in the element must be defined:

$$\mathbf{g}_i = \frac{\partial \mathbf{X}}{\partial \xi^i} \equiv \mathbf{X}_{,i} \quad i = 1, 2, 3. \quad (4)$$

The covariant basis vectors in Equation (4), by means of Equation (3), can be separated as follows:

$$\mathbf{g}_\alpha = \mathbf{a}_\alpha + \xi^3 \frac{h}{2} \hat{\mathbf{n}}_{,\alpha}, \quad (5)$$

$$\mathbf{g}_3 = \frac{h}{2} \hat{\mathbf{n}}. \quad (6)$$

where the α subindex can take values of 1, 2. Furthermore, the vector $\mathbf{a}_\alpha$ is given by

$$\mathbf{a}_\alpha = \mathbf{X}_{,\alpha}.$$

A differential line element in the typical shell element, in terms of the curvilinear coordinates, can be expressed as

$$\begin{Bmatrix} dX_1 \\ dX_2 \\ dX_3 \end{Bmatrix}^T = \begin{Bmatrix} d\xi^1 \\ d\xi^2 \\ d\xi^3 \end{Bmatrix}^T \begin{bmatrix} \frac{\partial X_1}{\partial \xi^1} & \frac{\partial X_2}{\partial \xi^1} & \frac{\partial X_3}{\partial \xi^1} \\ \frac{\partial X_1}{\partial \xi^2} & \frac{\partial X_2}{\partial \xi^2} & \frac{\partial X_3}{\partial \xi^2} \\ \frac{\partial X_1}{\partial \xi^3} & \frac{\partial X_2}{\partial \xi^3} & \frac{\partial X_3}{\partial \xi^3} \end{bmatrix} = \{d\xi\}^T [J], \quad (7)$$

where $[J]$ is the Jacobian matrix with determinant J . In conjunction with the covariant basis vectors, a contravariant set of basis vectors $\mathbf{g}^i$ is defined as follows:

$$\mathbf{g}^1 = \frac{\mathbf{g}^2 \times \mathbf{g}^3}{J}, \quad \mathbf{g}^2 = \frac{\mathbf{g}^3 \times \mathbf{g}^1}{J}, \quad \mathbf{g}^3 = \frac{\mathbf{g}^1 \times \mathbf{g}^2}{J}.$$

2.2. Displacement Field

By definition, the displacement of a particle with position vector $\mathbf{X}$, in the reference configuration, to a point $\mathbf{x} = \chi(\mathbf{X}, t)$ in the current configuration is given by the vector $\mathbf{u}(\mathbf{X}, t)$

$$\mathbf{u}(\mathbf{X}, t) = \mathbf{x} - \mathbf{X} = \chi(\mathbf{X}, t) - \mathbf{X}. \quad (8)$$

The displacement vector $\mathbf{u}(\mathbf{X}, t)$ can be expanded using the Taylor's series in terms of the thickness coordinate (ξ^3) as follows:

$$\mathbf{u}(\mathbf{X}, t) = \sum_{k=0}^{\infty} \frac{(\xi^3)^k}{k!} \mathbf{u}^k(\xi^\alpha, t) \Big|_{\xi^3=0},$$

$$\text{where } \mathbf{u}^k(\xi^\alpha, t) = \frac{\partial^k \mathbf{u}^k(\xi^i, t)}{\partial (\xi^3)^k} \Big|_{\xi^3=0}.$$

In this work, following the ones of Arciniega & Reddy [11] and Payette & Reddy [14], the displacement field is expanded upto the quadratic term in thickness direction:

$$\mathbf{u}(\mathbf{X}, t) = \underline{\mathbf{u}}(\zeta^\alpha, t) + \zeta^3 \frac{h}{2} \boldsymbol{\varphi}(\zeta^\alpha, t) + \left(\zeta^3\right)^2 \frac{h}{2} \boldsymbol{\Psi}(\zeta^\alpha, t), \quad (9)$$

where the generalized displacements $\underline{\mathbf{u}}$, $\boldsymbol{\varphi}$ and $\boldsymbol{\Psi}$ are defined as follows:

$$\underline{\mathbf{u}}(\zeta^\alpha, t) = u_i(\zeta^\alpha, t) \hat{\mathbf{E}}_i, \quad \boldsymbol{\varphi}(\zeta^\alpha, t) = \varphi_i(\zeta^\alpha, t) \hat{\mathbf{E}}_i, \quad \boldsymbol{\Psi}(\zeta^\alpha, t) = \Psi(\zeta^\alpha, t) \mathbf{n}(\zeta^\alpha),$$

and $\hat{\mathbf{E}}_i$ denotes the unit basis vectors in the fixed orthogonal Cartesian coordinate system.

The vector $\underline{\mathbf{u}}$ is associated with the displacements at mid-surface, $\boldsymbol{\varphi}$ is the difference vector, and it gives the change in the mid-surface director, and the vector $\boldsymbol{\Psi}$ is defined by scaling the normal vector $\mathbf{n}$ with the parameter Ψ . Both vectors $\underline{\mathbf{u}}$ and $\boldsymbol{\varphi}$ have three components (i.e., six degrees of freedom), and the term Ψ is the seventh parameter and it denotes the thickness stretch, which is used to mitigate the phenomenon of Poisson's locking that appears in six-parameter formulation [14,39].

For a harmonic motion, the displacement field is assumed to be a periodic function in time and, in general, its solution can be treated as the product of a function of $\mathbf{X}$ and a time function (see text of Reddy [40]):

$$\mathbf{u}(\mathbf{X}, t) = \mathbf{U}(\mathbf{X}) e^{-i\omega t}, \quad (10)$$

where $\mathbf{U}(\mathbf{X})$ is the vibration mode vector associated with the frequency of natural (or free) vibration, ω .

2.3. Strains

The strains in the element are obtained by means of the Green–Lagrange strain tensor, which is defined as

$$\mathbf{E} = \frac{1}{2} \left(\mathbf{u}_{,i} \cdot \mathbf{g}_j + \mathbf{g}_i \cdot \mathbf{u}_{,j} + \underline{\mathbf{u}}_{,i} \cdot \underline{\mathbf{u}}_{,j} \right) \mathbf{g}^i \otimes \mathbf{g}^j. \quad (11)$$

In terms of the covariant components, it can be written as [12]

$$\mathbf{E} = E_{\alpha\beta} \mathbf{g}^\alpha \otimes \mathbf{g}^\beta + E_{\alpha 3} \mathbf{g}^\alpha \otimes \mathbf{g}^3 + E_{3\alpha} \mathbf{g}^3 \otimes \mathbf{g}^\alpha + E_{33} \mathbf{g}^3 \otimes \mathbf{g}^3. \quad (12)$$

Considering only the linear part, the covariant components in the above equation are given as

$$E_{\alpha\beta} = \mathbf{u}_{,\alpha} \cdot \mathbf{g}_\beta + \mathbf{g}_\alpha \cdot \mathbf{u}_{,\beta}, \quad (13)$$

$$E_{\alpha 3} = \mathbf{u}_{,\alpha} \cdot \mathbf{g}_3 + \mathbf{g}_\alpha \cdot \mathbf{u}_{,3}, \quad (14)$$

$$E_{3\alpha} = E_{\alpha 3}, \quad (15)$$

$$E_{33} = \mathbf{u}_{,3} \cdot \mathbf{g}_3 + \mathbf{g}_3 \cdot \mathbf{u}_{,3}. \quad (16)$$

If the scalar products are computed, the covariant components can be expanded in terms of the thickness coordinate ζ^3 as

$$E_{\alpha\beta} = \varepsilon_{\alpha\beta}^{(0)}(\zeta^\alpha, t) + \zeta^3 \varepsilon_{\alpha\beta}^{(1)}(\zeta^\alpha, t) + \left(\zeta^3\right)^2 \varepsilon_{\alpha\beta}^{(2)}(\zeta^\alpha, t) + \left(\zeta^3\right)^3 \varepsilon_{\alpha\beta}^{(3)}(\zeta^\alpha, t), \quad (17)$$

$$E_{\alpha 3} = \varepsilon_{\alpha 3}^{(0)}(\zeta^\alpha, t) + \zeta^3 \varepsilon_{\alpha 3}^{(1)}(\zeta^\alpha, t) + \left(\zeta^3\right)^2 \varepsilon_{\alpha 3}^{(2)}(\zeta^\alpha, t) + \left(\zeta^3\right)^3 \varepsilon_{\alpha 3}^{(3)}(\zeta^\alpha, t), \quad (18)$$

$$E_{33} = \varepsilon_{33}^{(0)}(\zeta^\alpha, t) + \zeta^3 \varepsilon_{33}^{(1)}(\zeta^\alpha, t). \quad (19)$$

This expansion is based on the assumed displacement field, which was taken to be a quadratic function of ζ^3 .

Neglecting all the quadratic and higher-order terms of the covariant components (the underlined terms in Equations (17) and (18)), the coefficients of covariant components in Equations (17)–(19) are determined as

$$\begin{aligned}\varepsilon_{\alpha\beta}^{(0)}(\xi^\alpha, t) &= \frac{1}{2}(\underline{\mathbf{u}}_{,\alpha} \cdot \mathbf{a}_\beta + \mathbf{a}_\alpha \cdot \underline{\mathbf{u}}_{,\beta}), \\ \varepsilon_{\alpha\beta}^{(1)}(\xi^\alpha, t) &= \frac{h}{4}(\underline{\mathbf{u}}_{,\alpha} \cdot \mathbf{n}_{,\beta} + \underline{\boldsymbol{\varphi}}_{,\alpha} \cdot \mathbf{a}_\beta + \mathbf{a}_\alpha \cdot \underline{\boldsymbol{\varphi}}_{,\beta} + \mathbf{n}_{,\alpha} \cdot \underline{\mathbf{u}}_{,\beta}), \\ \varepsilon_{\alpha 3}^{(0)}(\xi^\alpha, t) &= \frac{h}{4}(\underline{\mathbf{u}}_{,\alpha} \cdot \mathbf{n} + \mathbf{a}_\alpha \cdot \underline{\boldsymbol{\varphi}}), \\ \varepsilon_{\alpha 3}^{(1)}(\xi^\alpha, t) &= \frac{h}{2}\left[\mathbf{a}_\alpha \cdot \boldsymbol{\Psi} + \frac{h}{4}(\underline{\boldsymbol{\varphi}}_{,\alpha} \cdot \mathbf{n} + \mathbf{n}_{,\alpha} \cdot \underline{\boldsymbol{\varphi}})\right], \\ \varepsilon_{33}^{(0)}(\xi^\alpha, t) &= \frac{h^2}{4}(\boldsymbol{\varphi} \cdot \mathbf{n}), \\ \varepsilon_{33}^{(1)}(\xi^\alpha, t) &= \frac{h^2}{2}(\boldsymbol{\Psi} \cdot \mathbf{n}).\end{aligned}$$

3. Constitutive Equations

This formulation considers only isotropic linear elastic materials. For these materials, the fourth-order elasticity tensor, in terms of contravariant vectors, is defined as

$$\mathbf{C} = C^{ijkl} \mathbf{g}_i \otimes \mathbf{g}_j \otimes \mathbf{g}_k \otimes \mathbf{g}_l, \quad (20)$$

where the contravariant components are determined by

$$C^{ijkl} = \lambda g^{ij} g^{kl} + \mu (g^{ik} g^{jl} + g^{il} g^{jk}). \quad (21)$$

In the above expression, λ and μ are the Lamé parameters, which can be expressed in terms of Young's modulus, E , and Poisson's ratio, ν , as

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)}.$$

Here, g^{ij} denotes the contravariant components of the Riemannian metric tensor in the reference configuration, given as $g^{ij} = \mathbf{g}^i \cdot \mathbf{g}^j$.

The mechanical behavior of material is assumed to follow Hooke's Law; then, the relation between the Second Piola–Kirchhoff stress tensor and the Green–Lagrange strain tensor is given by

$$\mathbf{S} = \mathbf{C}\mathbf{E}, \quad (22)$$

where the contravariant components of $\mathbf{S}$ could be determined by $S^{ij} = C^{ijkl} E_{kl}$.

4. Equations of Motion

The equations of motion for the finite element model are obtained by means of the Hamilton's principle [41]:

$$\int_0^T [\delta K - (\delta U + \delta V)] dt = 0, \quad (23)$$

where δK represents the virtual kinetic energy, δU is the virtual strain energy, and δV is the virtual potential energy due to the external loads.

Hamilton's principle allows us to directly derive the finite element equations (i.e., the mass matrix is obtained with the virtual kinetic energy, the stiffness matrix with the virtual strain energy, and the

load vector with the virtual potential energy). For natural vibration, it is only necessary to determine the stiffness and mass matrices. The matrix calculation procedure is explained in Section 5.

The virtual kinetic energy is given by

$$\delta K = \int_V \rho \delta \dot{\mathbf{u}} \cdot \dot{\mathbf{u}} dV, \quad (24)$$

where ρ is the density. Substituting the displacement field and the differential volume element $dV = J dA d\zeta^3$ in the above equation, expressions for the mass matrix coefficients can be obtained. Integrating along ζ^3 yields

$$\begin{aligned} \delta K = \int_A \left\{ I_0 (\delta \dot{\mathbf{u}} \cdot \dot{\mathbf{u}}) + I_1 \left[\frac{h}{2} (\dot{\mathbf{u}} \cdot \delta \dot{\boldsymbol{\varphi}} + \dot{\boldsymbol{\varphi}} \cdot \delta \dot{\mathbf{u}}) \right] + I_2 \left[\frac{h}{2} (\dot{\mathbf{u}} \cdot \delta \dot{\Psi} \mathbf{n} + \frac{h}{2} \dot{\boldsymbol{\varphi}} \cdot \delta \dot{\boldsymbol{\varphi}} + \dot{\Psi} \mathbf{n} \cdot \delta \dot{\mathbf{u}}) \right] + \right. \\ \left. + I_3 \left[\frac{h^2}{4} (\dot{\boldsymbol{\varphi}} \cdot \delta \dot{\Psi} \mathbf{n} + \dot{\Psi} \mathbf{n} \cdot \delta \dot{\boldsymbol{\varphi}}) \right] + I_4 \left[\frac{h^2}{4} (\dot{\Psi} \delta \dot{\Psi} ||\mathbf{n}||^2) \right] \right\} dA, \end{aligned} \quad (25)$$

where I_0, I_1, I_2, I_3, I_4 are the mass inertias defined as follows:

$$\begin{aligned} I_0 &= \int_{-1}^1 \rho J d\zeta^3, & I_1 &= \int_{-1}^1 \rho \zeta^3 J d\zeta^3, & I_2 &= \int_{-1}^1 \rho (\zeta^3)^2 J d\zeta^3, \\ I_3 &= \int_{-1}^1 \rho (\zeta^3)^3 J d\zeta^3, & I_4 &= \int_{-1}^1 \rho (\zeta^3)^4 J d\zeta^3, \end{aligned}$$

and $||\mathbf{n}||$ is the norm of the normal vector $\mathbf{n}$.

The virtual strain energy is computed by

$$\delta U = \int_V \delta \mathbf{E} \cdot \mathbf{E} : \mathbf{C} dV. \quad (26)$$

Substituting the Green–Lagrange tensor and the fourth-order elasticity tensor, an expression to determine the stiffness matrix is obtained:

$$\delta U = \int_A \left[A^{ijkl} \delta \varepsilon_{ij}^{(0)} \varepsilon_{kl}^{(0)} + B^{ijkl} \left(\delta \varepsilon_{ij}^{(1)} \varepsilon_{kl}^{(0)} + \delta \varepsilon_{ij}^{(0)} \varepsilon_{kl}^{(1)} \right) + D^{ijkl} \delta \varepsilon_{ij}^{(1)} \varepsilon_{kl}^{(1)} \right] J d\zeta^3 dA, \quad (27)$$

where the contravariant components $A^{ijkl}, B^{ijkl}, D^{ijkl}$ are the effective extensional, extensional-bending coupling, and bending fourth-order stiffness tensor components [14]. These components are calculated as

$$A^{ijkl} = \int_{-1}^1 C^{ijkl} J d\zeta^3, \quad B^{ijkl} = \int_{-1}^1 \zeta^3 C^{ijkl} J d\zeta^3, \quad D^{ijkl} = \int_{-1}^1 (\zeta^3)^2 C^{ijkl} J d\zeta^3.$$

5. Finite Element Model

For natural vibration, the generalized displacements in Equation (9) are assumed of the form

$$\underline{\mathbf{u}}(\zeta^\alpha, t) = \underline{\mathbf{u}}(\zeta^\alpha) e^{-i\omega t}, \quad (28)$$

$$\boldsymbol{\varphi}(\zeta^\alpha, t) = \boldsymbol{\varphi}(\zeta^\alpha) e^{-i\omega t}, \quad (29)$$

$$\Psi(\zeta^\alpha, t) = \Psi(\zeta^\alpha) e^{-i\omega t}, \quad (30)$$

where ω represents the frequency of natural vibration, $i = \sqrt{-1}$, and the finite element approximations of the displacements are given by

$$\underline{\mathbf{u}}(\zeta^\alpha) = \sum_{j=1}^n \psi_j(\zeta^\alpha) \underline{\mathbf{u}}, \quad \boldsymbol{\varphi}(\zeta^\alpha) = \sum_{j=1}^n \psi_j(\zeta^\alpha) \boldsymbol{\varphi}, \quad \Psi(\zeta^\alpha) = \sum_{j=1}^n \psi_j(\zeta^\alpha) \Psi, \quad (31)$$

$\psi_j(\xi^\alpha)$ being the Lagrange interpolation functions of order p .

Substituting the approximations in the virtual kinetic energy of Equation (25) and in the virtual strain energy of Equation (27), the mass and the stiffness matrices are obtained. It is important to mention that the terms involved in the stiffness matrix are extensive, and a computer program of algebraic manipulation is required; in this case, they were obtained using Maple.

Finally, the finite element model to estimate the natural frequencies and the vibration modes of a shell is an eigenvalue problem of the form

$$([\mathbf{K}^e] - \omega^2[\mathbf{M}^e]) \{\Delta^e\} = \{0\}, \quad (32)$$

where $[\mathbf{K}^e]$ and $[\mathbf{M}^e]$ denote the stiffness and mass matrices of the e element, respectively, ω denotes the natural frequency, and $\{\Delta^e\}$ is the vibration mode vector associated with ω . In this work, the solution to the resulting eigenvalue problem is obtained by means of Eigen, a C++ template library for linear algebra [42].

6. Numerical Results

This section is divided into three different parts. In the first one, a convergence study is presented. Then, in the second one, a numerical comparison with the results available on the literature is developed. Finally, in the third one, the relation between some geometrical parameters with the natural frequencies is studied for shell like structures with variable thickness.

In order to describe the boundary conditions, the nomenclature used to label them is given by the letters C for clamped edge, F for free edge, and S for simply supported edge. For open geometries, the description of the boundary conditions starts with the nearest edge to the xz plane, and the others are listed using the left-hand rule. Figure 3 shows two examples of this terminology applied to open geometries, a CFCF condition for a plate, and an SCSF condition for a cylindrical panel.

Figure 3. Examples of boundary conditions for open geometries.

In cases where closed geometries are analyzed, the applied boundary conditions are described listing the nearest edge to the reference system and then the other edge. Figure 4 shows two examples of this convention, an annular plate under CF condition and an open ended cylinder with SC condition.

Figure 4. Examples of boundary conditions for closed geometries.

6.1. Convergence Study

First, a convergence study is performed to determine the mesh size and the element degree needed to obtain the eigenvalues (i.e., natural frequencies) with good accuracy. In this work, square meshes are used for all the cases studied. In general, an element with polynomial degree $p = 4$ is used and, if the convergence is slow, the polynomial degree is increased to $p = 8$. In this study, the relative error between each mesh size is calculated as follows:

$$\epsilon_r = \left| \frac{\omega^{i-1} - \omega^i}{\omega^i} \right| \times 100, \quad (33)$$

where ϵ_r is the relative error in percentage, ω^{i-1} is the natural frequency obtained with the previous mesh, and ω^i is the natural frequency calculated with the current mesh i . This study is divided in two parts. The first one presents results for plates and the second one for shells.

6.1.1. Plates

In the convergence study, the material properties considered are: $\nu = 0.30$, $E = 210$ GPa, and $\rho = 7850$ kg/m³. The natural frequencies for rectangular plates, with dimensions $a \times b$, are reported using a dimensionless parameter defined in reference [35] as

$$\lambda = \frac{2b^2 f}{\pi} \left[\frac{Eh^2}{12\rho(1-\nu^2)} \right]^{1/2}, \quad (34)$$

where f is the natural frequency expressed in Hz and h represents the thickness of the geometry. In the case of a plate with variable thickness, h takes the value of the smallest thickness.

First, the present formulation is tested to model square plates with ratio $a/b = 1$, and constant thickness. For this geometry, the convergence study is performed for the following boundary conditions:

I. All edges simply-supported (SSSS)

$$u_3(x, 0) = 0, \quad u_3(0, y) = 0, \quad u_3(x, b) = 0, \quad u_3(a, y) = 0. \quad (35)$$

II. Two opposite edges simply-supported and the other two clamped (SCSC)

$$u_3(x, 0) = 0, \quad \mathbf{u}(0, y) = 0, \quad u_3(x, b) = 0, \quad \mathbf{u}(a, y) = 0. \quad (36)$$

III. All edges clamped (CCCC)

$$\mathbf{u}(x, 0) = 0, \quad \mathbf{u}(0, y) = 0, \quad \mathbf{u}(x, b) = 0, \quad \mathbf{u}(a, y) = 0. \quad (37)$$

Table 1 contains results using different mesh sizes for Cases I, II, and III. In all the studies, the relative error for the last mesh refinement (from 36 to 64 elements) is less than 0.12%. For that reason, a mesh size of 36 elements is used on the verification section to model plates with constant thickness.

Table 1. Convergence of dimensionless frequency parameter for square plates of ratio $b/h = 10$.

B.C.	b/h	Mesh Size	Dimensionless Frequency Parameter									
			λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9	λ_{10}
SSSS	10	4	1.8850	4.6013	4.6013	7.0535	8.1524	8.7411	8.7411	8.8824	8.8935	9.2255
		16	1.8768	4.5881	4.5881	7.0230	8.1452	8.6764	8.6764	8.7754	8.7886	9.2255
		36	1.8758	4.5866	4.5866	7.0193	8.1452	8.6764	8.6764	8.7731	8.7864	9.2255
		64	1.8757	4.5864	4.5864	7.0187	8.1452	8.6764	8.6764	8.7728	8.7861	9.2255
SCSC	10	4	2.7475	5.0590	5.9661	6.2146	8.2245	9.1435	10.6854	10.9426	10.9804	11.7858
		16	2.7289	5.0420	5.9548	6.1604	8.1746	9.0403	10.6534	10.7096	10.9684	11.7122
		36	2.7236	5.0379	5.9524	6.1481	8.1632	9.0368	10.6470	10.6881	10.9667	11.7026
		64	2.7212	5.0362	5.9515	6.1425	8.1584	9.0357	10.6450	10.6795	10.9663	11.6989
CCCC	10	4	3.4000	6.5599	6.5599	9.2440	11.0969	11.2003	12.5671	12.5671	13.4031	13.4031
		16	3.3754	6.5062	6.5062	9.1816	10.8716	10.9702	12.5399	12.5399	13.2043	13.2043
		36	3.3676	6.4928	6.4928	9.1648	10.8503	10.9478	12.5351	12.5351	13.1818	13.1818
		64	3.3638	6.4864	6.4864	9.1569	10.8415	10.9384	12.5340	12.5340	13.1721	13.1721

In addition, the present formulation is tested to model plates with variable thickness in one direction along its mid-surface. Thickness variation is considered as follows:

$$h(y) = h_1 \left[1 - \chi \frac{y}{2b} \right], \quad (38)$$

where h_1 is the maximum thickness, χ is the tapered ratio, and b is the length of the side along y axis. Thus, the dimensionless frequency parameter λ , given by Equation (34), is calculated using the maximum thickness. Additionally, for this variation, the minimum thickness in the plate can be calculated setting $y = b$ in Equation (38), which gives

$$h_0 \equiv h(b) = h_1 \left[1 - \frac{\chi}{2} \right]. \quad (39)$$

Table 2 contains the frequency parameter values obtained in the convergence study performed by applying each case of the boundary conditions given in Equations (35)–(37) to a plate with variable thickness. From those values, it is concluded that the results obtained using 36 elements are converged, and this mesh size will be used for the numerical verification.

In addition to the plates with linear variable thickness, a square plate with parabolic thickness variation is analyzed. The thickness variation used is the same presented by Bacciocchi et al. [35], and it is given as follows:

$$h(y) = h_1 \left[1 + \frac{1}{2} \left(\frac{y}{b} \right)^2 - \frac{y}{b} \right]. \quad (40)$$

This plate is subject to the boundary conditions given in Case III, Equation (37), with a ratio $b/h_0 = 20$ and $h_0/h_1 = 0.5$. Table 2 also presents the convergence study of plates with linear and parabolic thickness variation, LTV and PTV denote a linear and parabolic thickness variation, respectively. The values obtained for the last refinement are in the order of 0.1%. For that reason, a mesh with 36 elements is used for the numerical validation.

6.1.2. Shells

Here, two cases for truncated half hemispheres and three cases for cylinders are studied. In Figure 5, the revolution profiles of (a) a truncated hemisphere, (b) a cylinder with linear variation, and (c) a cylinder with linear symmetric variation [30] are shown. The important geometric parameters are shown in these figures. Note that an angle θ , not shown, lies in the xy -plane for the hemisphere and in the yz -plane for the cylinder.

Table 2. Convergence of dimensionless frequency parameter for square plates of variable thickness.

B.C.	b/h_0	Mesh Size	Dimensionless Frequency Parameter									
			λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9	λ_{10}
SSSS (LTV)	20	4	1.4395	3.4873	3.5268	5.4898	6.7130	6.9748	8.0993	8.5927	8.7670	8.7840
		16	1.4317	3.4744	3.5049	5.4478	6.6048	6.7935	8.0989	8.5679	8.5920	8.6508
		36	1.4304	3.4720	3.5028	5.4424	6.6003	6.7904	8.0989	8.5599	8.5920	8.6431
		64	1.4301	3.4714	3.5023	5.4411	6.5991	6.7898	8.0989	8.5581	8.5920	8.6412
SCSC (LTV)	20	4	2.0928	3.8518	4.8478	5.9363	6.4744	6.9981	8.9493	9.4589	10.4564	10.6023
		16	2.0773	3.8307	4.7784	5.9253	6.3929	6.8794	8.4973	9.2892	10.0036	10.5687
		36	2.0730	3.8265	4.7678	5.9229	6.3816	6.8740	8.4780	9.2778	9.9834	10.5607
		64	2.0710	3.8247	4.7629	5.9219	6.3768	6.8721	8.4696	9.2735	9.9751	10.5578
CCCC (LTV)	10	4	3.0223	5.8861	5.8973	8.3730	10.0989	10.2055	12.2775	12.2962	12.5543	12.5610
		16	2.9998	5.8341	5.8439	8.3088	9.8615	9.9473	12.0541	12.0675	12.5249	12.5317
		36	2.9836	5.8057	5.8155	8.2721	9.8170	9.9016	12.0050	12.0183	12.5057	12.5126
		64	2.9804	5.8001	5.8098	8.2649	9.8086	9.8930	11.9957	12.0090	12.5042	12.5112
CCCC (PTV)	20	4	2.3831	4.4976	4.7480	6.7065	7.7330	8.4742	9.9484	10.2233	12.6720	12.8649
		16	2.3655	4.4543	4.6832	6.6283	7.5201	8.0129	9.7064	9.7584	11.3271	12.1759
		36	2.3533	4.4315	4.6585	6.5960	7.4785	7.9704	9.6596	9.7117	11.2627	12.1013
		64	2.3508	4.4267	4.6539	6.5896	7.4706	7.9628	9.6507	9.7029	11.2503	12.0891

Figure 5. Revolution profiles for (a) hemispherical, and (b,c) cylindrical shells.

The dimensionless frequency parameter for the hemispheres is denoted by Λ (see Bacciocchi et al. [35])

$$\Lambda = 2\pi fr \left(\frac{\rho}{G} \right)^{1/2}, \quad (41)$$

where G is the shear modulus. In addition, for these kinds of geometries, the following boundary conditions are considered:

IV. All free edges (FF)

$$\mathbf{u}(r, \varphi_0, \theta) \neq 0, \quad \mathbf{u}(r, \varphi_1, \theta) \neq 0. \quad (42)$$

On the the other hand, for cylinders, the boundary conditions are:

V. All clamped edges (CC)

$$\mathbf{u}(0, y, r) = 0, \quad \mathbf{u}(L, y, r) = 0. \quad (43)$$

VI. A clamped edge and a free edge (CF)

$$\mathbf{u}(0, y, r) = 0, \quad \mathbf{u}(L, y, r) \neq 0. \quad (44)$$

The first case studied is a truncated hemisphere with constant thickness, where its geometrical parameters are taken from the work by Kang and Leissa [32], and those correspond to $\varphi_0 = \pi/6$, $\varphi_1 = \pi/2$ and $r/h = 5$. The second case analyzed is a truncated hemisphere with linear variable thickness, and its geometric parameters are given by $\varphi_0 = \pi/6$, $\varphi_1 = \pi/2$ and $r/h_m = 5$, where h_m corresponds to the mean thickness defined as $h_m = (h_0 + h_1)/2$. The thickness variation is determined as [35]

$$h(\varphi) = h_0 \left[1 + 2 \left(\frac{\varphi - \varphi_0}{\varphi_1 - \varphi_0} \right) \right]. \quad (45)$$

Note that this expression works for a thickness ratio $h_1/h_0 = 3$, where $h_1 = h(\varphi_1)$.

Table 3 contains the results of the convergence study, using elements with $p = 4$, for the truncated hemispheres with constant and variable thicknesses. In both cases, the errors for the last increment are in general below of 0.43%. Thus, the study shows that a good convergence can be achieved with a mesh size of 49 elements.

Table 3. Convergence of dimensionless parameter Λ for a truncated hemisphere with FF boundary conditions.

Mesh Size		Dimensionless Frequency Parameter									
		Λ_1	Λ_2	Λ_3	Λ_4	Λ_5	Λ_6	Λ_7	Λ_8	Λ_9	Λ_{10}
$r/h = 5$ (constant)	16	0.3334	0.3382	0.8548	0.8583	0.9198	0.9198	1.5180	1.5671	1.5671	1.8278
	36	0.3296	0.3296	0.8510	0.8510	0.8586	0.8621	1.5180	1.5433	1.5433	1.5669
	49	0.3293	0.3293	0.8507	0.8507	0.8568	0.8568	1.5180	1.5245	1.5245	1.5669
	64	0.3292	0.3292	0.8507	0.8507	0.8556	0.8556	1.5168	1.5180	1.5188	1.5669
$r/h_m = 5$ (variable)	16	0.3462	0.3511	0.7477	0.7510	1.0126	1.0126	1.5319	1.5690	1.5690	1.7936
	36	0.3419	0.3419	0.7445	0.7445	0.9485	0.9518	1.5319	1.5688	1.5688	1.7362
	49	0.3415	0.3415	0.7442	0.7442	0.9465	0.9465	1.5319	1.5688	1.5688	1.7280
	64	0.3414	0.3414	0.7442	0.7442	0.9453	0.9453	1.5319	1.5688	1.5688	1.7207

The geometrical parameters of the cylinder with constant thickness are taken from Dai et al. [34], which correspond to $L = 0.502$ m, $r = 0.0635$ m, and $h = 0.00163$ m, while the material properties are $E = 210$ GPa, $\nu = 0.28$, and $\rho = 7800$ kg/m³. The study of convergence is realized for two types of boundary conditions, Cases V and VI given by Equations (43) and (44), respectively, and using elements with polynomial order $p = 8$. The results are presented in Table 4. In both cases, the relative errors are below of 0.1 order for the last increment. For that reason, it is concluded that a convergence can be achieved with a mesh size of nine elements.

Table 4. Convergence of natural frequencies of cylinder with constant thickness.

B.C.	Mesh Size	Natural Frequency											
		f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}	f_{11}	f_{12}
CC	1	943.46	1113.20	1902.29	1902.82	1907.98	2049.30	3109.91	3229.87	3256.72	3888.11	3889.17	4391.09
	4	904.81	905.10	913.04	932.30	1405.74	1416.29	1889.70	1889.70	1921.01	2026.41	2027.24	2059.93
	9	897.71	897.83	902.03	902.03	1393.46	1393.53	1510.55	1510.55	1685.00	1685.00	1885.93	1885.93
	16	896.96	896.96	901.15	901.15	1392.09	1392.09	1500.38	1500.93	1675.61	1676.11	1884.77	1884.77
CF	1	471.25	471.40	556.39	781.57	968.38	1154.96	1614.93	2031.64	2055.70	2056.21	2184.61	
	4	316.28	317.29	469.22	469.22	785.62	808.72	930.65	939.13	939.41	948.56	1462.88	
	9	315.38	315.38	468.52	468.52	768.22	768.37	914.46	914.58	937.50	937.50	1451.27	
	16	315.31	315.31	468.26	468.26	767.59	767.59	913.77	913.77	937.01	937.01	1450.34	

The cylinders with linear and linear symmetric thickness variations have the geometrical parameters reported in the work by Ganesan & Sivadas [30], which are: $h_m = 0.508$ mm, $h_0 = 2h_m/3$, $r/h_m = 100$ and $L/r = 2$. For these cases, the material properties are: $E = 203.5$ GPa, $\nu = 0.285$, and $\rho = 7846$ kg/m³. Additionally, the boundary conditions applied correspond to Case VI, given by Equation (44), where the edge with the maximum thickness is clamped. For both models, the mesh is made using elements $p = 8$. Table 5 presents the convergence studies for these cylinders. It is concluded that good convergence is achieved in both cases using 25 elements.

Table 5. Convergence of natural frequencies for a cylinder of ratio $r/h_m = 100$ and CF boundary conditions.

Thickness Variation	Mesh Size	Natural Frequency										
		f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}	f_{11}
Linear	4	1207.35	1421.01	1562.11	1565.90	1926.52	2504.34	2731.78	2731.91	3823.94	3866.67	3879.78
	16	1132.68	1132.69	1162.82	1162.82	1467.54	1488.30	1567.27	1567.27	1987.23	1987.23	2644.84
	25	1132.38	1132.38	1160.26	1160.50	1462.59	1462.59	1567.05	1567.05	1905.66	1905.66	2455.58
	36	1132.28	1132.28	1160.12	1160.12	1460.54	1460.96	1566.91	1566.91	1896.33	1896.33	2416.55
Linear symmetric	4	1268.97	1316.57	1320.46	1577.64	2199.21	2259.86	2260.21	3135.01	3738.82	3796.26	3882.94
	16	1107.47	1107.58	1313.82	1313.82	1380.46	1380.46	1873.68	1901.42	2258.84	2258.84	2501.53
	25	1107.12	1107.12	1313.50	1313.50	1376.88	1377.28	1866.73	1866.73	2258.27	2258.27	2411.96
	36	1107.08	1107.08	1313.58	1313.58	1376.71	1376.71	1864.07	1864.62	2258.51	2258.51	2401.38

6.2. Numeric Validation

In order to validate the numerical results obtained by means of the present finite element model, a comparison with results previously reported in the literature is performed. The number of spectral elements used for comparison are the ones obtained in the previous section, when convergence is achieved for each case. The natural frequencies obtained by means of the present formulation are reported in the column labeled as Present. In addition, for each case of study, a three-dimensional model is made in the commercial code ANSYS, using an eight node brick element (SOLID185). The number of brick elements used for the solid finite element models are shown below of the label 3D in each table.

6.2.1. Plates

In this section, comparisons for plates with constant, linear, and parabolic variations in thickness are presented. Tables 6–8 present the results for the boundary conditions listed on Cases I, II, and III, applied to squares plates with constant thickness. The values shown in parentheses represent the relative error in percentage. In general, a good agreement is observed with the results previously reported and those obtained by means of solid elements, with relative errors smaller than 5.8%.

Table 6. Comparison of results for a square plate of ratio $b/h = 10$ and SSSS boundary conditions.

Dimensionless Frequency Parameter	Formulation						
	Present	Reddy & Phan [43]	Liew [44] (FOPT)	Liew [45] (3D)	Shufrin [46] (FOPT)	Shufrin [46] (HOPT)	3D 172,800
λ_1	1.8758	1.9311 (2.86)	1.9311 (2.86)	1.9342 (3.02)	1.9311 (2.86)	1.9317 (2.89)	1.8424 (1.81)
λ_2	4.5866	4.6077 (0.46)	4.605 (0.40)	4.6222 (0.77)	4.6050 (0.40)	4.6088 (0.48)	4.4577 (2.89)
λ_3	4.5866	4.6077 (0.46)	4.605 (0.40)	4.6222 (0.77)	4.6050 (0.40)	4.6088 (0.48)	4.4577 (2.89)
λ_4	7.0193	7.0724 (0.75)	7.0642 (0.64)	7.1030 (1.18)	7.0642 (0.64)	7.0732 (0.76)	6.7263 (4.36)
λ_5	8.1452	-	-	-	-	-	8.1454 (0.002)
λ_6	8.6764	-	-	-	-	-	8.3994 (3.30)
λ_7	8.6764	-	-	-	-	-	8.4148 (3.11)
λ_8	8.7731	8.6153 (1.83)	8.6055 (1.95)	8.6617 (1.29)	8.6054 (1.95)	8.6188 (1.79)	8.6739 (1.14)
λ_9	8.7864	8.6153 (1.99)	8.6055 (2.10)	8.6617 (1.44)	8.6054 (2.10)	8.6188 (1.94)	8.6739 (1.30)
λ_{10}	9.2255	-	-	-	-	-	9.2259 (0.004)
λ_{11}	10.4887	10.8101 (2.97)	10.7932 (2.82)	10.8790 (3.59)	10.7932 (2.82)	10.8145 (3.01)	10.3310 (1.53)
λ_{12}	10.9272	10.8101 (1.08)	10.7932 (1.24)	10.8790 (0.44)	10.7932 (1.24)	10.8145 (1.04)	10.3310 (5.77)
λ_{13}	10.9272	-	-	-	-	-	10.4821 (4.25)

Table 7. Comparison of results for a square plate of ratio $b/h = 10$ and SCSC boundary conditions.

Dimensionless Frequency Parameter	Formulation			
	Present	Liew [44] (FOPT)	Liew [45] (3D)	3D 172,800
λ_1	2.7236	2.6998 (0.88)	2.7188 (0.18)	2.6808 (1.60)
λ_2	5.0379	4.9714 (1.34)	5.0110 (0.54)	4.9098 (2.61)
λ_3	5.9524	5.9904 (0.63)	6.0522 (1.65)	5.9492 (0.05)
λ_4	6.1481	-	6.5234 (5.75)	5.9955 (2.54)
λ_5	8.1632	7.9730 (2.39)	8.0513 (1.39)	7.8849 (3.53)
λ_6	9.0368	8.7873 (2.84)	8.8528 (2.08)	8.6754 (4.17)
λ_7	10.6470	10.2501 (3.87)	10.3860 (2.51)	10.3285 (3.08)
λ_8	10.6881	-	-	10.6326 (0.52)
λ_9	10.9667	-	-	10.9622 (0.04)
λ_{10}	11.7026	11.3338 (3.25)	11.4510 (2.20)	11.1587 (4.874)
λ_{11}	11.7732	12.024 (2.09)	11.5170 (2.22)	11.7670 (0.05)
λ_{12}	12.5019	-	12.1810 (2.63)	11.9813 (4.34)
λ_{13}	13.1770	-	13.0470 (1.00)	13.1662 (0.08)

Table 8. Comparison of results for a square plate of ratio $b/h = 10$ and CCCC boundary conditions.

Dimensionless Frequency Parameter	Formulation					
	Present	Liew [44] (FOPT)	Liew [45] (3D)	Shufrin [46] (FOPT)	Shufrin [46] (HOPT)	3D 400,000
λ_1	3.3676	3.2921 (2.29)	3.3215 (1.39)	3.2978 (2.12)	3.3046 (1.91)	3.3176 (1.51)
λ_2	6.4928	6.2761 (3.45)	6.3457 (2.32)	6.2877 (3.26)	6.3123 (2.86)	6.3419 (2.38)
λ_3	6.4928	6.2761 (3.45)	6.3457 (2.32)	6.2877 (3.26)	6.3123 (2.86)	6.3419 (2.38)
λ_4	9.1648	8.7934 (4.22)	8.9030 (2.94)	8.8127 (4.00)	8.8594 (3.45)	8.8982 (3.00)
λ_5	10.8503	10.3570 (4.76)	10.4990 (3.35)	10.4299 (4.03)	10.4946 (3.39)	10.4996 (3.34)
λ_6	10.9478	10.4561 (4.70)	10.5980 (3.30)	10.4299 (4.97)	10.4946 (4.32)	10.5988 (3.29)
λ_7	12.5351	12.5246 (0.08)	12.5220 (0.10)	12.5568 (0.17)	12.6471 (0.89)	12.5194 (0.13)
λ_8	12.5351	12.5246 (0.08)	12.5220 (0.10)	12.5568 (0.17)	12.6471 (0.89)	12.5194 (0.13)
λ_9	13.1818	-	12.7060 (3.74)	-	-	12.7053 (3.75)

Table 8. Cont.

Dimensionless Frequency Parameter	Formulation					
	Present	Liew [44] (FOPT)	Liew [45] (3D)	Shufrin [46] (FOPT)	Shufrin [46] (HOPT)	3D 400,000
λ_{10}	13.1818	-	12.7060 (3.74)	-	-	12.7053 (3.75)
λ_{11}	14.8717	-	14.8700 (0.01)	-	-	14.8712 (0.003)
λ_{12}	16.1611	-	-	15.8491 (1.97)	15.9868 (1.09)	15.5110 (4.19)
λ_{13}	16.1611	-	-	-	-	15.5110 (4.19)

Tables 9–11 present the results for the boundary conditions listed for Cases I, II, and III, applied to square plates with linear thickness variation. The highest relative errors between the presented formulation and solid elements is on the order of 3.92%. However, the highest errors are obtained when the present formulation is compared with the FSDT presented by Shufrin & Eisenberg [47], having a value of 4.11%.

Table 9. Comparison of results for a square plate of ratio $b/h_0 = 20$ and SSSS boundary conditions.

Dimensionless Frequency Parameter	Formulation				
	Present	Chopra [48]	Mizusawa [49]	Bacciocchi [35]	3D 400,000
λ_1	1.4304	1.4798 (3.33)	1.4504 (1.38)	1.404 (1.88)	1.3943 (2.59)
λ_2	3.4720	3.6354 (4.49)	3.4743 (0.06)	3.4743 (0.07)	3.3959 (2.24)
λ_3	3.5028	3.6755 (4.70)	3.5058 (0.09)	3.5058 (0.09)	3.4176 (2.49)
λ_4	5.4424	5.9040 (7.82)	5.484 (0.76)	5.4838 (0.75)	5.2862 (2.95)
λ_5	6.6003	7.0800 (6.78)	6.5347 (1.00)	6.5345 (1.01)	6.4379 (2.52)
λ_6	6.7904	7.3250 (7.30)	6.7039 (1.29)	6.7038 (1.29)	6.5949 (2.97)
λ_7	8.0989	-	-	-	8.0718 (0.34)
λ_8	8.5599	-	-	-	8.2568 (3.67)
λ_9	8.5920	-	8.5302 (0.72)	8.5303 (0.72)	8.3459 (2.95)
λ_{10}	8.6431	-	-	8.5921 (0.59)	8.58 (0.74)
λ_{11}	8.9568	9.5500 (6.21)	-	-	(8.944) (0.14)
λ_{12}	9.3862	9.6350 (2.58)	-	-	9.3532 (0.35)

Table 10. Comparison of results for a square plate of ratio $b/h_0 = 20$ and SCSC boundary conditions.

Dimensionless Frequency Parameter	Formulation			
	Present	Blevins [50]	Mizusawa [49]	3D 324,000
λ_1	2.0730	2.1460 (3.40)	2.0457 (1.33)	2.0307 (2.08)
λ_2	3.8265	-	3.7950 (0.83)	3.7564 (1.86)
λ_3	4.7678	-	4.6714 (3.40)	4.6667 (2.17)
λ_4	5.9229	-	-	5.9004 (0.38)
λ_5	6.3816	-	6.2870 (1.50)	6.2264 (2.49)
λ_6	6.8740	-	6.7723 (1.50)	6.7200 (2.29)
λ_7	8.4780	-	8.2283 (3.03)	8.2623 (2.61)
λ_8	9.2778	-	9.1022 (1.93)	8.9973 (3.12)
λ_9	9.9834	-	-	9.6869 (3.06)
λ_{10}	10.5607	-	-	10.5382 (0.213)

Table 11. Comparison of results for a square plate of ratio $b/h_0 = 10$ and CCCC boundary conditions.

Dimensionless Frequency Parameter	Formulation			
	Present	Shufrin [47] (FSDT)	Shufrin [47] (HSDT)	3D 324,000
λ_1	2.9836	2.9359 (1.62)	2.9401 (1.48)	2.9470 (1.24)
λ_2	5.8057	5.6634 (2.51)	5.6789 (2.23)	5.6972 (1.91)
λ_3	5.8155	5.669 (2.58)	5.6848 (2.30)	5.7037 (1.96)
λ_4	8.2721	8.0155 (3.20)	8.0466 (2.80)	8.0724 (2.47)
λ_5	9.8170	9.5017 (3.32)	9.5434 (2.87)	9.5529 (2.76)
λ_6	9.9016	9.5269 (3.93)	9.5699 (3.47)	9.6371 (2.74)
λ_7	12.0050	11.5315 (4.11)	11.5932 (3.55)	11.6360 (3.17)
λ_8	12.0183	11.5459 (4.09)	11.6079 (3.54)	11.6494 (3.17)
λ_9	12.5057	-	-	12.5044 (0.01)
λ_{10}	12.5126	-	-	12.5107 (0.02)
λ_{11}	14.7274	-	-	14.2251 (3.53)
λ_{12}	14.8112	-	-	14.2932 (3.62)

Table 11. Cont.

Dimensionless Frequency Parameter	Formulation			
	Present	Shufrin [47] (FSDT)	Shufrin [47] (HSDT)	3D 324,000
λ_{13}	14.9194	14.6648 (1.74)	14.7608 (1.07)	14.8173 (0.69)
λ_{14}	15.3667	-	-	14.8633 (3.39)
λ_{15}	16.6314	-	-	16.0038 (3.92)

Table 12 shows a comparison for the natural frequencies for a square plate with the parabolic thickness variation given in Equation (40), and its four edges clamped. The relative errors between the present formulation and solid elements have a maximum value of 2.97%. The maximum relative error is found when the present formulation is compared with the FSDT presented in the work by Bacciocchi et al. [35], with a value of 3.73%.

Table 12. Comparison of results for a square plate of ratio $b/h_0 = 20$, parabolic variation and CCCC boundary conditions.

Dimensionless Frequency Parameter	Formulation				
	Present	Shufrin [47] (FSDT)	Shufrin [47] (HSDT)	Bacciocchi [35]	3D 324,000
λ_1	2.3533	2.2835 (3.06)	2.2850 (2.99)	2.2818 (3.13)	2.2901 (2.76)
λ_2	4.4315	4.3327 (2.28)	4.3375 (2.17)	4.3298 (2.35)	4.3476 (1.93)
λ_3	4.6585	4.5231 (2.99)	4.5294 (2.85)	4.5207 (3.05)	4.5450 (2.50)
λ_4	6.5960	6.4168 (2.79)	6.4291 (2.60)	6.4107 (2.89)	6.4458 (2.33)
λ_5	7.4785	7.2995 (2.45)	7.3118 (2.28)	7.2945 (2.52)	7.3336 (1.98)
λ_6	7.9704	7.7170 (3.28)	7.7358 (3.03)	7.7157 (3.30)	7.7704 (2.57)
λ_7	9.6596	9.3726 (3.06)	9.3971 (2.79)	9.3631 (3.17)	9.4242 (2.50)
λ_8	9.7117	9.3996 (3.32)	9.4270 (3.02)	9.3922 (3.40)	9.4588 (2.67)
λ_9	11.2627	-	-	-	11.0079 (2.31)
λ_{10}	12.1013	-	-	-	11.7525 (2.97)
λ_{11}	12.5463	12.1072 (3.63)	12.1505 (3.26)	12.0957 (3.73)	12.1901 (2.92)
λ_{12}	12.6196	-	-	-	12.6188 (0.01)

6.2.2. Shells

Within this section, comparisons between the presented formulation and results available in the literature and those obtained by means of solid elements in ANSYS, for truncated hemispheres and cylinders, are presented. In these comparisons, values of natural frequencies and dimensionless parameters written in bold font represent even frequencies, that is, frequencies with identical value but different vibration mode.

Table 13 presents comparisons for a truncated hemisphere with constant and linear variable thickness. For constant thickness, the relative errors are below 5.10%. For linear thickness variation, frequencies of the present formulation are approximated of those reported in references [35,51] with maximum errors of 6.04% and 5.70%, respectively. The comparison with the 3D model gives a maximum error of 5.59%.

Table 13. Comparison of results for a truncated hemisphere with FF boundary conditions.

Dimensionless Frequency Parameter	Formulation						
	Constant Thickness $r/h = 5$			Linear Variation Thickness $r/h_m = 5$			
	Present	Kang [51]	3D 807,942	Present	Kang [51]	Baccocchi [35] (FSDT)	3D 855,000
Λ_1	0.3293	0.3241 (1.60)	0.3242 (1.58)	0.3415	0.3485 (2.00)	0.3491 (2.17)	0.3487 (2.05)
Λ_2	0.8507	0.8379 (1.53)	0.8380 (1.51)	0.7442	0.7052 (5.54)	0.7041 (5.70)	0.7054 (5.51)
Λ_3	0.8568	0.8397 (2.03)	0.8396 (2.05)	0.9465	0.9343 (1.31)	0.9364 (1.08)	0.9345 (1.29)
Λ_4	1.5180	1.4750 (2.91)	1.4753 (2.89)	1.5319	1.5360 (0.26)	1.5298 (0.14)	1.5357 (0.25)
Λ_5	1.5245	-	1.4753 (3.33)	1.5688	1.5680 (0.05)	1.5685 (0.02)	1.5681 (0.05)
Λ_6	1.5245	1.5160 (0.56)	1.5160 (0.56)	1.7280	1.6600 (4.10)	1.6649 (3.79)	1.6666 (3.69)
Λ_7	1.5669	1.5580 (0.57)	1.5577 (0.59)	1.7346	1.6710 (3.80)	1.6758 (3.51)	1.6714 (3.78)
Λ_8	1.9525	1.9290 (1.22)	1.9288 (1.23)	1.8993	1.8570 (2.28)	1.8622 (1.99)	1.8579 (2.23)
Λ_9	1.9766	1.9300 (2.42)	1.9298 (2.43)	2.0516	2.0900 (1.84)	2.1012 (2.36)	2.0893 (1.81)
Λ_{10}	2.2298	2.205 (1.12)	2.2057 (1.09)	2.3824	2.374 (0.35)	2.384 (0.05)	2.3743 (0.34)
Λ_{11}	2.3182	-	2.2057 (5.10)	2.5573	2.4230 (5.54)	2.4116 (6.04)	2.4219 (5.59)
Λ_{12}	2.3182	2.2100 (4.90)	2.2104 (4.88)	2.5816	2.4850 (3.89)	2.4786 (4.15)	2.4844 (3.91)
Λ_{13}	2.4050	2.3910 (0.58)	2.3914 (0.57)	2.6447	2.5070 (5.49)	2.5150 (5.16)	2.5075 (5.47)
Λ_{14}	2.4341	2.4180 (0.67)	2.4184 (0.65)	2.6880	2.6430 (1.70)	2.6451 (1.62)	2.6439 (1.67)

Table 14 presents a comparison for the natural frequencies of a cylinder with constant thickness, subjected to CC and CF boundary conditions. For these boundary conditions, the maximum relative errors are obtained when the present formulation is compared with the three-dimensional models; having values of 1.21% and 1.43%, respectively.

A comparison of the natural frequencies for cylinders with linear and linear symmetric thickness variations subjected to CF boundary conditions is presented in Table 15. For the linear thickness variation, the relative errors are smaller than 2.5%, when the results obtained using the present formulation are compared with those reported by Ganesan & Sivadas [30], and below 8% when the comparison is made with solid elements. On the other hand, for the cylinder with linear symmetric thickness variation, the results obtained with the present formulation have relative errors lower than 5%, except for the natural frequency f_{19} , when those are compared with the results presented by Ganesan & Sivadas [30]. The frequencies obtained with solid elements present errors below 4.4%.

Table 14. Comparison of results for a cylinder with constant thickness.

Natural Frequency	Formulation					
	CC			CF		
	Present	Dai [34]	3D 750,000	Present	Dai [34]	3D 750,000
f_1	896.96	896.56 (0.04)	901.05 (0.45)	315.31	316.02 (0.23)	318.68 (1.06)
f_2	901.15	898.18 (0.33)	904.78 (0.40)	468.26	467.34 (0.20)	467.92 (0.07)
f_3	1392.09	1388.9 (0.23)	1395.00 (0.21)	767.59	768.45 (0.11)	778.72 (1.43)
f_4	1500.38	1501.6 (0.08)	1518.80 (1.21)	913.77	915.13 (0.15)	921.79 (0.87)
f_5	1500.93	-	1518.80 (1.18)	937.01	936.25 (0.08)	937.3 (0.03)
f_6	1675.61	1676.00 (0.02)	1689.40 (0.82)	1450.34	1462.80 (0.85)	1453.5 (0.22)
f_7	1676.11	-	1689.40 (0.79)	1461.22	-	1482.4 (1.43)
f_8	1884.77	1880.9 (0.21)	1883.70 (0.06)	1461.79	-	1482.4 (1.39)
f_9	2020.03	2014.1 (0.29)	2018.70 (0.07)	1516.70	1519 (0.15)	1535.5 (1.22)
f_{10}	2049.35	-	2057.30 (0.386)	1517.25	-	1535.6 (1.20)
f_{11}	2049.75	-	2057.30 (0.37)	1614.93	1615.2 (0.02)	1615 (0.004)
f_{12}	2182.71	-	2182.50 (0.01)	1709.24	-	1723.2 (0.81)
f_{13}	2387.65	2386 (0.07)	2413.00 (1.05)	1709.72	-	1723.2 (0.78)
f_{14}	2474.29	2472.6 (0.07)	2495.90 (0.87)	2045.12	2043.3 (0.09)	2044.6 (0.03)

Table 15. Comparison of results for cylinders of ratio $r/h_m = 100$, with thickness variation and CF boundary conditions.

Natural Frequency	Formulation			Natural Frequency	Formulation		
	Linear Variation				Linear Symmetric Variation		
	Present	Ganesan [30]	3D 1,107,000		Present	Ganesan [30]	3D 1,107,000
f_1	1132.4	1129.6 (0.25)	1157.0 (2.13)	f_1	1107.12	1112.7 (0.50)	1136.7 (2.60)
f_2	1160.3	1154.2 (0.53)	1227.0 (5.44)	f_2	1313.50	1316.2 (0.20)	1320.3 (0.51)
f_3	1160.5	-	1227.0 (5.42)	f_5	1376.88	1382.7 (0.42)	1435.2 (4.06)
f_4	1462.6	1452.2 (0.72)	1570.6 (6.88)	f_6	1377.28	-	1435.2 (4.04)
f_5	1567.1	1566.7 (0.02)	1578.1 (0.70)	f_7	1866.73	1865.0 (0.09)	1950.7 (4.30)
f_6	1905.7	1884.3 (1.13)	2070.4 (7.96)	f_9	2258.27	2259.4 (0.05)	2258.3 (0.001)
f_7	2455.6	2396.4 (2.47)	2652.0 (7.41)	f_{11}	2411.96	2390.9 (0.88)	2522.6 (4.39)

Table 15. Cont.

Natural Frequency	Formulation			Natural Frequency	Formulation		
	Linear Variation				Linear Symmetric Variation		
	Present	Ganesan [30]	3D 1,107,000		Present	Ganesan [30]	3D 1,107,000
f_8	2730.0	2729.8 (0.01)	2729.2 (0.03)	f_{13}	2805.22	2925.4 (4.11)	2863.7 (2.04)
f_9	2808.2	-	2852.7 (1.56)	f_{19}	3090.96	3523.9 (12.29)	3182.8 (2.89)
f_{10}	2974.3	2977.5 (0.11)	3052.8 (2.57)	f_{23}	3767.56	-	3808 (1.06)
f_{11}	3080.7	-	3096.9 (0.52)	f_{24}	3767.56	-	3808.1 (1.06)
f_{12}	3080.8	-	3096.9 (0.52)				

In addition to the comparison presented in Table 15, for the cylinder with linear thickness variation, the vibration modes calculated using the presented formulation and solid elements in ANSYS are reported in Table 16. The plotted contours correspond to the resultant displacement vector. In this case, the fundamental frequency is even; therefore, two modes are shown in the table with illustrative purposes. From now on, for even frequencies, a single vibration mode will be shown. Note that the axes are rotated, and that the scaling factors may be different. A detailed inspection shows that almost all the vibration modes are almost equal, except for 4 and 5, which are interchanged. This difference is attributed to the closeness of the values of frequencies obtained, since $f_4 = 1570.6$ in ANSYS and $f_5 = 1567.1$ using the presented formulation, which are almost identical. Since the eigenvalues are almost the same, the eigenvectors associated are very similar (vibration modes).

Table 16. Vibration modes of a cylinder with linear thickness variation and CF boundary conditions.

Vibration Mode	Formulation	
	Present	3D
1	$f = 1132.4$	$f = 1157.0$
	$f = 1132.4$	$f = 1157.0$

Table 16. Cont.

Vibration Mode	Formulation	
	Present	3D
2	$f = 1160.3$	$f = 1227.0$

3	$f = 1160.5$	$f = 1227.0$

4	$f = 1462.6$	$f = 1570.6$

5	$f = 1567.1$	$f = 1578.1$

6	$f = 1905.7$	$f = 2070.4$

Table 16. Cont.

Vibration Mode	Formulation	
	Present	3D
7	$f = 2455.6$	$f = 2652.0$

8	$f = 2730.0$	$f = 2729.2$

9	$f = 2808.2$	$f = 2852.7$

10	$f = 2974.3$	$f = 3052.8$

6.3. Numeric Results

This section presents a study of the geometric parameters on the natural frequencies. Although the formulation is capable of modeling arbitrary shapes, in this study, the natural frequencies for the following geometries are obtained:

1. Hemisphere with linear variable thickness, $h_1/h_0 = 3$.
2. Cylinder with linear variable thickness, $h_1/h_0 = 2$.
3. Fraction of cylinder with linear variable thickness, $h_1/h_0 = 2$.

In addition, for each case of study, a three-dimensional model is made in the commercial code ANSYS, using the element SOLID185. The number of brick elements used for the solid finite element models are shown below of the label 3D in each table.

6.3.1. Hemisphere

A hemisphere with linear thickness variation is analyzed for different ratios of radius to mean thickness: $r/h_m = 50, 20, 10, 5, 2.5$. For these cases, the thickness ratio $h_1/h_0 = 3$ remains constant and FF boundary conditions are applied. Table 17 presents the natural frequencies, written using the dimensionless parameter defined for the hemispheres. The results obtained with the present formulation, using a mesh with 16 elements and a polynomial degree $p = 8$, are shown in columns labeled as 7-PL. Note that the ratios r/h_m are more closely to be thick shells, and, for these cases, the parameter Λ increases as this ratio decreases. Values of the relative errors suggest that the model has a good performance for ratios r/h_m less than 20.

Table 17. Dimensionless natural frequencies parameter for an hemisphere with linear thickness variation and FF boundary conditions, $h_1/h_0 = 3$.

Dimensionless Frequency Parameter	r/h_m									
	50		20		10		5		2.5	
	7-PL	3D 840,000	7-PL	3D 630,000	7-PL	3D 968,000	7-PL	3D 855,000	7-PL	3D 720,000
Λ_1	0.0245	0.0284 (13.89)	0.0978	0.1092 (10.44)	0.1846	0.1987 (7.07)	0.3414	0.3487 (2.10)	0.6005	0.5772 (4.03)
Λ_2	0.0695	0.0614 (13.14)	0.2662	0.2374 (12.16)	0.4716	0.4303 (9.60)	0.7441	0.7054 (5.49)	0.9446	0.9289 (1.69)
Λ_3	0.0775	0.0844 (8.20)	0.2973	0.3143 (5.41)	0.5375	0.5534 (2.88)	0.9445	0.9345 (1.08)	1.4447	1.4161 (2.02)
Λ_4	0.1531	0.1559 (1.81)	0.5703	0.5774 (1.23)	1.0087	1.0178 (0.90)	1.5319	1.5357 (0.25)	1.4946	1.4161 (5.54)
Λ_5	0.1685	0.1620 (3.99)	0.6204	0.5883 (5.47)	1.0783	1.0180 (5.92)	1.5688	1.5681 (0.05)	1.4946	1.4783 (1.11)
Λ_6	0.2470	0.2567 (3.79)	0.9001	0.9144 (1.57)	1.5350	1.5361 (0.07)	1.7173	1.6666 (3.04)	1.6346	1.6342 (0.02)
Λ_7	0.2851	0.2704 (5.42)	1.0093	0.9609 (5.03)	1.5511	1.5470 (0.26)	1.7328	1.6714 (3.68)	1.9918	2.0419 (2.46)
Λ_8	0.3566	0.3661 (2.57)	1.2819	1.2872 (0.41)	1.5739	1.5679 (0.38)	1.8992	1.8579 (2.22)	2.1352	2.1141 (1.00)
Λ_9	0.3602	0.3661 (1.60)	1.2880	1.2872 (0.06)	1.7065	1.6401 (4.05)	2.0516	2.0893 (1.81)	2.2496	2.2282 (0.96)
Λ_{10}	0.4198	0.4043 (3.84)	1.4379	1.3882 (3.58)	1.7295	1.7184 (0.65)	2.3824	2.3743 (0.34)	2.3438	2.2866 (2.50)
Λ_{11}	0.4928	0.4887 (0.83)	1.5228	1.5236 (0.05)	1.8170	1.8052 (0.65)	2.5573	2.4219 (5.59)	2.4515	2.3295 (5.24)
Λ_{12}	0.5705	0.5538 (3.03)	1.5488	1.5444 (0.28)	1.9320	1.8895 (2.25)	2.5816	2.4844 (3.91)	3.2142	3.1248 (2.86)
Λ_{13}	0.5738	0.5538 (3.61)	1.6482	1.6459 (0.14)	2.0753	2.0343 (2.01)	2.6048	2.5075 (3.88)	3.2998	3.2291 (2.19)
Λ_{14}	0.6450	0.6239 (3.39)	1.6825	1.6728 (0.58)	2.2262	2.1944 (1.45)	2.6870	2.6439 (1.63)	3.3207	3.2354 (2.63)
Λ_{15}	0.6901	0.6239 (10.61)	1.6855	1.6757 (0.58)	2.2297	2.1946 (1.60)	2.7410	2.6470 (3.55)	3.3207	3.2472 (2.26)

6.3.2. Cylinder

Cylinders with linear thickness variation, and subjected to CC and CF boundary conditions are studied for different radius to thickness ratios, in order to determine a valid work range for the present formulation. For these geometries, the radius-thickness ratio is calculated using the minimum thickness, and the natural frequencies are represented by the dimensionless parameter Θ , which is obtained by [50]

$$\Theta = 2\pi fr \left[\frac{\rho(1-\nu^2)}{E} \right]^{1/2}. \quad (46)$$

The number of spectral p shell elements used for this geometry is shown in Tables below the label **7-PL**. In all the cases, elements with a polynomial degree $p = 8$ were used.

Table 18 presents the natural frequencies for a cylinder with both edges clamped and for different r/h ratios. The ratio length to radius remains fixed and its value is $L/r = 2$. From these results, a raise in the parameter Θ can be observed as the shell becomes thicker. The highest error values are obtained when thicker shells are analyzed ($r/h = 5$), having a maximum error of 6.0%.

Table 18. Dimensionless natural frequencies parameter Θ for cylinders with linear thickness variation and CC boundary conditions, $L/r = 2$.

Dimensionless Frequency Parameter	r/h									
	100		50		20		10		5	
	7-PL 25	3D 847,800	7-PL 25	3D 840,000	7-PL 25	3D 628,000	7-PL 16	3D 642,560	7-PL 16	3D 628,000
Θ_1	0.1815	0.1812 (0.15)	0.2439	0.2429 (0.44)	0.3580	0.3547 (0.94)	0.4823	0.4773 (1.05)	0.6513	0.6324 (2.99)
Θ_2	0.1939	0.1937 (0.09)	0.2677	0.2663 (0.51)	0.4111	0.4096 (0.38)	0.5345	0.5246 (1.89)	0.6896	0.6828 (0.99)
Θ_3	0.2025	0.2022 (0.13)	0.2788	0.2782 (0.23)	0.4215	0.4158 (1.36)	0.6212	0.6192 (0.33)	0.8806	0.8407 (4.74)
Θ_4	0.2306	0.2301 (0.19)	0.3330	0.3314 (0.50)	0.5668	0.5583 (1.52)	0.7428	0.7258 (2.35)	0.9177	0.9244 (0.72)
Θ_5	0.2631	0.2629 (0.07)	0.3825	0.3821 (0.09)	0.5949	0.5943 (0.11)	0.9223	0.9240 (0.18)	1.0719	1.0647 (0.67)
Θ_6	0.2837	0.2807 (1.07)	0.4239	0.4211 (0.68)	0.6679	0.6619 (0.90)	0.9412	0.9281 (1.41)	1.2873	1.2179 (5.70)
Θ_7	0.3131	0.3119 (0.39)	0.4307	0.4279 (0.64)	0.6890	0.6800 (1.32)	0.9657	0.9466 (2.02)	1.2954	1.2514 (3.52)
Θ_8	0.3233	0.3212 (0.64)	0.4496	0.4477 (0.43)	0.7510	0.7471 (0.52)	1.0100	1.0078 (0.22)	1.3159	1.2826 (2.59)
Θ_9	0.3366	0.3357 (0.25)	0.4658	0.4621 (0.80)	0.7619	0.7497 (1.62)	1.0386	1.0187 (1.95)	1.3370	1.3116 (1.94)
Θ_{10}	0.3516	0.3402 (3.33)	0.5294	0.5271 (0.44)	0.8033	0.7905 (1.62)	1.0462	1.0287 (1.71)	1.4614	1.3950 (4.76)
Θ_{11}	0.3632	0.3578 (1.51)	0.5353	0.5282 (1.35)	0.9056	0.9028 (0.32)	1.1016	1.0929 (0.79)	1.5314	1.5372 (0.37)
Θ_{12}	0.3763	0.3762 (0.03)	0.5447	0.5391 (1.04)	0.9234	0.9239 (0.05)	1.1303	1.1019 (2.58)	1.5567	1.5565 (0.01)
Θ_{13}	0.3974	0.3969 (0.13)	0.5840	0.5838 (0.02)	0.9861	0.9716 (1.49)	1.4065	1.3644 (3.09)	1.6369	1.6246 (0.76)
Θ_{14}	0.4234	0.4069 (4.06)	0.6313	0.6270 (0.68)	0.9895	0.9717 (1.84)	1.4118	1.3684 (3.17)	1.7816	1.6808 (6.00)
Θ_{15}	0.4298	0.4069 (5.62)	0.6453	0.6397 (0.88)	0.9895	0.9750 (1.49)	1.4141	1.3684 (3.34)	1.7872	1.6968 (5.33)

Table 19 presents the dimensionless parameter Θ for a cantilevered cylinder with linear thickness variation, where the edge with maximum thickness is clamped. In this case, for a fixed aspect ratio L/r , the natural frequencies rise for thick shells, and the relative errors also increase.

Table 19. Dimensionless natural frequencies parameter Θ for cylinders with linear thickness variation and CF boundary conditions, $L/r = 2$.

Dimensionless Frequency Parameter	r/h									
	100		50		20		10		5	
	7-PL 25	3D 847,800	7-PL 25	3D 840,000	7-PL 25	3D 628,000	7-PL 16	3D 642,560	7-PL 16	3D 628,000
Θ_1	0.0795	0.0798 (0.37)	0.1112	0.1106 (0.54)	0.1803	0.1765 (2.18)	0.2180	0.2131 (2.32)	0.3171	0.3006 (5.47)
Θ_2	0.0949	0.0960 (1.15)	0.1247	0.1241 (0.55)	0.1804	0.1789 (0.84)	0.3172	0.3068 (3.38)	0.3476	0.3443 (0.98)
Θ_3	0.0967	0.0966 (0.07)	0.1248	0.1241 (0.57)	0.2784	0.2723 (2.21)	0.3296	0.3286 (0.31)	0.5299	0.5334 (0.66)
Θ_4	0.1269	0.1290 (1.61)	0.1663	0.1660 (0.16)	0.3221	0.3217 (0.10)	0.5305	0.5167 (2.66)	0.5843	0.5527 (5.71)
Θ_5	0.1635	0.1634 (0.06)	0.1763	0.1756 (0.41)	0.4212	0.4124 (2.13)	0.5324	0.5167 (3.03)	0.7913	0.7609 (3.99)
Θ_6	0.1679	0.1707 (1.61)	0.2460	0.2450 (0.40)	0.4327	0.4274 (1.24)	0.5324	0.5314 (0.20)	0.8315	0.8288 (0.32)
Θ_7	0.2064	0.2057 (0.32)	0.2493	0.2450 (1.72)	0.4774	0.4690 (1.78)	0.6038	0.5943 (1.59)	0.8529	0.8395 (1.59)
Θ_8	0.2103	0.2096 (0.34)	0.2857	0.2840 (0.62)	0.5210	0.5182 (0.52)	0.6285	0.6125 (2.61)	0.9819	0.9315 (5.40)
Θ_9	0.2176	0.2190 (0.62)	0.2967	0.2940 (0.90)	0.5306	0.5309 (0.04)	0.7838	0.7790 (0.62)	0.9848	0.9516 (3.50)
Θ_{10}	0.2421	0.2416 (0.22)	0.3182	0.3181 (0.04)	0.5957	0.5826 (2.24)	0.8079	0.7851 (2.90)	1.0035	0.9783 (2.58)
Θ_{11}	0.2438	0.2426 (0.49)	0.3397	0.3275 (3.73)	0.5999	0.5826 (2.96)	0.8160	0.8030 (1.62)	1.0035	1.0102 (0.66)
Θ_{12}	0.2789	0.2737 (1.93)	0.3442	0.3432 (0.31)	0.6187	0.6068 (1.97)	0.8266	0.8030 (2.94)	1.0174	1.0102 (0.71)
Θ_{13}	0.2976	0.2936 (1.38)	0.3586	0.3547 (1.09)	0.7530	0.7517 (0.17)	0.8266	0.8148 (1.45)	1.0174	1.1043 (7.87)
Θ_{14}	0.3170	0.3169 (0.02)	0.3618	0.3547 (2.00)	0.7619	0.7530 (1.18)	0.9636	0.9608 (0.29)	1.1225	1.3225 (15.12)
Θ_{15}	0.3266	0.3263 (0.10)	0.4475	0.4222 (5.98)	0.7721	0.7599 (1.62)	0.9875	0.9866 (0.09)	1.3384	1.3225 (1.20)

6.3.3. Fraction of Cylinder

A quarter of cylinder, with different radius to thickness (r/h) and length to radius (L/r) ratios, is analyzed. The geometry is in cantilever, where the edge with the maximum thickness is clamped. The natural frequencies are presented using the dimensionless parameter Θ . Table 20 shows the results for different r/h ratios. It is observed that the highest error is presented for relatively thick shells ($r/h = 5$) with a variation of 10.02%, when the present formulation is compared with solid elements.

Table 20. Dimensionless parameter Θ for a quarter of cylinder with linear thickness variation.

Dimensionless Frequency Parameter	r/h									
	100		50		20		10		5	
	7-PL 16	3D 645,344	7-PL 16	3D 703,360	7-PL 16	3D 603,680	7-PL 16	3D 639,576	7-PL 16	3D 743,400
Θ_1	0.0170	0.0164 (4.18)	0.0240	0.0222 (8.17)	0.0464	0.0445 (4.34)	0.0136	0.0131 (4.37)	0.0186	0.0169 (10.02)
Θ_2	0.0296	0.0291 (1.79)	0.0416	0.0411 (1.34)	0.0496	0.0457 (8.51)	0.0414	0.0375 (10.30)	0.0530	0.0520 (1.94)
Θ_3	0.0527	0.0526 (0.35)	0.0742	0.0721 (2.97)	0.1381	0.1344 (2.73)	0.0600	0.0588 (2.04)	0.0864	0.0809 (6.79)
Θ_4	0.0580	0.0575 (0.95)	0.0832	0.0820 (1.40)	0.1721	0.1663 (3.44)	0.0670	0.0661 (1.41)	0.0901	0.0834 (8.04)
Θ_5	0.0638	0.0633 (0.75)	0.0988	0.0969 (1.89)	0.1855	0.1834 (1.18)	0.1025	0.0993 (3.26)	0.1877	0.1802 (4.15)
Θ_6	0.0883	0.0888 (0.53)	0.1646	0.1626 (1.23)	0.1939	0.1937 (0.08)	0.1636	0.1624 (0.74)	0.2112	0.2074 (1.85)
Θ_7	0.1064	0.1060 (0.45)	0.1709	0.1676 (2.01)	0.2830	0.2789 (1.44)	0.1791	0.1761 (1.75)	0.2493	0.2502 (0.36)
Θ_8	0.1116	0.1111 (0.42)	0.1711	0.1697 (0.79)	0.2977	0.2933 (1.51)	0.2427	0.2428 (0.06)	0.3068	0.3066 (0.05)
Θ_9	0.1425	0.1423 (0.16)	0.1946	0.1943 (0.17)	0.3336	0.3295 (1.24)	0.2751	0.2714 (1.36)	0.3272	0.3179 (2.91)
Θ_{10}	0.1621	0.1636 (0.90)	0.2266	0.2260 (0.28)	0.4195	0.4119 (1.85)	0.2804	0.2781 (0.82)	0.3761	0.3702 (1.60)
Θ_{11}	0.1644	0.1640 (0.26)	0.2446	0.2412 (1.41)	0.4703	0.4642 (1.32)	0.2941	0.2884 (1.95)	0.4884	0.4765 (2.51)
Θ_{12}	0.1651	0.1654 (0.20)	0.2535	0.2512 (0.92)	0.4984	0.4924 (1.22)	0.3065	0.3064 (0.03)	0.5238	0.5262 (0.45)
Θ_{13}	0.1933	0.1931 (0.06)	0.2626	0.2602 (0.93)	0.5387	0.5348 (0.73)	0.3834	0.3772 (1.64)	0.5518	0.5347 (3.20)
Θ_{14}	0.2118	0.2120 (0.10)	0.3215	0.3170 (1.40)	0.5684	0.5610 (1.33)	0.3908	0.3859 (1.26)	0.5821	0.5688 (2.33)
Θ_{15}	0.2252	0.2254 (0.08)	0.3467	0.3431 (1.05)	0.6135	0.6133 (0.03)	0.4133	0.4099 (0.84)	0.6773	0.6601 (2.59)

From the previous results, the geometry with $r/h = 10$ is chosen to vary the ratio L/r . The ratio $r/h = 10$ is near to be a thick shell and, in general, the frequencies of the present formulation have good behavior for greater ratios. The natural frequencies for different aspect ratios L/r of a partial cylinder are presented in Table 21. From these results, an increment on the frequencies is observed when the ratio L/r is decreased. The relative errors increase when the geometries have aspect ratios less than 4, with maximum values at $L/r = 1$.

The vibration modes for a cantilevered quarter of cylinder, with ratio $L/r = 5$, are shown in Table 22. In addition, the modes obtained using solid elements in the commercial code ANSYS are shown. In general, the modes of each natural frequency are in agreement. Note that the Cartesian system is rotated 90° clockwise around the y -axis.

Table 21. Dimensionless parameter Θ for a quarter of cylinder with linear thickness variation, $r/h = 10$.

Dimensionless Frequency Parameter	L/r											
	1		2		3		4		5		10	
	7-PL 16	3D 642,560	7-PL 16	3D 862,400	7-PL 16	3D 812,160	7-PL 16	3D 832,480	7-PL 16	3D 785,000	7-PL 16	3D 539,000
Θ_1	0.3450	0.3122 (10.49)	0.1015	0.0969 (4.70)	0.0467	0.0447 (4.46)	0.0266	0.0254 (4.40)	0.0171	0.0164 (4.37)	0.00429	0.00411 (4.38)
Θ_2	0.3508	0.3232 (8.55)	0.1391	0.1241 (12.13)	0.0861	0.0768 (12.09)	0.0619	0.0553 (11.80)	0.0476	0.0429 (11.06)	0.01660	0.01620 (2.46)
Θ_3	0.5966	0.5489 (8.68)	0.3330	0.3326 (0.12)	0.1746	0.1741 (0.25)	0.1069	0.1063 (0.60)	0.0727	0.0717 (1.32)	0.02184	0.02155 (1.36)
Θ_4	0.8582	0.8559 (0.27)	0.3704	0.3543 (4.54)	0.2063	0.2028 (1.75)	0.1249	0.1230 (1.53)	0.0829	0.0818 (1.44)	0.02672	0.02430 (9.96)
Θ_5	1.0280	0.9713 (5.84)	0.3877	0.3718 (4.27)	0.2206	0.2128 (3.66)	0.1530	0.1479 (3.42)	0.1170	0.1133 (3.30)	0.05322	0.05164 (3.05)
Θ_6	1.1272	1.0630 (6.04)	0.4021	0.3886 (3.49)	0.3287	0.3172 (3.63)	0.2804	0.2750 (1.97)	0.1988	0.1972 (0.79)	0.05630	0.05595 (0.63)
Θ_7	1.1641	1.0938 (6.43)	0.6800	0.6597 (3.07)	0.4243	0.4149 (2.28)	0.2858	0.2831 (0.96)	0.2074	0.2037 (1.81)	0.08887	0.08778 (1.25)
Θ_8	1.4464	1.3788 (4.90)	0.7830	0.7599 (3.04)	0.4459	0.4391 (1.54)	0.3097	0.3012 (2.82)	0.2880	0.2882 (0.07)	0.09391	0.09356 (0.37)
Θ_9	1.7101	1.6620 (2.90)	0.8028	0.7825 (2.59)	0.4976	0.4860 (2.39)	0.3962	0.3965 (0.09)	0.2959	0.2894 (2.24)	0.10435	0.10389 (0.44)
Θ_{10}	1.7427	1.7084 (2.01)	0.8611	0.8393 (2.59)	0.5732	0.5730 (0.05)	0.4272	0.4209 (1.50)	0.3227	0.3181 (1.43)	0.13021	0.12872 (1.16)
Θ_{11}	1.8404	1.8158 (1.36)	0.8677	0.8604 (0.86)	0.5771	0.5777 (0.10)	0.4322	0.4300 (0.51)	0.3383	0.3355 (0.84)	0.16278	0.16216 (0.38)
Θ_{12}	1.8993	1.8412 (3.16)	0.9233	0.9239 (0.07)	0.6919	0.6782 (2.03)	0.4472	0.4400 (1.63)	0.3436	0.3434 (0.05)	0.17148	0.17143 (0.03)
Θ_{13}	2.0950	2.0922 (0.13)	1.1355	1.1057 (2.70)	0.7089	0.6961 (1.82)	0.4708	0.4654 (1.16)	0.3972	0.3902 (1.77)	0.17619	0.17446 (0.99)
Θ_{14}	2.3301	2.2226 (4.84)	1.2046	1.1702 (2.94)	0.7404	0.7262 (1.96)	0.5751	0.5645 (1.87)	0.4625	0.4563 (1.35)	0.21804	0.21820 (0.08)
Θ_{15}	2.3632	2.2491 (5.08)	1.3029	1.2680 (2.75)	0.8067	0.7868 (2.54)	0.6493	0.6391 (1.61)	0.4859	0.4813 (0.94)	0.22795	0.22612 (0.81)

Table 22. Vibration modes for a cantilevered quarter of cylinder with linear thickness variation, $L/r = 5$.

Table 22. Cont.

Vibration Mode	Formulation	
	7-PL	3D
3	$\Theta = 0.0727$	$\Theta = 0.0717$

4	$\Theta = 0.0829$	$\Theta = 0.0818$

5	$\Theta = 0.1170$	$\Theta = 0.1133$

6	$\Theta = 0.1988$	$\Theta = 0.1972$

7	$\Theta = 0.2074$	$\Theta = 0.2037$

8	$\Theta = 0.2880$	$\Theta = 0.2882$

Table 22. Cont.

Vibration Mode	Formulation	
	7-PL	3D
9	$\Theta = 0.2959$	$\Theta = 0.2894$

10	$\Theta = 0.3227$	$\Theta = 0.3181$

Finally, based on the results presented in Tables 17–21, the following limits for the presented formulation are set:

Truncated hemisphere: $20 < r/h_m \leq 2.5$,

Cylinder: $100 \leq r/h \leq 5$,

Fraction of cylinder: $4 < L/r \leq 10$.

These parameter ranges demonstrate the capability of the spectral element presented to model thin and moderate thick shells with constant and variable thickness.

7. Conclusions

In the present work, a seven-parameter spectral finite element has been used to determine natural frequencies in shell structures. This formulation allows for modeling thin and moderate thick structures with constant and variable thicknesses, and avoids the use of shear correction factor. The spectral element is validated to model plates, truncated hemispheres, cylinders, and cylindrical panels. Other types of shell geometries can be analyzed. The numerical verification is presented for different types of boundary conditions.

The natural frequencies obtained with the present formulation are in agreement with analytic results, having relative errors less than 9%. For the cases where exact solutions are not available, a comparison with the results obtained by means of solid elements is performed, having a maximum error of 15.12% for a thick shell ($L/r = 2$ and $r/h = 5$).

It is important to note that the presented formulation could be used to model arbitrary geometries and boundary conditions, contrary to other solutions found in the literature where those are restricted to specific cases. Furthermore, the computational time is smaller, compared to the one used to model the same problems using solid elements, even when the current formulation calculated all the frequencies and vibration modes in the finite element model, while using solid elements in the commercial code only forty were computed.

In a numerical model, the number of natural frequencies obtained corresponds to the number of unrestricted degrees of freedom on the finite element model. The fundamental frequency can be computed with a coarse mesh, but a refinement is needed to determine higher frequencies accurately. Refined meshes also yield an accurate fundamental frequency. The determination of the vibration modes requires a close examination of the numerical results.

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