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New Results on Tracking and Adaptive Tracking Control of High-Order Nonlinear Systems with Polynomial Conditions

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Abstract

Many practical dynamical models are inherently nonlinear and may exhibit complex growth conditions and uncertainties, which pose considerable challenges to controller design. In this paper, the tracking control issue for a category of high-order nonlinear systems with polynomial-type growth conditions is studied. Through the construction of a new Lyapunov function and the development of a dynamic-gain-based homogeneous control method, a robust tracking controller is designed to achieve the required tracking accuracy while ensuring the global boundedness of all closed-loop signals. As an extension, the system with unknown nonlinear functions is further considered, and a new adaptive tracking controller is constructed. Simulation examples demonstrate the efficacy of the presented control methods.

Keywords: high-order nonlinear system; tracking control; adaptive control

1. Introduction

Nonlinear systems often have complex structures and exhibit distinct dynamical behaviors. Such systems are commonly encountered in engineering practice and are often subject to complex dynamics and uncertainties, making many control methods developed for linear systems inapplicable [1,2]. Since nonlinear systems differ from one another in their nonlinearities, uncertainties, and structural characteristics, it is challenging to design a universal control scheme that is applicable to all classes of nonlinear systems. Moreover, nonlinear systems are frequently affected by exogenous disturbances, which can significantly degrade the behavior of the system. Recently, a variety of control schemes have been proposed, including sliding-mode control [3–5], backstepping control [6–8], prescribed-time control [9,10], and intelligent control [11,12]. Nevertheless, as modern industry advances rapidly, increasingly complex nonlinear system models continue to emerge, posing new challenges for controller design. Consequently, new control methods and theoretical tools that can effectively address these issues are urgently needed.

Recently, trajectory tracking for high-order systems with polynomial growth conditions and uncertainties remains challenging [13]. Due to the coupling of high-order powers, complex nonlinearities, and uncertainties, traditional control methods often yield complicated controllers and are even inapplicable in some cases. The goal of the tracking control is to design an appropriate controller that ensures closed-loop stability and satisfactory tracking accuracy. When a system contains uncertainties, the unknown quantities are often estimated, and their estimates are then used in the controller design [14,15]. In particular,

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when the system contains unknown parameters, adaptive control methods have been developed to construct suitable controllers [16,17]. When nonlinear systems contain unknown nonlinear functions, intelligent control methods based on artificial neural networks or fuzzy logic systems are often employed to model the uncertain nonlinearities and construct suitable control laws. For example, an adaptive backstepping control scheme using a neural network is proposed in [18] for systems with unknown nonlinear dynamics, unknown signs of control gains, and constraints dependent on the system state. Moreover, for uncertain systems, an adaptive fuzzy backstepping control strategy is proposed in [19] to ensure system stability and the convergence of the tracking error. When nonlinear systems are subject to unknown disturbances, a disturbance observer is usually used to estimate disturbances, and the resulting estimates are then incorporated into the controller design [20]. For example, an equivalent input disturbance method based on a nonlinear state observer is proposed in [21] to reject unknown exogenous disturbances while preserving the inherent nonlinear dynamics. In [22], a reduced-order extended state observer is employed for nonlinear systems affected by multiple perturbations, parametric uncertainties, and unmodeled nonlinear dynamics. When some system states are unmeasurable, output-feedback control methods are commonly employed. In such methods, a state observer is usually constructed, and the estimated states together with the measured output are utilized to design the controller [23]. For example, an output-feedback predefined-time fuzzy control method is proposed in [24] for strict-feedback stochastic nonlinear systems subject to state constraints. In [25], for nonlinear systems subject to input limitations and unmeasured states, an adjustable performance-guaranteed control law is constructed, ensuring that the output tracks the reference within a given finite time. For more results, see also [26–29]. However, the existing approaches still suffer from several limitations. Notably, the repeated differentiation of virtual control functions can cause the explosion of complexity. Furthermore, any change in the nonlinear terms or system structure typically requires a redesign of the nonlinear controller. A natural question then arises: for higher-order nonlinear systems with polynomial growth conditions, can a much simpler tracking controller be designed? Furthermore, is it possible to design an adaptive controller?

In the following, the above problem will be studied. The contributions of this work are listed as follows.

(i) In contrast to existing studies [8,11,14,16,17], this paper proposes a new solution for a more general class of nonlinear systems subject to high-order powers and complex polynomial growth nonlinearities. By introducing a dynamic-gain-based transformation, constructing a suitable Lyapunov function, and developing a modified homogeneous domination approach, this paper presents a new tracking control strategy. The strategy utilizes the dynamic gain to dominate the nonlinear terms and reduces the complexity of the resulting controller.

(ii) Compared with the reported methods [22,24,25,28], this work further develops a novel adaptive tracking control scheme. Specifically, for systems with unknown parameters, a new transformation is introduced, and a Lyapunov function with two integral terms is constructed. By presenting a new adaptive mechanism and using the homogeneous domination approach, a dynamic adaptive tracking controller using one parameter estimate is successfully constructed. Once the required system conditions hold, the explicit form of the adaptive controller can be directly obtained.

2. Preliminary Results

This article investigates the system

$$\begin{cases} \dot{x}_1 = x_2^{p_1} + f_1(\theta, x_1, w_1(t)), \\ \dot{x}_2 = x_3^{p_2} + f_2(\theta, x_1, x_2, w_2(t)), \\ \quad \vdots \\ \dot{x}_n = u^{p_n} + f_n(\theta, x_1, \ldots, x_n, w_n(t)), \\ y = x_1, \end{cases} \tag{1}$$

in which $x_i \in \mathbb{R}, 1 \leq i \leq n$, and $y \in \mathbb{R}$ are the state variables and output, respectively; $u \in \mathbb{R}$ is the control input; $w_i(t)$ is the external disturbance; $f_i(\cdot)$ denotes a continuous function; θ is a constant parameter vector; and p_i is an odd integer with $p_i \geq 1$.

The following assumptions are required for subsequent design.

Assumption 1. *For $1 \leq i \leq n$, the function f_i satisfies:*

$$|f_i(\cdot)| \quad \leq \bar{\theta}\Big(\textstyle\sum_{j=1}^{i-1} |x_j|^{\frac{1}{p_j \cdots p_{i-1}}} + |x_i| + \sum_{j=1}^{i} |x_j|^{\frac{\gamma_i + w}{\gamma_j}} \Big) + \bar{w}_i, \tag{2}$$

where $\bar{\theta} \geq 0$ is an upper bound for θ, $\bar{w}_i \geq 0$ is a constant, $p_0 = 1$, $\gamma_1 = 1, \gamma_i = \frac{\gamma_{i-1} + w}{p_{i-1}}$, $w = \frac{w_a}{w_b} > 0$ (w_a is an even integer, and w_b is an odd integer).

Assumption 2. *The reference signal y_r satisfies $|y_r| + |\dot{y}_r| \leq M_0$, where $M_0 \geq 0$ is a constant.*

The control objectives are as follows:

1. When $\bar{\theta}$ is known, design a dynamic tracking controller for the given reference signal y_r such that the system output tracks y_r and all closed-loop signals remain bounded.

2. When $\bar{\theta}$ is unknown, construct an adaptive tracking controller to achieve the same control objectives as Objective 1.

Remark 1. *Assumption 1 is less restrictive than many commonly used growth conditions for high-order nonlinear systems. It imposes a unified polynomial growth condition on the nonlinear function f_i, which is allowed to depend on the first i state variables $x_1, \ldots, x_i$, without requiring its explicit expression. Unlike neural-network or fuzzy system methods that approximate unknown nonlinearities online [11,14,18,19,24], the proposed method directly constructs the controller using the growth properties of the nonlinear terms. When $w = 0$, Assumption 1 reduces to a weighted homogeneous growth condition. When $w \neq 0$, it can further accommodate nonlinearities with higher-order polynomial growth. The system powers p_i and the growth order w are assumed to be known structural parameters. The growth coefficient $\bar{\theta}$ is treated as known in Theorem 1 and unknown in Theorem 2, while the exact nonlinear functions and disturbance bounds are not required in the controller implementation. Therefore, the proposed condition relaxes several conventional requirements on precise nonlinear models and known growth coefficients and expands the class of nonlinear systems that can be dealt with.*

Remark 2. *There exist several practical examples that satisfy the form of system (1). For example, these include the flexible-joint robot system, the quadrotor unmanned aerial vehicle system, and the circuit system [30–32]. In practice, numerous dynamic models possess distinct nonlinear terms. Nevertheless, they can be converted into the form of system (1) via the appropriate coordinate transformations. System (1) actually generalizes these models and thus offers a unified form. In this work, the more general condition, namely the polynomial growth condition, is considered. It encompasses many different practical systems as special cases. Next, two control approaches for system (1) will be provided.*

Lemma 1 ([12]). *For real numbers $A_1 > 0, B_1 > 0$, there holds*

$$|sA_1^{l_1}B_1^{l_2}| \le l_0|A_1|^{l_1+l_2} + \frac{l_2}{l_1+l_2}\left(\frac{l_1}{l_0(l_1+l_2)}\right)^{\frac{l_1}{l_2}}|s|^{\frac{l_1+l_2}{l_2}}|B_1|^{l_1+l_2}, \tag{3}$$

where s, l_0, l_1, and l_2 are positive constants.

Lemma 2 ([29]). *For $T \in \mathbb{R}$, $G \in \mathbb{R}$, and a constant $p \ge 1$, there hold the inequalities*

$$|T+G|^p \le 2^{p-1}|T^p + G^p|; \tag{4}$$

$$(|T|+|G|)^{1/p} \le |T|^{1/p} + |G|^{1/p} \le 2^{(p-1)/p}(|T|+|G|)^{1/p}. \tag{5}$$

3. Methodology

In this section, the main method is outlined as follows. First, a dynamic-gain-based transformation is introduced so that the original system is transformed into a time-varying system. Then, for the convenience of control design, a new coordinate transformation is utilized, and a Lyapunov function is further constructed. Subsequently, several key inequalities are derived. These results facilitate the controller design for system (1). The detailed procedure is provided as follows.

Define the constants

$$h_1 = 1, h_j = \frac{h_{j-1}+a}{p_{j-1}}, 2 \le j \le n+1, \tag{6}$$

where $a \ge 1$ is a constant. Introduce the transformation

$$z_1 = \frac{x_1 - y_r}{S^{h_1}(t)}, z_i = \frac{x_i}{S^{h_i}(t)}, \nu = \frac{u}{S^{h_{n+1}}(t)}, 2 \le i \le n, \tag{7}$$

where $S(t) \ge 1$ is a dynamic gain to be determined. By (7), the derivative of z_1 along system (1) satisfies

$$\dot{z}_1 = S^a(t)z_2^{p_1} - h_1\frac{\dot{S}(t)}{S(t)}z_1 + \frac{f_1 - \dot{y}_r}{S(t)}, \tag{8}$$

Similarly, it is deduced that

$$\dot{z}_i = S^a(t)z_{i+1}^{p_i} - h_i\frac{\dot{S}(t)}{S(t)}z_i + \frac{f_i}{S^{h_i}(t)}, i = 2, \ldots, n, \tag{9}$$

where $z_{n+1} = \nu$. Define $E(z,\nu) = [z_2^{p_1}, \ldots, z_n^{p_{n-1}}, \nu^{p_n}] \in \mathbb{R}^n$, $H = \mathrm{diag}(h_1, h_2, \ldots, h_n) \in \mathbb{R}^{n\times n}$, and $F = [\frac{f_1-\dot{y}_r}{S^{h_1}(t)}, \frac{f_2}{S^{h_2}(t)}, \ldots, \frac{f_n}{S^{h_n}(t)}]^T \in \mathbb{R}^n$. Using (8) and (9), it follows that

$$\dot{z} = S^a(t)E(z,\nu) - \frac{\dot{S}(t)}{S(t)}Hz + F. \tag{10}$$

For system (10), introduce the following transformation

$$\begin{array}{ll} \eta_1 = z_1, & \eta_i = z_i^{p_1\cdots p_{i-1}} - \pi_{i-1}^{p_1\cdots p_{i-1}}, \\ \pi_1^{p_1} = -c_1\eta_1, & \pi_{i-1}^{p_1\cdots p_{i-1}} = -c_i\eta_{i-1}, 2 \le i \le n, \end{array} \tag{11}$$

where $c_1, \ldots, c_n$ are positive constants. Construct the following Lyapunov function

$$V(z) = \sum_{i=1}^{n-1} \beta_{n-1} \cdots \beta_i W_i + W_n, W_i = \int_{\pi_{i-1}}^{z_i} \left(l^{p_1 \cdots p_{i-1}} - \pi_{i-1}^{p_1 \cdots p_{i-1}} \right)^{2 - \frac{1}{p_1 \cdots p_{i-1}}} dl, \tag{12}$$

where $\beta_i, 1 \leq i \leq n-1$ are positive constants.

Before constructing the controller, the following lemmas are provided. They help to construct the controller for system (1). Their proofs are given in Appendices A–C.

Lemma 3. *Consider the system*

$$\dot{\tilde{z}} = A(t)E(\tilde{z}, \tilde{v}) - B(t)H\tilde{z}, \tag{13}$$

where $\tilde{z} \in \mathbb{R}^n$ and $\tilde{v} \in \mathbb{R}$ are the state vector and control input, respectively; $A(t) \geq 1$ and $B(t) \geq 0$ are time-varying functions; $E(\tilde{z}, \tilde{v}) = [\tilde{z}_2^{p_1}, \ldots, \tilde{z}_n^{p_{n-1}}, \tilde{v}^{p_n}] \in \mathbb{R}^n$ is a vector. There exists a transformation of the form (11) and a controller $\tilde{v} = \pi_n(\eta_n)$ such that

$$\frac{\partial V}{\partial \tilde{z}} \dot{\tilde{z}} \leq -m_1 A(t) \sum_{i=1}^{n} \eta_i^2 - m_2 B(t) \sum_{i=1}^{n} \eta_i^2, \tag{14}$$

where m_1 and m_2 are positive constants.

Lemma 4. *The nonlinear term F_i satisfies*

$$|F_i| \leq S^{a-b}(t)\bar{c} \left[\sum_{j=1}^{i} |\eta_j|^{\frac{1}{p_1 \cdots p_{i-1}}} + \sum_{j=1}^{i} |\eta_j|^{\frac{\gamma_i + w}{p_1 \cdots p_{j-1} \gamma_j}} \right] + \frac{\hat{\omega}_i}{S^{h_i}(t)}, \tag{15}$$

where b, $\bar{c}$, and $\hat{\omega}_i$ are positive constants.

Lemma 5. *There are constants $m_3 > 0$, $\delta_j > 0$ and $d > 0$ such that*

$$\frac{\partial V}{\partial z} F \leq S^{a-b}(t) m_3 \left[\sum_{j=1}^{n} \eta_j^2 + \sum_{j=1}^{n} \eta_j^{2+\delta_j} \right] + d. \tag{16}$$

In the next section, Lemmas 3–5 will be used to prove the main results.

4. Main Results

When $\bar{\theta}$ given in Assumption 1 is known, the following theorem is provided to outline the main results of this study.

Theorem 1 ($\bar{\theta}$ is a known constant). *For the system in (1) under Assumptions 1 and 2, there exists a dynamic controller*

$$\begin{cases} u = -c_n^{\frac{1}{p_1 \cdots p_{n-1}}} S^{h_{n+1}}(t) \eta_n^{\frac{1}{p_1 \cdots p_{n-1}}}, \\ \dot{S}(t) = \max\left\{ -(1-\alpha_1) S^{a+1}(t) + \alpha_2 S^{a-b+1}(t) \phi_1, 0 \right\}, S(0) \geq 1, \end{cases} \tag{17}$$

where $c_n > 0$, $a \geq 1, b > 0$, $0 < \alpha_1 < 1$ and $\alpha_2 > 0$ are constants; h_{n+1} is defined in (6). $\phi_1(\eta) = 1 + \sum_{j=1}^{n} |\eta_j|^{\delta_j}$, where $\delta_j > 0$ is a constant, η_j is defined in (11). The controller ensures that the tracking error satisfies $\lim_{t \to \infty} |y - y_r| \leq \varepsilon$, where $\varepsilon > 0$ is a constant. Moreover, all closed-loop signals remain globally uniformly bounded.

Proof of Theorem 1. Using (6) and (7), system (1) is transformed into (10). Choose the Lyapunov function V as that in (12). Then, the derivative of V along the system (10) satisfies

$$\dot{V} = \frac{\partial V}{\partial z}\left[S^a(t)E - \frac{\dot{S}(t)}{S(t)}Hz\right] + \frac{\partial V}{\partial z}F. \tag{18}$$

By (14) in Lemma 3 and (16) in Lemma 5, there exists a controller $\nu = -c_n^{\frac{1}{p_1\cdots p_{n-1}}}\eta_n^{\frac{1}{p_1\cdots p_{n-1}}}$ such that the inequality (18) satisfies

$$\dot{V} \le -m_1 S^a(t)\sum_{j=1}^{n}\eta_j^2 - m_2\frac{\dot{S}(t)}{S(t)}\sum_{j=1}^{n}\eta_j^2 + S^{a-b}(t)m_3\left(\sum_{j=1}^{n}\eta_j^2 + \sum_{j=1}^{n}\eta_j^{2+\delta_j}\right) + d. \tag{19}$$

By definition of $\phi_1(\eta)$, (19) is rewritten as

$$\dot{V} \le -m_1 S^a(t)\sum_{j=1}^{n}\eta_j^2 - m_2\frac{\dot{S}(t)}{S(t)}\sum_{j=1}^{n}\eta_j^2 + S^{a-b}(t)m_3\phi_1\sum_{j=1}^{n}\eta_j^2 + d. \tag{20}$$

Define $m_0 = \min\{m_1, m_2\}$, introduce the parameter $\alpha_1 \in (0,1)$ and decompose the first term as $-m_1\alpha_1 S^a(t)\sum_{j=1}^{n}|\eta_j|^2 - m_1(1-\alpha_1)S^a(t)\sum_{j=1}^{n}|\eta_j|^2$. Then, (20) becomes

$$\begin{aligned}\dot{V} &\le -m_1\alpha_1 S^a(t)\textstyle\sum_{j=1}^{n}\eta_j^2 - \left[m_2\frac{\dot{S}(t)}{S(t)} + m_1(1-\alpha_1)S^a(t) - m_3 S^{a-b}(t)\phi_1\right]\sum_{j=1}^{n}\eta_j^2 + d\\ &\le -m_1\alpha_1 S^a(t)\textstyle\sum_{j=1}^{n}\eta_j^2 - m_0\left[\frac{\dot{S}(t)}{S(t)} + (1-\alpha_1)S^a(t) - \frac{m_3}{m_0}S^{a-b}(t)\phi_1\right]\sum_{j=1}^{n}\eta_j^2 + d.\end{aligned} \tag{21}$$

Now, define $\alpha_2 = \frac{m_3}{m_0}$, and choose the dynamic gain $S(t)$ such that

$$\dot{S}(t) = \max\left\{-(1-\alpha_1)S^{a+1}(t) + \alpha_2 S^{a-b+1}(t)\phi_1,\ 0\right\}, S(0) \ge 1. \tag{22}$$

Using (22), one obtains

$$\frac{\dot{S}(t)}{S(t)} + (1-\alpha_1)S^a(t) - \frac{m_3}{m_0}S^{a-b}(t)\phi_1 \ge 0. \tag{23}$$

Substituting (23) into (21), it follows that

$$\dot{V} \le -m_1\alpha_1 S^a(t)\sum_{j=1}^{n}\eta_j^2 + d. \tag{24}$$

By the definition of V, it is deduced that $V \le \alpha_2\sum_{j=1}^{n}\eta_j^2$, where $\alpha_2 > 0$ is a constant. By (22), there holds $S^a(t) \ge 1$, which together with (24) gives

$$\dot{V} \le -\alpha V + d. \tag{25}$$

where $\alpha = m_1\alpha_1/\alpha_2$. By (25), there holds

$$V(t) \le \left(V(0) - \frac{d}{\alpha}\right)e^{-\alpha t} + \frac{d}{\alpha}. \tag{26}$$

Inequality (26) and $V \ge W_i \ge \iota_i(z_i - \pi_{i-1})^{2p_1\cdots p_{i-1}}$ ($\iota_i > 0$ is a constant) imply that $z_1, \ldots, z_n$ are bounded. By the transformation (11), $\eta_1, \ldots, \eta_n$ are bounded as well. Since ϕ_1 is a continuous function, it follows that ϕ_1 is also bounded. By the update law of the dynamic gain $S(t)$ in (17), it follows that $S(t)$ is bounded as well. Hence, by (7), (17) and the boundedness of $y_r(t)$, it follows that the control input u and all system states $x_1, \ldots, x_n$ are bounded. Since $V \ge \frac{1}{2}\eta_1^2$ and $\eta_1 = z_1$, there is $|z_1| \le (\frac{2d}{\alpha})^{\frac{1}{2}}$. Using $z_1 =$

$\frac{x_1-y_r}{S^{h_1}(t)}$ and Assumption 2, there is a constant $\varepsilon > 0$ such that the tracking error satisfies $\lim_{t\to\infty}|x_1 - y_r| \le \varepsilon$. □

When $\bar{\theta}$ is unknown, define the unknown constant $\varrho = \frac{2\bar{m}_3}{\bar{m}_1}$, where $\bar{m}_1, \bar{m}_3$ are constants arising in the subsequent control design. The parameter estimation error is $\tilde{\varrho}(t) = \hat{\varrho}(t) - \varrho$, where $\hat{\varrho}(t)$ is an estimation of ϱ. Define a function $M(\hat{\varrho}(t))$, which satisfies $M(\cdot) \ge 1$ and $\frac{dM(\cdot)}{d\hat{\varrho}} \ge 0$. Introduce the transformation

$$\chi_1 = \frac{x_1 - y_r}{M^{h_1}(\hat{\varrho})}, \chi_i = \frac{x_i}{M^{h_i}(\hat{\varrho})}, \tau = \frac{u}{M^{h_{n+1}}(\hat{\varrho})}, \tag{27}$$

where h_i is defined in (6). Then system (1) is transformed into

$$\dot{\chi} = M^a(\hat{\varrho})\bar{E}(\chi,\tau) - \frac{dM(\hat{\varrho})/d\hat{\varrho}}{M(\hat{\varrho})}\dot{\hat{\varrho}}H\chi + \bar{F}, \tag{28}$$

where $\bar{E}(\chi,\tau) = [\chi_2^{p_1}, \ldots, \chi_n^{p_{n-1}}, \tau^{p_n}] \in \mathbb{R}^n$ is a vector; $H = \mathrm{diag}(h_1, h_2, \ldots, h_n) \in \mathbb{R}^{n\times n}$ is a diagonal matrix, $\bar{F} = [\frac{f_1-\dot{y}_r}{M^{h_1}(\hat{\varrho})}, \ldots, \frac{f_n}{M^{h_n}(\hat{\varrho})}]^T$. For system (28), introduce the transformation

$$\begin{aligned} \eta_1 = \chi_1, \pi_1^{p_1} = -g_1(\eta_1 + \eta_1^{\frac{\gamma_2 p_1}{\gamma_1}}), \\ \eta_i = \chi_i^{p_1\cdots p_{i-1}} - \pi_{i-1}^{p_1\cdots p_{i-1}}, \quad \pi_{i-1}^{p_1\cdots p_{i-1}} = -g_{i-1}(\eta_{i-1} + \eta_{i-1}^{\frac{\gamma_{i-1}+w}{\gamma_{i-1}}}), 2 \le i \le n. \end{aligned} \tag{29}$$

where $g_j > 0, 1 \le j \le n-1$ are constants. Next, an adaptive control scheme is proposed as the following theorem.

Theorem 2 ($\bar{\theta}$ is an unknown constant). *For system (1), if Assumptions 1 and 2 are satisfied, then there exists an adaptive controller*

$$\begin{cases} u = -g_n^{\frac{1}{p_1\cdots p_{n-1}}} M^{h_{n+1}}(\hat{\varrho}(t))(\eta_n + \eta_n^{\frac{\gamma_n+w}{\gamma_n}})^{\frac{1}{p_1\cdots p_{n-1}}}, \\ \dot{\hat{\varrho}}(t) = \frac{\Gamma}{2}M^{a-b}(\hat{\varrho})\left[\sum_{j=1}^n \eta_j^2 + \sum_{j=1}^n \eta_j^{2+\bar{\delta}_j}\right] - \mu\hat{\varrho}(t), \hat{\varrho}(0) \ge 0, \end{cases} \tag{30}$$

where $g_n > 0, a > 0, b > 0, \bar{\delta}_j > 0$ and $\mu > 0$ are design constants; $\Gamma > 0$ is the adaptive gain; γ_n, w are defined in Assumption 1; h_{n+1} is given in (6). $M(\hat{\varrho})$ is chosen as a monotone non-decreasing function satisfying $M(\hat{\varrho}) \ge \max\{1, \hat{\varrho}^{1/b}\}$ and $\frac{dM(\hat{\varrho})}{d\hat{\varrho}} \ge 0$; η_j is given by (29). The controller ensures that all closed-loop signals remain globally bounded. Moreover, the tracking error satisfies $\lim_{t\to\infty}|y - y_r| \le \bar{\varepsilon}$, where $\bar{\varepsilon} > 0$ is a constant.

Proof of Theorem 2. By the definition of $\bar{F}_i$, it follows that

$$\bar{F}_1 = \frac{f_1 - \dot{y}_r}{M^{h_1}(\hat{\varrho})}, \bar{F}_i = \frac{f_i}{M^{h_i}(\hat{\varrho})}, 2 \le i \le n. \tag{31}$$

By Assumptions 1 and 2, $\bar{F}_1$ and $\bar{F}_i$ in (31) satisfy

$$\begin{aligned} |\bar{F}_1| &\le \frac{\bar{\theta}\left(|x_1| + |x_1|^{\frac{\gamma_1+w}{\gamma_1}}\right) + M_0 + \bar{w}_1}{M^{h_1}(\hat{\varrho})}, \\ |\bar{F}_i| &\le \frac{\bar{\theta}\left(\sum_{j=1}^{i-1}|x_j|^{\frac{1}{p_j\cdots p_{i-1}}} + |x_i| + \sum_{j=1}^{i}|x_j|^{\frac{\gamma_i+w}{\gamma_j}}\right) + \bar{w}_i}{M^{h_i}(\hat{\varrho})}, 2 \le i \le n. \end{aligned} \tag{32}$$

Using (27), (29) and (32), $\bar{F}_i$ satisfies

$$|\bar{F}_i| \leq M^{a-b}(\hat{\varrho})\bar{c}\left[\sum_{j=1}^{i}|\eta_j|^{\frac{1}{p_1\cdots p_{i-1}}}+\sum_{j=1}^{i}|\eta_j|^{\frac{\gamma_i+w}{p_1\cdots p_{j-1}\gamma_j}}\right]+\frac{\hat{\omega}_i}{M^{h_i}(\hat{\varrho})}, \tag{33}$$

where $\bar{c}>0$ is an unknown constant. Define $\sigma=(\gamma_n+w)p_n\cdots p_1$, and construct the function $\bar{W}_i=\int_{\pi_{i-1}}^{\chi_i}\left(l^{p_1\cdots p_{i-1}}-\pi_{i-1}^{p_1\cdots p_{i-1}}\right)^{2-\frac{1}{p_1\cdots p_{i-1}}}dl+\int_{\pi_{i-1}}^{\chi_i}\left(l^{p_1\cdots p_{i-1}}-\pi_{i-1}^{p_1\cdots p_{i-1}}\right)^{\frac{2\sigma-\gamma_i-w}{p_1\cdots p_{i-1}\gamma_i}}dl$. Choose the Lyapunov function

$$U(\chi,\tilde{\varrho})=\sum_{i=1}^{n}\beta_n\cdots\beta_i\bar{W}_i(\chi)+\frac{\bar{m}_1}{2\Gamma}\tilde{\varrho}^2, \tag{34}$$

where $\bar{m}_1,\beta_j\,(1\leq j\leq n)$ are positive constants. Similar to (16), it follows from (33) and (34) that

$$\frac{\partial U}{\partial\chi}\bar{F}\leq M^{a-b}(\hat{\varrho})\bar{m}_3\left[\sum_{j=1}^{n}\eta_j^2+\sum_{j=1}^{n}\eta_j^{2+\bar{\delta}_j}\right]+\bar{d}, \tag{35}$$

where $\bar{m}_3$, $\bar{\delta}_j$, and $\bar{d}$ are positive constants. Noting that $M(\cdot)\geq 1$, $\frac{dM(\cdot)}{d\hat{\varrho}}\geq 0$, and $\dot{\hat{\varrho}}\geq 0$, and by following steps similar to those in the proof of Lemma 1, there exists $\tau=-g_n^{\frac{1}{p_1\cdots p_{n-1}}}(\eta_n+\eta_n^{\frac{\gamma_n+w}{\gamma_n}})^{\frac{1}{p_1\cdots p_{n-1}}}$ such that

$$\frac{\partial U}{\partial\chi}\left[M^a(\hat{\varrho})\bar{E}(\chi,\tau)-\frac{dM(\hat{\varrho})/d\hat{\varrho}}{M(\hat{\varrho})}\dot{\hat{\varrho}}H\chi\right]\leq-\bar{m}_1M^a(\hat{\varrho})\left[\sum_{j=1}^{n}\eta_j^2+\sum_{j=1}^{n}\eta_j^{2+\bar{\delta}_j}\right]. \tag{36}$$

The derivative of U along the system (28) satisfies

$$\dot{U}=\frac{\partial U}{\partial\chi}\left[M^a(\hat{\varrho})\bar{E}-\frac{dM(\hat{\varrho})/d\hat{\varrho}}{M(\hat{\varrho})}\dot{\hat{\varrho}}H\chi\right]+\frac{\partial U}{\partial\chi}\bar{F}+\frac{\bar{m}_1}{\Gamma}\tilde{\varrho}\dot{\hat{\varrho}} \tag{37}$$

Substituting (35) and (36) into (37), it yields that

$$\dot{U}\leq-\bar{m}_1M^a(\hat{\varrho})\left[\sum_{j=1}^{n}\eta_j^2+\sum_{j=1}^{n}\eta_j^{2+\bar{\delta}_j}\right]+M^{a-b}(\hat{\varrho})\bar{m}_3\left[\sum_{j=1}^{n}\eta_j^2+\sum_{j=1}^{n}\eta_j^{2+\bar{\delta}_j}\right]+\bar{d}+\frac{\bar{m}_1}{\Gamma}\tilde{\varrho}\dot{\hat{\varrho}}. \tag{38}$$

Using $\varrho=\frac{2\bar{m}_3}{\bar{m}_1}$, and noting $-\bar{m}_1=-\frac{\bar{m}_1}{2}-\frac{\bar{m}_1}{2}$, (38) becomes

$$\begin{aligned}\dot{U}\;\leq\;&-\tfrac{\bar{m}_1}{2}M^a(\hat{\varrho})\left[\textstyle\sum_{j=1}^{n}\eta_j^2+\sum_{j=1}^{n}\eta_j^{2+\bar{\delta}_j}\right]-\tfrac{\bar{m}_1}{2}M^a(\hat{\varrho})\left[\textstyle\sum_{j=1}^{n}\eta_j^2+\sum_{j=1}^{n}\eta_j^{2+\bar{\delta}_j}\right]\\&+\tfrac{\bar{m}_1}{2}M^{a-b}(\hat{\varrho})\varrho\left[\textstyle\sum_{j=1}^{n}\eta_j^2+\sum_{j=1}^{n}\eta_j^{2+\bar{\delta}_j}\right]+\tfrac{\bar{m}_1}{\Gamma}\tilde{\varrho}\dot{\hat{\varrho}}+\bar{d}\\\leq\;&-\tfrac{\bar{m}_1}{2}M^a(\hat{\varrho})\left[\textstyle\sum_{j=1}^{n}\eta_j^2+\sum_{j=1}^{n}\eta_j^{2+\bar{\delta}_j}\right]-\tfrac{\bar{m}_1}{2}\left(M^a(\hat{\varrho})-M^{a-b}(\hat{\varrho})\varrho\right)\left[\textstyle\sum_{j=1}^{n}\eta_j^2+\sum_{j=1}^{n}\eta_j^{2+\bar{\delta}_j}\right]\\&+\tfrac{\bar{m}_1}{\Gamma}\tilde{\varrho}\left(\dot{\hat{\varrho}}-\tfrac{\Gamma}{2}M^{a-b}(\hat{\varrho})\left[\textstyle\sum_{j=1}^{n}\eta_j^2+\sum_{j=1}^{n}\eta_j^{2+\bar{\delta}_j}\right]\right)+\bar{d}.\end{aligned} \tag{39}$$

Now, choose $M(\hat{\varrho})\geq\hat{\varrho}^{1/b}$ and

$$\dot{\hat{\varrho}}(t)=\frac{\Gamma}{2}M^{a-b}(\hat{\varrho})\left[\sum_{j=1}^{n}\eta_j^2+\sum_{j=1}^{n}\eta_j^{2+\bar{\delta}_j}\right]-\mu\hat{\varrho}(t),\hat{\varrho}(0)\geq 0. \tag{40}$$

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Substituting (40) into (39), it follows that

$$\dot{U} \leq -\frac{\bar{m}_1}{2}M^a(\hat{\varrho})\left[\sum_{j=1}^{n}\eta_j^2+\sum_{j=1}^{n}\eta_j^{2+\bar{\delta}_j}\right]-\frac{\bar{m}_1\mu}{\Gamma}\tilde{\varrho}\hat{\varrho}(t)+\bar{d}. \tag{41}$$

By the definition of U and noting that $M^a(\hat{\varrho}) \geq 1$ and $\frac{\bar{m}_1\mu}{\Gamma}\tilde{\varrho}\hat{\varrho}(t) \leq -\frac{\bar{m}_1\mu\tilde{\varrho}^2}{2\Gamma}+\frac{\bar{m}_1\mu\varrho^2}{2\Gamma}$, (41) becomes

$$\dot{U} \leq -\bar{\alpha}_1 U+\bar{d}_1. \tag{42}$$

where $\bar{\alpha}_1$ and $\bar{d}_1$ are positive constants. Using (42) and an analysis similar to that of Theorem 1, it can be proved that Theorem 2 holds. □

Remark 3. *For system (1), there have been some control results reported. Specifically, based on the adding a power integrator method, various control problems including stabilization control, tracking control, and output-feedback control have been considered [33,34]. By exploring intelligent control methods, finite-time control, adaptive control, and other strategies have also been proposed [35,36]. Some works have also investigated homogeneous domination approaches [28,29]. However, those results are not applicable to system (1) with polynomial growth condition, disturbances, and parameter uncertainties. In this work, two control approaches are presented. The first approach adopts the dynamic gain to dominate the nonlinear terms and reduces the complexity of the resulting controller. The second approach provides a new adaptive mechanism to deal with parameter uncertainty. Furthermore, it requires only one parameter estimation and circumvents the issue of over-parameterization.*

Remark 4. *For bounded disturbances $w_i(t)$, there holds the inequality (2) by virtue of Assumption 1. Based on the transformation (7), system (1) is transformed into (10). Then, by constructing the Lyapunov function V given in (12), and using Lemmas 3–5, the time derivative of V along the closed-loop system satisfies (25) which further yields (26). Following the arguments in Theorem 1, it follows that $z_1,\ldots,z_n$, and $S(t)$ are bounded. By the transformation (7), all system signals $x_1,\ldots,x_n$, $S(t)$ and u in the closed-loop system are proved to be bounded. Accordingly, the robustness of the control loop is established. The robustness for the second scheme in Theorem 2 can be proved similarly.*

Remark 5. *The control parameters affect the closed-loop system in the following manner. Specifically, in the controller (17), the feedback gains c_n, and $c_j, 1 \leq j < n$ (defined in η_n, see (11)) affect the tracking accuracy. Larger $c_1,\ldots,c_n$ can reduce the tracking error. However, the magnitude of the control input u will increase accordingly. The parameters α_1 and α_2 govern the evolution of the dynamic gain $S(t)$. A smaller α_1 prevents unbounded growth of $S(t)$ so as to avoid excessive control effort. A larger α_2 enables faster growth of $S(t)$ to compensate for strong nonlinearities. The constants a, b, p_i, h_i are determined by the polynomial growth condition in Assumption 1 and the system powers, and cannot be arbitrarily adjusted. The initial condition $S(0)$ affects the transient response. Larger $S(0)$ helps suppress initial tracking error, but may also induce a large control impulse at the initial stage. In the controller (30), the gains Γ, μ affect the growing speed of $\hat{\varrho}(t)$. A Larger Γ and a smaller μ accelerate the parameter estimation $\hat{\varrho}(t)$ to grow faster. The role of $g_i, 1 \leq i \leq n$ is identical to that of c_i in (17), and the role of $\hat{\varrho}(0)$ is similar to that of $S(0)$. The effects of $a, b, p_i, h_{n+1}, \gamma_{n+1}, \bar{\delta}_i$ are also determined by system conditions in Assumption 1 and the system powers, and these parameters cannot be tuned manually.*

5. Simulation Examples

In this paper, all simulations are performed using MATLAB 2010b (The MathWorks, Inc., Natick, MA, USA) on a laptop equipped with a 2.2 GHz i7-1360P processor and 16 GB RAM.

Example 1. *Consider the following controlled pendulum model [37]*

$$mL\ddot{q} + kL\dot{q} + mg\sin q = u, \tag{43}$$

where q is the angular displacement of the rod from the vertical axis, $\dot{q}$ is the angular velocity, u is the control input. The other parameters are provided in Table 1. By introducing the transformation $x_1 = mLq$, $x_2 = mL\dot{q}$, (43) is transformed into

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = u - \dfrac{k}{m}x_2 - mg\sin\left(\dfrac{x_1}{mL}\right). \end{cases} \tag{44}$$

Define $f_1 = 0$ and $f_2 = -\dfrac{k}{m}x_2 - mg\sin\left(\dfrac{x_1}{mL}\right)$. It follows that Assumption 1 holds for system (44). Define the system output as $y = x_1$, and choose the reference signal as $y_r(t) = \cos(t)$. This shows that Assumption 2 also holds. Then, using the control method in Theorem 1, the dynamic tracking controller is designed as

$$\begin{cases} u = -c_2 S^{h_{n+1}}(t)\eta_2, \\ \dot{S}(t) = \max\left\{-(1-\alpha_1)S^{a+1}(t) + \alpha_2 S^{a-b+1}(t)\phi_1, 0\right\}, S(0) = 1, \end{cases} \tag{45}$$

where $\eta_1 = \frac{x_1 - y_r}{S(t)}$, $\eta_2 = \frac{x_2}{S^2} + c_1\eta_1$, and $\phi_1(\eta) = 1 + \eta_1^{\delta_1} + \eta_2^{\delta_2}$.

Table 1. Physical meanings of symbols.

Symbols	Physical Meanings
$k = \frac{1}{4}\,\mathrm{N/m^2}$	Frictional coefficient
$m = \frac{1}{4}\,\mathrm{kg}$	Mass of the bob
$L = 4\,\mathrm{m}$	Length of the rod
$g = 10\,\mathrm{m/s^2}$	Gravitational acceleration

In the simulation, the system parameters in (45) are chosen as $a = 1$, $b = \frac{3}{7}$, $h_1 = 1$, $h_2 = 2$, $h_3 = 3$, $c_1 = 6.5$, $c_2 = 13$, $\alpha_1 = 0.1$, $\alpha_2 = 2$, $\delta_1 = \delta_2 = \frac{4}{5}$. The system's initial conditions are $x_1(0) = 0$, $x_2(0) = 0.1$, $S(0) = 1$.

To further verify the effectiveness of the control strategy, some comparisons are made with the fuzzy control method from [38]. Figure 1 shows that the system output under the proposed method achieves higher tracking accuracy in tracking the reference signal. Figure 2 reveals that the tracking error $e_1 = y - y_r$ of the proposed method is smaller than that of the fuzzy control method, demonstrating stronger error suppression capability. Figure 3 illustrates that the state x_2 under both methods undergoes rapid adjustment during the initial stage. After the transient process, the state responses from the two methods largely coincide and remain stable. As shown in Figure 4, the control inputs for both methods exhibit distinct transient peaks during the initial phase. Once the transient response decays, u decreases rapidly and exhibits similar bounded periodic oscillations.

To quantitatively compare the tracking performance and control effort of the two control methods, the following performance metrics are considered: Integral of Squared Error (ISE), Integral of Absolute Error (IAE), Integral of Time multiplied by Absolute Error (ITAE), and Root Mean Square of the controller (RMSC). To ensure the validity of the steady-state performance evaluation, RMSC is calculated over the last two complete reference periods. The comprehensive performance metrics are shown in Table 2. Table 2 shows that the metrics ISE, IAE, and ITAE for the proposed method are smaller than those of the fuzzy control method, which indicates that the proposed method has smaller cumulative and time-weighted tracking

errors. At the same time, the values of RMSC for the two methods are very close, indicating that the proposed method improves tracking accuracy without significantly increasing the control effort. This example illustrates that the proposed method not only satisfactorily achieves the control objective, but also yields superior tracking performance.

Table 2. Comparisons for some performance metrics.

Control Method (Unit)	IAE (kg·m·s)	ISE ($kg^2 \cdot m^2 \cdot s$)	ITAE ($kg \cdot m \cdot s^2$)	RMSC (N)
The proposed method	0.6577	0.1254	7.1604	1.1273
Fuzzy control	1.2280	0.2299	14.3814	1.1192

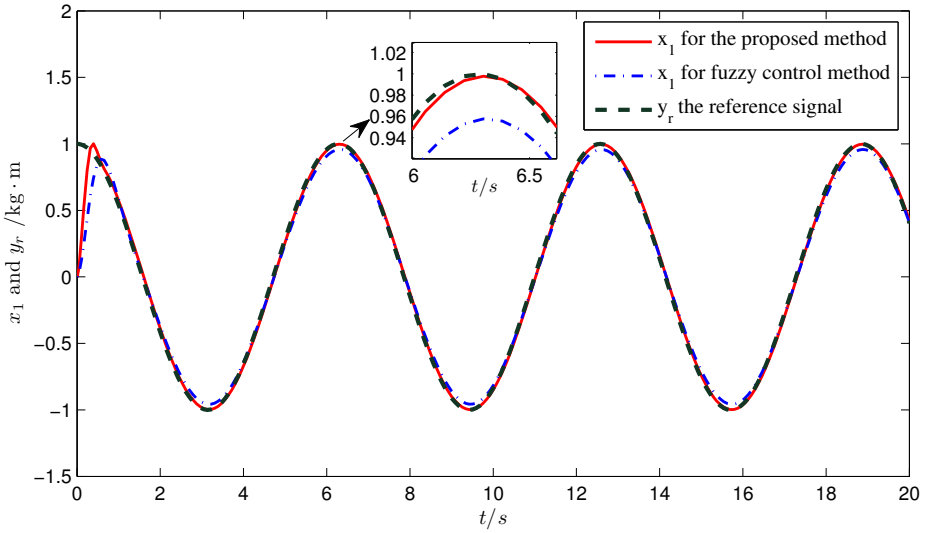


Figure 1. Trajectories of y_r and x_1 under two methods.

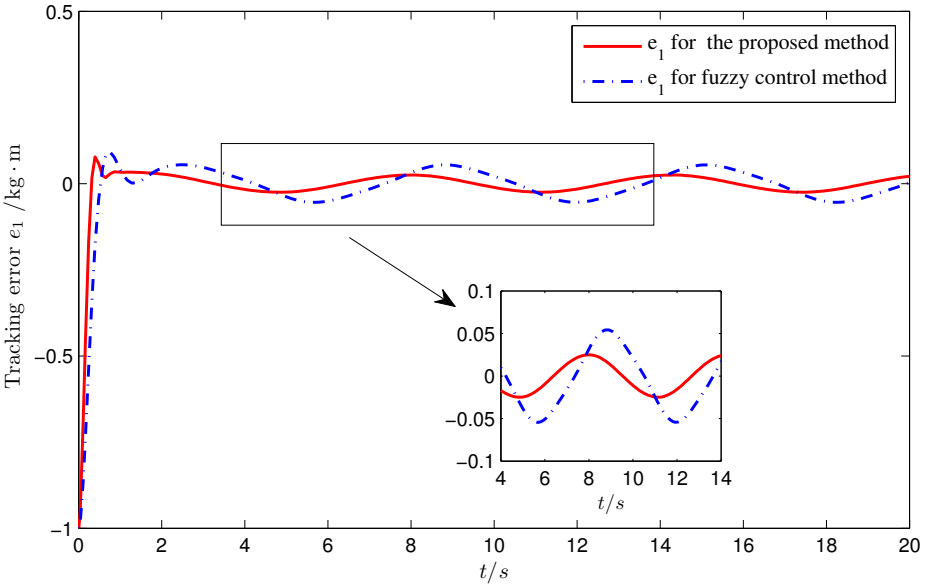


Figure 2. Trajectories of e_1 under two methods.

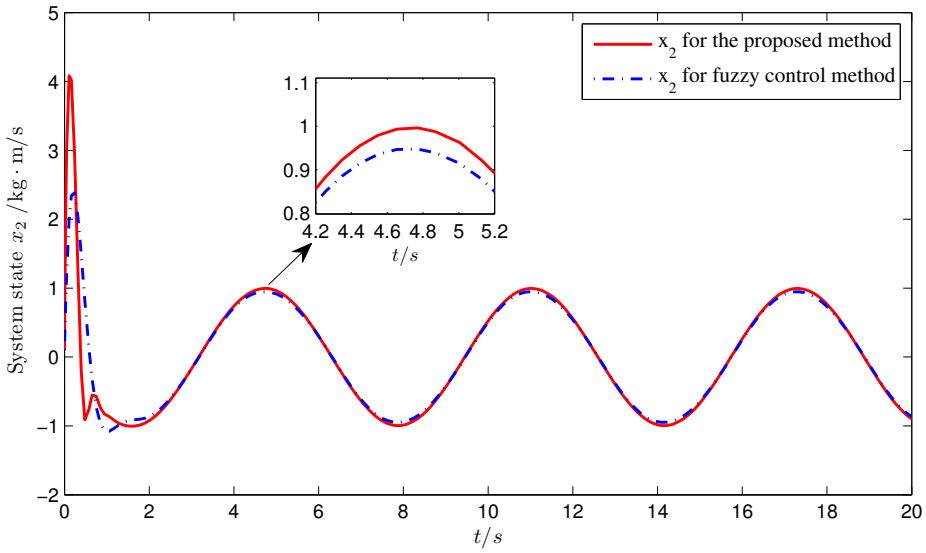


Figure 3. Trajectories of x_2 under two methods.

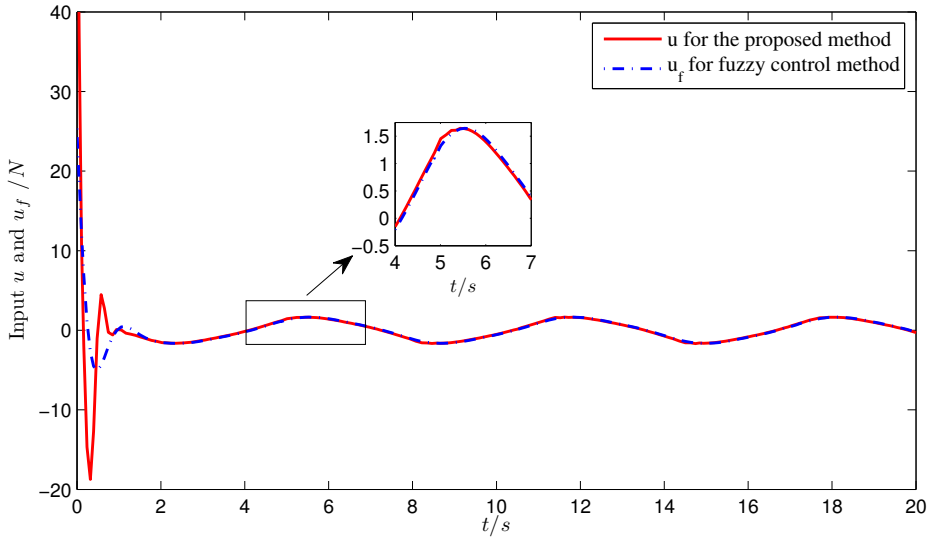


Figure 4. Trajectories of u under two methods.

Example 2. *Consider the following uncertain nonlinear system*

$$\begin{cases} \dot{x}_1 = x_2 + \theta_1 x_1^{5/3} \sin x_2 + \theta_2 x_1, \\ \dot{x}_2 = u + \theta_3 x_2^{7/5} \cos x_2 + w_1(t), \\ y = x_1, \end{cases} \tag{46}$$

where x_1, x_2 are the system states, u is the control input, y is the system output, $w_1(t)$ is the external disturbance, $\theta_j, 1 \le j \le 3$ are unknown constants. Define $f_1 = \theta_1 x_1^{5/3} \sin x_2 + \theta_2 x_1$ and $f_2 = \theta_3 x_2^{7/5} \cos x_2 + w_1(t)$. It then follows that Assumption 1 holds for system (46). Choose the reference signal $y_r(t) = \sin t$. It follows that Assumption 2 holds as well. Then, using the method in Theorem 2, the adaptive controller is designed as

$$\begin{cases} u = -g_2 M^{h_3}(\hat{\varrho}(t))(\eta_2 + \eta_2^{\frac{\gamma_2 + w}{\gamma_2}}), \\ \dot{\hat{\varrho}}(t) = \frac{\Gamma}{2} M^{a-b}(\hat{\varrho}) \left[\sum_{j=1}^{2} \eta_j^2 + \sum_{j=1}^{n} \eta_j^{2+\bar{\delta}_j} \right] - \mu \hat{\varrho}(t), \end{cases} \tag{47}$$

where $M(\hat{\varrho}) = \hat{\varrho}^{\frac{1}{b}}$, $\eta_1 = \frac{x_1 - y_r}{M^{h_1}(\hat{\varrho})}$, $\eta_2 = \frac{x_2}{M^{h_2}(\hat{\varrho})} + g_1 \left(\eta_1 + \eta_1^{\gamma_2/\gamma_1} \right)$.

In this example, the system parameters are chosen as $\theta_1 = 0.5$, $\theta_2 = 0.5$, $\theta_3 = 1$, $\gamma_1 = 1$, $\gamma_2 = \frac{5}{3}$, $\gamma_3 = \frac{7}{3}$, and $w = \frac{2}{3}$. The control parameters in (47) are chosen as $a = 1$, $b = 0.2$, $h_1 = 1$, $h_2 = 2$, $h_3 = 3$, $g_1 = 6$, $g_2 = 8$, $\Gamma = 11$, $\mu = 0.0001$, $\bar{\delta}_1 = 2$, and $\bar{\delta}_2 = \frac{2}{5}$. The initial conditions are $x_1(0) = 0.3$, $x_2(0) = -0.5$, $\hat{\varrho}(0) = 1$. To further evaluate the robustness of the designed controller, in the following, three cases are studied:

Case A1: $w_1(t) = 0$;

Case A2: $w_1(t) = 1$;

Case A3: $w_1(t) = sin(t)$.

Figures 5–9 show the system responses of the closed-loop system (46) and (47) under Cases A1–A3. Figure 5 depicts the trajectories of the system output x_1 and the reference signal y_r. After a brief transient period, the system output under Cases A1–A3 can track the periodic reference signal well, and the corresponding curves are almost superimposed. Figure 6 demonstrates that the trajectories of state x_2 under Cases A1–A3 achieve stable periodic motions and remain bounded after rapid adjustments in the initial stage. As shown in Figure 7, under Cases A1–A3, the trajectories of the control input u undergo transient changes at the initial moment, and then rapidly decrease and exhibit bounded periodic motions. Figure 8 illustrates that the parameter estimate $\hat{\varrho}$ adjusts rapidly during the initial stage. After that, it increases slowly and remains positive and bounded. This indicates that the adaptive law adjusts the control gain online in response to changes in the system states and the tracking error under different disturbance conditions. Figure 9 shows that the trajectories of $e = y - y_r$ under Cases A1–A3 decrease rapidly and remain within a neighborhood of zero.

Further comparisons of the performance indices (see Table 3) show that the tracking-error metrics vary only slightly under the considered disturbance conditions, whereas the value of RMSC increases. These results indicate that the considered external disturbances primarily increase the controller's compensation effort without significantly degrading the system's tracking accuracy. In summary, under Cases A1–A3, the system output can track the reference signal, while the other signals x_2, u, $\hat{\varrho}$ remain bounded. The performance indices further show that the considered constant and sinusoidal disturbances produce only minor changes in the system performance, which indicates that the proposed controller has satisfactory robustness.

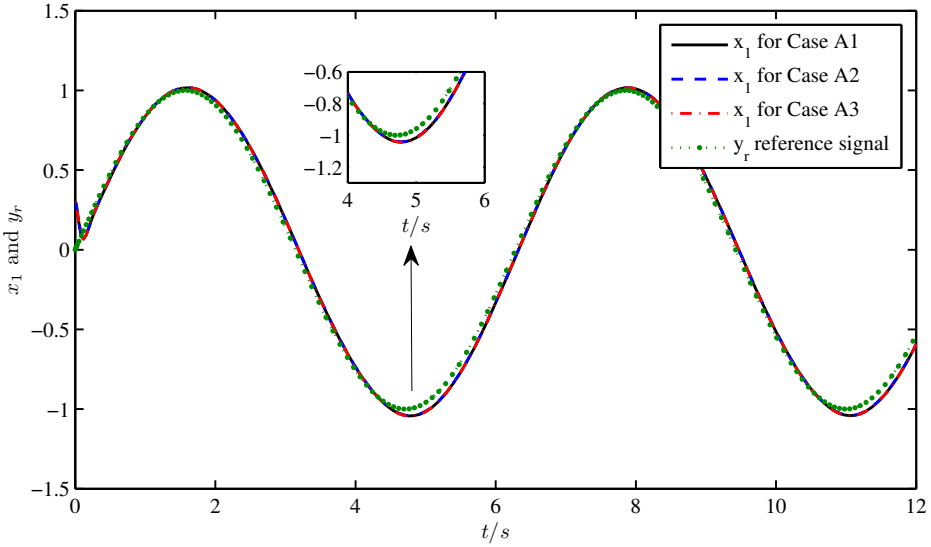


Figure 5. Trajectories of y_r and x_1 under Cases A1–A3.

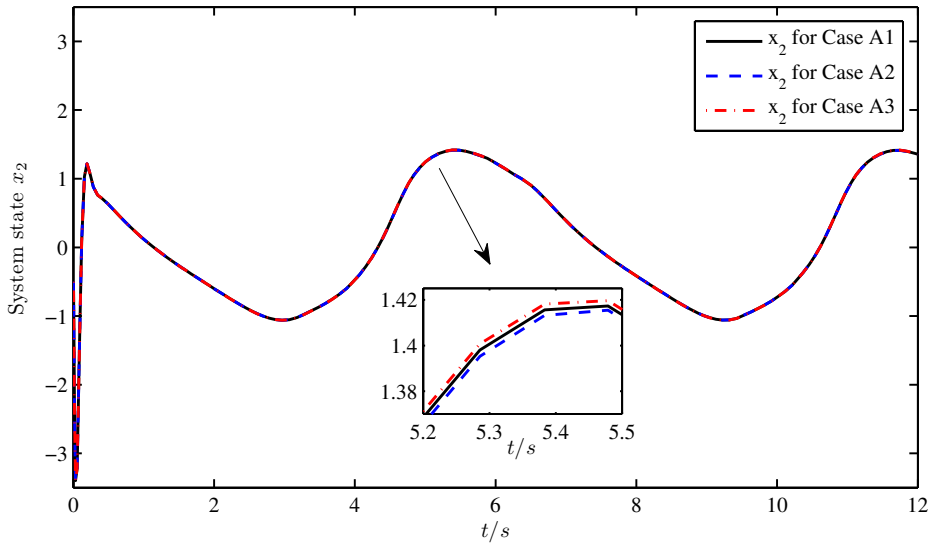


Figure 6. Trajectories of x_2 under Cases A1–A3.

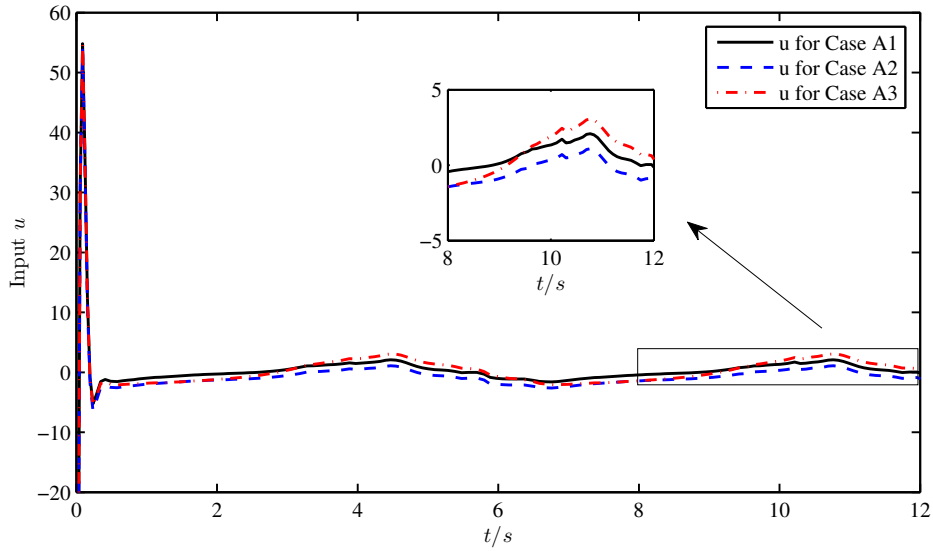


Figure 7. Trajectories of u under Cases A1–A3.

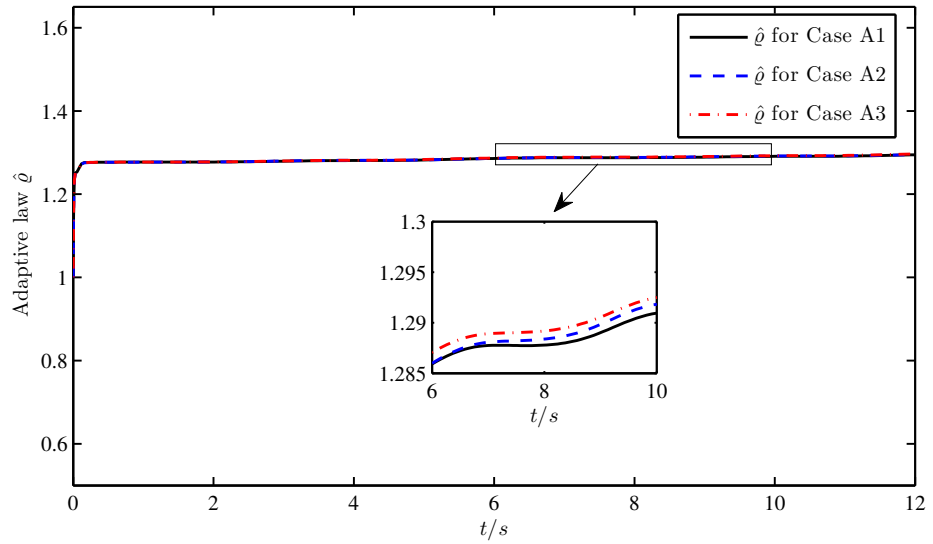


Figure 8. Trajectories of $\hat{\varrho}$ under Cases A1–A3.

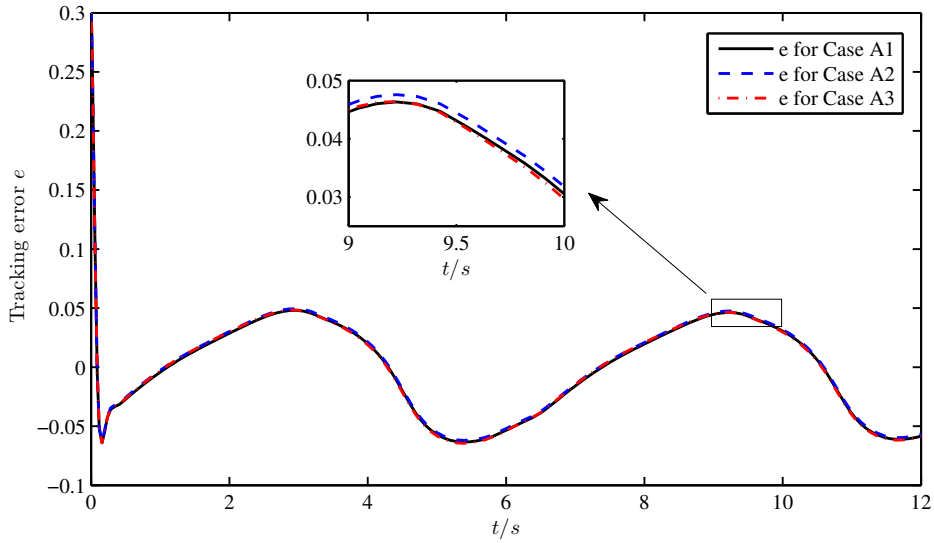


Figure 9. Trajectories of e under Cases A1–A3.

Table 3. Performance metrics under Cases A1–A3.

Value of the Disturbance (Unit)	ISE (s)	IAE (s)	ITAE (s^2)	RMSC ([-])
Case A1: $w_1(t) = 0$	0.0416	0.9628	14.0140	1.0498
Case A2: $w_1(t) = 1$	0.0413	0.9630	14.0094	1.4059
Case A3: $w_1(t) = sin(t)$	0.0421	0.9676	14.0572	1.6168

6. Discussion

This paper presents two dynamic control design schemes. Some discussions are presented as follows.

- Assumption 1 characterizes polynomial growth condition of the system, while Assumption 2 ensures that the reference signal is bounded. These conditions are relatively general. In practical scenarios, the nonlinear terms of practical systems may have one or several terms described in (2) of Assumption 1. Nevertheless, there are some limitations of the proposed approaches. For example, the proposed methods are somewhat conservative since they mainly depend on some general assumptions. In addition, the methods require the lower-triangular structure of the system. Hence, they are not applicable to upper-triangular systems.
- Some practical implementation issues need to be discussed. For example, the considered system contains numerous unknown parameters. In such cases, one can determine the bounds of all these parameters and estimate the bounds of nonlinear terms in accordance with Assumption 1. Furthermore, certain designed control gains may take large values. In practical applications, these gains can be gradually reduced to achieve satisfactory transient performance. In addition, from the control design steps, one can observe that the designed controller is derived using only basic algebraic operations. This indicates that the computational complexity is very low and the presented algorithm is suitable for real devices with low computational performance.
- Compared with the existing methods, the main challenges encountered in this paper are as follows. (i) The high-order powers and polynomial growth conditions of the studied system render many methods, such as backstepping control and dynamic surface control, difficult to apply. For example, conventional quadratic Lyapunov functions are not suitable for handling high-order power terms in the system. Further-

more, the polynomial growth conditions generally result in complicated controller structures. (ii) When dynamic-gain-based transformations are adopted, the original system becomes a time-varying system involving $S(t)$ and its derivative $\dot{S}(t)$, see system (10). Accordingly, constructing an appropriate $S(t)$ and designing the corresponding controller for the resulting system constitute nontrivial tasks. (iii) When the studied system also contains unknown parameters, designing an adaptive tracking controller becomes challenging. In this case, the transformation (7) and the Lyapunov function V used in Theorem 1 are not applicable. Developing a new transformation and constructing a suitable Lyapunov function for control design are not easy, and hence the adaptive control design becomes more challenging.

- The main novelty and practical advantages of the proposed controllers are as follows. First, the explicit form of the designed controllers can be obtained directly once the system conditions are satisfied. There is no need to calculate step by step. Second, the constructed adaptive controller introduces only one parameter estimate and avoids the over-parameterization problem.

7. Conclusions

This paper considers the robust output tracking control issue for a category of high-order nonlinear systems affected by polynomial growth conditions and external disturbances. For the case where the system has a known nonlinear growth condition, a dynamic-gain tracking controller is developed using a dynamic scaling transformation and the homogeneous domination technique. The designed control scheme guarantees that all signals in the closed-loop system remain bounded and that the prescribed tracking precision is attained. For the case where the system has an unknown nonlinear growth condition, an adaptive tracking controller is constructed by constructing an appropriate Lyapunov function and an adaptive control strategy. The efficacy of the designed control scheme is verified through numerical simulations. Based on the current framework, future work can focus on some valuable directions. First, the proposed strategy can be generalized to time-delay systems, which are widely encountered in practical engineering. Second, combining the proposed method with the finite-time control theories is also a promising research direction, which can improve the control performance of the system.

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Appendix A

Proof of Lemma 3. System (13) can be rewritten as

$$\begin{cases} \dot{\tilde{z}}_1 = A(t)\tilde{z}_2^{p_1} - B(t)h_1\tilde{z}_1, \\ \dot{\tilde{z}}_2 = A(t)\tilde{z}_3^{p_2} - B(t)h_2\tilde{z}_2, \\ \quad \vdots \\ \dot{\tilde{z}}_n = A(t)\tilde{v}^{p_n} - B(t)h_n\tilde{z}_n. \end{cases} \tag{A1}$$

Next, it is proved using a recursive control design.

Step 1. Choose the function $V_1(\tilde{z}_1) = W_1(\tilde{z}_1)$, then its derivative along system (A1) satisfies

$$\dot{V}_1 = \eta_1 A(t)\tilde{z}_2^{p_1} - B(t)h_1\eta_1^2 = \eta_1 A(t)\left(\tilde{z}_2^{p_1} + \pi_1^{p_1} - \pi_1^{p_1}\right) - B(t)h_1\eta_1^2. \tag{A2}$$

Now, choose the virtual controller $\pi_1^{p_1} = -c_1\eta_1$, where $c_1 > 0$ is a constant. Substituting it into (A2), it follows that

$$\dot{V}_1 = -c_1 A(t)\eta_1^2 + \eta_1\left(\tilde{z}_2^{p_1} - \pi_1^{p_1}\right) - B(t)h_1\eta_1^2. \tag{A3}$$

Step k ($k \geq 2$). Similar to (A3) in step 1, suppose for step $k-1$, there are constants $\rho_0, \ldots, \rho_{k-1}$ such that

$$\begin{aligned} \dot{V}_{k-1} \quad &\leq -(n-k+2)\rho_0 \cdots \rho_{k-2}A(t)\textstyle\sum_{i=1}^{k-1}\eta_i^2 - \sigma_{k-1}B(t)\sum_{i=1}^{k-1}\eta_i^2 \\ &+\eta_{k-1}^{2-\frac{1}{p_1\cdots p_{k-1}}}\left(\tilde{z}_k^{p_{k-1}} - \pi_{k-1}^{p_{k-1}}\right). \end{aligned} \tag{A4}$$

where $\rho_0, \ldots, \rho_{k-2}$, and σ_{k-1} are positive constants. In step k, we choose the function $V_k = \beta_k V_{k-1} + W_k$. Similar to the work [29], using (A4), and Lemmas 1 and 2, one can construct the virtual controller $\pi_k^{p_1\cdots p_k} = -c_k\eta_k$ such that

$$\dot{V}_k \quad \leq -(n-k+1)\rho_0 \cdots \rho_{k-1}A(t)\textstyle\sum_{i=1}^{k}\eta_i^2 - \rho_k B(t)\sum_{i=1}^{k}\eta_i^2 + \eta_k^{2-\frac{1}{p_1\cdots p_k}}\left(\tilde{z}_{k+1}^{p_k} - \pi_k^{p_k}\right). \tag{A5}$$

where ρ_{k-1} and σ_k are positive constants. Finally, in step n, using a derivation similar to that for (A5), one can construct $\tilde{v}^{p_1\cdots p_n} = -c_n\eta_n$ such that

$$\frac{\partial V_n}{\partial \tilde{z}}\dot{\tilde{z}} = \dot{V}_n \leq -\rho_0 \cdots \rho_n A(t)\sum_{i=1}^{n}\eta_i^2 - \sigma_n B(t)\sum_{i=1}^{n}\eta_i^2. \tag{A6}$$

Define $m_1 = \rho_0 \cdots \rho_n$, $m_2 = \sigma_n$. From (A6), it follows that Lemma 3 holds. □

Appendix B

Proof of Lemma 4. By Assumption 1, (4) and (5) of Lemma 2, and (7), it follows that

$$\begin{aligned} |f_i| &= \bar{\theta}\big[\textstyle\sum_{j=1}^{i-1}|x_j|^{\frac{1}{p_j\cdots p_{i-1}}} + |x_i| + \sum_{j=1}^{i}|x_j|^{\frac{\gamma_i+w}{\gamma_j}}\big] + \bar{\omega}_i, \\ &= \bar{\theta}\big[\textstyle\sum_{j=1}^{i-1}S(t)^{\frac{h_j}{p_j\cdots p_{i-1}}}|z_j|^{\frac{1}{p_j\cdots p_{i-1}}} + S(t)^{h_i}|z_i| + \sum_{j=1}^{i}S(t)^{h_j\cdot\frac{\gamma_i+w}{\gamma_j}}|z_j|^{\frac{\gamma_i+w}{\gamma_j}}\big] + |y_r|^{\frac{1}{p_1\cdots p_{i-1}}} + \bar{\omega}_i \\ &\leq \bar{\theta}\big[\textstyle\sum_{j=1}^{i-1}S(t)^{\frac{h_j}{p_j\cdots p_{i-1}}}|z_j|^{\frac{1}{p_j\cdots p_{i-1}}} + S(t)^{h_i}|z_i| + \sum_{j=1}^{i}S(t)^{h_j\cdot\frac{\gamma_i+w}{\gamma_j}}|z_j|^{\frac{\gamma_i+\omega}{\gamma_j}}\big] + \hat{\omega}_i, \end{aligned} \tag{A7}$$

where $\hat{\omega}_i \geq M_0 + \bar{\omega}_i$ is a constant. By the definition of F_i and (A7), there is

$$|F_i| \leq \bar{\theta}\big[\sum_{j=1}^{i-1}S(t)^{\frac{h_j}{p_j\cdots p_{i-1}}-h_i}|z_j|^{\frac{1}{p_j\cdots p_{i-1}}} + |z_i| + \sum_{j=1}^{i}S(t)^{h_j\cdot\frac{\gamma_i+w}{\gamma_j}-h_i}|z_j|^{\frac{\gamma_i+w}{\gamma_j}}\big] + \frac{\hat{\omega}_i}{S^{h_i}(t)}. \tag{A8}$$

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From the definition of h_i and γ_i, it follows that

$$\frac{h_j}{p_j \cdots p_{i-1}} - h_i \leq 0 \leq h_j \frac{\gamma_i + w}{\gamma_j} - h_i \leq a - \frac{a-w}{p_0 \cdots p_{i-1}\gamma_i}. \tag{A9}$$

Then, using (A7), (A8) and (A9), it yields that

$$|F_i| \leq S^{a-b}(t)\bar{c}\left[\sum_{j=1}^{i-1} |z_j|^{\frac{1}{p_j \cdots p_{i-1}}} + |z_i| + \sum_{j=1}^{i} |z_j|^{\frac{\gamma_i + w}{\gamma_j}}\right] + \frac{\hat{\omega}_i}{S^{h_i}(t)}, \tag{A10}$$

where $b = \frac{a-w}{p_0 \cdots p_{i-1}\gamma_i}$. Using the transformations in (11), there holds the inequality (15), where $\bar{c} = \bar{\theta} \cdot \max_{2 \leq j \leq n}\{1 + c_{j-1}^{\frac{1}{p_1 \cdots p_{i-1}}}, 2^{\frac{\gamma_i + w}{p_i \cdots p_{j-1}\gamma_j} - 1} c_{j-1}^{\frac{\gamma_i + w}{p_1 \cdots p_{j-1}\gamma_j}}\}$. □

Appendix C

Proof of Lemma 5. By the definition of V, there exist the inequalities

$$\frac{\partial V}{\partial z} F \quad \leq \quad \sum_{i=1}^{n} \left|\frac{\partial V}{\partial z_i}\right| |F_i| \left|\frac{\partial V_n}{\partial z_i}\right| \leq \iota_i \sum_{j=1}^{n} |\eta_j|^{2 - \frac{1}{p_1 \cdots p_{i-1}}}, \tag{A11}$$

where $\iota_i > 0$ is a constant. Using (15) in Lemma 4 and (A11), there holds

$$\begin{aligned} &\tfrac{\partial V}{\partial z} F \leq \textstyle\sum_{i=1}^{n} \left[\iota_i \sum_{j=i}^{n} |\eta_j|^{2 - \frac{1}{p_1 \cdots p_{i-1}}}\right] \left[S^{a-b}\bar{c}\left[\sum_{j=1}^{i} |\eta_j|^{\frac{1}{p_1 \cdots p_{i-1}}} + \sum_{j=1}^{i} |\eta_j|^{\frac{\gamma_{i+1} p_i}{p_i \cdots p_{j-1}\gamma_j}}\right] + \frac{\hat{\omega}_i}{S^{h_i}(t)}\right] \\ &\leq \textstyle\sum_{i=1}^{n} \sum_{j=i}^{n} \iota_i \bar{c} S(t)^{a-b} \left(|\eta_j|^{2 - \frac{1}{p_1 \cdots p_{i-1}}} \cdot \sum_{k=1}^{i} \left(|\eta_k|^{\frac{1}{p_1 \cdots p_{i-1}}} + |\eta_k|^{\frac{\gamma_{i+1} p_i}{p_1 \cdots p_{k-1}\gamma_k}}\right)\right) \\ &+ \textstyle\sum_{i=1}^{n} \sum_{j=1}^{n} \iota_i |\eta_j|^{2 - \frac{1}{p_1 \cdots p_{i-1}}} \cdot \frac{\hat{\omega}_i}{S^{h_i}(t)}. \end{aligned} \tag{A12}$$

By (3) in Lemma 1, there are positive constants μ_{j1}, ς_{k1} such that

$$|\eta_j|^{2 - \frac{1}{p_1 \cdots p_{i-1}}} |\eta_k|^{\frac{1}{p_1 \cdots p_{i-1}}} \leq \mu_{j1}|\eta_j|^2 + \varsigma_{k1}|\eta_k|^2. \tag{A13}$$

Similarly, there are the inequalities

$$|\eta_j|^{2 - \frac{1}{p_1 \cdots p_{i-1}}} |\eta_k|^{\frac{\gamma_{i+1} p_i}{p_1 \cdots p_{k-1}\gamma_k}} \leq \mu_{j2}|\eta_j|^2 + \varsigma_{k2}|\eta_k|^{2+\delta_k}, \tag{A14}$$

and

$$|\eta_j|^{2 - \frac{1}{p_1 \cdots p_{i-1}}} \cdot \frac{\hat{\omega}_i}{S^{h_i}(t)} \leq \mu_{j3}|\eta_j|^2 + \frac{\hat{\omega}_i^2}{(\lambda_i S^{h_i}(t))^{2p_{i-1} \cdots p_1}}, \tag{A15}$$

where $\mu_{j2}, \varsigma_{k2}, \delta_k, \mu_{j3}, \lambda_i$ are positive constants. Substituting (A13), (A14) and (A15) into (A12), it follows that (16) holds. □

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