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Oscillation Theory for Non-Linear Neutral Delay Differential Equations of Third Order

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Abstract: In this article, we study a class of non-linear neutral delay differential equations of third order. We first prove criteria for non-existence of non-Kneser solutions, and criteria for non-existence of Kneser solutions. We then use these results to provide criteria for the under study differential equations to ensure that all its solutions are oscillatory. An example is given that illustrates our theory.

Keywords: third-order differential equations; kneser solution; differential inequalities; neutral; delays; non-linear; oscillation criteria

1. Introduction

The interest in studying delay differential equations is caused by the fact that they appear in models of several areas in science. In [1–3], systems of differential equations with delays are used to study the dynamics and stability properties of electrical power systems. The concept of delays is also used to study stability properties of macroeconomic models, see [4–6]. Finally, properties of delay differential equations are used in the study of singular differential equations of fractional order, see [7–9], and other type of fractional operators such as the fractional nabla applied to difference equations where the memory effect appears, see [10,11].

Neutral time delay differential equations (NDDEs) are equations where the delays appear in both the state variables and their time derivatives. They have wide applications in engineering, see [12], in ecology, see [13], in physics, see [14], in electrical power systems, see [15], and applied mathematics, see [16]. This type of NDDEs also appear in the study of vibrating masses attached to an elastic bar, in problems concerning electric networks containing lossless transmission lines (as in high speed computers), and in the solution of variational problems with time delays, see [17,18]. In this article, we consider the following class of non-linear NDDEs of third-order:

$$\left(r_2(t)\left(\left(r_1(t)\left(z'(t)\right)^{\alpha_1}\right)'\right)^{\alpha_2}\right)' + q(t)\,f(x(g(t))) = 0, \quad t \geq t_0, \tag{1}$$

where $z(t) := x(t) + p(t)\,x(\tau(t))$. Throughout this paper, we will assume that

(A1) α_1 and α_2 are quotients of odd positive integers;

(A2) $r_1, r_2 \in C^1([t_0,\infty),(0,\infty))$ and

$$\int_{t_0}^{\infty} \frac{1}{r_i^{1/\alpha_i}(t)}\,\mathrm{d}t = \infty \quad \text{for } i = 1,2; \tag{2}$$

(A3) $p, q \in C([t_0, \infty), [0, \infty))$, $q(t)$ does not vanish identically and $0 \leq p(t) < 1$;
(A4) $\tau, g \in C^1([t_0, \infty), \mathbb{R})$, $\tau(t) \leq t$, $g(t) < t$, $\tau \circ g = g \circ \tau$, $\tau'(t) \geq \tau_0 \geq 0$ and $\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} g(t) = \infty$;
(A5) $f \in C(\mathbb{R}, \mathbb{R})$ and there exists a constant $k > 0$ such that $f(x) \geq kx^\beta$ for $x \neq 0$, where $\beta = \alpha_1\alpha_2$.

For the sake of clarity and brevity, we define the operators

$$\begin{aligned} L_0 z(t) &= z(t),\ L_1 z(t) = r_1(t)\left(z'(t)\right)^{\alpha_1}, \\ L_2 z(t) &= r_2(t)\left(\left(L_1 z(t)\right)'\right)^{\alpha_2} = r_2(t)\left(\left(r_1(t)\left(z'(t)\right)^{\alpha_1}\right)'\right)^{\alpha_2} \end{aligned}$$

and

$$L_3 z(t) = \left(L_2 z(t)\right)' = \left(r_2(t)\left(\left(r_1(t)\left(z'(t)\right)^{\alpha_1}\right)'\right)^{\alpha_2}\right)'.$$

By a solution of (1), we mean a function $x \in C([T_x, \infty), \mathbb{R})$, $T_x \geq t_0$, which has the property $L_i z(t) \in C^1([T_x, \infty), \mathbb{R})$, $i = 0, 1, 2$ and satisfies (1) on $[T_x, \infty)$. We consider only those solutions of (1) which satisfy $\sup\{|x(t)| : T \leq t < \infty\} > 0$, for any $T \geq T_x$. We assume that (1) possesses such a solution. A solution x of (1) is said to be non-oscillatory if it is positive or negative, ultimately; otherwise, it is said to be oscillatory. The equation itself is termed oscillatory if all its solutions oscillate.

The study of qualitative behavior of NDDEs has received great attention in recent times. The theory of oscillation is one of the most important branches of qualitative theory of differential equations. See [19–21] for principles and basic results of oscillation theory. For more recent results of oscillatory properties of solutions of NDDEs and non-linear differential equations, we refer the reader to the works [22–55].

Baculikova and Dzurina [26] discussed oscillatory criteria of equations

$$\left(r_2(t)\left(z''(t)\right)\right)' + q(t)\, x(g(t)) = 0,$$

under the condition

$$\int_{t_0}^{\infty} \frac{1}{r_2(t)} \mathrm{d}s = \infty.$$

As a special case of (1), Chatzarakis et al. [27] considered the oscillation for equation

$$\left(r_2(t)\left(\left(r_1(t)\left(z'(t)\right)\right)'\right)\right)' + q(t) f(x(g(t))) = 0, \tag{3}$$

where $0 \leq p(v) \leq p_0 < 1$ and under the condition

$$\int_{t_0}^{\infty} \frac{1}{r_i(t)} \mathrm{d}t = \infty \ \text{ for } i = 1, 2.$$

Dzurina et al. [33] completed oscillation results for equation (3), by establishing sufficient conditions for nonexistence of so-called Kneser solutions. In this paper we extend and improve the results in [25,34,35,37,38] by proving new criteria which ensure that all solutions of (1) are oscillatory.

The article is organized as follows. In Section 2 we present the necessary mathematical background used throughout the paper. In Section 3 we prove criteria for non-existence of non-Kneser solutions, and in Section 4 we provide criteria for non-existence of Kneser solutions. In Section 5 we use the results in the previous sections to provide criteria for (1) to ensure that all its solutions are oscillatory. An example is given in the same section that illustrates our theory.

Remark 1. *All functional inequalities and properties such as increasing, decreasing, positive, and so on, are assumed to hold eventually, i.e., they are satisfied for all $t \geq t_1 \geq t_0$, where t_1 large enough.*

2. Preliminary Results

We use the following notations for the simplicity:

$$
\begin{aligned}
R_i(t,u) &: = \int_u^t \frac{1}{r_i^{1/\alpha_i}(s)} \mathrm{d}s, \quad i = 1,2, \\
\widetilde{R}(v,u) &: = \int_u^v \left(\frac{R_2(v,s)}{r_1(s)} \right)^{1/\alpha_1} \mathrm{d}s, \\
G_u(t) &: = \frac{1}{r_1^{\frac{1}{\alpha_1}}(t)} \left(\int_u^t \frac{1}{r_2^{\frac{1}{\alpha_2}}(s)} \mathrm{d}s \right)^{\frac{1}{\alpha_1}}, \quad \widetilde{G}_u(t) = \int_u^t G_u(s)\, ds
\end{aligned}
$$

and

$$Q(t) = \min\{kq(t), kq(\tau(t))\},$$

for $t \geq u \geq t_0$. Next we present the following six Lemmas that will be used as tools to prove our main results in the next sections.

Lemma 1 ([23]). *Assume that $h_1, h_2 \in [0,\infty)$ and $\gamma > 0$. Then*

$$(h_1 + h_2)^\gamma \leq \mu \left(h_1^\gamma + h_2^\gamma \right),$$

where

$$\mu = \begin{cases} 1 & \text{for} \quad 0 < \gamma \leq 1, \\ 2^{\gamma-1} & \text{for} \quad \gamma > 1. \end{cases}$$

Lemma 2. *Assume that x is a positive solution of (1). Then, $L_3 z(t) \leq 0$ and there are only two possible classes for the corresponding function z :*

$$
\begin{aligned}
\textbf{Case } (1) &: \; z(t) > 0, \; L_1 z(t) < 0 \text{ and } L_2 z(t) > 0; \\
\textbf{Case } (2) &: \; z(t) > 0, \; L_1 z(t) > 0 \text{ and } L_2 z(t) > 0.
\end{aligned}
$$

Proof. Let x be a positive solution of (1). Then, there exists a $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(g(t)) > 0$ for $t \geq t_1$. Therefore, $z(t) > 0$ and (1) implies that

$$L_3 z(t) = -q(t) f(x(g(t))) \leq 0.$$

Hence, $L_2 z(t)$ is a non-increasing function and of one sign. We claim that $L_2 z(t) > 0$ for $t \geq t_1$. Suppose that $L_2 z(t) < 0$ for $t \geq t_2 \geq t_1$, then there exists a $t_3 \geq t_2$ and constant $K_1 > 0$ such that

$$\frac{d}{dt} L_1 z(t) < -K_1 (r_2(t))^{-1/\alpha_2},$$

for $t \geq t_3$. By integrating the last inequality from t_3 to t, we get

$$L_1 z(t) < L_1 z(t_3) - K_1 \int_{t_3}^t (r_2(s))^{-1/\alpha_2} \mathrm{d}s.$$

Letting $t \to \infty$, we have $\lim_{t\to\infty} E_1(t) = -\infty$. Then there exists a $t_4 \geq t_3$ and constant $K_2 > 0$ such that

$$z'(t) < -K_2 (r_1(t))^{-1/\alpha_1},$$

for $t \geq t_4$. By integrating this inequality from t_4 to ∞, we get $\lim_{t\to\infty} z(t) = -\infty$, which contradicts $z(t) > 0$. Now we have $E_2(t) > 0$ for $t \geq t_1$. Therefore, $E_1(t)$ is increasing function and of one sign. The proof is complete. □

Definition 1. *The set of all functions z satisfy that Case (1) is denoted by $\aleph$. The set of all functions z that satisfy Case (2) is denoted by $\widetilde{\aleph}$. Solutions x whose corresponding function $z \in \aleph$ are called Kneser-solutions.*

Lemma 3 ([25] Lemmas 3–5)**.** *Assume that $r_i'(t) > 0$ for $i = 1, 2$, x is a positive solution of (1) with corresponding function $z \in \widetilde{\aleph}$ for all $t \geq t_2 \geq t_0$. Then, the following results are achieved.*
(1) *For each $\eta \in (0, 1)$, there exists a $T_\eta \geq t_1$ such that, for all $t \geq T_\eta$,*

$$tz'(\tau(t)) \geq \eta\tau(t)z'(t).$$

(2) *For each $t \in [t_1, \infty)$,*

$$z(t) \geq \frac{1}{2}(t - t_1) z'(t). \tag{4}$$

(3) *For each $t \in [t_1, \infty)$,*

$$z'(t) \geq (t - t_1) z''(t).$$

Lemma 4. *Assume that $r_i'(t) > 0$ for $i = 1, 2$, x is a positive solution of (1) with corresponding function $z \in \widetilde{\aleph}$ for all $t \geq t_1 \geq t_0$. Then, the following facts are verified:*

$$z(t) \geq \frac{1}{2}(t - t_0)^{1+1/\alpha_1} \left(\frac{(L_1 z(t))'}{r_1(t)}\right)^{1/\alpha_1}, \tag{5}$$

$$(L_2 z(t))' \leq -kq(t)(1 - p(g(t)))^\beta z^\beta(g(t)), \tag{6}$$

$$z'(t) \geq \left(\frac{R_2(t, t_1)}{r_1(t)}\right)^{1/\alpha_1} (L_2 z(t))^{1/\beta} \tag{7}$$

and there exists a $t_2 \geq t_1$ such that

$$z(t) \geq \widetilde{R}(t, t_2)(L_2 z(t))^{1/\beta}, \tag{8}$$

for all $t \geq t_2$.

Proof. Let x be a positive solution of (1) with corresponding function $z \in \widetilde{\aleph}$ for all $t \geq t_1 \geq t_0$. Then, $z(t) > 0$, $L_1 z(t) > 0$ and $L_2 z(t) > 0$ for $t \geq t_1$. Since $L_3 z(t) \leq 0$ and $r_2'(t) > 0$, we get

$$\left(r_1(t)(z'(t))^{\alpha_1}\right)'' \leq 0.$$

Thus,

$$\begin{aligned} r_1(t)(z'(t))^{\alpha_1} &= r_1(t_1)(z'(t_1))^{\alpha_1} + \int_{t_1}^{t} \left(r_1(s)(z'(s))^{\alpha_1}\right)' ds \\ &\geq (t - t_1)\left(r_1(t)(z'(t))^{\alpha_1}\right)' \end{aligned}$$

and so

$$z'(t) \geq \left(\frac{t - t_1}{r_1(t)}\right)^{1/\alpha_1} \left((L_1 z(t))'\right)^{1/\alpha_1}.$$

It follows from (4) that

$$z(t) \geq \frac{1}{2}(t-t_1)^{1+1/\alpha_1}\left(\frac{(L_1 z(t))'}{r_1(t)}\right)^{1/\alpha_1}.$$

Since $z(t) > x(t)$ and $z'(t) > 0$, we obtain $x(t) \geq (1-p(t))z(t)$, and hence

$$(L_2 z(t))' \leq -kq(t)x^{\beta}(g(t)) \leq -kq(t)(1-p(g(t)))^{\beta} z^{\beta}(g(t)).$$

Now, we have

$$\begin{aligned} L_1 z(t) &= L_1 z(T) + \int_{t_1}^{t}\left(\frac{L_2 z(s)}{r_2(s)}\right)^{1/\alpha_2} \mathrm{d}s \geq (L_2 z(t))^{1/\alpha_2}\int_{t_1}^{t}\left(\frac{1}{r_2(s)}\right)^{1/\alpha_2}\mathrm{d}s \\ &\geq R_2(t,t_1)(L_2 z(t))^{1/\alpha_2} \end{aligned}$$

and so

$$z'(t) \geq \left(\frac{R_2(t,t_1)}{r_1(t)}\right)^{1/\alpha_1}(L_2 z(t))^{1/\beta},$$

for $t \geq t_2 \geq t_1$. By integrating the latter inequality from t_2 to t and using $(L_2 z(t))' < 0$, we get

$$\begin{aligned} z(t) &\geq z(t_2) + (L_2 z(t))^{1/\beta}\int_{t_2}^{t}\left(\frac{R_2(s,t_1)}{r_1(s)}\right)^{1/\alpha_1}\mathrm{d}s \\ &\geq \widetilde{R}(t,t_2)(L_2 z(t))^{1/\beta}. \end{aligned}$$

The proof is complete. □

Lemma 5. *Assume that x is a positive solution of (1) with corresponding function $z \in \aleph$ for all $t \geq t_1 \geq t_0$. Then:*

$$z(u) \geq \widetilde{R}(v,u)(L_2 z(v))^{1/\beta}, \tag{9}$$

for $u \leq v$, and

$$\left(L_2 z(t) + \frac{p_0^{\beta}}{\tau_0}L_2 z(\tau(t))\right)' + \frac{1}{\mu}Q(t)z^{\beta}(g(t)) \leq 0. \tag{10}$$

Proof. Suppose that x is positive solution of (1). Then, there exists a $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(g(t)) > 0$ for $t \geq t_1$. From Lemma 1, we obtain

$$z^{\beta}(t) \leq \mu\left(x^{\beta}(t) + p_0^{\beta}x^{\beta}(\tau(t))\right), \tag{11}$$

it follows from the monotonicity of $L_2 z(t)$ that

$$-L_1 z(u) \geq L_1 z(v) - L_1 z(u) = \int_u^v \left(\frac{L_2 z(s)}{r_2(s)}\right)^{1/\alpha_2}\mathrm{d}s \geq (L_2 z(v))^{1/\alpha_2} R_2(v,u),$$

for $v \geq u \geq t_1$. Integrating the last inequality from u to v, we obtain

$$z(u) \geq (L_2 z(v))^{1/\beta}\int_u^v\left(\frac{R_2(v,s)}{r_1(s)}\right)^{1/\alpha_1}\mathrm{d}s = (L_2 z(v))^{\frac{1}{\beta}}\widetilde{R}(v,u).$$

From (1), (A1) and (A5), we have

$$\frac{p_0^\beta}{\tau'(t)}\left(L_2 z(\tau(t))\right)' + k p_0^\beta q(\tau(t)) x^\beta (g(\tau(t))) \leq 0$$

and so,

$$\frac{p_0^\beta}{\tau_0}\left(L_2 z(\tau(t))\right)' + k p_0^\beta q(\tau(t)) x^\beta (\tau(g(t))) \leq 0. \tag{12}$$

Combining (1) with (12), we get

$$L_3 z(t) + \frac{p_0^\beta}{\tau_0}\left(L_2 z(\tau(t))\right)' + k q(t) x^\beta (g(t)) + k p_0^\beta q(\tau(t)) x^\beta (\tau(g(t))) \leq 0.$$

Hence,

$$L_3 z(t) + \frac{p_0^\beta}{\tau_0}\left(L_2 z(\tau(t))\right)' + Q(t)\left(x^\beta (g(t)) + p_0^\beta x^\beta (\tau(g(t)))\right) \leq 0. \tag{13}$$

From (11) and (13) becomes

$$L_3 z(t) + \frac{p_0^\beta}{\tau_0} L_3 z(\tau(t)) + \frac{1}{\mu} Q(t) z^\beta (g(t)) \leq 0,$$

that is,

$$\left(L_2 z(t) + \frac{p_0^\beta}{\tau_0} L_2 z(\tau(t))\right)' + \frac{1}{\mu} Q(t) z^\beta (g(t)) \leq 0.$$

The proof of the Lemma is complete. □

3. Criteria for Nonexistence of Non-Kneser Solutions

For simplicity, we use the following notations:

$$\theta(t) = k q(t)\left(1 - p(g(t))\right)^\beta.$$

In the following, we establish a Hille and Nehari type criterion for nonexistence of non-Kneser solutions.

Lemma 6. *Assume that $r_i'(t) > 0$ for $i = 1, 2$, x is a positive solution of (1) with corresponding function $z \in \widetilde{\aleph}$ for all $t \geq t_1 \geq t_0$. If $P < \infty$ and $D < \infty$, then*

$$P \leq L - L^{\frac{1+\beta}{\beta}} \tag{14}$$

and

$$P + D \leq 1, \tag{15}$$

where

$$\begin{aligned} P &:= \liminf_{t\to\infty} \left(\widetilde{R}(g(t), t_0)\right)^\beta \int_t^\infty \theta(s)\, \mathrm{d}s, \\ D &:= \limsup_{t\to\infty} \frac{1}{\widetilde{R}(g(t), t_0)} \int_{t_0}^t \left(\widetilde{R}(g(t), s)\right)^{\beta+1} \theta(s)\, \mathrm{d}s \end{aligned}$$

and

$$L := \liminf_{t\to\infty} \left(\widetilde{R}\left(g\left(t\right),t_0\right)\right)^{\beta} \frac{L_2 z\left(t\right)}{z^{\beta}\left(g\left(t\right)\right)}.$$

Proof. Assume that x is a positive solution of (1) and $z \in \widetilde{\aleph}$. By Lemma 4, we get that (5)–(8) hold. Now, we define the function

$$\omega\left(t\right) = \frac{L_2 z\left(t\right)}{z^{\beta}\left(g\left(t\right)\right)}.$$

Them ω is positive for $t \geq t_1$, and satisfies

$$\omega'\left(t\right) = \frac{\left(L_2 z\left(t\right)\right)'}{z^{\beta}\left(g\left(t\right)\right)} - \beta \frac{L_2 z\left(t\right)}{z^{\beta+1}\left(g\left(t\right)\right)} z'\left(g\left(t\right)\right) g'\left(t\right).$$

Thus, from (6) and (7), there exists a $T \geq t_1$ such that

$$\omega'\left(t\right) \leq -kq\left(t\right)\left(1-p\left(g\left(t\right)\right)\right)^{\beta} - \beta \widetilde{R}'\left(g\left(t\right),T\right) g'\left(t\right) \frac{L_2^{1+1/\beta} z\left(t\right)}{z^{\beta+1}\left(g\left(t\right)\right)},$$

for $t \geq T$. This implies that

$$\omega'\left(t\right) \leq -\theta\left(t\right) - \beta \widetilde{R}'\left(g\left(t\right),T\right) g'\left(t\right) \omega^{1+1/\beta}\left(t\right). \tag{16}$$

Using (8), we get

$$\left(\widetilde{R}\left(g\left(t\right),T\right)\right)^{\beta} \omega\left(t\right) \leq 1,$$

which with (2), gives

$$\lim_{t\to\infty} \omega\left(t\right) = 0. \tag{17}$$

On the other hand, we define the function

$$U = \limsup_{t\to\infty} \left(\widetilde{R}\left(g\left(t\right),t_0\right)\right)^{\beta} \frac{L_2 z\left(t\right)}{z^{\beta}\left(g\left(t\right)\right)}. \tag{18}$$

From the definitions of $\omega\left(t\right)$, L and U, we see that

$$0 \leq L \leq U \leq 1. \tag{19}$$

Now, let $\varepsilon > 0$, then from the definition of P and L, we can pick $t_3 \geq T$ sufficiently large such that

$$\left(\widetilde{R}\left(g\left(t\right),T\right)\right)^{\beta} \int_t^{\infty} \theta\left(s\right) ds \geq P - \epsilon \text{ and } \left(\widetilde{R}\left(g\left(t\right),T\right)\right)^{\beta} \omega\left(t\right) \geq L - \epsilon \quad \text{for } t \geq t_3.$$

By integrating (16) from t to ∞ and using (17), we have

$$\omega\left(t\right) \geq \int_t^{\infty} \theta\left(s\right) ds + \beta \int_t^{\infty} \widetilde{R}'\left(g\left(s\right),T\right) g'\left(s\right) \omega^{1+1/\beta}\left(s\right) \mathrm{d}s. \tag{20}$$

Multiplying the latter inequality by $\left(\widetilde{R}\left(g\left(t\right),T\right)\right)^{\beta}$, we obtain

$$
\begin{aligned}
\left(\widetilde{R}\left(g\left(t\right),T\right)\right)^{\beta}\omega\left(t\right) \geq\;& \left(\widetilde{R}\left(g\left(t\right),T\right)\right)^{\beta}\int_{t}^{\infty}\theta\left(s\right)\mathrm{d}s \\
&+\beta\left(\widetilde{R}\left(g\left(t\right),T\right)\right)^{\beta}\int_{t}^{\infty}\frac{\widetilde{R}'\left(g\left(s\right),T\right)g'\left(s\right)}{\left(\widetilde{R}\left(g\left(s\right),T\right)\right)^{\beta+1}}\left(\left(\widetilde{R}\left(g\left(s\right),T\right)\right)^{\beta}\omega\left(s\right)\right)^{\frac{1+\beta}{\beta}}\mathrm{d}s \\
\geq\;& \left(P-\epsilon\right)+\left(L-\epsilon\right)^{\frac{1+\beta}{\beta}}\left(\widetilde{R}\left(g\left(t\right),T\right)\right)^{\beta}\int_{t}^{\infty}\frac{\beta\widetilde{R}'\left(g\left(s\right),T\right)g'\left(s\right)}{\left(\widetilde{R}\left(g\left(s\right),T\right)\right)^{\beta+1}}\mathrm{d}s \\
\geq\;& \left(P-\epsilon\right)+\left(L-\epsilon\right)^{\frac{1+\beta}{\beta}}.
\end{aligned}
$$

Taking the limit inferior on both sides as $t\to\infty$, we get

$$
L\geq\left(P-\epsilon\right)+\left(L-\epsilon\right)^{\frac{1+\beta}{\beta}}.
$$

Since $\epsilon>0$ is arbitrary, we obtain

$$
P\leq L-L^{\frac{1+\beta}{\beta}}.
$$

Next, multiplying (16) by $\left(\widetilde{R}\left(g\left(t\right),T\right)\right)^{\beta+1}$ and integrating it from t_3 to t, we get

$$
\begin{aligned}
\int_{t_3}^{t}\left(\widetilde{R}\left(g\left(s\right),s\right)\right)^{\beta+1}\omega'\left(s\right)ds \leq\;& -\int_{t_3}^{t}\left(\widetilde{R}\left(g\left(s\right),s\right)\right)^{\beta+1}\theta\left(s\right)\mathrm{d}s \\
&-\beta\int_{t_3}^{t}\widetilde{R}'\left(g\left(s\right),s\right)g'\left(s\right)\left(\left(\widetilde{R}\left(g\left(s\right),s\right)\right)^{\beta}\omega\left(s\right)\right)^{\frac{1+\beta}{\beta}}\mathrm{d}s.
\end{aligned}
$$

Integrating by parts, we find

$$
\begin{aligned}
\left(\widetilde{R}\left(g\left(t\right),t\right)\right)^{\beta+1}\omega\left(t\right) \leq\;& \left(\widetilde{R}\left(g\left(t\right),t_3\right)\right)^{\beta+1}\omega\left(t_3\right)-\int_{t_3}^{t}\left(\widetilde{R}\left(g\left(s\right),s\right)\right)^{\beta+1}\theta\left(s\right)\mathrm{d}s \\
&+\int_{t_3}^{t}\widetilde{R}'\left(g\left(t\right),s\right)g'\left(s\right)\left(\left(\beta+1\right)V-\beta V^{\frac{1+\beta}{\beta}}\right)\mathrm{d}s,
\end{aligned}
$$

where $V=\left(\widetilde{R}\left(g\left(t\right),T\right)\right)^{\beta}\omega\left(s\right)$. Using the inequality

$$
a\phi-b\phi^{\frac{1+\beta}{\beta}}\leq\frac{\beta^{\beta}}{\left(\beta+1\right)^{\beta+1}}a^{\beta+1}b^{-\beta}\quad\text{for } a\geq0,\ b>0 \text{ and } \phi\geq0, \tag{21}
$$

with $\phi=V$, $a=\left(\beta+1\right)$ and $b=\beta$, we see that

$$
\begin{aligned}
\left(\widetilde{R}\left(g\left(t\right),t\right)\right)^{\beta+1}\omega\left(t\right) \leq\;& \left(\widetilde{R}\left(g\left(t\right),t_3\right)\right)^{\beta+1}\omega\left(t_3\right)-\int_{t_3}^{t}\left(\widetilde{R}\left(g\left(s\right),s\right)\right)^{\beta+1}\theta\left(s\right)\mathrm{d}s \\
&+\widetilde{R}\left(g\left(t\right),t\right)-\widetilde{R}\left(g\left(t\right),t_3\right).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\left(\widetilde{R}\left(g\left(t\right),t\right)\right)^{\beta}\omega\left(t\right) \leq\;& \frac{\left(\widetilde{R}\left(g\left(t\right),t_3\right)\right)^{\beta+1}\omega\left(t_3\right)}{\widetilde{R}\left(g\left(t\right),t\right)}-\frac{1}{\widetilde{R}\left(g\left(t\right),t\right)}\int_{t_3}^{t}\left(\widetilde{R}\left(g\left(s\right),s\right)\right)^{\beta+1}\theta\left(s\right)\mathrm{d}s \\
&+1-\frac{\widetilde{R}\left(g\left(t\right),t_3\right)}{\widetilde{R}\left(g\left(t\right),t\right)}.
\end{aligned}
$$

Taking the limit superior on both sides as $t \to \infty$ and using (18), we get

$$U \leq 1 - D.$$

Thus, from (19), we arrive at

$$P \leq L - L^{\frac{1+\beta}{\beta}} \leq L \leq U \leq 1 - D, \tag{22}$$

which completes the proof. □

Theorem 1. *Assume that $r_i'(t) > 0$ for $i = 1, 2$, and x is a positive solution of (1). If*

$$P = \liminf_{t\to\infty} \left(\widetilde{R}(g(t), t_0)\right)^{\beta} \int_t^{\infty} \theta(s)\, ds > \frac{\beta^{\beta}}{(\beta+1)^{\beta+1}}, \tag{23}$$

then the class $\widetilde{\aleph}$ is empty.

Proof. Let x be a positive solution of (1) and $z \in \widetilde{\aleph}$. First, let $P = \infty$. As in the proof of Lemma 6, we obtain that (19) and (20). Then, from (20), we have

$$\widetilde{R}(g(t), t)^{\beta} \omega(t) \geq \widetilde{R}(g(t), t)^{\beta} \int_t^{\infty} \theta(s)\, ds.$$

Taking the limit inferior as $t \to \infty$ and using (19), we get

$$1 \geq L \geq P = \infty,$$

this is a contradiction.

On the other hand, let $P < \infty$. From Lemma 6, we have $P \leq L - L^{\frac{1+\beta}{\beta}}$. Using inequality (21) with $\phi = L$ and $a = b = 1$, we get that

$$p \leq \frac{\beta^{\beta}}{(\beta+1)^{\beta+1}},$$

which contradicts (23). The proof is complete. □

By using the comparison principles, we show that the class $\widetilde{\aleph}$ is empty.

Theorem 2. *Assume that $r_i'(t) > 0$ for $i = 1, 2$, and x is a positive solution of (1). If the first-order delay equation*

$$y'(t) + \theta(t) \frac{(g(t) - t_0)^{\beta+\alpha_2} \theta(t)}{2^{\beta} (r_1(g(t)))^{\alpha_2} r_2(g(t))} y(g(t)) = 0 \tag{24}$$

is oscillatory, then the class $\widetilde{\aleph}$ is empty.

Proof. Assume on the contrary that $z \in \widetilde{\aleph}$. Using Lemma 4, we obtain that (5) and (6). From (5) and (6), we get

$$x^{\beta}(g(t)) \geq (1 - p(g(t)))^{\beta} \frac{(g(t) - t_0)^{\beta+\alpha_2}}{2^{\beta} (r_1(g(t)))^{\alpha_2} r_2(g(t))} r_2(g(t)) \left((L_1 z(g(t)))'\right)^{\alpha_2}. \tag{25}$$

Combining (1) with (25), one can see that $y(t) = L_2 z(t)$ is a positive solution of the differential inequality

$$y'(t) + \frac{(g(t) - t_1)^{\beta+\alpha_2} \theta(t)}{2^{\beta} (r_1(g(t)))^{\alpha_2} r_2(g(t))} y(g(t)) \leq 0.$$

In view of [22], Theorem 1, the associated delay differential Equation (24), also has a positive solution. This contradiction completes the proof. □

In the following Theorem, we are concerned with the oscillation of solutions of (1) by using a Riccati transformation technique.

Theorem 3. *Assume that x is a positive solution of (1). If that there exists a positive function $\rho(t)$such that*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left(\rho(s)\theta(s)-\frac{\beta^{\beta}}{(\beta+1)^{\beta+1}}\left(\frac{\rho'(s)}{\rho(s)}\right)^{\beta+1}\eta^{-\beta}(s)\right)\mathrm{d}s=\infty, \tag{26}$$

then the class $\widetilde{\aleph}$ is empty, where

$$\eta(t)=\beta\frac{\rho(t)g'(t)}{\rho^{1+1/\beta}(t)}\left(\frac{R_2(g(t),t_2)}{r_1(g(t))}\right)^{1/\alpha_1}.$$

Proof. Let x be a positive solution of (1) and $z\in\widetilde{\aleph}$. By Lemma 4, we have that (5)–(8) hold. Now, we define

$$\widetilde{\omega}(t)=\rho(t)\frac{L_2z(t)}{z^{\beta}(g(t))}.$$

Then, from (6) and (7), we have

$$\begin{aligned}
\widetilde{\omega}'(t) &\le \frac{\rho'(t)}{\rho(t)}\widetilde{\omega}(t)-\rho(t)\theta(t)-\beta\rho(t)\frac{L_2z(t)}{z^{\beta+1}(g(t))}z'(g(t))g'(t)\\
&\le \frac{\rho'(t)}{\rho(t)}\widetilde{\omega}(t)-\rho(t)\theta(t)\\
&\quad -\beta\rho(t)\frac{L_2z(t)}{z^{\beta+1}(g(t))}\left(\frac{R_2(g(t),t_2)}{r_1(g(t))}\right)^{1/\alpha_1}(L_2z(g(t)))^{1/\beta}g'(t)\\
&\le \frac{\rho'(t)}{\rho(t)}\widetilde{\omega}(t)-\rho(t)\theta(t)\\
&\quad -\beta\frac{\rho(t)g'(t)}{\rho^{1+1/\beta}(t)}\left(\frac{R_2(g(t),t_2)}{r_1(g(t))}\right)^{1/\alpha_1}\widetilde{\omega}^{1+1/\beta}(t).
\end{aligned}$$

Using inequality (21) with $\phi=\widetilde{\omega}$, $a=\rho'/\rho$ and $b=\eta$, we obtain

$$\frac{\rho'}{\rho}\widetilde{\omega}-\eta\widetilde{\omega}^{\frac{\beta+1}{\beta}}\le\frac{\beta^{\beta}}{(\beta+1)^{\beta+1}}\left(\frac{\rho'}{\rho}\right)^{\beta+1}\eta^{-\beta}.$$

Therefore, we get

$$\widetilde{\omega}'(t)\le-\rho(t)\theta(t)+\frac{\beta^{\beta}}{(\beta+1)^{\beta+1}}\left(\frac{\rho'(t)}{\rho(t)}\right)^{\beta+1}\eta^{-\beta}(t).$$

By integrating the above inequality from t_2 to t we have

$$\widetilde{\omega}(t)\le\widetilde{\omega}(t_2)-\int_{t_2}^{t}\left(\rho(s)\theta(s)-\frac{\beta^{\beta}}{(\beta+1)^{\beta+1}}\left(\frac{\rho'(s)}{\rho(s)}\right)^{\beta+1}\eta^{-\beta}(s)\right)\mathrm{d}s.$$

Taking the superior limit as $t\to\infty$ and using (26), we get $\widetilde{\omega}(t)\to-\infty$, which contradicts that $\widetilde{\omega}(t)>0$. This completes the proof. □

4. Criteria for Nonexistence of Kneser Solutions

Theorem 4. *Assume that x is a positive solution of (1). If there exists a function $\psi \in C([t_0,\infty),(0,\infty))$ satisfying $g(t) < \psi(t)$ and $\tau^{-1}(\psi(t)) < t$, such that the first-order delay differential equation*

$$y'(t) + \frac{1}{\mu}\frac{\tau_0}{\tau_0 + p_0} Q(t) \left(\widetilde{R}(\psi(t), g(t))\right)^{\beta} y\left(\tau^{-1}(\psi(t))\right) = 0 \tag{27}$$

is oscillatory, then the class $\aleph$ is empty.

Proof. Assume on the contrary that x is a Kneser solution of (1) and $z \in \aleph$. Then, we assume that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(g(t)) > 0$ for $t \geq t_1 \geq t_0$. From Lemma 5, we get that (9) and (10) hold.

$$z^{\beta}(g(t)) \geq L_2 z(\psi(t)) \left(\widetilde{R}(\psi(t), g(t))\right)^{\beta},$$

which by virtue of (10) yields that

$$\left(L_2 z(t) + \frac{p_0^{\beta}}{\tau_0} L_2 z(\tau(t))\right)' + \frac{1}{\mu} Q(t) L_2 z(\psi(t)) \left(\widetilde{R}(\psi(t), g(t))\right)^{\beta} \leq 0. \tag{28}$$

Now, we define the function

$$y(t) = L_2 z(t) + \frac{p_0^{\beta}}{\tau_0} L_2 z(\tau(t)).$$

From the fact that $L_2 z(t)$ is non-increasing, we have

$$y(t) \leq L_2 z(\tau(t)) \left(1 + \frac{p_0^{\beta}}{\tau_0}\right)$$

or equivalently,

$$L_2 z(\psi(t)) \geq \frac{\tau_0}{\tau_0 + p_0^{\beta}} y\left(\tau^{-1}(\psi(t))\right). \tag{29}$$

Using (29) in (28), we see that y is a positive solution of the first-order delay differential inequality

$$y'(t) + \frac{1}{\mu}\frac{\tau_0}{\tau_0 + p_0^{\beta}} Q(t) \left(\widetilde{R}(\psi(t), g(t))\right)^{\beta} y\left(\tau^{-1}(\psi(t))\right) \leq 0. \tag{30}$$

Under these conditions, it has already been shown in [22], Theorem 1, that the associated delay differential Equation (27) also has a positive solution, that is a contradiction. Thus, the class $\aleph$ is empty and the proof is complete. □

Corollary 1. *Assume that x is a positive solution of (1). If there exists a function $\psi(t) \in C([t_o,\infty),(0,\infty))$ satisfying $g(t) < \psi(t)$ and $\tau^{-1}(\psi(t)) < t$, such that*

$$\liminf_{t\to\infty} \int_{\tau^{-1}(\psi(t))}^{t} Q(s) \left(\widetilde{R}(\psi(s), g(s))\right)^{\beta} \mathrm{d}s > \frac{\tau_0 + p_0^{\beta}}{\mathrm{e}\mu\tau_0}, \tag{31}$$

then the class $\aleph$ is empty.

Theorem 5. *Assume that x is a positive solution of (1). If there exists a function $\varphi(t) \in C([t_0,\infty),(0,\infty))$ satisfying $\varphi(t) < t$ and $g(t) < \tau(\varphi(t))$, such that*

$$\limsup_{t\to\infty}\left(\widetilde{R}(\tau(\varphi(t)),g(t))\right)^{\beta}\int_{\varphi(t)}^{t}Q(s)\,\mathrm{d}s > \frac{\tau_o + p_0^{\beta}}{\mu\tau_o}, \tag{32}$$

then the class $\aleph$ is empty.

Proof. Assume on the contrary that x is a Kneser solution of (1) and $z \in \aleph$. Then, we assume that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(g(t)) > 0$ for $t \ge t_1 \ge t_0$. From Lemma 5, we get that (9) and (10) hold. Integrating (10) from $\varphi(t)$ to t and using the fact that $L_3 z(t) \le 0$, we see that

$$\begin{aligned} L_2 z(\varphi(t)) + \frac{p_0^{\beta}}{\tau_0}L_2 z(\tau(\varphi(t))) &\ge L_2 z(t) + \frac{p_0^{\beta}}{\tau_0}L_2 z(\tau(t)) + \frac{1}{\mu}\int_{\varphi(t)}^{t}Q(s)\,z^{\beta}(g(s))\,\mathrm{d}s \\ &\ge \frac{1}{\mu}\int_{\varphi(t)}^{t}Q(s)\,z^{\beta}(g(s))\,\mathrm{d}s \\ &\ge \frac{1}{\mu}z^{\beta}(g(t))\int_{\varphi(t)}^{t}Q(s)\,\mathrm{d}s. \end{aligned}$$

Since $\tau(\varphi(t)) < \tau(t)$ and $L_2 z(t)$ is non-increasing, we get

$$L_2 z(\tau(\varphi(t)))\left(1 + \frac{p_0^{\beta}}{\tau_0}\right) \ge \frac{1}{\mu}z^{\beta}(g(t))\int_{\varphi(t)}^{t}Q(s)\,\mathrm{d}s. \tag{33}$$

Using (9) with $u = g(t)$ and $v = \tau(\varphi(t))$ in (33), we arrive at

$$L_2 z(\tau(\varphi(t)))\left(1 + \frac{p_0^{\beta}}{\tau_0}\right) \ge \frac{1}{\mu}L_2 z(\tau(\varphi(t)))\left(\widetilde{R}(\tau(\varphi(t)),g(t))\right)^{\beta}\int_{\varphi(t)}^{t}Q(s)\,\mathrm{d}s,$$

that is,

$$\frac{\tau_0 + p_0^{\beta}}{\mu\tau_0} \ge \left(\widetilde{R}(\tau(\varphi(t)),g(t))\right)^{\beta}\int_{\varphi(t)}^{t}Q(s)\,\mathrm{d}s.$$

Finally, by taking the lim sup on both sides of the latter inequality, we arrive at a contradiction to (32). The proof is complete. □

By setting $\varphi(t) = \tau(t)$ in (32), the following result is an immediate consequence.

Corollary 2. *Assume that x is a positive solution of (1). If*

$$\limsup_{t\to\infty}\left(\widetilde{R}(\tau(\tau(t)),g(t))\right)^{\beta}\int_{\tau(t)}^{t}Q(s)\,\mathrm{d}s > \frac{\tau_0 + p_0^{\beta}}{\mu\tau_0}, \tag{34}$$

then the class $\aleph$ is empty.

5. Oscillation Criteria

Based on the fact that there are only two cases for the corresponding function z, we can use the results in the previous two sections to infer new criteria for oscillation of all solutions of Equation (1). Any of the criteria (23), (24) and (26) ensures that $\widetilde{\aleph} = \emptyset$, whereas one of the criteria (27), (31) and (34) ensures that $\aleph = \emptyset$. This guarantees that $\aleph = \widetilde{\aleph} = \emptyset$, and we can ensure that there are no non-oscillatory solutions. Hence through these results we proved the following oscillation Theorem:

Theorem 6. *Assume the non-linear NDDE of third order (1). Then:*

1. *If $r_i'(t) > 0$ for $i = 1,2$.*
   - *If there exists a function $\psi \in C([t_0,\infty),(0,\infty))$ satisfying $g(t) < \psi(t)$ and $\tau^{-1}(\psi(t)) < t$, such that the first-order delay differential Equations (24) and (27) are oscillatory, then (1) is oscillatory;*
   - *If (23) and (31) hold, then (1) is oscillatory;*
   - *If (23) and (34) hold, then (1) is oscillatory.*
2. *If there exists a positive function $\rho(t)$such that (26) and*
   - *(31) hold, then (1) is oscillatory;*
   - *(34) hold, then (1) is oscillatory.*

Example

Consider the third-order NDDE

$$\left(\left((x(t)+p_0 x(\delta t))''\right)^{\alpha_2}\right)' + \frac{q_0}{t^{2\alpha_2+1}} x^{\alpha_2}(\lambda t) = 0, \quad t \geq 1, \tag{35}$$

where p_0 and q_0 are positive constants and $\delta, \lambda \in (0,1)$. Please note that $r_1(t) = r_2(t) = 1$, $\alpha_1 = 1$, $\beta = \alpha_2$ and $f(x) = x^{\alpha_2}$. It is easy to verify that $R_i(t,u) = t-u$ for $i = 1,2$,

$$\begin{aligned} \widetilde{R}(v,u) &= \int_u^v (v-s)\,\mathrm{d}s = \frac{1}{2}(v-u)^2, \\ \theta(t) &= q_0(1-p_0)^{\alpha_2}\frac{1}{t^{2\alpha_2+1}}, \end{aligned}$$

and $Q(t) = q_0/t^{2\alpha_2+1}$. Then, we have

$$\begin{aligned} \liminf_{t\to\infty}\left(\widetilde{R}(g(t),t_0)\right)^{\beta}\int_t^{\infty}\theta(s)\,ds &= \frac{q_0(1-p_0)^{\alpha_2}}{2^{\alpha_2+1}}\liminf_{t\to\infty}\frac{(\lambda t-t_0)^{2\alpha_2}}{t^{2\alpha_2}} \\ &= \frac{q_0(1-p_0)^{\alpha_2}}{2^{\alpha_2+1}}\lambda^{2\alpha_2}. \end{aligned}$$

Thus, the condition (23) becomes

$$q_0 > \frac{2^{\alpha_2+1}\alpha_2^{\alpha_2+1}}{\lambda^{2\alpha_2}(1-p_0)^{\alpha_2}(\alpha_2+1)^{\alpha_2+1}}. \tag{36}$$

Next, by choosing $\psi(t) = (\lambda+\delta)t/2$, condition (31) reduces to

$$\begin{aligned} \liminf_{t\to\infty}\int_{\tau^{-1}(\psi(t))}^{t} Q(s)\left(\widetilde{R}(\psi(s),g(s))\right)^{\beta}\mathrm{d}s &= \liminf_{t\to\infty}\int_{(\delta+\lambda/2\delta)t}^{t}\frac{kq_0}{2^{\alpha_2}}\left(\frac{\delta-\lambda}{2}\right)^{2\alpha_2}\frac{1}{s}\mathrm{d}s \\ &= \frac{kq_0}{2^{\alpha_2}}\left(\frac{\delta-\lambda}{2}\right)^{2\alpha_2}\ln\frac{2\delta}{\delta+\lambda} \end{aligned}$$

or equivalently,

$$\frac{kq_0}{2^{\alpha_2}}\left(\frac{\delta-\lambda}{2}\right)^{2\alpha_2}\ln\frac{2\delta}{\delta+\lambda} > \frac{\tau_0+p_0^{\alpha_2}}{\mathrm{e}\mu\tau_0}. \tag{37}$$

Hence, by Theorem 6, every solution of Equation (35) is oscillatory if (36) and (37) hold.

Remark 2. *By using our results, we obtain sufficient conditions to ensure that all solutions of (1) are oscillatory. Whereas, the related results [37,38,40,41,43] created conditions that ensure that solutions are either oscillatory or tend to zero. So, our new criteria improve and complement a number of existing results.*

6. Conclusions

We considered a class of non-linear NDDEs of third order. By using Riccati transformation and comparison principles that compare the third-order equation with a first-order equation, we proved criteria for non-existence of non-Kneser solutions, and criteria for non-existence of Kneser solutions. We then used these results to conclude to a Theorem that provides criteria for (1) in order to ensure that all its solutions are oscillatory. These criteria extend and improve several other results in the literature. An example was given to support our theory.

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