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Optical Soliton Solutions of the Cubic-Quartic Nonlinear Schrödinger and Resonant Nonlinear Schrödinger Equation with the Parabolic Law

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Received: 23 October 2019; Accepted: 20 December 2019; Published: 27 December 2019



Featured Application: The optical soliton solutions obtained in this research paper may be of concern and useful in many fields of science, such as mathematical physics, applied physics, nonlinear science, and engineering.

Abstract: In this paper, the cubic-quartic nonlinear Schrödinger and resonant nonlinear Schrödinger equation in parabolic law media are investigated to obtain the dark, singular, bright-singular combo and periodic soliton solutions. Two powerful methods, the $\left(m + \frac{G'}{G}\right)$ improved expansion method and the $\exp(-\varphi(\xi))$ expansion method are utilized to construct some novel solutions of the governing equations. The obtained optical soliton solutions are presented graphically to clarify their physical parameters. Moreover, to verify the existence solutions, the constraint conditions are utilized.

Keywords: cubic-quartic Schrödinger equation; cubic-quartic resonant Schrödinger equation; parabolic law

1. Introduction

In the current century, many entropy problems have been expressed by using mathematical models that are nonlinear partial differential equations. New results in the last few years have shown that the relation between non-standard entropies and nonlinear partial differential equations can be applied on new nonlinear wave equations inspired by quantum mechanics. Nonlinear models of the celebrated Klein–Gordon and Dirac equations have been found to admit accurate time dependent soliton-like solutions with the shapes of the so-called q-plane waves. Such q-plane waves are generalizations of the complex exponential plane wave solutions of the linear Klein–Gordon and Dirac equation forming and its application in the differential equation has been noticeable in the last few years. The physical phenomena of nonlinear partial differential equations (NLPDEs) may connect to many areas of sciences, for example plasma physics, optical fibers, nonlinear optics, fluid mechanics, chemistry, biology, geochemistry, and engineering sciences. The nonlinear Schrödinger equations describe wave propagation in optical fibers with nonlinear impacts [2–4].



Various numeric and analytic techniques have been used to seek solutions for nonlinear differential equations such as the homotopy perturbation scheme [5], the Adams–Bashforth–Moulton method [6], the shooting technique with fourth-order Runge–Kutta scheme [7–10], the group preserving method [11], the finite forward difference method [12,13], the Adomian decomposition method [14,15], the sine-Gordon expansion method [16–18], the modified auxiliary expansion method [19], the modified $\exp(-\varphi(\xi))$ expansion function method [20,21], the improved Bernoulli sub-equation method [22,23], the Riccati–Bernoulli sub-ODE method [24], the modified exponential function method [25], the improved $\tan(\phi(\xi)/2)$ [26,27], the Darboux transformation method [28,29], the double $\left(\frac{G'}{G}, \frac{1}{G'}\right)$ expansion method [30,31], the $\left(\frac{1}{G'}\right)$ expansion method [32,33], the decomposition Sumudu-like-integral transform method [34], and the inverse scattering method [35].

In recent years, many researchers have carried out investigations on the governing models in optical fibers. The nonlinear Schrödinger equation, involving cubic and quartic-order dispersion terms, has been investigated to seek the exact optical soliton solutions via the undetermined coefficients method [36], the modified Kudryashov approach [37], the complete discrimination system method [38], the generalized tanh function method [39], the sin-cosine method, as well as the Bernoulli equation approach [40], the semi-inverse variation method [41], a simple equation method [3], and the extended sinh-Gordon expansion method [42].

Now, optical solitons are the exciting research area of nonlinear optics studies, and this research field has led to tremendous advances in their extensive applications. It is identified that the dynamics of nonlinear optical solitons and Madelung fluids are based on the generalized nonlinear Schrödinger dispersive equation and resonant nonlinear Schrödinger dispersive equation. In the research of chirped solitons in Hall current impacts in the field of quantum mechanics, a specific resonant term must be given [43].

Dispersion and nonlinearity are the two key elements for the propagation of solitons over intercontinental ranges. Normally, group velocity dispersion (GVD) leveling with self-phase modulation in a sensitive way allows such solitons to maintain long distance travel. Now, it could occur that GVD is minuscule and therefore completely overlooked, so in this condition, the dispersion impact is rewarded for by third-order (3OD) and fourth-order (4OD) dispersion impacts. This is generally referred to as solitons that are cubic-quartic (CQ). This term was implemented in 2017 for the first time. This model was later extensively researched through different points of view such as the semi-inverse variation principle [41], Lie symmetry [44], conservation rules [45], and the system of undetermined coefficients [37]. Consider the nonlinear Schrödinger and resonant nonlinear Schrödinger equations in the appearance of 3OD and 4OD without both GVD and disturbance. The equations are as follows:

$$iu_t + i\alpha u_{xxx} + \beta u_{xxxx} + cF\left(|u|^2\right)u = 0,$$
(1)

$$iu_t + i\alpha u_{xxx} + \beta u_{xxxx} + cF\left(|u|^2\right)u + c_3\left(\frac{|u|_{xx}}{|u|}\right)u = 0.$$
(2)

In Equations (1) and (2), u(x, t) is the complex valued wave function and x (space) and t (time) are independent variables. The coefficients α and β are real constants, while c_3 is the Bohm potential that occurs in Madelung fluids. The Bohm potential term of disturbance generates quantum behavior, so that quantum characteristics are closely related to their special characteristics. Therefore, we have the chirped NLSE's disturbance expression giving us the introduction of the theory of hidden variables. Therefore, it will be more crucial to retrieve accurate solutions for the development of quantum mechanics from disturbed chiral (resonant) NLSE [46]. Furthermore, the functional F is a real valued algebraic function that represents the source of nonlinearity and $F(|u|^2) u : C \to C$. In more detail, the function $F(|u|^2) u$ is p-times continuously differentiable, so that:

$$F\left(|u|^2\right)u\in \bigcup_{m,n=1}^{\infty}C^p\left((-n,n)\times(-m,m):R^2\right).$$

Suppose that $F(u) = c_1 u + c_2 u^2$, so Equations (1) and (2) can be rewritten as:

$$iu_t + i\alpha u_{xxx} + \beta u_{xxxx} + \left(c_1|u|^2 + c_2|u|^4\right)u = 0,$$
(3)

$$iu_t + i\alpha u_{xxx} + \beta u_{xxxx} + \left(c_1|u|^2 + c_2|u|^4\right)u + c_3\left(\frac{|u|_{xx}}{|u|}\right)u = 0.$$
(4)

Equation (3) was investigated by making $c_2 = 0$ in [47] via the Kudryashov approach. The conservation laws to obtain the conserved densities for Schrödinger's nonlinear cubic-quarter equation have been analyzed in Kerr and power-law media [45]. The undetermined coefficients method has been employed to construct bright soliton and singular soliton solutions of Equation (1), when nonlinearity has been taken into consideration in the instances of the Kerr law and power law [37]. In this study, we use two methods to investigate soliton solutions of the cubic-quartic nonlinear Schrödinger equation and cubic-quartic resonant nonlinear Schrödinger equation with the parabolic law, namely Equations (3) and (4).

2. Instructions for the Methods

Assume a nonlinear partial differential equation (NLPDE) as follows:

$$P(U, U_x, U_t, U_{xx}, U_{tt}, U_{tx}, \dots) = 0,$$
(5)

and define the traveling wave transformation as follows,

$$U(x, y, t) = \phi(\zeta), \quad \zeta = x - \nu t.$$
(6)

Putting Equation (6) into Equation (5), the outcome is:

$$N\left(\phi,\phi',\phi'',\ldots\right)=0. \tag{7}$$

For the $m + \frac{G'(\zeta)}{G(\zeta)}$ expansion method, we take the trial solution for Equation (7) as follows:

$$\phi(\zeta) = \sum_{i=-n}^{n} a_i (m+F)^i = a_{-n} (m+F)^{-n} + \ldots + m a_0 + a_1 (m+F) + \ldots + a_n (m+F)^n, \quad (8)$$

where a_i , i = 0, 1, ..., n and m are nonzero constants. According to the principles of balance, we find the value of n. In this manuscript, we define F to be a function as:

$$F = \frac{G'(\zeta)}{G(\zeta)},\tag{9}$$

where $G(\zeta)$ satisfy $G'' + (\lambda + 2m) G' + \mu G = 0$.

Putting Equation (8) into Equation (7) by using Equation (9), then collecting all terms with the same order of $(m + F)^n$, we get the system of algebraic equations for ν , a_n , n = 0, 1, ..., n, λ , and μ . As a result, solving the obtained system, we get the explicit and exact solutions of Equation (5).

For the (exp – $\varphi(\xi)$) expansion method, we use the trial solution as follows:

$$\phi(\xi) = \sum_{i=0}^{n} b_i (\exp(-\varphi(\xi)))^i, \quad i = 1, 2, \dots, n$$
(10)

where b_i are non-zero constants. The auxiliary ODE $\varphi(\xi)$ is defined as follows:

$$\varphi'(\xi) = \exp\left(-\varphi(\xi)\right) + \mu \exp\left(\varphi(\xi)\right) + \lambda.$$
(11)

Solving Equation (11), we have:

Case 1. When $\Delta > 0$ and $\mu \neq 0$, we get the hyperbolic function solution:

$$\varphi\left(\xi\right) = \ln\left(\frac{-\lambda - \sqrt{\Delta} \tanh\left(\frac{1}{2}\sqrt{\Delta}\left(\xi + c\right)\right)}{2\mu}\right).$$
(12)

Case 2. When $\Delta < 0$ and $\mu \neq 0$, we get the trigonometric function solution:

$$\varphi\left(\xi\right) = \ln\left(\frac{-\lambda + \sqrt{-\Delta}\tan\left(\frac{1}{2}\sqrt{-\Delta}\left(\xi + c\right)\right)}{2\mu}\right).$$
(13)

Case 3. When $\Delta > 0$, $\mu = 0$, and $\lambda \neq 0$, we get hyperbolic function solution

$$\varphi\left(\xi\right) = -\ln\left(\frac{\lambda}{-1 + \cosh\left(\lambda\left(\xi + c\right)\right) + \sinh\left(\lambda\left(\xi + c\right)\right)}\right).$$
(14)

Case 4. When $\Delta = 0$, $\mu \neq 0$ and $\lambda \neq 0$, we get the rational function solution:

$$p\left(\xi\right) = \ln\left(\frac{-2 - 2\lambda\left(\xi + c\right)}{\lambda^2\left(\xi + c\right)}\right).$$
(15)

Case 5. When $\Delta = 0$, $\mu = 0$, and $\lambda = 0$, we get:

$$\varphi\left(\xi\right) = \ln\left(\xi + c\right),\tag{16}$$

where *c* is the non-zero constant of integration and $\Delta = \lambda^2 - 4\mu$.

3. Application to the $\left(m + \frac{G'}{G}\right)$ Expansion Method

In this section, we use the $\left(m + \frac{G'}{G}\right)$ expansion method for the cubic-quartic nonlinear Schrödinger and cubic-quartic resonant nonlinear Schrödinger equations.

3.1. The Cubic-Quartic Nonlinear Schrödinger Equation

To solve Equation (3), by the $\left(m + \frac{G'}{G}\right)$ expansion method, we use the following transformation:

$$u(x,t) = U(\xi)e^{i\theta}, \qquad \xi = x - \nu t, \qquad \theta = -\kappa x + \omega t.$$
(17)

In the above equation, $\theta(x, t)$ symbolize the phase component of the soliton, κ represent the soliton frequency, while ω denote the wave number, and ν symbolize the velocity of the soliton. Substitute wave transformation into Equation (3), and separate the outcome equation into real and imaginary parts. We can write the real part as follows:

$$-\left(\alpha\kappa^{3} - \beta\kappa^{4} + \omega\right)U + c_{1}U^{3} + c_{2}U^{5} + 3\alpha\kappa U'' - 6\beta\kappa^{2}U'' + \beta U^{(4)} = 0,$$
(18)

and the imaginary part can be written as:

$$\left(3\alpha\kappa^2 - 4\beta\kappa^3 + \nu\right)U' - \left(\alpha - 4\beta\kappa\right)U^{(3)} = 0.$$
(19)

From Equation (19) $U' \neq 0$ and $U''' \neq 0$, then:

$$\nu = 4\beta\kappa^3 - 3\alpha\kappa^2, \quad \alpha = 4\beta\kappa. \tag{20}$$

Hence, Equation (18) can be rewritten as:

$$\left(3\beta\kappa^{4}+\omega\right)U-c_{1}U^{3}-c_{2}U^{5}-12\beta\kappa^{2}U''+6\beta\kappa^{2}U''-\beta U^{(4)}=0.$$
(21)

Multiplying both sides of Equation (21) by U' and taking its integration with respect to ξ , we get:

$$\beta \left(-12 (U'')^2 + 24U''' U' \right) + 6c_1 U^4 + 4c_2 U^6 + 72\beta \kappa^2 (U')^2 - \left(36\beta \kappa^4 + 12\omega \right) U^2 = 0.$$
 (22)

Finding the balance, we gain n = 1. Replacing this value of balance into Equation (8), we get:

$$U(\xi) = a_{-1}(m+F)^{-1} + a_0 + a_1(m+F).$$
(23)

By substituting Equation (23) into Equation (3) by using Equation (9), we get the following solutions:

Case 1. When $a_0 = \frac{\lambda a_1}{2}$, $\kappa = \pm \frac{\sqrt{(2m+\lambda)^2 - 4\mu}}{\sqrt{6}}$, $c_1 = \frac{8\beta\left((2m+\lambda)^2 - 4\mu\right)}{a_1^2}$, $c_2 = -\frac{24\beta}{a_1^4}$, $a_{-1} = 0$, and $\Delta = (\lambda + 2m)^2 - 4\mu$, we get an exponential function solution as follows:

$$u(x,t) = e^{i\left(\sqrt{\frac{\Delta}{6}}x + \frac{5}{12}\beta\Delta^{2}t\right)} \left(\frac{\lambda a_{1}}{2} + a_{1}\left(m + \frac{1}{2}\left(-2m + \left(1 - \frac{2A_{1}}{A_{1} + A_{2}e^{\sqrt{\Delta}\left(x - \frac{2}{3}\sqrt{\frac{2}{3}}\beta(\Delta)^{3/2}t\right)}\right)\sqrt{\Delta} - \lambda\right)\right)\right),$$
(24)

which is a dark solution, as shown in Figure 1, A_1 and A_2 are non-zero numbers, and $\Delta > 0$. Figure 1 shows that Equation (24) is a dark soliton under the suitable values of parameters.



Figure 1. 3D surface of Equation (24), which is a dark optical soliton solution plotted when $A_1 = 1$, $A_2 = 0.3$, $\beta = 0.2$, $a_1 = 0.4$, $\lambda = 1$, m = 1, $\mu = -1$, and t = 2 for 2D.

Case 2. When
$$a_0 = -\frac{\lambda a_{-1}}{2m(m+\lambda)-2\mu}$$
, $a_1 = 0$, $a_2 = \frac{12\omega}{5((2m+\lambda)^2-4\mu)^2}$, $\kappa = \frac{\sqrt{(2m+\lambda)^2-4\mu}}{\sqrt{6}}$, $c_1 = \frac{96(-m(m+\lambda)+\mu)^2\omega}{5((2m+\lambda)^2-4\mu)a_{-1}^2}$, $c_2 = -\frac{288(-m(m+\lambda)+\mu)^4\omega}{5((2m+\lambda)^2-4\mu)^2a_{-1}^4}$, and $\Delta = (\lambda+2m)^2 - 4\mu$, we obtain an exponential function solution:

$$u(x,t) = \frac{a_{-1}e^{i\left(-\frac{x\sqrt{(2m+\lambda)^{2}-4\mu}}{\sqrt{6}}+t\omega\right)}}{m+\frac{1}{2}\left(-2m+\left(1-\frac{2A_{1}}{\sqrt{\Delta}\left(x+\frac{8\sqrt{\frac{2}{3}\omega}}{5\sqrt{\Delta}}T\right)}\right)\sqrt{\Delta}-\lambda\right)} + \frac{\lambda a_{-1}e^{i\left(-\frac{x\sqrt{(2m+\lambda)^{2}-4\mu}}{\sqrt{6}}+t\omega\right)}}{2m(m+\lambda)-2\mu}, (25)$$

which is a soliton solution, as shown in Figure 2, A_1 and A_2 are non-zero numbers, and $\Delta > 0$. With the suitable values, Figure 2 presents that Equation (25) is a singular soliton.



Figure 2. 3D surface of Equation (25), which is a singular solution solution plotted when $A_1 = 2$, $A_2 = 3$, $\beta = 6$, $a_{-1} = 6$, $\lambda = 1$, m = 1, $\mu = -1$, and t = 2 for 2D.

Case 3. When $a_{-1} = -\frac{i\sqrt{3}c_1(m(m+\lambda)-\mu)}{\sqrt{c_2((2m+\lambda)^2-4\mu)}}$, $a_0 = \frac{i\sqrt{3}c_1\lambda}{2\sqrt{c_2((2m+\lambda)^2-4\mu)}}$, $a_1 = 0$, $\omega = -\frac{5c_1^2}{32c_2}$, $\kappa = \pm \frac{\sqrt{(2m+\lambda)^2-4\mu}}{\sqrt{6}}$, $\gamma = -\frac{3c_1^2}{8c_2((2m+\lambda)^2-4\mu)^2}$, and $\Delta = (\lambda+2m)^2 - 4\mu$, we have an exponential function solution:

function solution:

$$\mathbf{u}(x,t) = \mathbf{e}^{\mathbf{i}\left(-\frac{5c_1^2}{32c_2}t + \frac{\sqrt{\Delta}}{\sqrt{6}}x\right)} \left(\frac{\mathbf{i}\sqrt{3}\sqrt{c_1}\lambda}{2\sqrt{c_2\Delta}} - \frac{\mathbf{i}\sqrt{3}\sqrt{c_1}\left(m\left(m+\lambda\right) - \mu\right)}{\left(m + \frac{1}{2}\left(-2m + \left(1 - \frac{2A_1}{A_1 + A_2\mathbf{e}^{\sqrt{\Delta}\left(x + \frac{c_1^2t}{2\sqrt{6}c_2\sqrt{\Delta}}\right)}}\right)\sqrt{\Delta} - \lambda\right)\right)\sqrt{c_2\Delta}} \right),$$
(26)

which is a soliton solution, as shown in Figure 3, A_1 and A_2 are non-zero numbers, and $\Delta > 0$. Considering some values of parameters, Figure 3 shows singular soliton solution.



Figure 3. 3D surface of Equation (26), which is a singular soliton solution plotted when $A_1 = 0.3$, $A_2 = 2$, $c_1 = 0.3$, $c_2 = 2$, $\lambda = 1$, m = 1, $\mu = -1$, and t = 2 for 2D.

3.2. The Cubic-Quartic Resonant Nonlinear Schrödinger Equation

To solve Equation (4), by the $\left(m + \frac{G'}{G}\right)$ expansion method, we consider wave transformation Equation (17). Replacing Equation (17) into Equation (4) and separating the outcome equation into real and imaginary parts, we can write the real part as follows:

$$\left(\kappa^{3} (\alpha - \beta \kappa) + \omega\right) U - c_{1} U^{3} - c_{2} U^{5} - (c_{3} + 3\kappa (\alpha - 2\beta \kappa)) U'' - \beta U^{(4)} = 0,$$
(27)

and the imaginary part can be written as:

$$\left(3\alpha\kappa^2 - 4\beta\kappa^3 + \nu\right)U' - (\alpha - 4\beta\kappa)U''' = 0.$$
(28)

From Equation (28) $U' \neq 0$ and $U''' \neq 0$, then:

$$\nu = 4\beta\kappa^3 - 3\alpha\kappa^2, \qquad \alpha = 4\beta\kappa. \tag{29}$$

Hence, Equation (27) can be rewritten as:

$$\left(3\beta\kappa^{4}+\omega\right)U-c_{1}U^{3}-c_{2}U^{5}-\left(c_{3}+6\beta\kappa^{2}\right)U''-\beta U^{(4)}=0.$$
(30)

Multiplying both sides of Equation (30) by U' and integrating it once with respect to ξ , we get:

$$\left(36\beta\kappa^{4} + 12\omega\right)U^{2} - 6c_{1}U^{4} - 4c_{2}U^{6} - \left(12c_{3} + 72\beta\kappa^{2}\right)\left(U'\right)^{2} + \beta\left(12(U'')^{2} - 24U'U'''\right) = 0.$$
 (31)

Finding the balance, we gain n = 1. Putting this value into Equation (8), we get the same result of Equation (23). Substituting Equation (23) with Equation (9) into Equation (4), we get the following solutions:

Case 1. When
$$a_{-1} = -\frac{2(m(m+\lambda)-\mu)a_0}{\lambda}$$
, $a_1 = 0$, $\omega = \frac{1}{2}\beta\left(-6\kappa^4 + \left((2m+\lambda)^2 - 4\mu\right)^2\right)$, $c_1 = \frac{2\beta\lambda^2\left((2m+\lambda)^2 - 4\mu\right)}{a_0^2}$, $c_2 = -\frac{3\beta\lambda^4}{2a_0^4}$, and $c_3 = \beta\left(-6\kappa^2 + (2m+\lambda)^2 - 4\mu\right)$, we obtain the following solutions:

Solution 1. In the case $\Delta > 0$, we have an exponential function solution:

$$u(x,t) = e^{i\left(-\kappa x + \frac{1}{2}\beta\left(-6\kappa^{4} + \left((2m+\lambda)^{2} - 4\mu\right)^{2}\right)t\right)} \left(a_{0} - \frac{2\left(m\left(m+\lambda\right) - \mu\right)a_{0}}{\left(m + \frac{1}{2}\left(-2m + \left(1 - \frac{2A_{1}}{A_{1} + A_{2}e^{\sqrt{\Delta}\left(x + 8\beta\kappa^{3}t\right)}}\right)\sqrt{\Delta} - \lambda\right)\right)\lambda}\right).$$
(32)

Considering some values of parameters, Figure 4 shows singular soliton solution.



Figure 4. 3D figure of Equation (32), which is a singular soliton solution plotted when $A_1 = 1$, $A_2 = 3$, $\lambda = 1$, m = 1, $\mu = -1$, $\beta = 0.2$, $a_0 = 0.2$, $\kappa = 0.01$, and t = 2 for 2D.

Solution 2. In the case $\Delta < 0$, we have a trigonometric function solution:

$$u(x,t) = e^{-i\kappa x + \frac{1}{2}i\beta\left(-6\kappa^{4} + \left((2m+\lambda)^{2} - 4\mu\right)^{2}t\right)} \left(a_{0} + \frac{4a_{0}\left(m^{2} + m\lambda - \mu\right)\left(A_{2}\cos\left(\alpha\right) + A_{1}\sin\left(\alpha\right)\right)}{\lambda\left(\left(-A_{1}\sqrt{-\Delta} + A_{2}\lambda\right)\cos\left(\alpha\right) + \left(A_{2}\sqrt{-\Delta} + A_{1}\lambda\right)\sin\left(\alpha\right)\right)}\right),$$
(33)

which is $\alpha = \frac{1}{2}\sqrt{-\Delta} (x + 8\beta\kappa^3 t)$.

Periodic singular solution is plotted in Figure 5. Case 2. When $a_{-1} = 0$, $a_1 = \frac{2a_0}{\lambda}$, $\omega = \frac{1}{2}\beta \left(-6\kappa^4 + \left((2m+\lambda)^2 - 4\mu\right)^2\right)$, $c_1 = \frac{2\beta\lambda^2\left((2m+\lambda)^2 - 4\mu\right)}{a_0^2}$, $c_2 = -\frac{3\beta\lambda^4}{2a_0^4}$, and $c_3 = \beta \left(-6\kappa^2 + (2m+\lambda)^2 - 4\mu\right)$, we obtain the following solutions:

Solution 1. In the case $\Delta > 0$, we get dark solution, as shown in Figure 6 :

$$u(x,y) = e^{i\left(-\kappa x + \frac{1}{2}\beta\left(-6\kappa^{4} + \left((2m+\lambda)^{2} - 4\mu\right)^{2}\right)t\right)} \\ \left(a_{0} + \frac{2\left(m + \frac{1}{2}\left(-2m + \left(1 - \frac{2A_{1}}{A_{1} + A_{2}e^{\sqrt{\Delta}(x+8\beta\kappa^{3}t)}}\right)\sqrt{\Delta} - \lambda\right)\right)a_{0}}{\lambda}\right).$$
(34)

Figure 6 shows the dark structure this solution.



Figure 5. 3D surface of Equation (33), which is a periodic singular soliton solution plotted when $A_1 = 1, A_2 = 2, \lambda = 1, m = \frac{1}{2}, \mu = 2, \beta = 0.1, a_0 = 2, \kappa = 0.1$, and t = 2 for 2D.



Figure 6. 3D surface of Equation (34), which is a dark soliton solution plotted when $A_1 = 1$, $A_2 = 3$, $\lambda = 1$, m = 1, $\mu = -1$, $\beta = 0.2$, $a_0 = 0.2$, $\kappa = 0.01$, and t = 2 for 2D.

Solution 2. In the case $\Delta < 0$, we have a trigonometric function solution:

$$u(x,t) = e^{-ix\kappa + \frac{1}{2}i\beta\left(-6\kappa^{4} + \left((2m+\lambda)^{2} - 4\mu\right)^{2}\right)}t$$

$$\left(\frac{\sqrt{-\Delta}\left(A_{1}\cos\left(\frac{1}{2}\sqrt{-\Delta}\left(x+8\beta\kappa^{3}t\right)\right) - A_{2}a_{0}\sin\left(\frac{1}{2}\sqrt{-\Delta}\left(x+8\beta\kappa^{3}t\right)\right)\right)}{\lambda\left(A_{2}\cos\left(\frac{1}{2}\sqrt{-\Delta}\left(x+8\beta\kappa^{3}t\right)\right) + A_{1}\sin\left(\frac{1}{2}\sqrt{-\Delta}\left(x+8\beta\kappa^{3}t\right)\right)\right)}\right).$$
(35)

Periodic singular solution is plotted in Figure 7.



Figure 7. 3D figure of Equation (35), which is a periodic singular soliton solution plotted when $A_1 = 1, A_2 = 2, \lambda = 1, m = \frac{1}{2}, \mu = 2, \beta = 0.1, a_0 = 0.2, \kappa = 0.1$, and t = 2 for 2D.

4. Application to the Exp $(-\varphi(\xi))$ Expansion Method

In this section, we apply the $\exp(-\varphi(\xi))$ expansion method to the cubic-quartic nonlinear Schrödinger and resonant nonlinear Schrödinger equations.

4.1. The Cubic-Quartic Nonlinear Schrödinger Equation

To apply this method on the cubic-quartic nonlinear Schrödinger equation, Equation (3), we utilize the same wave transformation of Equation (17). As a result, we get Equation (22). Finding the balance, we gain n = 1. By inserting the value of the balance into Equation (10), we get:

$$U(\xi) = b_0 + b_1 e^{-\varphi(\xi)}.$$
(36)

Substituting Equation (40) into Equation (22) and setting each summation of the coefficients of the exponential identities of the same power to be zero, we discuss the following cases of the solutions.

Case 1. When $b_0 = \frac{\lambda b_1}{2}$, $c_1 = \frac{8a_2(\lambda^2 - 4\mu)}{b_1^2}$, $c_2 = -\frac{24a_2}{b_1^4}$, $\kappa = -\frac{\sqrt{\lambda^2 - 4\mu}}{\sqrt{6}}$, and $\omega = \frac{5}{12}a_2(\lambda^2 - 4\mu)^2$, we get the following solutions:

Solution 1. In the case $\lambda^2 - 4\mu > 0$ and $\mu \neq 0$, we have a hyperbolic function solution:

$$u(x,t) = e^{i\left(\frac{\sqrt{\lambda^{2}-4\mu}}{\sqrt{6}}x + \frac{5}{12}a_{2}\left(\lambda^{2}-4\mu\right)^{2}t\right)} \left(\frac{\lambda b_{1}}{2} + \frac{2\mu b_{1}}{-\lambda - \sqrt{\lambda^{2}-4\mu}\tanh\left[\frac{1}{2}\left(c + x - \frac{2}{3}\sqrt{\frac{2}{3}}a_{2}\left(\lambda^{2}-4\mu\right)^{3/2}t\right)\sqrt{\lambda^{2}-4\mu}\right]}\right).$$
(37)

This is a dark soliton solution, as shown in Figure 8.



Figure 8. 3D surface of Equation (37), which is a bright singular combo soliton solution plotted when $b_1 = 0.2$, $\beta = 0.2$, c = 1, $\lambda = 3$, $\mu = 1$, and t = 2 for 2D.

Solution 2. When $\lambda^2 - 4\mu > 0$, $\mu = 0$, and $\lambda \neq 0$, we have hyperbolic function solutions:

$$u(x,t) = e^{i\left(\frac{5}{12}\beta\lambda^4 t + \frac{\sqrt{\lambda^2}}{\sqrt{6}}x\right)} \left(\frac{\lambda b_1}{2} + \frac{\lambda b_1}{-1 + \cosh\left(\lambda\left(c + x - \frac{2}{3}\sqrt{\frac{2}{3}}\beta(\lambda^2)^{3/2}t\right)\right) + \sinh\left(\lambda\left(c + x - \frac{2}{3}\sqrt{\frac{2}{3}}\beta(\lambda^2)^{3/2}t\right)\right)}\right).$$
(38)

This is a bright singular combo soliton solution, as shown in Figure 9.



Figure 9. 3D surface of Equation (38), which is a bright singular combo soliton solution plotted when $b_1 = 0.04$, $\beta = 0.2$, c = 0.2, $\lambda = 1$, $\mu = 0$, and t = 2 for 2D.

Case 2. When $b_0 = \frac{\sqrt{3}\sqrt{c_1}\lambda}{2\sqrt{-c_2(\lambda^2 - 4\mu)}}$, $b_1 = \frac{\sqrt{3}\sqrt{c_1}}{\sqrt{-c_2(\lambda^2 - 4\mu)}}$, $\kappa = -\frac{\sqrt{\lambda^2 - 4\mu}}{\sqrt{6}}$, $\omega = -\frac{5c_1^2}{32c_2}$, and $\beta = -\frac{3c_1^2}{8c_2(\lambda^2 - 4\mu)^2}$, we get the following solutions:

Solution 1. When $\lambda^2 - 4\mu > 0$ and $\mu \neq 0$, we get a dark solution, as shown in Figure 10:

$$u(x,t) = e^{i\left(-\frac{5c_1^2}{32c_2}t + \frac{\sqrt{\lambda^2 - 4\mu}}{\sqrt{6}}x\right)} \left(\frac{\sqrt{3}\sqrt{c_1}\left(\lambda^2 - 4\mu + \lambda\sqrt{\lambda^2 - 4\mu}\tanh\left(\frac{c_1^2t}{4\sqrt{6c_2}} + \frac{1}{2}(c+x)\sqrt{\lambda^2 - 4\mu}\right)\right)}{2\sqrt{-c_2(\lambda^2 - 4\mu)}\left(\lambda + \sqrt{\lambda^2 - 4\mu}\tanh\left(\frac{c_1^2t}{4\sqrt{6c_2}} + \frac{1}{2}(c+x)\sqrt{\lambda^2 - 4\mu}\right)\right)}\right).$$
(39)



Figure 10. 3D surface of Equation (39), which is a dark soliton solution plotted when $b_1 = 0.2$, $\beta = 0.2$, $c_1 = 0.1$, $c_2 = -0.1$, c = 1, $\lambda = 3$, $\mu = 1$, and t = 2 for 2D.

Solution 2. When $\lambda^2 - 4\mu > 0$ and $\mu = 0$, we get hyperbolic function solution:

$$u(x,t) = e^{i\left(-\frac{5c_1^2}{32c_2}t + \frac{\sqrt{\lambda^2 - 4\mu}}{\sqrt{6}}x\right)} \left(\frac{\sqrt{3}\sqrt{c_1}\lambda \coth\left(\frac{1}{24}\lambda\left(12(c+x) + \frac{\sqrt{6}c_1^2}{c_2\sqrt{\lambda^2}}t\right)\right)}{2\sqrt{-c_2\lambda^2}}\right).$$
 (40)

This is a singular soliton solution, as shown in Figure 11.



Figure 11. 3D figure of Equation (40), which is a singular soliton solution plotted when $b_1 = 4$, $\beta = 0.2$, $c_1 = 0.1$, $c_2 = -0.1$, c = 0.2, $\lambda = 1$, $\mu = 0$, and t = 2 for 2D.

4.2. The Cubic-Quartic Resonant Nonlinear Schrödinger Equation

To apply the exp $(-\varphi(\xi))$ expansion method on the cubic-quartic resonant nonlinear Schrödinger equation, Equation (4), we utilize the same wave transformation of Equation (17). As a result, we get Equation (31). Finding the balance, we gain n = 1. Via inserting the value of the balance into Equation (10), we get the same result of Equation (36). Substituting Equation (36) into Equation (31) and setting each summation of the coefficients of the exponential identities of the same power to be zero, we discuss the following cases of solutions.

Case 1. When $b_1 = \frac{2b_0}{\lambda}$, $c_1 = \frac{2\beta\lambda^2(\lambda^2 - 4\mu)}{b_0^2}$, $c_2 = -\frac{3\beta\lambda^4}{2b_0^4}$, $\kappa = \frac{\sqrt{-c_3 + \beta(\lambda^2 - 4\mu)}}{\sqrt{6}\sqrt{\beta}}$, and $\omega = \frac{-c3^2 + 2c3\beta(\lambda^2 - 4\mu) + 5\beta^2(\lambda^2 - 4\mu)^2}{12\beta}$, we get the following solutions:

Solution 1. When $\lambda^2 - 4\mu > 0$ and $\mu \neq 0$, we get hyperbolic function solution:

$$u(x,t) = e^{i\left(-\frac{\sqrt{-c_3 + \beta(\lambda^2 - 4\mu)}}{\sqrt{6}\sqrt{\beta}}x + \frac{\left(-c_3^2 + 2c_3\beta\left(\lambda^2 - 4\mu\right) + 5\beta^2\left(\lambda^2 - 4\mu\right)^2\right)}{12\beta}t\right)} \\ \left(b_0 + \frac{4\mu b_0}{\lambda\left(-\lambda - \sqrt{\lambda^2 - 4\mu}\tanh\left[\frac{1}{2}\left(c + x + \frac{2\sqrt{\frac{2}{3}t(-c_3 + \beta(\lambda^2 - 4\mu))^{3/2}}{3\sqrt{\beta}}\right)\sqrt{\lambda^2 - 4\mu}\right]\right)}\right).$$
(41)

This is a dark soliton solution, as shown in Figure 12.



Figure 12. 3D surface of Equation (41), which is a dark soliton solution plotted when $b_0 = 0.2$, $\beta = 0.2$, $c_3 = 1$, c = 1, $\lambda = 3$, $\mu = 1$, and t = 2 for 2D.

Solution 2. When $\lambda^2 - 4\mu < 0$ and $\mu \neq 0$, we get trigonometric function solution:

$$u(x,t) = e^{i\left(-\frac{x\sqrt{-c_{3}+\beta(\lambda^{2}-4\mu)}}{\sqrt{6}\sqrt{\beta}} + \frac{t\left(-c_{3}^{2}+2c_{3}\beta\left(\lambda^{2}-4\mu\right)+5\beta^{2}\left(\lambda^{2}-4\mu\right)^{2}\right)}{12\beta}\right)} \\ \left(b_{0} - \frac{4\mu b_{0}}{\lambda^{2} - \lambda\sqrt{-\lambda^{2}+4\mu}} \tan\left[\frac{1}{2}\left(c+x + \frac{2\sqrt{\frac{2}{3}}(-c_{3}+\beta(\lambda^{2}-4\mu))^{3/2}}{3\sqrt{\beta}}t\right)\sqrt{-\lambda^{2}+4\mu}\right]}\right).$$
(42)

This is a periodic singular soliton solution, as shown in Figure 13.



Figure 13. 3D surface of Equation (42), which is a periodic singular soliton solution plotted when $b_0 = 0.4$, $\beta = 0.1$, $c_3 = -5$, c = 1, $\lambda = 1$, $\mu = 1$, and t = 2 for 2D.

Solution 3. When $\lambda^2 - 4\mu > 0$ and $\mu = 0$, we get hyperbolic function solution:

$$u(x,t) = e^{i\left(-\frac{x\sqrt{-c_3+\beta\lambda^2}}{\sqrt{6}\sqrt{\beta}} + \frac{t\left(-c_3^2 + 2c_3\beta\lambda^2 + 5\beta^2\lambda^4\right)}{12\beta}\right)} \left(b_0 + \frac{2b_0}{\cosh\left[\lambda\left(c + x + \frac{2\sqrt{\frac{2}{3}}t\left(-c_3+\beta\lambda^2\right)^{3/2}}{3\sqrt{\beta}}\right)\right] + \sinh\left[\lambda\left(c + x + \frac{2\sqrt{\frac{2}{3}}t\left(-c_3+\beta\lambda^2\right)^{3/2}}{3\sqrt{\beta}}\right)\right]}\right).$$
(43)

This is a bright singular combo soliton solution, as shown in Figure 14.



Figure 14. 3D surface of Equation (43), which is a singular soliton solution plotted when $b_0 = 0.4$, $\beta = 0.2, c_3 = -1, c = 0.2, \lambda = 3, \mu = 0$, and t = 2 for 2D.

Case 2. When $b_0 = \frac{\sqrt{2}\sqrt{\beta\lambda^2(\lambda^2 - 4\mu)}}{\sqrt{c_1}}$, $b_1 = \frac{2\sqrt{2}\sqrt{\beta\lambda^2(\lambda^2 - 4\mu)}}{\sqrt{c_1\lambda}}$, $c_2 = -\frac{3c_1^2}{8\beta(\lambda^2 - 4\mu)^2}$, $c_3 = \beta \left(-6\kappa^2 + \lambda^2 - 4\mu\right)$, and $\omega = \frac{1}{2}\beta \left(-6\kappa^4 + (\lambda^2 - 4\mu)^2\right)$, we get the following solutions:

Solution 1. When $\lambda^2 - 4\mu > 0$ and $\mu \neq 0$, we get hyperbolic function solution:

$$u(x,t) = \sqrt{2\beta\lambda^{2}(\lambda^{2}-4\mu)} e^{-i\kappa x + \frac{1}{2}i\beta\left(-6\kappa^{4} + (\lambda^{2}-4\mu)^{2}\right)t}$$

$$\frac{\left(1 - \frac{4\mu}{\lambda^{2} + \lambda\sqrt{\lambda^{2}-4\mu}\tanh\left[\frac{1}{2}(c+x+8\beta\kappa^{3}t)\sqrt{\lambda^{2}-4\mu}\right]}\right)}{\sqrt{c_{1}}}.$$
(44)

This is a dark soliton solution, as shown in Figure 15.



Figure 15. 3D surface of Equation (44), which is a dark soliton solution plotted when $\beta = 3$, $c_1 = 5$, c = 0.03, $\lambda = 1$, $\mu = 0.1$, $\kappa = 0.1$, and t = 2 for 2D.

Solution 2. When $\lambda^2 - 4\mu < 0$ and $\mu \neq 0$, we get trigonometric function solution:

$$u(x,t) = \sqrt{2\beta\lambda^{2}(\lambda^{2}-4\mu)}e^{-i\kappa x+\frac{1}{2}i\beta\left(-6\kappa^{4}+(\lambda^{2}-4\mu)^{2}\right)t}$$

$$\frac{\left(1-\frac{4\mu}{\lambda^{2}-\lambda\sqrt{-\lambda^{2}+4\mu}\tan\left[\frac{1}{2}(c+x+8t\beta\kappa^{3})\sqrt{-\lambda^{2}+4\mu}\right]}\right)}{\sqrt{c_{1}}}.$$
(45)

This is a periodic singular soliton solution, as shown in Figure 16.



Figure 16. 3D surface of Equation (45), which is a periodic singular soliton solution plotted when $\beta = -3$, $c_1 = 5$, c = 0.03, $\lambda = 0.1$, $\mu = 1$, $\kappa = 0.1$, and t = 2 for 2D.

Solution 3. When $\lambda^2 - 4\mu > 0$ and $\mu = 0$, we get the hyperbolic function solution:

$$u(x,t) = \frac{\sqrt{2}e^{-i\kappa x + \frac{1}{2}i\beta\left(-6\kappa^4 + \lambda^4\right)t}\sqrt{\beta\lambda^4}\coth\left(\frac{1}{2}\left(c + x + 8\beta\kappa^3t\right)\lambda\right)}{\sqrt{c_1}},\tag{46}$$

which is a periodic singular solution, as shown in Figure 17.



Figure 17. 3D surface of Equation (46), which is a singular soliton solution plotted when $\beta = 3$, $c_1 = 5$, c = 0.03, $\lambda = 0.1$, $\mu = 0$, $\kappa = 0.1$, and t = 2 for 2D.

5. Conclusions

In this research, the new dark, singular, bright singular combo soliton, and periodic singular solutions of the cubic-quantic nonlinear Schrödinger equation and the cubic-quantic resonant nonlinear Schrödinger equation were shown. Figures 1, 6, 8, 10, 12 and 15 are dark soliton solutions, Figures 2–4, 11, 14 and 17 are singular soliton solutions, Figures 5, 7, 13 and 16 are periodic singular solutions, and Figure 9 is bright singular combo soliton solution. The (m + G'/G) expansion and $\exp(-\varphi(\xi))$ expansion methods were utilized to study these two models with the parabolic law. The new solutions verified the main equations after we substituted them into Equations (3) and (4) for the existence of the equation.

Conte and Musette introduced that wave transformation, which we considered in this paper, protects the Painleve conditions and its properties [48]. Therefore, it can be seen that all results verified their physical properties and presented their estimated wave behaviors. Therefore, one can observe that the wave transformation considered in this paper in Equation (17) satisfies these conditions. We substituted all solutions to the main equations Equations (3) and (4), and they verified it; the constraint conditions Equations (20) and (29) were also used to verify this existence. The optical solution solutions obtained in this research paper may be of concern and useful in many fields of science, such as mathematical physics, applied physics, nonlinear science, and engineering.

Author Contributions: All authors contributed equally.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

Abbreviations

The following abbreviations are used in this manuscript:

NLPDE Nonlinear partial differential equation

ODE Ordinary differential equation

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