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Numerical Simulation of 3D Condensation Nanofluid Film Flow with Carbon Nanotubes on an Inclined Rotating Disk

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Abstract: Here, we discuss three-dimensional dusty nanofluid thin film flow with nonlinear thermal radiation, where carbon nanotubes flow past an inclined rotating disk with a constant angular velocity of Ω . This novel mathematical model is unique and is discussed here for the first time. Downward draining flow and lateral flow arise due to inclination. The demonstrated geometry is characterized in terms of time-independent continuity, momentum, and energy balance. Similarity transformations convert the partial differential equation into a system of ordinary differential equations. The obtained equations are analyzed numerically using the bvp4c MATLAB function. The thermal field of the dust phase was smaller than that of the nanofluid phase, and this difference was exacerbated by increasing the thermal radiation. To validate the model presented here, it is compared to a previous model; the models showed high concordance.

Keywords: dusty carbon nanotubes; thin film; nonlinear thermal radiation; condensation velocity

1. Introduction

Fluids comprised of very small-sized particles or impurities show enhanced thermal properties and are termed dusty fluids. The efficiency of machines is affected by the suspension of dust particles, but dusty fluids are still beneficial for increasing thermal conductivity. Dust particles have been used in scientific and engineering applications including cosmic dust, which is formed by mixing gas and dust particles, as well as dust collection, acoustics, the transportation of suspended powdered materials through pipes, oceanography, sedimentation processes, the transport of slag or slurry, facilitating the flow of blood through arteries, spray cooling, rain erosion, etc. [1]. Farbar and Morley [2] discussed heat transportation via a mixture of gas and solids. Saffman [3] analyzed the stability of the boundary layer flow of dusty gas. Hazem et al. [4] studied heat transfer in the context of unsteady Couette flow of a dusty fluid with an ion slip effect, in the presence of uniform suction/injection. Koneri et al. [5] discussed the influence of nonlinear thermal radiation on Maxwell fluid flow with uniformly distributed dust particles. Sandeep et al. [6] performed radiative flow and heat transfer analysis of a dusty nanofluid with electrically conducting dust particles (Cu, CuO); they studied how increased heat transfer promoted particle interactions. Pop et al. [7] performed a two-dimensional

boundary layer flow analysis of dusty fluids over a shrinking surface. Boundary layer approximation was utilized to formulate the problem, and the solution was obtained using the `bvp4c` MATLAB function. Ghadikolaei et al. [8] performed boundary layer flow analyses of a dusty fluid, together with an analysis of porosity and the magnetic field effect, using titanium oxide nanoparticles. Siddiqua et al. [9] explored the Casson fluid flow of dust particles in a vertical wavy cone, including the heat transfer effect. Gireesha et al. [10] investigated two-phase transient fluid flow with dust particles; they used Kirchhoff's voltage law (KVL) to estimate the viscosity and thermal conductivity. Instead of using only a base fluid or the bulk of the colloidal suspension, heat transfer achieved by suspending small-sized solid particles was investigated to improve thermal conduction. Such fluids are called nanofluids and are formed by the colloidal suspension of nanoparticles that can function as an efficient heat exchanger [11–21]. Heat and mass transfer analysis of dusty fluid flows over various geometries have been reported [22–24]. Experiments have suggested that the capacity for the transportation of heat differs among nanoparticles. The shape and size of nanoparticles have been shown to be important in this regard. The thermal conduction of nanofluids increased by sixfold in the presence of carbon nanotubes (CNTs) in comparison to nanoparticles. CNTs are allotropes of carbon; they have a one-dimensional structure and are cylindrical in shape, which enhances their mechanical and electrical properties [25]. Choi et al. [26] discussed the enhancement of thermal conduction in cylindrical CNTs. Carbon atoms show three types of hybridization. Carbon chains exist within a network, and carbon nanotubes are characterized by a hexagonal arrangement of carbon atoms. CNTs have high thermal conductivity. Here, we considered only single-walled carbon nanotubes (SWCNTs). A literature review revealed no previous studies on dust particles with CNTs.

The heat transfer in fluid flow problems wherein a thin film is stretched over surfaces has been examined by many researchers. These investigations indicted numerous applications, including strand casting, the production of plastic films and sheeting, the drawing of polymer surfaces, thermoplastic coating, condensation, etc. The liquid obtained as a result of condensation plays a vital role in chemical engineering processes [27]. Sparrow and Gregg [28] studied condensation by applying inertial forces (centrifugal force) to a cooled rotating disk. Beckett et al. [29] extended this work by adding vapor drag. Recently, Chary and Sarma [30] studied suction on a plate. Elsewhere, a liquid film was analyzed during a spray-cooling process [31]. The deposition of chemical vapors when a thin film is deposited on a cooled rotating disk was explored in [32]. Wang discussed the effects of film condensation on an inclined rotating disk [33]. The phenomenon of heat transfer during extrusion and coating has intrigued researchers for a long time [34–37].

A literature review revealed few articles discussing nanofluid flow over an inclined rotating disk, especially nanofluid thin film flow past a rotating disk. The present study is unique in that we considered nanofluid thin film flow in the context of SWCNTs amalgamated with dust particles flowing over a rotating disk. The effects of nonlinear thermal radiation with thermal stratification and heat generation/absorption are also considered. The results were analyzed numerically using the `bvp4c` finite-difference MATLAB function. The Nusselt number and the skin friction coefficients are computed and discussed. Finally, the major findings are graphically illustrated, and the numerical values of physical parameters provided.

2. Mathematical Modeling

We assumed a time-independent dusty flow of nanofluid past a rotating disk with an angular velocity of Ω . The disk was inclined at angle β . The dusty particles present in the fluid were assumed to be uniformly distributed. These particles were initially at rest, with a consistent density throughout the stream. Moreover, the particles were uniform in size. A nanofluid film of thickness δ was formed by spraying with velocity α . The disk radius was assumed to be large in comparison to the film thickness, such that the end effects were assumed to be negligible. The force of gravity $\bar{g}$ was acting vertically downward. $\bar{T}_w$ was the temperature of the disk, and $\bar{T}_0$ was the temperature of the surface of the

film. Pressure was taken as a function of z only (Figure 1). The phenomenological laws for CNTs are outlined below.

Figure 1. Geometrical sketch of the model.

The governing equations based on the above suppositions are given by [38,39]:

Nanofluid Phase:

$$\nabla \cdot V = 0, \quad (1)$$

$$(V \cdot \nabla u) = \frac{\mu_{nf}}{\rho_{nf}} (\nabla^2 u) + \tilde{g} \sin\left(\frac{\beta}{\Omega}\right) + \frac{\rho_p}{\rho_{nf} \tau_v} (u_p - u), \quad (2)$$

$$(V \cdot \nabla v) = \left(\frac{\mu_{nf}}{\rho_{nf}}\right) (\nabla^2 v) + \frac{\rho_p}{\rho_{nf}} \tau_v (v_p - v), \quad (3)$$

$$(V \cdot \nabla w) = \left(\frac{\mu_{nf}}{\rho_{nf}}\right) (\nabla^2 w) + \tilde{g} \sin\left(\frac{\beta}{\Omega}\right) + \frac{\rho_p}{\rho_{nf}} \tau_v (w_p - w), \quad (4)$$

$$V \cdot \nabla \tilde{T} = \frac{K_{nf}}{\rho_{nf} C_{nf}} (\nabla^2 \tilde{T}) + \frac{1}{\rho_{nf} C_{nf}} \frac{\partial q_r}{\partial z} + \frac{(\rho_p C_p)}{\rho_{nf} C_{nf} \tau_T} (\tilde{T}_p - \tilde{T}) + \frac{Q_0}{\rho_{nf} C_{nf}} (\tilde{T} - \tilde{T}_\infty). \quad (5)$$

Dust Phase:

$$\nabla \cdot V_p = 0, \quad (6)$$

$$(V \cdot \nabla u_p) = \tilde{g} \sin\left(\frac{\beta}{\Omega}\right) + \frac{1}{\tau_v} (u - u_p), \quad (7)$$

$$(V \cdot \nabla v_p) = \frac{1}{\tau_v} (v - v_p), \quad (8)$$

$$(V \cdot \nabla w_p) = \tilde{g} \sin\left(\frac{\beta}{\Omega}\right) + \frac{1}{\tau_v} (w - w_p), \quad (9)$$

$$V \cdot \nabla \tilde{T} = \frac{1}{\tau_T} (\tilde{T} - \tilde{T}_p), \quad (10)$$

The Rosseland approximation of radiative heat flux q_r is given by:

$$q_r = \frac{16\psi}{3\tilde{k}} \frac{\partial \tilde{T}^4}{\partial z}, \quad (11)$$

where ψ is the Stefan–Boltzmann constant, and $\tilde{k}$ is the average of absorption coefficient. Expanding $\tilde{T}^4$ about $\tilde{T}_0$ with the help of Taylor's series, we have

$$\tilde{T}^4 = \tilde{T}_0^4 + 4(\tilde{T} - \tilde{T}_0)\tilde{T}_0^3 + \dots \quad (12)$$

$$\tilde{T}^4 = \tilde{T}_0^4 + 4\tilde{T}_0^3\tilde{T} - 4\tilde{T}_0^4. \quad (13)$$

Simplifying this, we obtain

$$\tilde{T}^4 = 4\tilde{T}\tilde{T}_0^3 - 3\tilde{T}_0^4. \quad (14)$$

From Equations (11) and (14), we get

$$\frac{\partial q_r}{\partial z} = -\frac{16\tilde{T}_0^3\psi}{3\tilde{k}} \frac{\partial^2 \tilde{T}^4}{\partial z^2}, \quad (15)$$

with boundary conditions

$$u = u_p = -\Omega y, v = v_p = \Omega x, w = w_p = 0, \tilde{T} = \tilde{T}_w = \tilde{T}_0 + ax, \text{ at } z = 0. \quad (16)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u_p}{\partial z} = 0, \frac{\partial v}{\partial z} = \frac{\partial v_p}{\partial z} = 0, w = w_p = -W, \tilde{T} = \tilde{T}_\infty = \tilde{T}_0 + bx, \text{ at } z = h. \quad (17)$$

The thermophysical properties of water and both types of carbon nanotubes i.e., SWCNTs and MWCNTs, are given in Table 1.

Table 1. Thermophysical characteristics of water.

Thermophysical Properties	Base Fluid (H ₂ O)	Single-Walled Carbon Nanotubes (SWCNTs)
$C\left(\frac{J}{kg}\right)$	4179	425
$\rho\left(\frac{kg}{m^3}\right)$	997.1	2600
$k\left(\frac{W}{mk}\right)$	0.613	6600

Similarity Transformation

The following transformations were used to convert the abovementioned nonlinear partial differential equations to ordinary differential equations.

$$\begin{aligned} u &= -\Omega y g(\eta) + \Omega x f'(\eta) + g k(\eta) \sin\left(\frac{\beta}{\Omega}\right), \\ u_p &= -\Omega y G(\eta) + \Omega x F'(\eta) + g K(\eta) \sin\left(\frac{\beta}{\Omega}\right), \\ v &= \Omega x g(\eta) + \Omega y f'(\eta) + g s(\eta) \sin\left(\frac{\beta}{\Omega}\right) \\ v_p &= \Omega x G(\eta) + \Omega y F'(\eta) + g S(\eta) \sin\left(\frac{\beta}{\Omega}\right) \\ w &= -2\sqrt{\Omega v_{nf}} f(\eta), \theta(\eta) = \frac{\tilde{T} - \tilde{T}_w}{\tilde{T}_0 - \tilde{T}_\infty} \\ w_p &= -2\sqrt{\Omega v_{nf}} F(\eta), \theta_p(\eta) = \frac{\tilde{T}_p - \tilde{T}_w}{\tilde{T}_0 - \tilde{T}_\infty} \end{aligned} \quad (18)$$

$$\eta = z \sqrt{\frac{\Omega}{v_{nf}}} \quad (19)$$

The phenomenological laws for CNTs are as follows [38]:

$$\mu_{nf} = \frac{\mu_f}{(1-\phi)^{2.5}}, \quad (20)$$

$$A_1 = \frac{\rho_{nf}}{\rho_f} = (1-\phi) + \frac{(\rho_{CNT})}{\rho_f} \phi, \quad (21)$$

$$A_2 = \frac{\rho_{nf} C_{nf}}{\rho_f C_f} = (1-\phi) + \frac{\rho_{CNT} C_{CNT}}{\rho_f C_f} \phi \quad (22)$$

$$A_3 = \frac{K_{nf}}{K_f} = \frac{(1-\phi) + 2\phi \frac{k_{CNT}}{k_{CNT} - k_f} \ln \frac{k_{CNT} + k_f}{2k_f}}{(1-\phi) + 2\phi \frac{k_f}{k_{CNT} - k_f} \ln \frac{k_{CNT} + k_f}{2k_f}}. \quad (23)$$

Equation (1) holds, and Equations (2)–(9) are transformed into Equations (24)–(33).

$$f'^2 - 2ff'' - g^2 - \frac{\beta_v}{A_1} l(F' - f') = f''' \quad (24)$$

$$2(f'g - fg') - \frac{\beta_v}{A_1} l(G - g) = g'' \quad (25)$$

$$kf' - 2fk' - 1 - \frac{\beta_v}{A_1} l(K - k) = k'', \quad (26)$$

$$kg + sf' - 2s'f - \frac{\beta_v}{A_1} l(S - s) = s'', \quad (27)$$

$$\left(\frac{A_1 A_3}{A_2 Pr} - \frac{A_1 Rd}{A_2 Pr} \right) \theta'' - (S_1 + \theta) f' + 2f\theta' + \frac{\delta \theta}{A_2} - \frac{A_1 Rd}{A_2 Pr} \left[\frac{3\theta'^2}{\theta_w + S_1} \left(1 + \frac{1}{\theta_w + S_1} \right)^2 \right] + \beta_t \gamma l(\theta - \theta_p) = 0 \quad (28)$$

Dusty Fluid:

$$F'^2 - 2FF'' - G^2 - \beta_v(f' - F') = 0, \quad (29)$$

$$2(F'G - FG') - \beta_v(g - G) = 0, \quad (30)$$

$$KF' - GS - 2FK' - \beta_v(k - K) = 0, \quad (31)$$

$$KG + SF' - 2S'F - \beta_v(s - S) = 0, \quad (32)$$

$$(S_1 + \theta_p)F' - 2\theta_p' - \beta_t(\theta_p - \theta) = 0, \quad (33)$$

The boundary conditions after transformation are

$$\begin{aligned} f(0) = 0, f'(0) = 0, g(0) = 1, k(0) = 0, s(0) = 0, S(0) = 0, \theta(0) = 1 - S_1, \\ f''(\delta) = 0, F''(\delta) = 0, k'(\delta) = 0, K'(\delta) = 0, s'(\delta) = 0, \theta(\delta) = 0, \theta_p(\delta) = 0. \end{aligned} \quad (34)$$

where δ is the normalized thickness constant.

$$\delta = h \sqrt{\frac{\Omega}{v_{nf}}} \quad (35)$$

The condensation or spraying velocity is defined as

$$f(\delta) = \frac{W}{2\sqrt{\Omega v_{nf}}} = \alpha. \quad (36)$$

Here, the dimensionless parameters are defined as:

$$Rd = \frac{16\psi\tilde{T}_0^3}{3k_f\tilde{k}}, Pr = \frac{\mu_f C_f}{k_f}, \beta_v = \frac{1}{\tau_v\Omega}, S_1 = \frac{b}{a}, \beta_t = \frac{1}{\tau_T\Omega}, \gamma = \frac{C_p}{C_f}, \delta_1 = \frac{Q_0}{\rho_f C_f \Omega}, l = \frac{\rho_p}{\rho_f} \quad (37)$$

Skin Friction and Local Nusselt Number

The drag force is:

$$Re^{1/2}C_f = \frac{\mu_{nf}\left(\frac{\partial u}{\partial y}\right)_{y=0}}{\rho_f W^2} = 2\alpha \frac{f''(0)}{(1-\phi)^{2.5}}. \quad (38)$$

The non-dimensional form of Nu (the local Nusselt number) is:

$$Nu = \frac{k_{nf}\left(\frac{\partial \tilde{T}}{\partial z}\right)_w}{k_f(\tilde{T}_0 - \tilde{T}_w)} = \delta_1 A_3 \theta'(0). \quad (39)$$

3. Results and Discussion

The formulated problem (Equations (24)–(33)) and boundary conditions (Equation (34)) were treated numerically using the `bvp4c` function because of its efficiency and accuracy. The function was applied to the three-stage Lobatto IIIa formula, which gave fourth-order C^1 continuous solutions depending on the collocation formula. The residual of the continuous solution controls the error and adjusts the mesh size. Figures 2–12 show the effects of various parameters on the temperature and velocity profiles. Here, the solid line shows the thermal and flow field profiles for the dusty phase, and the dashed line shows those for the nanofluid phase. The ranges of the parameters were taken as $0.2 \leq \beta_v \leq 1.3$, $0.01 \leq \phi \leq 0.03$, $0.01 \leq \delta \leq 0.03$, $0.50 \leq l \leq 0.52$, $0.50 \leq \alpha \leq 0.70$, $1.2 \leq Rd \leq 1.4$, $0.5 \leq \theta_w \leq 0.7$, and $0.5 \leq \delta_1 \leq 1.5$. Figure 2 shows the impact of the momentum dust parameter β_v on the fluid with dusty particles, while keeping the mass concentration of the dust particles fixed as $l = 0.5$, $\delta = 0.5$, $Rd = 0.5$, and $\theta_w = 0.5$. Figure 2 shows the moment the dust phase is aggrandized for increases in β_v . The velocity increased until it approached the nanofluid velocity. This increase occurred because the dust particles had a higher momentum. Figure 3 shows the impact of nanoparticle volume fraction on the velocity profile for the nanofluid and dust phases. Increasing the nanoparticle volume fraction causes an increase in particle–particle interactions, which lowers the velocity. Moreover, due to the presence of dust particles, the decrease in velocity was greater than in the nanofluid phase. Figure 4 illustrates the impact of film thickness on the velocity profile with constant spraying or condensation velocity; it shows that the velocity increased with increasing film thickness, but because of the presence of dusty particles, the increase in velocity of the dusty fluid was lower compared to that of the nanofluid. The velocity decreased with the increasing mass concentration of the dusty particles for the dusty and nanofluid phases, as shown in Figure 5. The additional dust particles created a resistive force that influenced the speed of the fluid. Figure 6 shows the draining velocity behavior for various spraying velocities in the nanofluid and dusty phases. The inclination of the disk causes gravitational flow. Gravity causes the downward flow of the fluid. The free surface flow on an inclined plane will be strongly affected by the spray rate or rotation of the disk. An influence of the momentum of the dust on the velocity of the nanofluid and dusty phases, $g(\eta)$ and $G(\eta)$, respectively, is observed. It can be seen in the figure that the velocity of dusty phase $G(\eta)$ showed a smaller increase with increasing dust momentum in comparison to the nanofluid phase. Increasing the dust momentum increased the

momentum of the nanoparticles. Figure 8 shows that the temperature profiles of the dust phase θ and fluid phase θ_p increase with increasing thermal radiation. This behavior is visualized because of the increasing thermal radiation; radiation is absorbed into the system, which increases the thermal boundary layer of the nanofluid and dusty fluid. Figure 9 shows that wall temperature θ_w increases because it is the ratio of the wall temperature to the ambient temperature; as a result, the temperature will also increase. The effects of heat source parameter δ_1 on θ and θ_p are presented in Figure 10. The increased temperature occurred because nanoparticles enhance the temperature of the nanofluid; when a heat source is also present in the system, it will cause a greater increase in the temperature of the nanofluid. Figure 10 shows that the presence of dust particles dampened the increase in temperature compared to the nanofluid phase. Figure 11 shows the effects of thermal stratification on the temperature profiles of θ_p and θ . Increased thermal stratification caused an increase in the density of the fluid, and caused it to divide into lower- and higher-density fluid regions. The higher-density fluid was in the lower region and that with less density was in the upper region. In the case of dusty particles, the influence of the stratification parameter is decreased to be less than that of the nanofluid phase. Figure 12 shows the impact of the nanoparticle volume fraction on the temperature profile. Increasing the nanoparticle volume fraction caused an increase in particle–particle interactions, such that the thermal boundary layer thickness also increased, leading to a rise in temperature.

Figure 2. Impact of β_v with dusty particles on velocity profile F' .

Figure 3. Impact of ϕ on velocity profile f' and F' of fluid with dusty particles.

Figure 4. Impact of δ on velocity profile f' and F' of fluid with dusty particles.

Figure 5. Impact of l on velocity profile f' and F' of fluid with dusty particles.

Figure 6. Impact of α on draining velocity k and K of fluid with dusty particles.

Figure 7. Impact of β_v on lateral velocity profile g and G of fluid with dusty particles.

Figure 8. Impact of Rd on temperature profile θ and θ_p of fluid with dusty particles.

Figure 9. Impact of θ_w on temperature profile θ and θ_p of fluid with dusty particles.

Figure 10. Impact of δ_1 on temperature profile θ and θ_p of fluid with dusty particles.

Figure 11. Impact of S_1 on temperature profile θ and θ_p of fluid with dusty particles.

Figure 12. Impact of ϕ on temperature profile θ and θ_p of fluid with dusty particles.

Table 2 shows the numerical values for the rate of heat transfer as according to the Prandtl number $Pr, \gamma, \theta_w, Rd, \beta_t$, while the values of $\delta_1 = 0.5, S_t = 0.5, S_1 = 0.5$ remain fixed. The ratio of momentum diffusivity to thermal diffusivity is designated as the Pr . The enhancement of Pr caused a decrease in thermal diffusion, leading to a reduction in the temperature of the fluid. Bilal et al. [38] reported that increasing the thermal dust content had a negligible increasing effect on shear stress. Similarly, a negligible increase in the heat transfer rate was seen with increases in thermal dust content. Increasing γ, θ_w, Rd increased the heat transfer rate, as shown in Table 2. Table 3 shows the drag force by dust momentum, spraying velocity, and dust particle impact mass concentration. It is clear that increasing the dust momentum, spraying velocity, skin friction coefficient, and dust particle mass concentration causes the drag force to decrease. Table 4 shows the values of $f(\eta)$ versus different estimates of η in the absence of a drag force or thermal interactions between fluid particles. Excellent concordance between the values was achieved, thus validating our model.

Table 2. Numerical values of Nusselt number against the different values of parameters, i.e., $\delta_1 = 0.5, S_t = 0.5$, and $S_1 = 0.5$.

Pr	γ	θ_w	Rd	β_t	$Nu = \delta_1 A_3 \theta'(0)$
6.9	0.5	0.5	0.5	0.5	0.2471
7.0					0.2554
7.1					0.2639
		0.6	0.5	0.5	0.3168
					0.3894
					0.4650
		0.7	0.5	0.5	0.3903
					0.5936
					0.9109
		0.8	0.5	0.5	0.9767
					0.6708
					0.4349
		0.6	0.6	0.6	0.2417
					0.2435
					0.24530

Table 3. Numerical values of drag force $C_f Re_x^{1/2}$ against the distinct estimates of parameters. i.e., $\delta_1 = 0.5$, $S_t = 0.5$, and $S_1 = 0.5$.

γ	β_v	l	α	$Re^{\frac{1}{2}} C_f = 2\alpha f''(0)/(1-\phi)^{2.5}$
1.2	0.5	0.1	0.1	0.1057
1.3				0.1014
1.4				0.0988
	0.7			0.5857
	0.8			0.5890
	0.9			0.5900
		0.2		0.0873
		0.3		0.0431
			0.2	0.6789
			0.3	0.9191

Table 4. Comparative values between the numerical and analytical solution (HAM) for $f(\eta)$ versus varied estimates of η with Zahir et al. [39].

η	HAM Result	Numerical Result	Present
0	0.000000	-1.739080×10^{-9}	-1.739080×10^{-9}
0.1	0.004739	0.004707	0.004730
0.2	0.018245	0.018116	0.018234
0.3	0.039447	0.039159	0.039439
0.4	0.067284	0.066775	0.067252
0.5	0.100707	0.099923	0.100699
0.6	0.138694	0.137585	0.138685
0.7	0.180252	0.178775	0.180239
0.8	0.224425	0.222546	0.224386
0.9	0.270298	0.267994	0.270284
1.0	0.316999	0.314258	0.316991

4. Conclusions

This article described a numerical study of nanofluid thin film dusty flow, where SWCNTs flowed past an inclined rotating disk. Equations were analyzed using bvp4c. The main observations can be summarized as follows:

- The effect of dust momentum on the velocity profile and wall shear stress increased with greater dust particle momentum.
- The thermal dust parameter increased the Nusselt number, which decreased with increases in the ratio of wall temperature to ambient temperature.
- The presence of dust particles increased the temperature and velocity boundary layer.
- The velocity profile and rotational coefficient showed a reciprocal trend.
- With increases in radiation, the thermal and flow field of nanofluid increased compared to those of dusty fluid.

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Abbreviations

β_v	Momentum dust parameter
β_t	Thermal dust parameter
q_r	Radiative heat flux $\text{W}\cdot\text{m}^{-2}$
Rd	Radiation parameter
Pr	Prandtl number
Ω	Angular velocity (sec^{-1})
Nu	Nusselt number
l	Mass concentration of dust phase
S_1	Thermal stratification parameter
τ_t	Thermal relaxation time of dust phase
τ_v	Momentum relaxation time of dust phase
Q_0	Heat generation absorption coefficient (m^2s^{-2})
μ_f	Dynamic viscosity of fluid ($\text{Pa}\cdot\text{s}$)
μ_{nf}	Dynamic viscosity of nanofluid ($\text{Pa}\cdot\text{s}$)
ρ_f	Density of the fluid (kgm^{-3})
k_{nf}	Thermal conductivity of the nanofluid ($\text{W}\cdot\text{m}^{-1}$)
ρ_{nf}	Density of the nanofluid (kgm^{-3})
k_{CNT}	Thermal conductivity of carbon nanotubes (W/mk)
k_f	Thermal conductivity of the fluid (W/mk)
δ	Density of nanofluid (kgm^{-3})
δ_1	Heat source parameter
γ	Specific heat ratio
θ_w	Temperature ratio parameter (K)
α	Condensation or normalized velocity (m^2/s)
$V(u, v, w)$	Velocity vector (ms^{-1})
$V_p(u_p, v_p, w_p)$	Velocity vector for dust phase (ms^{-1})
ρ_p	Density of dust particles (kgm^{-3})
$\bar{T}$	Temperature of the fluid (K)
$\bar{T}_p$	Temperature of the fluid for dust phase (K)
Ψ	Stefan–Boltzmann constant ($\text{W}\cdot\text{m}^{-2}\cdot\text{K}^{-4}$)
$\bar{k}$	Average absorption coefficient (m^{-1})
$\bar{T}_w$	Temperature of the disk (K)
$\bar{T}_0$	Temperature of the surface of the film (K)
$\bar{T}$	Ambient fluid temperature (K)
β	The angle of inclination (radians)
φ	Nanoparticle volume fraction
$\bar{g}$	Gravitational acceleration (m/s^2)

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