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Analytical Solutions of (2+Time Fractional Order) Dimensional Physical Models, Using Modified Decomposition Method

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Abstract: In this article, a new analytical technique based on an innovative transformation is used to solve (2+time fractional-order) dimensional physical models. The proposed method is the hybrid methodology of Shehu transformation along with Adomian decomposition method. The series form solution is obtained by using the suggested method which provides the desired rate of convergence. Some numerical examples are solved by using the proposed method. The solutions of the targeted problems are represented by graphs which have confirmed closed contact between the exact and obtained solutions of the problems. Based on the novelty and straightforward implementation of the method, it is considered to be one of the best analytical techniques to solve linear and non-linear fractional partial differential equations.

Keywords: Shehu transformation; Adomian decomposition; analytical solution; Caputo derivatives; (2+time fractional-order) dimensional physical models

1. Introduction

Fractional calculus is considered to be a powerful tool for modeling complex phenomenon. Recently, the researchers have shown the greatest interest towards fractional calculus because of its numerous applications in different fields of sciences. Despite complicated background of fractional calculus, it came into being from simple question of L'Hospital. The first order represent slope of a function, what will it represent for fractional order $(\frac{1}{2})$? To find the answer of this question, the mathematicians have managed to open a new window of opportunities to improve the mathematical modeling of real world problems, which has given birth to many new questions and intriguing results. These newly established results have numerous implementation in many areas of engineering [1,2], such as fractional-order Buck master and diffusion problems [3], fractional-order telegraph model [4,5], fractional KdV-Burger-Kuramoto equation [6], fractal vehicular traffic flow [7], fractional Drinfeld-Sokolov-Wilson equation [8], fractional-order anomalous sub-diffusion model [9], fractional design of hepatitis B virus [10], fractional modeling chickenpox disease [11], fractional blood ethanol concentration model [12], fractional model for tuberculosis [13], fractional vibration equation [14], fractional Black-Scholes option pricing equations [15], fractionally damped beams [16], fractionally damped coupled system [17], fractional-order heat, wave and diffusion equations [18,19], fractional order pine wilt disease model [20], fractional diabetes model [21] etc.

Nowadays, the focus of the researchers is to develop different numerical and analytical techniques for the solution of fractional-order models. Therefore, different types of analytical and numerical methods have been developed and used for the solution of different fractional models. The analytical algorithm, the history of integral transform traced back to the time when Laplace started work an integral transform in 1780*s* and Joseph Fourier in 1822. Integral transformations are without question one of the most useful and effective methods in theoretical and applied mathematics, with numerous uses in quantum physics, mechanical engineering and several other areas of science. Moreover, the integral transform is used in chemistry, architecture, and other social sciences to evaluate various models [22]. In recent years, different integral transform such as Laplace transform [23–25] , Fourier transform [26,27], Hankel transform [28], Mellin transform [29], Z-transform [30], Wavelet transform [31], Elzaki transform [32,33], Mahgoub transform [34], Aboodh transform [35], Mohand transform [36], Sumudu transform [37,38], Hermite transform [39] etc have been used for the solution of different physical models.

Originality of the paper: In this article, we have applied a new analytical technique, which is based on generalization of sumudu and laplace transform with Adomian decomposition method (ADM) to solve (2+time fractional-order) dimensional physical models. In the present research we have analyzed the fractional view of some important physical problems by using Shehu decomposition method (SDM). Some important fractional-order problems are solved, which provide the best information about the targeted physical problems as compare to integer-order problems solution. The results of the integer-order problem are compared with the fractional-order problems. In conclusion, in the present research work, we provided and improved the existing physical models of integer-order by using the idea of fractional calculus. The modified mathematical models of fractional-order derivative are solved by using a new and sophisticated analytical method. Moreover, the proposed analytical method has provided the solutions of the problems that have a very close contact with the exact solutions of the problems. The methodology can be extended towards other fractional-order partial differential equations, that are frequently occurred in science and engineering.

The rest of the paper is organized as: In Section 2, we presented the basic definitions and theorem of the proposed method. In Section 3, we have discussed the implementation of proposed transformation. In Section 4 we evaluated the numerical examples by using the proposed technique and discussed the plots. In Section 5 we lastly summarized our results.

2. Preliminaries Concepts

In this section, we present some fundamental and appropriate definitions and preliminary concepts related to the fractional calculus and the Shehu transformation.

Definition 1. *Shehu transform*

Shehu transformation is new and similar to other integral transformation which is defined for functions of exponential order [40]. We take a function in the set A define by

$$A = \{u(\tau) : \exists, \rho_1, \rho_2 > 0, |u(\tau)| < Me^{\frac{|\tau|}{\rho_i}}, \ if \ \ \tau \in [0, \infty), \tag{1}$$

The Shehu transform which is represented by $S(.)$ for a function $u(\tau)$ is defined as

$$S\{u(\tau)\} = V(s, \mu) = \int_0^{\infty} u(\tau) e^{\frac{-s\tau}{\mu}} u(\tau) d\tau, \ \ \tau > 0, \ \ s > 0. \tag{2}$$

The Shehu transform of a function $u(\tau)$ is $V(s,\mu)$: then $u(\tau)$ is called the inverse of $V(s,\mu)$ which is expressed as

$$S^{-1}\{V(s,\mu)\} = u(\tau), \ \ for \ \ \tau \geq 0, \ \ S^{-1} is \ \ inverse \ \ Shehu \ \ transform. \tag{3}$$

Definition 2. *Shehu transform for nth derivatives*
The Shehu transformation for nth derivatives is defined as [40]

$$S\left\{u^{(n)}(\tau)\right\} = \frac{s^n}{\mu^n}V(s,\mu) - \sum_{k=0}^{n-1}\left(\frac{s}{\mu}\right)^{n-k-1} u^{(k)}(0). \tag{4}$$

Definition 3. *Caputo operator of fractional partial derivative*
The fractional Caputo operator is represented as [41]

$$D_\tau^\beta f(\tau) = \begin{cases} \frac{\partial^n f(\tau)}{\partial \tau^n}, & \beta = n \in N, \\ \frac{1}{\Gamma(n-\beta)}\int_0^\tau (\tau-\phi)^{n-\beta-1} f^{(n)}(\phi)\partial\phi, & n-1 < \beta \leq n, \quad n \in N. \end{cases} \tag{5}$$

Definition 4. *Shehu transform for fractional order derivatives*
The Shehu transformation for the fractional order derivatives is expressed as

$$S\left\{u^{(\beta)}(\tau)\right\} = \frac{s^\beta}{\mu^\beta}V(s,\mu) - \sum_{k=0}^{n-1}\left(\frac{s}{\mu}\right)^{\beta-k-1} u^{(k)}(0), \quad 0 < \beta \leq n, \tag{6}$$

In Table 1 show different special functions of Shehu transformation.

Table 1. The Shehu transform of some special functions.

Functional Form	Shehu Transform Form
1	$\frac{u}{s}$
t	$\frac{u^2}{s^2}$
e^τ	$\frac{u}{s-au}$
$\sin(\tau)$	$\frac{u^2}{s^2+u^2}$
$\cos(\tau)$	$\frac{us}{s^2+u^2}$
$\frac{\tau^n}{n!}$ for $n = 0,1,2,\cdots$	$(\frac{u}{s})^{n+1}$
$\frac{\tau^n}{\Gamma(n+1)}$ for $n = 0,1,2,\cdots$	$(\frac{u}{s})^{n+1}$

Theorem 1. *If the function $u(\tau)$ is piecewise continues at every finite interval of $0 \leq \tau \leq \beta$ and of exponential order α for $\tau > \beta$, then there's the Shehu transform $u(s,\mu)$ [40].*

Proof. For any natural number β, we deduct algebraically:

$$\int_0^\infty exp(-\frac{s\tau}{\mu})u(\tau)d\tau = \int_0^\beta exp(-\frac{s\tau}{\mu})u(\tau)d\tau + \int_\beta^\infty exp(-\frac{s\tau}{\mu})u(\tau)d\tau, \tag{7}$$

since the function $u(\tau)$ continues in a piecewise manner at every finite interval $0 \leq \tau \leq \beta$, there's the first integral on the right hand side. We suggest the following situation to validate this statement,

$$
\begin{aligned}
&|\int_{\alpha}^{\infty} exp(-\frac{s\tau}{\mu})u(\tau)d\tau| \leq \int_{\alpha}^{\infty} |exp(-\frac{s\tau}{\mu})u(\tau)|d\tau \\
&\leq \int_{\alpha}^{\infty} exp(-\frac{s\tau}{\mu})|u(\tau)|d\tau \\
&\leq \int_{\alpha}^{\infty} exp(-\frac{s\tau}{\mu})Nexp(\beta\tau)d\tau \\
&= N\int_{\alpha}^{\infty} exp(-\frac{(s-\beta u)\tau}{u})d\tau \\
&= -\frac{Nu}{(s-\beta u)} \lim_{\gamma\longrightarrow\infty} \left[exp(-\frac{(s-\beta u)\tau}{u})d\tau\right]_0^{\gamma} \\
&= \frac{Nu}{(s-\beta u)}.
\end{aligned} \tag{8}
$$

The proof is complete. □

3. Implementation of Shehu Transform

In this section, we have considered a time fractional (2+time fractional-order) dimensional physical model in the form

$$u_{\tau}^{\beta}(\Im,\Re,\tau) = \kappa u_{\Im\Im}(\Im,\Re,\tau) + \pounds u(\Im,\Re,\tau) + \aleph u(\Im,\Re,\tau), \quad \beta \in [1,2] \tag{9}$$

with initial condition

$$u(\Im,\Re,0) = u(\Im,\Re), \tag{10}$$

while κ is a non-linear operator and £ linear operator.

Applying the Shehu transform to both sides of the Equation (9) we obtain

$$S\left\{u_{\tau}^{\beta}(\Im,\Re,\tau)\right\} = S\left\{\kappa u_{\Im\Im}(\Im,\Re,\tau) + \pounds u(\Im,\Re,\tau) + \aleph u(\Im,\Re,\tau)\right\}, \quad \beta \in [1,2]. \tag{11}$$

Using the differential property of Shehu transformation we have,

$$\frac{s^{\beta}}{\mu^{\beta}}\left\{V(s,\mu) - \frac{\mu}{s}u(0) - \frac{\mu^2}{s^2}u^{'}(0)\right\} = S\left\{\kappa u_{\Im\Im}(\Im,\Re,\tau) + \pounds u(\Im,\Re,\tau) + \aleph u(\Im,\Re,\tau)\right\}. \tag{12}$$

Simplifying Equation (12), we obtain

$$V(s,\mu) = +\frac{\mu^{\beta}}{s^{\beta}}S\left\{\kappa u_{\Im\Im}(\Im,\Re,\tau) + \pounds u(\Im,\Re,\tau) + \aleph u(\Im,\Re,\tau)\right\} + \frac{\mu}{s}u(0) + \frac{\mu^2}{s^2}u^{'}(0). \tag{13}$$

Applying the inverse Shehu transformation, we get

$$u(\Im,\Re,\tau) = S^{-1}\left\{\frac{\mu^{\beta}}{s^{\beta}}S\left\{\kappa u_{\Im\Im}(\Im,\Re,\tau) + \pounds u(\Im,\Re,\tau) + \aleph u(\Im,\Re,\tau)\right\}\right\} + u(0) + \tau u^{'}(0). \tag{14}$$

The nonlinear term $\aleph u(\Im,\Re,\tau)$ is evaluated by using the procedure of Adomian polynomial decomposition given by

$$\aleph u(\Im,\Re,\tau) = \sum_{m=0}^{\infty} A_m(u_0,u_1,\cdots), \ \ m = 0,1,\cdots \tag{15}$$

where,

$$A_m(u_0,u_1,\cdots) = \frac{1}{m!}\left[\frac{d^m}{d\lambda^m}\aleph\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\right]_{\lambda=0}, \quad m > 0. \tag{16}$$

With the help of Equation (16), Equation (15) can be written as

$$u(\Im,\Re,\tau) = S^{-1}\left\{\frac{\mu^\beta}{s^\beta}S\left\{\kappa u_{\Im\Im}(\Im,\Re,\tau) + \pounds u(\Im,\Re,\tau) + \sum_{m=0}^{\infty} A_m\right\}\right\} + u(0) + \tau u'(0). \tag{17}$$

Finally, we obtain the recursive relation as

$$\begin{aligned} &u_0(\Im,\Re,\tau) = u(0) + \tau u'(0),\ \ m = 0 \\ &u_m(\Im,\Re,\tau) = S^{-1}\left\{\frac{\mu^\beta}{s^\beta}S\left\{\kappa u_{(m-1)\Im\Im}(\Im,\Re,\tau) + \pounds u_{(m-1)}(\Im,\Re,\tau) + \aleph u_{(m-1)}(\Im,\Re,\tau) + \sum_{m=0}^{\infty} A_m\right\}\right\},\ \ m \geq 1. \end{aligned} \tag{18}$$

4. Applications and Discussion

Example 1. *Consider the (2+time fractional-order) dimensional hyperbolic wave model:*

$$u_\tau^\beta(\Im,\Re,\tau) = \frac{1}{12}\Im^2 u_{\Im\Im}(\Im,\Re,\tau) + \frac{1}{12}\Re^2 u_{\Re\Re}(\Im,\Re,\tau), \quad \beta \in (1,2) \tag{19}$$

with initial conditions

$$u(\Im,\Re,0) = \Im^4, \qquad u_\tau(\Im,\Re,0) = \Re^4. \tag{20}$$

If $\beta = 2$, then the exact solution of Equation (19) is

$$u(\Im,\Re,\tau) = \Im^4\sinh(t) + \Re^4\cosh(t), \tag{21}$$

Taking the Shehu transform of Equation (19) we obtain

$$\frac{s^\beta}{\mu^\beta}\left\{V(s,\mu) - \frac{\mu}{s}u(0) - \frac{\mu^2}{s^2}u'(0)\right\} = S\left\{\frac{1}{12}\Im^2 u_{\Im\Im}(\Im,\Re,\tau) + \frac{1}{12}\Re^2 u_{\Re\Re}(\Im,\Re,\tau)\right\}. \tag{22}$$

Simplifying Equation (22), we get

$$V(s,\mu) = \frac{\mu^\beta}{s^\beta}S\left\{\frac{1}{12}\Im^2 u_{\Im\Im}(\Im,\Re,\tau) + \frac{1}{12}\Re^2 u_{\Re\Re}(\Im,\Re,\tau)\right\} + \frac{\mu}{s}u(0) + \frac{\mu^2}{s^2}u'(0). \tag{23}$$

Applying inverse Shehu transform, we get

$$u(\Im,\Re,\tau) = u(0) + u'(0)\tau + S^{-1}\left\{\frac{\mu^\beta}{s^\beta}S\left\{\frac{1}{12}\Im^2 u_{\Im\Im}(\Im,\Re,\tau) + \frac{1}{12}\Re^2 u_{\Re\Re}(\Im,\Re,\tau)\right\}\right\}. \tag{24}$$

Thus we get the following recursive scheme

$$u_0(\Im,\Re,\tau) = u(0) + u'(0)\tau = \Im^4 + \Re^4\tau, \tag{25}$$

$$u_{m+1}(\Im,\Re,\tau) = S^{-1}\left\{\frac{\mu^\beta}{s^\beta}S\left\{\frac{1}{12}\Im^2 u_{m\Im\Im}(\Im,\Re,\tau) + \frac{1}{12}\Re^2 u_{m\Re\Re}(\Im,\Re,\tau)\right\}\right\}. \tag{26}$$

Using Equation (26), for $m = 0,1,2,3,\cdots$ we get the following values

$$
\begin{aligned}
u_1(\Im,\Re,\tau) &= \Im^4\frac{\tau^{\beta}}{\beta!}+\Re^4\frac{\tau^{\beta+1}}{(\beta+1)!},\\
u_2(\Im,\Re,\tau) &= \Im^4\frac{\tau^{2\beta}}{(2\beta)!}+\Re^4\frac{\tau^{2\beta+1}}{(2\beta+1)!},\\
u_3(\Im,\Re,\tau) &= \Im^4\frac{\tau^{3\beta}}{(3\beta)!}+\Re^4\frac{\tau^{3\beta+1}}{(3\beta+1)!},\\
u_4(\Im,\Re,\tau) &= \Im^4\frac{\tau^{4\beta}}{(4\beta)!}+\Re^4\frac{\tau^{4\beta+1}}{(4\beta+1)!},\\
&\vdots
\end{aligned}
\tag{27}
$$

Now using the values of $u_0, u_1, u_2, u_3, \cdots$, we get Shehu transformation solution for example 1

$$
u(\Im,\Re,\tau) = \Im^4+\Re^4\tau+\Im^4\frac{\tau^{\beta}}{\beta!}+\Re^4\frac{\tau^{\beta+1}}{(\beta+1)!}+\Im^4\frac{\tau^{2\beta}}{(2\beta)!}+\Re^4\frac{\tau^{2\beta+1}}{(2\beta+1)!}+\Im^4\frac{\tau^{3\beta}}{(3\beta)!}+
\Re^4\frac{t^{3\beta+1}}{(3\beta+1)!}+\Im^4\frac{\tau^{4\beta}}{(4\beta)!}+\Re^4\frac{\tau^{4\beta+1}}{(4\beta+1)!}+\cdots
\tag{28}
$$

After simplification, we get

$$
u(\Im,\Re,\tau) = \Im^4\left\{1+\frac{\tau^{\beta}}{\beta!}+\frac{\tau^{2\beta}}{(2\beta)!}+\frac{\tau^{3\beta}}{(3\beta)!}+\frac{\tau^{4\beta}}{(4\beta)!}+\cdots\right\}+\Re^4\left\{\tau+\frac{\tau^{\beta+1}}{(\beta+1)!}+\frac{\tau^{2\beta+1}}{(2\beta+1)!}+
\frac{\tau^{3\beta+1}}{(3\beta+1)!}+\frac{\tau^{4\beta+1}}{(4\beta+1)!}+\cdots\right\}.
\tag{29}
$$

In particular, when $\beta \to 2$, the analytical solution of Shehu transform become as

$$
u(\Im,\Re,\tau) = \Im^4\left\{1+\frac{\tau^2}{2!}+\frac{\tau^4}{(4)!}+\frac{\tau^6}{(6)!}+\frac{\tau^8}{(8)!}+\cdots\right\}+\Re^4\left\{\tau+\frac{\tau^3}{(3)!}+\frac{\tau^5}{(5)!}+\frac{\tau^7}{(7)!}+\frac{\tau^9}{(9)!}+\cdots\right\},
\tag{30}
$$

which provide the close form solution as

$$
u(\Im,\Re,\tau) = \Im^4\cosh(\tau)+\Re^4\sinh(\tau).
\tag{31}
$$

Figures 1 and 2 represent the exact and analytical solutions of Example 1. The solutions-graphs have confirmed the closed contact between the exact solution and the analytical solution obtained by the proposed method. In Figure 3, the solution of Example 1 are calculated at different fractional-order β of the derivative. It is investigated that the solutions at different fractional-orders β are convergent to an integer-order solution of Example 1. Figure 4 represent the solution verses time graph for Example 1. It is observed that as the time fractional-order varies toward time integer-order, the time fractional-order solutions also approaches to the solution of an integer-order problem of Example 1. All the above solution analysis of Example 1 indicate that SDM is an efficient and effective method to solve fractional-order partial differential equations that are frequently arising in science and engineering.

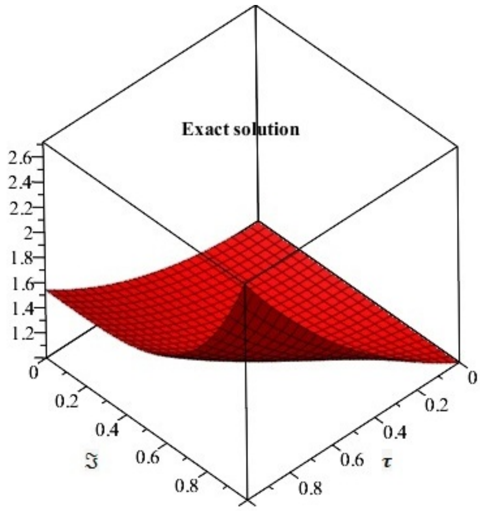

Figure 1. Represents the exact solution of Example 1 at $\beta = 2$.

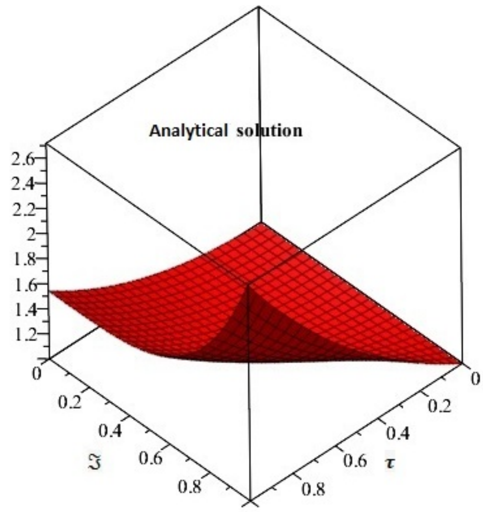

Figure 2. Represents the analytical solution of Example 1 at $\beta = 2$.

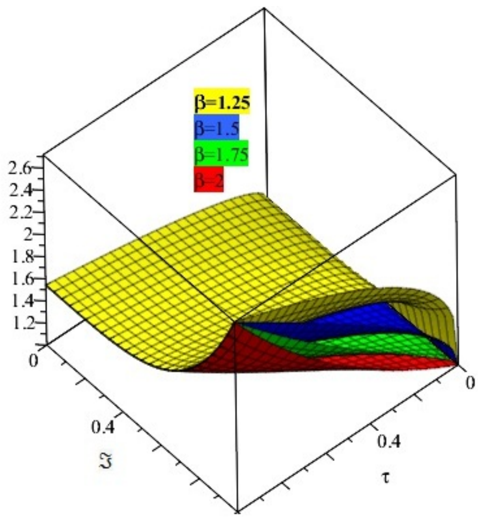

Figure 3. Represents the solution at different fractional order of Example 1.

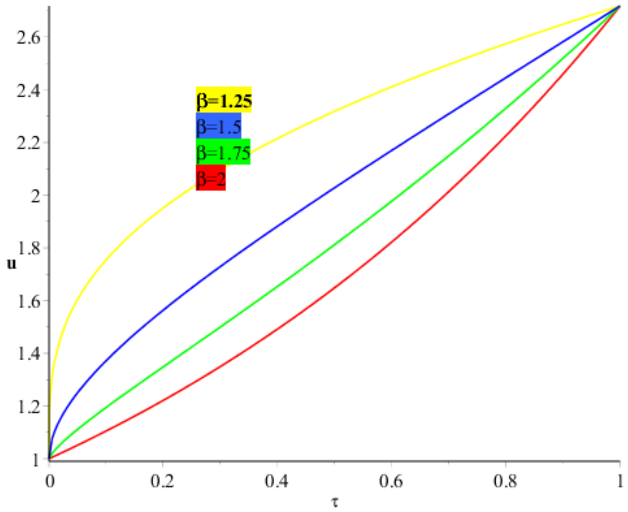

Figure 4. Represents the solution at different fractional order of Example 1.

In Table 2, the solutions of Shehu transform decomposition method (SDM) and Adomian decomposition method (ADM) are compared with each other. The comparison has shown that the solutions of proposed method are in strong agreement with the solution of ADM.

Table 2. Comparison of SDM and ADM [42] of Example 1 at $\tau = 0.1$.

		SDM (m = 5)	SDM (m = 3)	SDM (m= 5)	ADM (m = 5)	AE of SDM
$\Im$	$\Re$	$\beta = 1.75$	$\beta = 2$	$\beta = 2$	$\beta = 2$	$\beta = 2$
1	1	1.111568974	1.105195833	1.10519608	1.10519609	2.51×10^{-5}
2	2	17.78510358	17.68313333	17.6831373	17.6831374	4.02×10^{-4}
3	3	90.03708688	89.52086250	89.5208829	89.5208828	2.03×10^{-3}
4	4	284.5616573	282.9301334	282.930198	282.930199	6.44×10^{-3}
5	5	694.7306086	690.7473959	690.747553	690.747552	1.57×10^{-2}

Example 2. *Consider the (2+time fractional-order) dimensional Heat model:*

$$u_\tau^\beta(\Im, \Re, \tau) = u_{\Im\Im}(\Im, \Re, \tau) + u_{\Re\Re}(\Im, \Re, \tau), \quad \beta \in (0, 1] \tag{32}$$

with initial condition

$$u(\Im, \Re, 0) = \sin(\Im)\cos(\Re). \tag{33}$$

If $\beta = 1$, then the exact solution of Equation (32) is

$$u(\Im, \Re, \tau) = e^{-2\tau}\sin(\Im)\cos(\Re). \tag{34}$$

Taking Shehu transform of Equation (32)

$$\frac{s^\beta}{\mu^\beta}\left\{V(s, \mu) - \frac{\mu}{s}u(0)\right\} = S\left\{u_{\Im\Im}(\Im, \Re, \tau) + u_{\Re\Re}(\Im, \Re, \tau)\right\}, \tag{35}$$

Simplifying Equation (35), we get as

$$V(s, \mu) = \frac{\mu}{s}u(0) + \frac{\mu^\beta}{s^\beta}S\left\{u_{\Im\Im}(\Im, \Re, \tau) + u_{\Re\Re}(\Im, \Re, \tau)\right\}. \tag{36}$$

Applying inverse Shehu transform, we get

$$u(\Im,\Re,\tau) = u(0) + S^{-1}\left\{\frac{\mu^\beta}{s^\beta} S\left\{u_{\Im\Im}(\Im,\Re,\tau) + u_{\Re\Re}(\Im,\Re,\tau)\right\}\right\}. \tag{37}$$

Thus we get the following recursive scheme

$$u_0(\Im,\Re,\tau) = u(0) = \sin(\Im)\cos(\Re), \tag{38}$$

$$u_{m+1}(\Im,\Re,\tau) = S^{-1}\left\{\frac{\mu^\beta}{s^\beta} S\left\{u_{m\Im\Im}(\Im,\Re,\tau) + u_{m\Re\Re}(\Im,\Re,\tau)\right\}\right\}, \tag{39}$$

Using Equation (39), for $m = 0, 1, 2, 3, \cdots$ *we get the following values*

$$\begin{aligned}
u_1(\Im,\Re,\tau) &= -2\sin(\Im)\cos(\Re)\frac{\tau^\beta}{(\beta)!},\\
u_2(\Im,\Re,\tau) &= 4\sin(\Im)\cos(\Re)\frac{\tau^{2\beta}}{(2\beta)!},\\
u_3(\Im,\Re,\tau) &= -8\sin(\Im)\cos(\Re)\frac{\tau^{3\beta}}{(3\beta)!},\\
u_4(\Im,\Re,\tau) &= 16\sin(\Im)\cos(\Re)\frac{\tau^{4\beta}}{(4\beta)!},
\end{aligned} \tag{40}$$

$$\vdots$$

Now using the values of $u_0, u_1, u_2, u_3, \cdots$, *we get Shehu transformation solution for example 2*

$$\begin{aligned}
u(\Im,\Re,\tau) &= \sin(\Im)\cos(\Re) - 2\sin(\Im)\cos(\Re)\frac{\tau^\beta}{(\beta)!} + 4\sin(\Im)\cos(\Re)\frac{\tau^{2\beta}}{(2\beta)!} + \\
&- 8\sin(\Im)\cos(\Re)\frac{\tau^{3\beta}}{(3\beta)!} + 16\sin(\Im)\cos(\Re)\frac{\tau^{4\beta}}{(4\beta)!} + \cdots
\end{aligned} \tag{41}$$

After simplification, we get

$$u(\Im,\Re,\tau) = \sin(\Im)\cos(\Re)\left\{1 - 2\frac{\tau^\beta}{(\beta)!} + 4\frac{\tau^{2\beta}}{(2\beta)!} + -8\frac{\tau^{3\beta}}{(3\beta)!} + 16\frac{\tau^{4\beta}}{(4\beta)!} + \cdots\right\}, \tag{42}$$

which converge to the solution

$$u(\Im,\Re,\tau) = \sin(\Im)\cos(\Re)E_\beta(-2\tau^\beta), \tag{43}$$

For particular case $\beta \to 1$, *the Shehu transform solution become as*

$$u(\Im,\Re,\tau) = \sin(\Im)\cos(\Re)e^{-2\tau}. \tag{44}$$

Figures 5 and 6 show the exact and analytical solution of Example 2 respectively. The graphical representation have confirmed the closed contact of the obtained solution with the exact solution of Example 2. Similarly, Figures 7 and 8 represents the fractional-order solution of Example 2 for two and three space. Both graphs support the convergence phenomena of fractional-order problems to an integer-order problem of Example 2.

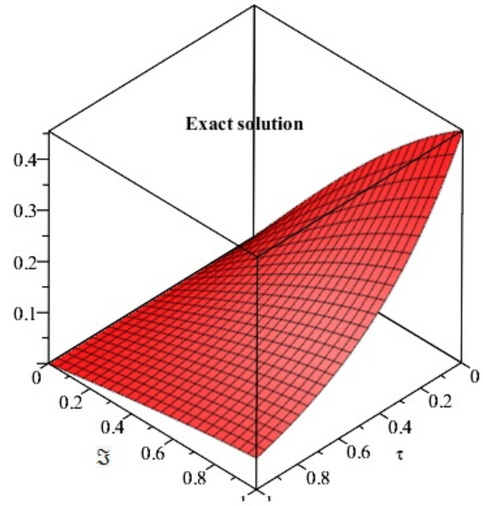

Figure 5. Represents the exact solution of Example 2 at $\beta = 1$.

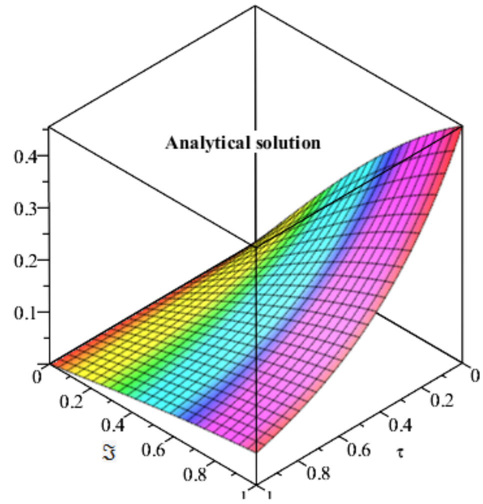

Figure 6. Represents the analytical solution of Example 2 at $\beta = 1$.

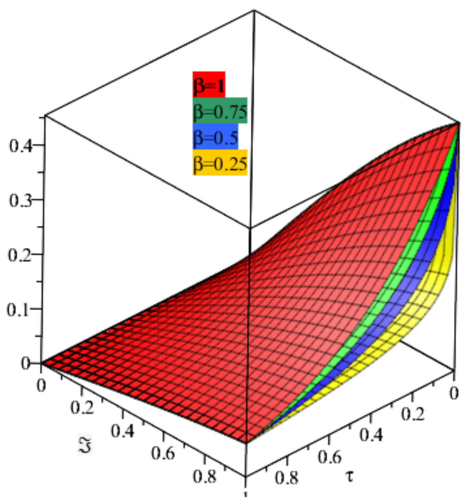

Figure 7. $u(\Im, \Re, \tau)$ Represents the solution at different fractional order of Example 2.

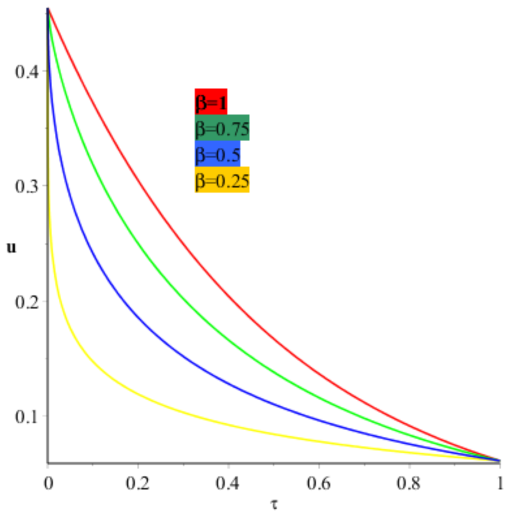

Figure 8. $u(\Im, \Re, \tau)$ Represents the solution at different fractional order of Example 2.

Example 3. *Consider the* $(2 + time fractional)$ *dimensional diffusion model:*

$$u_\tau^\beta(\Im, \Re, \tau) = u_{\Im\Im}(\Im, \Re, \tau) + u_{\Re\Re}(\Im, \Re, \tau), \quad \beta \in (0, 1] \tag{45}$$

with the initial condition

$$u(\Im, \Re, 0) = e^{\Im+\Re}. \tag{46}$$

If $\beta = 1$, *then the exact solution of Equation (45) is*

$$u(\Im, \Re, \tau) = e^{\Im+\Re+2\tau} \tag{47}$$

Taking Shehu transform of Equation (45)

$$\frac{s^\beta}{\mu^\beta}\left\{V(s, \mu) - \frac{\mu}{s}u(0)\right\} = S\left\{u_{\Im\Im}(\Im, \Re, \tau) + u_{\Re\Re}(\Im, \Re, \tau)\right\}. \tag{48}$$

Simplifying Equation (46), we get as

$$V(s, \mu) = \frac{\mu}{s}u(0) + \frac{\mu^\beta}{s^\beta}S\left\{u_{\Im\Im}(\Im, \Re, \tau) + u_{\Re\Re}(\Im, \Re, \tau)\right\}. \tag{49}$$

Applying inverse operator of Shehu transform, we get

$$u(\Im, \Re, \tau) = u(0) + S^{-1}\left\{\frac{\mu^\beta}{s^\beta}S\left\{u_{\Im\Im}(\Im, \Re, \tau) + u_{\Re\Re}(\Im, \Re, \tau)\right\}\right\}. \tag{50}$$

Thus we get the following recursive scheme

$$u_0(\Im, \Re, \tau) = u(0) = e^{\Im+\Re},$$

$$u_{m+1}(\Im, \Re, \tau) = S^{-1}\left\{\frac{\mu^\beta}{s^\beta}S\left\{u_{m\Im\Im}(\Im, \Re, \tau) + u_{m\Re\Re}(\Im, \Re, \tau)\right\}\right\}, \tag{51}$$

Using Equation (51), for $m = 0, 1, 2, 3, \cdots$ *we get the following values*

$$
\begin{aligned}
u_1(\Im,\Re,\tau) &= 2e^{\Im+\Re}\frac{\tau^{\beta}}{(\beta)!},\\
u_2(\Im,\Re,\tau) &= 4e^{\Im+\Re}\frac{\tau^{2\beta}}{(2\beta)!},\\
u_3(\Im,\Re,\tau) &= 8e^{\Im+\Re}\frac{\tau^{3\beta}}{(3\beta)!},\\
u_4(\Im,\Re,\tau) &= 16e^{\Im+\Re}\frac{\tau^{4\beta}}{(4\beta)!},\\
&\vdots
\end{aligned}
\tag{52}
$$

Now using the values of $u_0, u_1, u_2, u_3, \cdots$, we get Shehu transformation solution for Example 3

$$u(\Im,\Re,\tau) = e^{\Im+\Re} + 2e^{\Im+\Re}\frac{\tau^{\beta}}{(\beta)!} + 4e^{\Im+\Re}\frac{\tau^{2\beta}}{(2\beta)!} + 8e^{\Im+\Re}\frac{\tau^{3\beta}}{(3\beta)!} + 16e^{\Im+\Re}\frac{\tau^{4\beta}}{(4\beta)!} + \cdots. \tag{53}$$

After simplification, we get

$$u(\Im,\Re,\tau) = \sin(\Im)\cos(\Re)\left\{1 + 2\frac{\tau^{\beta}}{(\beta)!} + 4\frac{\tau^{2\beta}}{(2\beta)!} + 8\frac{\tau^{3\beta}}{(3\beta)!} + 16\frac{\tau^{4\beta}}{(4\beta)!} + \cdots\right\}. \tag{54}$$

The close form solution become as

$$u(\Im,\Re,\tau) = \sin(\Im)\cos(\Re)E_{\beta}(2\tau^{\beta}). \tag{55}$$

When $\beta \to 1$ the calculated result provide the exact solution in the close form

$$u(\Im,\Re,\tau) = \sin(\Im)\cos(\Re)e^{2\tau}. \tag{56}$$

Figures 9 and 10 show the exact and analytical solutions of Example 3. Both figures are almost coincident confirming the close contact of both exact and obtained solution. Figures 11 the SDM solutions at different fractional-order β are calculated for Example 3. The convergence phenomena of fractional-order solution towards exact solution is observed. The method is found to be very simple and straightforward to solve fractional-order different equations.

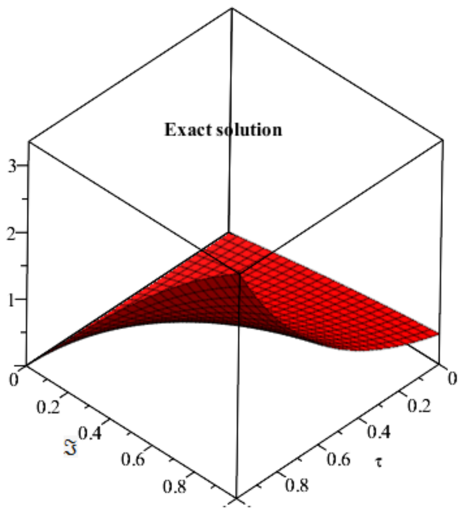

Figure 9. Exact solution of Example 3 at $\beta = 1$.

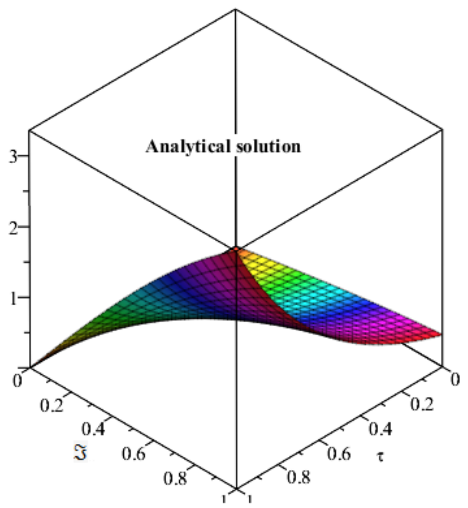

Figure 10. Represents the analytical solution of Example 3 at $\beta = 1$.

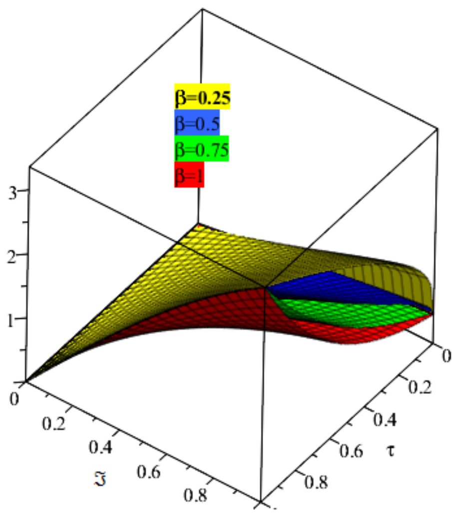

Figure 11. The solution graph at different fractional order β.

Example 4. *Consider the* $(2 + time fractional)$ *dimensional telegraph model:*

$$u_{\tau}^{\beta}(\Im, \Re, \tau) = \frac{1}{2}u_{\Im\Im}(\Im, \Re, \tau) + \frac{1}{2}u_{\Re\Re}(\Im, \Re, \tau) - 2u_t(\Im, \Re, \tau) - u(\Im, \Re, \tau), \quad \beta \in (1, 2], \tag{57}$$

with initial conditions

$$u(\Im, \Re, 0) = \sinh(\Im)\sinh(\Re), \qquad u_{\tau}(\Im, \Re, 0) = -2\sinh(\Im)\sinh(\Re). \tag{58}$$

If $\beta = 2$, *then the exact solution of Equation (57) is*

$$u(\Im, \Re, \tau) = \sinh(\Im)\sinh(\Re)e^{-2\tau}. \tag{59}$$

Taking Shehu transform of Equation (57)

$$\frac{s^{\beta}}{\mu^{\beta}}\left\{V(s, \mu) - \frac{\mu}{s}u(0) - \frac{\mu^2}{s^2}u'(0)\right\} = S\left\{\frac{1}{2}u_{\Im\Im}(\Im, \Re, \tau) + \frac{1}{2}u_{\Re\Re}(\Im, \Re, \tau) - 2u_t(\Im, \Re, \tau) - u(\Im, \Re, \tau)\right\}, \tag{60}$$

Simplifying Equation (60), we get as

$$V(s, \mu) = \frac{\mu^{\beta}}{s^{\beta}}S\left\{\frac{1}{2}u_{\Im\Im}(\Im, \Re, \tau) + \frac{1}{2}u_{\Re\Re}(\Im, \Re, \tau) - 2u_t(\Im, \Re, \tau) - u(\Im, \Re, \tau)\right\} + \frac{\mu}{s}u(0) + \frac{u^2}{s^2}u'(0), \tag{61}$$

Applying inverse of Shehu transform, we get

$$u(\Im,\Re,\tau) = u(0) + \tau u^{'}(0) + S^{-1}\left\{\frac{\mu^{\beta}}{s^{\beta}} S\left\{\frac{1}{2}u_{\Im\Im}(\Im,\Re,\tau) + \frac{1}{2}u_{\Re\Re}(\Im,\Re,\tau) - 2u_{t}(\Im,\Re,\tau) - u(\Im,\Re,\tau)\right\}\right\}. \quad (62)$$

Thus we get the following recursive scheme

$$u_0(\Im,\Re,\tau) = u(0) + \tau u^{'}(0) =$$
$$\sinh(\Im)\sinh(\Re) - 2t\sinh(\Im)\sinh(\Re),$$

$$u_{m+1}(\Im,\Re,\tau) = S^{-1}\left\{\frac{\mu^{\beta}}{s^{\beta}} S\left\{\frac{1}{2}u_{m\Im\Im}(\Im,\Re,\tau) + \frac{1}{2}u_{m\Re\Re}(\Im,\Re,\tau) - 2u_{m\tau}(\Im,\Re,\tau) - u_{m}(\Im,\Re,\tau)\right\}\right\}, \quad (63)$$

Using Equation (63), for $m = 0, 1, 2, 3, \cdots$ we get the following values

$$\begin{aligned}
u_1(\Im,\Re,\tau) &= 4\sinh(\Im)\sinh(\Re)\frac{\tau^{\beta}}{(\beta)!}, \\
u_2(\Im,\Re,\tau) &= -8\frac{\beta(\beta-1)!\sinh(\Im)\sinh(\Re)\tau^{2\beta}}{(2\beta-1)!(\beta)!}, \\
u_3(\Im,\Re,\tau) &= 16\frac{\beta(2\beta-1)(\beta-1)!(2\beta-2)!\sinh(\Im)\sinh(\Re)\tau^{3\beta-2}}{(\beta)!(2\beta-1)!(3\beta-2)!}, \\
&\vdots
\end{aligned} \quad (64)$$

Now using the values of $u_0, u_1, u_2, u_3, \cdots$, we get Shehu transformation solution for Example 4

$$u(\Im,\Re,\tau) = \sinh(\Im)\sinh(\Re) - 2\tau\sinh(\Im)\sinh(\Re) + 4\frac{\sinh(\Im)\sinh(\Re)\tau^{\beta}}{(\beta)!} - 8\frac{\beta(\beta-1)!\sinh(\Im)\sinh(\Re)\tau^{2\beta}}{(2\beta-1)!(\beta)!} + \frac{16\beta(2\beta-1)(\beta-1)!(2\beta-2)!\sinh(\Im)\sinh(\Re)\tau^{3\beta-2}}{(\beta)!(2\beta-1)!(3\beta-2)!} + \cdots. \quad (65)$$

After simplification, we get

$$u(\Im,\Re,\tau) = \sinh(\Im)\sinh(\Re)\left\{1 - 2\tau + 4\frac{\tau^{\beta}}{\beta!} - 8\frac{\beta(\beta-1)!\tau^{2\beta}}{(2\beta-1)!(\beta)!} + \frac{16\beta(2\beta-1)(\beta-1)!(2\beta-2)!\tau^{3\beta-2}}{(\beta)!(2\beta-1)!(3\beta-2)!} + \cdots\right\}. \quad (66)$$

For particular case $\beta \to 2$, the Shehu transform solution become as

$$u(\Im,\Re,\tau) = \sinh(\Im)\sinh(\Re)\left\{1 - 2\tau + 4\frac{\tau^{2}}{2!} - 8\frac{\tau^{3}}{3!} + 16\frac{\tau^{4}}{4!} + \cdots\right\}. \quad (67)$$

The calculated result provide the exact solution in the close form

$$u(\Im,\Re,\tau) = \sinh(\Im)\sinh(\Re)e^{-2\tau}. \quad (68)$$

Figures 12 and 13, display the exact and analytical solutions of Example 4. The solution graph of SDM is very similarly to the exact solution of Example 4. In Figure 14, we plotted the solutions of Example 4 at different fractional-order β. The fractional-order solutions are found to be convergent towards the exact solution of Example 4. It is investigated from the solution analysis that the present method is a sophisticated technique to solve fractional-order problems.

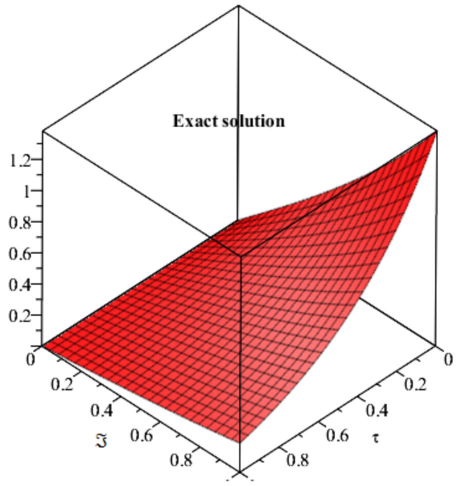

Figure 12. Exact solution of Example 4 at $\beta = 2$.

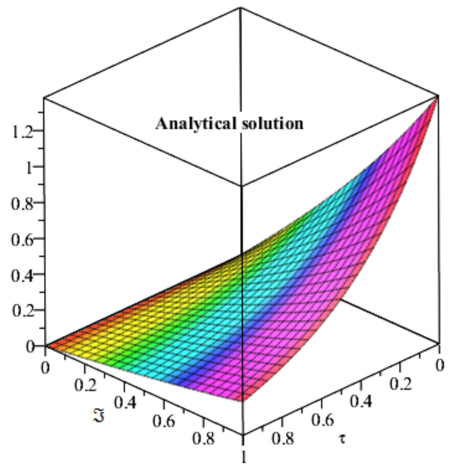

Figure 13. analytical solution of Example 4 at $\beta = 2$.

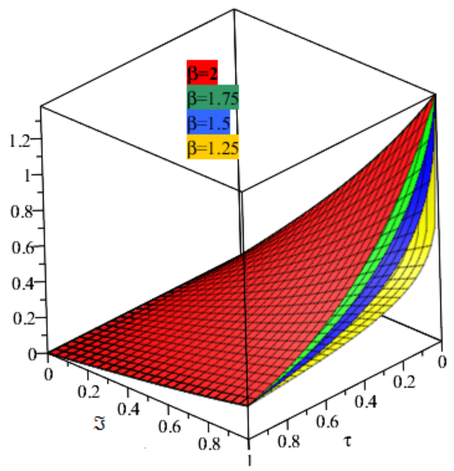

Figure 14. The solution graph at different fractional order β. of Example 4.

Example 5. *Consider the non-linear* $(2 + time fractional)$ *dimensional Burger's model:*

$$u_\tau^\beta(\Im, \Re, \tau) = u_{\Im\Im}(\Im, \Re, \tau) + u_{\Re\Re}(\Im, \Re, \tau) + u_\Im(\Im, \Re, \tau)u(\Im, \Re, \tau), \quad \beta \in (0, 1], \tag{69}$$

with initial condition

$$u(\Im, \Re, 0) = \Im + \Re. \tag{70}$$

If $\beta = 1$, then the exact solution of Equation (69) is

$$u(\Im, \Re, \tau) = \frac{\Im + \Re}{1 - \tau}. \tag{71}$$

Taking Shehu transform of Equation (69)

$$\frac{s^\beta}{\mu^\beta}\left\{V(s,\mu) - \frac{\mu}{s}u(0)\right\} = S\left\{u_{\Im\Im}(\Im,\Re,\tau) + u_{\Re\Re}(\Im,\Re,\tau) + u_{\Im}(\Im,\Re,\tau)u(\Im,\Re,\tau)\right\}, \tag{72}$$

The simplifying Equation (72), we get as

$$V(s,\mu) = \frac{\mu}{s}u(0) + \frac{\mu^\beta}{s^\beta}S\left\{u_{\Im\Im}(\Im,\Re,\tau) + u_{\Re\Re}(\Im,\Re,\tau) + u_{\Im}(\Im,\Re,\tau)u(\Im,\Re,\tau)\right\}, \tag{73}$$

By applying inverse of Shehu transform, we get

$$u(\Im,\Re,\tau) = u(0) + \tau u^{'}(0) + S^{-1}\left\{\frac{\mu^\beta}{s^\beta}S\left\{u_{\Im\Im}(\Im,\Re,\tau) + u_{\Re\Re}(\Im,\Re,\tau) + u_{\Im}(\Im,\Re,\tau)u(\Im,\Re,\tau)\right\}\right\}, \tag{74}$$

Thus we get the following recursive scheme

$$u_0(\Im,\Re,\tau) = u(0) = \Im + \Re, \tag{75}$$

$$u_{m+1}(\Im,\Re,\tau) = S^{-1}\left\{\frac{\mu^\beta}{s^\beta}S\left\{u_{m\Im\Im}(\Im,\Re,\tau) + u_{m\Re\Re}(\Im,\Re,\tau) + u_{m\Im}(\Im,\Re,\tau)u_m(\Im,\Re,\tau)\right\}\right\}. \tag{76}$$

For nonlinear term, use the Equation (12) in recursive scheme (76), we obtain

$$u_{m+1}(\Im,\Re,\tau) = S^{-1}\left\{\frac{\mu^\beta}{s^\beta}S\left\{u_{m\Im\Im}(\Im,\Re,\tau) + u_{m\Re\Re}(\Im,\Re,\tau) + \sum_{m=0}^{\infty}A_m(u_0, u_1, \cdots)\right\}\right\}. \tag{77}$$

Using Equation (77), for $m = 0, 1, 2, 3, \cdots$ we get the following values

$$\begin{aligned}
u_1(\Im,\Re,\tau) &= (\Im + \Re)\frac{t^\beta}{(\beta)!}, \\
u_2(\Im,\Re,\tau) &= 2(\Im + \Re)\frac{\tau^{2\beta}}{(2\beta)!}, \\
u_3(\Im,\Re,\tau) &= 4(\Im + \Re)\frac{\tau^{3\beta}}{(3\beta)!} + (\Im + \Re)(2\beta)!\frac{t^{3\beta}}{\beta!\beta!(3\beta)!}, \\
&\vdots
\end{aligned} \tag{78}$$

Now using the values of $u_0, u_1, u_2, u_3, \cdots$, we get Shehu transformation solution for example 5

$$u(\Im,\Re,\tau) = \Im + \Re + (\Im + \Re)\frac{\tau^\beta}{(\beta)!} + 2(\Im + \Re)\frac{\tau^{2\beta}}{(2\beta)!} + 4(\Im + \Re)\frac{\tau^{3\beta}}{(3\beta)!} + (\Im + \Re)(2\beta)!\frac{\tau^{3\beta}}{\beta!\beta!(3\beta)!} + \cdots. \tag{79}$$

After simplification, we get

$$u(\Im,\Re,\tau) = (\Im + \Re)\left\{1 + \frac{\tau^\beta}{(\beta)!} + 2\frac{\tau^{2\beta}}{(2\beta)!} + 4\frac{\tau^{3\beta}}{(3\beta)!} + (2\beta)!\frac{\tau^{3\beta}}{\beta!\beta!(3\beta)!} + \cdots\right\}. \tag{80}$$

For particular case $\beta \to 1$, the Shehu transform solution become as

$$u(\Im,\Re,\tau) = (\Im + \Re)\left\{1 + \tau + \tau^2 + \tau^3 + \cdots\right\}. \tag{81}$$

The calculated result provide the exact solution in the close form

$$u(\Im, \Re, \tau) = \frac{\Im + \Re}{1 - \tau}. \tag{82}$$

Figures 15 and 16 are plotted to discuss the exact and analytical solutions of Example 5. The SDM solutions are in good contact with the exact solution of the Example 5. Figures 17 and 18 are plotted to analyze the fractional-order solutions of Example 5 at fractional-order $\beta = 0.75$ and 0.50 respectively. The graphical analysis has verified the applicability of the proposed method.

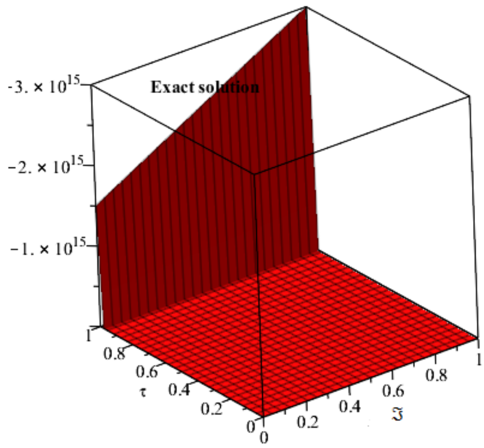

Figure 15. Exact solution of Example 5 at $\beta = 1$.

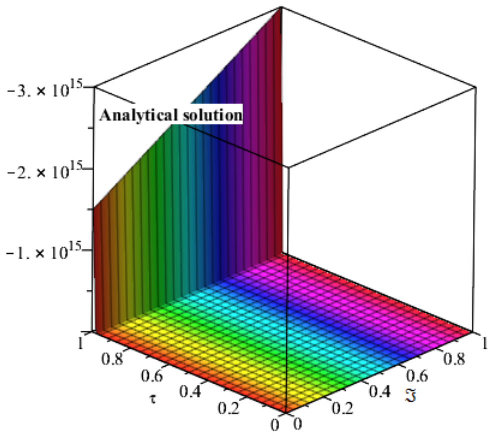

Figure 16. Represents the analytical solution of Example 5 at $\beta = 1$.

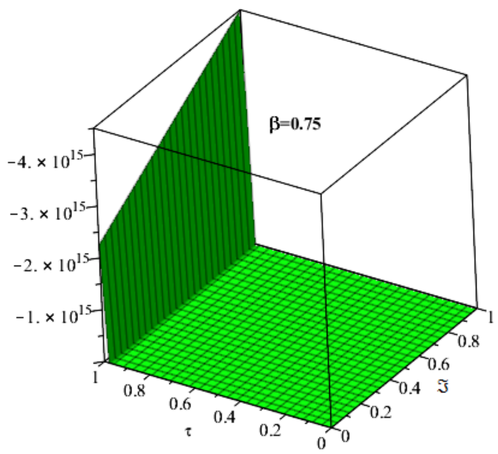

Figure 17. The solution of fractional-order $\beta = 0.75$ of Example 5.

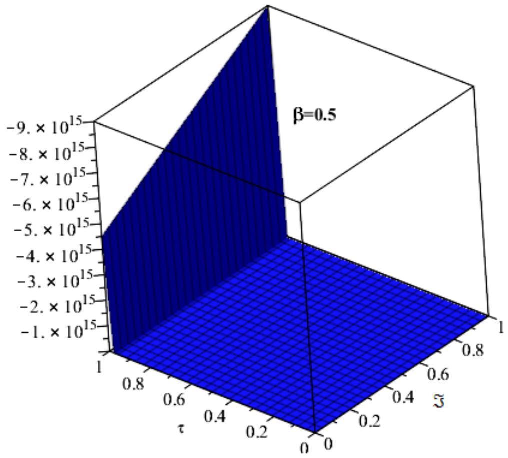

Figure 18. The solution of fractional-order $\beta = 0.5$ of Example 5.

5. Results and Discussion

In the present research work, we implemented a new analytical technique SDM for the solution of some important problems which are frequently arising in science and engineering, such as hyperbolic wave equation, heat equation, diffusion equation, telegraph and Burgers equations. The Caputo definition of fractional-derivative is used to define fractional-derivative. The proposed method is the combination of Shehu transformation and Adomian decomposition method which is known as Shehu decomposition method. For applicability and novelty of present method, we applied it different physical problems for applied sciences. These problems have been solved by using SDM for both fractional and integer-order of the targeted problems. In this connection some figure analysis have been done to demonstrate the obtained results in a sophisticated manner. It is investigated that SDM solution have a very close contact with the exact solution of the problems. It is also observed that the fractional-order problems are convergent towards the solution of an integer-order problem. Moreover, the high rate of convergence of the current method is noted during the simulation. It is calculated that the SDM can be considered as one of the best analytical technique to solve fractional partial differential equations.

6. Conclusions

In the present article, we presented some fractional-view analysis of physical problems, arising in science and engineering. A new and sophisticated analytical technique, which is known as Shehu transform decomposition method is implemented for both fractional and integer-orders of the problems. The Caputo definition of fractional derivative is used to express fractional-order derivative. For applicability and reliability of the proposed methods, some illustrative examples are presented from different areas of applied science. It has been investigated through graphical representation that the present technique provides an accurate and deserving analysis about the physical happening of the problems. It is observed through simulations of the present algorithm that as fractional-order of the derivative approaches to integer order of the problem then fractional-order solutions are convergent to integer-order solutions. Moreover, the present method is preferred as compared to other method because of its better rate of convergence. This direction motivates the researchers towards the implementation of the current method for other non-linear fractional partial differential equations.

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