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The Weak Disturbance Decoupling Problem for Nonlinear Systems with Unmodeled Dynamics and Disturbances and Its Application on Manipulators

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Abstract

This article considers a weak disturbance decoupling control problem for strict-feedback systems with unmodeled dynamics and disturbances. Utilizing a series of inequalities, the unmodeled dynamic is approximated. Using an adaptive backstepping control method, a weak disturbance decoupling controller is constructed to stabilize the closed-loop system. The primary objective of this paper is to design an adaptive controller that effectively attenuates the impact of disturbances in scenarios where such disturbances are present, while ensuring the bounded stability of system states in the absence of disturbances. Simulation and experiment results are presented to verify the effectiveness of the proposed weak disturbance decoupling strategy.

Keywords: adaptive control; backstepping; weak disturbance decoupling; unmodeled dynamics; manipulators

1. Introduction

Recently, the adaptive backstepping technique has become one of the many meaningful approaches to address nonlinear control problems. Utilizing the adaptive backstepping control technique, [1] solved the control problem for electrohydraulic actuator systems with uncertainties. In [2], using a fuzzy funnel control strategy, Liu et al. presented an adaptive backstepping finite-time scheme for non-affine nonlinear systems. With the Nussbaum gain technique, the virtual control coefficients issue was addressed for nonlinear systems [3,4]. In [5], the authors proposed a prescribed-time tracking control scheme for strict-feedback-like nonlinear systems. For stochastic nonlinear systems, the adaptive backstepping fuzzy tracking control scheme was studied in [6]. Furthermore, to overcome the overparameterization problem, the tuning function method is introduced to estimate unknown parameters by only one update law. To handle the issue of overparameterization, the adaptive fault-tolerant controller was designed by using the tuning functions for nonlinear systems in [7].

Since disturbances widely exist in many industrial control systems, there are some nonlinear control approaches to weaken the effect of disturbances, such as disturbance decoupling, almost disturbance decoupling (ADD) and weak disturbance decoupling (WDD). Since the conditions of disturbance decoupling control method are restrictive,

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the ADD control methods were first structured for a linear system in [8] and a nonlinear system in [9], respectively. The ADD method can be solved by milder conditions, such as linear, affine and power growth conditions. For nonlinear systems, an ADD controller is typically developed by selecting a Lyapunov function V, proving a dissipation inequality $\dot{V} \leq -y^T y + \gamma^2 q^T q$ and integrating it to obtain the gain inequality $\int_0^T \|y(t)\| dt \leq \gamma \int_0^T \|q(t)\| dt$ from the disturbance q to the output y. With uncertainties, the estimate often becomes $\dot{V} \leq -y^T y + \gamma^2 q^T q + \varepsilon_0$, which is acceptable only over finite horizons but leads to divergence as $T \to \infty$ [10]. To address this issue, ε_0 is replaced with a time-varying term $\alpha(t)$ satisfying $\int_0^\infty \alpha(t)\, dt < \infty$, and shows that such an $\alpha(t)$ exists for strict-feedback-like nonlinear systems, yielding $\dot{V} \leq -y^T y + \gamma^2 q^T q + \alpha(t)$. This motivates the concept of WDD, which guarantees $\int_0^\infty y^T(t) y(t)\, dt \leq \gamma \int_0^\infty q^T(t) q(t)\, dt + \varepsilon$ for some finite $\varepsilon > 0$, and WDD is achieved via adaptive backstepping for the considered system class [11]. In addition, in strict-feedback-like systems with unknown coefficients and internal dynamics, one must simultaneously cope with the coupling between the disturbance channel and the uncertainty-dependent input gain, which further complicates the controller design.

Meanwhile, there are some research and applications about ADD and WDD control strategies. Refs. [12,13] solved ADD problems for linear systems. The backstepping ADD controller was designed for SISO and MIMO strict-feedback nonlinear systems in [14,15], respectively. By utilizing fuzzy control stategy, the ADD issue was settled for strict-feedback nonlinear systems in [10]. Many other ADD problems were investigated for strict-feedback nonlinear systems in [16–24]. For strict-feedback-like systems, the ADD and output tracking ADD issues were developed in [25,26], respectively. Ref. [27] solved the finite-time ADD issues. Ref. [28] addressed the fuzzy approximate disturbance decoupling problems. Some ADD practical applications were also solved, such as the ball and beam system [29], the web tension control [30], the satellite attitude control [31] and the electrohydraulic load simulator system [32]. In [11], with a series of inequalities, the adaptive backstepping control method and high-order fully actuated control approach, the WDD control scheme was constructed in [11].

Most practical industry systems are affected by modeling errors, dynamic uncertainties and dynamic disturbances. Thus, the unmodeled dynamics often appear in real systems and the design of feedback controllers must make the closed-loop systems have expected control performance under the influence of these factors. Meanwhile, when accurate analytical modeling is challenging for complex nonlinear systems, nonparametric and data-driven identification/experimental characterization techniques attract increasing attention for extracting nonlinear dynamics from measured data and validating assumed models [33,34]. In recent years, there have been many studies conducted on the dynamic uncertainties nonlinear control problem. In [35], Jiang and Praly proposed robust adaptive control for nonlinear systems with dynamic uncertainties, which was considered for the first time by using dynamic signals. With fuzzy logic systems, nonlinear damping terms and a dynamic signal, the adaptive controllers were designed for unmodeled dynamics nonlinear systems in [36]. Aiming at nonlinear time-delay dynamic uncertainties systems, the adaptive neural intelligent method was addressed in [37]. An adaptive fuzzy decentralized approach was studied to handle the unmodeled dynamics for strong interconnected nonlinear systems in [38]. By applying a property of the hyperbolic tangent function, the adaptive neural tracking schemes have been developed for nonlinear systems in [39]. In [40], the neural network and the switching threshold event-triggered mechanisms were used for nonlinear systems. The adaptive finite-time controller and fuzzy finite-time tracking controller were constructed in [41,42], respectively. In [43], the fixed-time controller was investigated. In [44], with a small-gain approach, the inverse optimal adaptive control problem was

solved and a new criterion became a problem for a dynamic uncertainties nonlinear system. In view of mixed data and being event-driven, the controller was investigated for unmodeled dynamics nonlinear systems [45]. However, there are no results of the WDD tracking control problem of nonlinear systems with dynamic uncertainties and external disturbances, which inspires our research.

This article presents a weak disturbance decoupling control strategy for uncertain strict-feedback-like nonlinear systems with unmodeled dynamics, dynamic uncertainties and disturbances. A series of inequalities are used to approximate the unmodeled dynamics, Young's inequality is employed to handle the disturbance terms, and backstepping, together with an adaptive technique, is utilized to design controllers. By analyzing the system stability, the tracking performance of the system output is ensured for the case with no external disturbances and the disturbances are attenuated to a certain degree for the case of disturbances belonging to $\mathcal{L}_2$ norm.

To make the novelty more explicit, Table 1 compares this work with representative studies on unmodeled dynamics and on WDD design. Jiang and Praly in [35] and Tong et al. in [36] mainly established boundedness/ultimate boundedness properties for systems with unmodeled dynamics by using dynamic signals, but they did not address a WDD disturbance attenuation. Wang et al. in [11] studied WDD for HOFA nonlinear systems. Different from these works, uncertain strict-feedback-like systems with unmodeled dynamics are considered and an inequality-based approximation, together with adaptive backstepping, is developed to achieve WDD tracking under disturbances in $\mathcal{L}_2$.

Table 1. Comparison with representative related works.

Work	System Describtion	Treatment of Unmodeled Dynamics	Guarantee
Jiang & Praly (1998) [35]	Nonlinear systems with dynamic uncertainties	Dynamic signal + nonlinear damping	Global boundedness/output converges to a small neighborhood
Tong et al. (2009) [36]	Strict-feedback systems with nonlinear uncertainties, unmodeled dynamics, and dynamic disturbances	Dynamic signal + fuzzy approximation	Global boundedness/output converges to a small neighborhood
Wang et al. (2024) [11]	High-order fully actuated (HOFA) nonlinear systems	Not related to unmodeled dynamics	WDD design: stability with an output–disturbance gain inequality
This work	Uncertain strict-feedback-like systems with unmodeled dynamics and disturbances	Stable unmodeled dynamics	WDD tracking: boundedness for zero disturbance and attenuation for $q \in \mathcal{L}_2$

The specific contributions are summarized as follows.

(1) Compared with the existing unmodeled dynamic methods for nonlinear systems [35,38,41,42], this paper employs a series of inequalities and approximates the unmodeled dynamics.

(2) By utilizing an adaptive backstepping control strategy and a series of inequalities, a WDD controller is designed. In [35,38,41,42], the system states are bounded. In this paper, the boundedness of system states are guaranteed for the case of zero disturbance. The

influence of the disturbances on the system output is weakened to a certain degree for the case of disturbances belonging to $\mathcal{L}_2$ norm.

(3) Taking the displacement error of the trolley in a tower crane system as unmodeled dynamics, the adaptive backstepping PT controllers are tested on a 2 degree-of-freedom (DOF) manipulator, which is a simpified mechanical system of a tower crane experiment platform.

The remainder of this article is organized as follows. Section 2 focused on the problem formulation and preliminaries. Section 3 is devoted to the WDD controller design process. Simulation results are given in Section 4. Finally, the conclusion and future plan are provided in Section 5.

2. Problem Formulation and Preliminaries

Consider the following strict-feedback nonlinear system described by

$$\begin{cases} \dot{\varrho}(t) = \beta(\varrho) \\ \dot{x}_i(t) = g_i(\bar{x}_i)x_{i+1} + \theta^T(t)\phi_i(\bar{x}_i) + \Delta_i(\bar{x}_n, \varrho, t) + s_i^T(\bar{x}_i)q \\ \dot{x}_n(t) = g_n(\bar{x}_n)u + \theta^T(t)\phi_n(\bar{x}_n) + \Delta_n(\bar{x}_n, \varrho, t) + s_n^T(\bar{x}_n)q, \\ y(t) = x_1(t) \end{cases} \tag{1}$$

in which $\bar{x}_i = [x_1(t), \ldots x_i(t)]^T (i = 1, \ldots, n) \in R^i$ is composed of the measured states, ϱ is the unmeasured unmodeled dynamics states, $g_i(\bar{x}_i)$ is the known smooth function with $g_i(\bar{x}_i) \neq 0$, $\theta \in R^l$ is the unknown constant vector, $\phi_i(\bar{x}_i) \in R^l$ denotes the known smooth function with $\phi_i(0) = 0$, $\Delta_i(\cdot)$ depicts dynamic disturbance, $s_i^T(\bar{x}_i) \in R^{1\times r}$ represents a known smooth function with $s_i^T(0) = 0$, r is a positive integer, $q \in R^r$ is an unknown disturbance input, and $u \in R$ and $y(t) \in R$ are the system input and output, respectively. $\Delta_i(\cdot)$ and $\beta(\cdot)$ are continuous Lipschitz functions.

Assumption 1. *The dynamic disturbance function $\Delta_i(\cdot)$ satisfies*

$$|\Delta_i(\bar{x}_n, \varrho, t)| \leq \vartheta_{i1}^* \psi_{i1}(|\bar{x}_i|) + \vartheta_{i2}^* \psi_{i2}(|\varrho|) \tag{2}$$

where ϑ_{i1}^* and ϑ_{i2}^* are unknown positive constants, and $\psi_{i1}(\cdot)$ and $\psi_{i2}(\cdot)$ are known nonnegative smooth functions satisfying that $\psi_{i2}(0) = 0$.

Assumption 2. *There exists a continuous differentiable function $V(\varrho)$ for $\dot{\varrho} = \beta(\varrho)$,*

$$\begin{cases} \alpha_1(|\varrho|) \leq V(\varrho) \leq \alpha_2(|\varrho|) \\ \frac{\partial V(\varrho)}{\partial \varrho}\beta(\varrho) \leq -cV(\varrho) \end{cases} \tag{3}$$

where α_1 and α_2 are class K_∞-functions, $c > 0$ is a constant.

The second inequality of (3) implies that

$$\dot{V}(\varrho(t)) \leq -cV(\varrho(t)) \tag{4}$$

which is equivalent to

$$V(\varrho(t)) \leq D(t_0, t) \tag{5}$$

with $D(t_0, t) = e^{-c(t-t_0)}V(\varrho(t_0))$.

Assumption 3. $q(t) \in \mathcal{L}_2[0, \infty]$, $t \in (0, \infty)$.

Lemma 1 ([46]). *The following inequality is true*

$$\|x_i\| < \frac{\|x_i\|^2}{\sqrt{\|x_i\|^2 + \sigma_i^2(t)}} + \sigma_i(t) \tag{6}$$

with $\sigma_i(t)$ being arbitrary positive functions.

Definition 1. *The definition of weak disturbance decoupling tracking control is to construct a controller $u(\bar{x}_n)$ with $u(0) = 0$ such that the following two features can be satisfied with $T > 0$ being an arbitrary finite operation horizon (task duration).*

(1) When $q(t) = 0$, $y(t)$ and $x_i(t)$, $i = 2, 3, \ldots, n$ are bounded for any $t \in [0, T]$.

(2) The following inequality holds

$$\int_0^t \|y(\tau) - y_d(\tau)\| d\tau \leq \bar{\gamma} \int_0^t \|q(\tau)\| d\tau + \varepsilon \tag{7}$$

under zero initial conditions, when $q(t) \in \mathcal{L}_2$, where $\bar{\gamma} > 0$ and $0 \leq \varepsilon < \infty$ are constants.

3. Controller Design and Main Results

Using an adaptive backstepping technique, the design of a WDD controller for nonlinear systems for unmodeled dynamics and disturbances includes n steps. The key state transformation is introduced as follows

$$\begin{aligned} z_1 &= x_1 - y_d \\ z_j &= x_j - \alpha_{j-1} \end{aligned} \tag{8}$$

where y_d is the reference signal, α_{j-1} is the virtual controller, $\hat{\theta}$ is the estimation of θ with $\tilde{\theta} = \theta - \hat{\theta}$, $\vartheta^* = \max\{\vartheta_{11}^*, \vartheta_{12}^*, \ldots, \vartheta_{n1}^*, \vartheta_{n2}^*\}$, and $\hat{\vartheta}$ is the estimation of ϑ^* with $\tilde{\vartheta} = \vartheta^* - \hat{\vartheta}$.

Step 1: Based on $z_1 = x_1 - y_d$ and (1), it can be obtained that

$$\begin{aligned} \dot{z}_1 &= \dot{x}_1 - \dot{y}_d \\ &= g_1 x_2 + \theta^T \phi_1 + \Delta_1 + s_1^T q - \dot{y}_d \\ &= g_1 z_2 + g_1 \alpha_1 + \theta^T \phi_1 + \Delta_1 + s_1^T q - \dot{y}_d \end{aligned} \tag{9}$$

Consider the following Lyapunov function candidate

$$V_1 = \frac{1}{2} z_1^2 + \frac{1}{2} \tilde{\theta}^T \Gamma^{-1} \tilde{\theta} + \frac{1}{2\lambda} \tilde{\vartheta}^2 \tag{10}$$

where Γ and λ are positive constants.

By utilizing Assumption 1, $\dot{V}_1$ yields

$$\begin{aligned} \dot{V}_1 &= z_1 \dot{z}_1 - \tilde{\theta}^T \Gamma^{-1} \dot{\hat{\theta}} - \frac{1}{\lambda} \tilde{\vartheta} \dot{\hat{\vartheta}} \\ &= z_1 \left(g_1 z_2 + g_1 \alpha_1 + \theta^T \phi_1 - \dot{y}_d \right) + z_1 \Delta_1 + z_1 s_1^T q - \tilde{\theta}^T \Gamma^{-1} \dot{\hat{\theta}} - \frac{1}{\lambda} \tilde{\vartheta} \dot{\hat{\vartheta}} \end{aligned} \tag{11}$$

In view of Assumption 2, α_1 is a K_∞-function and α_1^{-1} is an increasing function. Define $\psi_{12}\left[\alpha_1^{-1}(D(t_0, t))\right] = \psi_{12} \circ \left(\alpha_1^{-1}(D(t_0, t))\right)$. Combining (3) with (5), it can be deduced that

$$\psi_{12}(|\varrho|) \leq \psi_{12}\left[\alpha_1^{-1}(D(t_0, t))\right] \tag{12}$$

Based on Assumption 1, $z_1\Delta_1$ in (11) is estimated by

$$\begin{aligned} z_1\Delta_1 &\leq \vartheta_{11}^*|z_1|\psi_{11}(|x_1|) + \vartheta_{12}^*|z_1|\psi_{12}(|\varrho|) \\ &\leq \vartheta^*|z_1|\psi_{11}(|x_1|) + \vartheta^*|z_1|\psi_{12}(|\varrho|) \end{aligned} \tag{13}$$

According to (12), (13) can be further deduced as

$$\begin{aligned} z_1\Delta_1 &\leq \vartheta^*|z_1|\Big(\psi_{11}(|x_1|) + \psi_{12}\Big[\Big(\alpha_1^{-1}(D(t_0,t))\Big)\Big]\Big) \\ &= \vartheta^*|z_1|\bar{\psi}_1 \end{aligned} \tag{14}$$

where $\bar{\psi}_1 = \psi_{11}(|x_1|) + \psi_{12}\Big[\alpha_1^{-1}(D(t_0,t))\Big]$.

With Lemma 1 and (14), $z_1\Delta_1$ is estimated by

$$z_1\Delta_1 \leq \vartheta^* z_1\hat{\psi}_1 + \vartheta^*\sigma_1(t) \tag{15}$$

where $\hat{\psi}_1 = \frac{z_1\bar{\psi}_1^2}{\sqrt{z_1^2\bar{\psi}_1^2+\sigma_1^2}}$ with $\sigma_1(t)$ chosen as an arbitrary positive function that satisfies (6).

With Young's inequality, it has

$$z_1 s_1^T q \leq \frac{a_1}{2\gamma} z_1^2 s_1^T s_1 + \frac{\gamma}{2a_1} q^T q \tag{16}$$

where $a_1 > 0$ is a constant.

By substituting (15) and (16) into (11) yields

$$\begin{aligned} \dot{V}_1 &< z_1\Big(g_1 z_2 + g_1\alpha_1 + \theta^T\phi_1 + \frac{a_1}{2\gamma} z_1 s_1^T s_1 + \vartheta^*\hat{\psi}_1 - \dot{y}_d\Big) + \vartheta^*\sigma_1 + \frac{\gamma}{2a_1} q^T q \\ &\quad -\tilde{\theta}^T\Gamma^{-1}\dot{\hat{\theta}} - \frac{1}{\lambda}\tilde{\vartheta}\dot{\hat{\vartheta}} \\ &= z_1 g_1 z_2 + g_1\alpha_1 + \hat{\theta}^T\phi_1 + \frac{a_1}{2\gamma} z_1 s_1^T s_1 + \hat{\vartheta}\hat{\psi}_1 - \dot{y}_d + \vartheta^*\sigma_1 + \frac{\gamma}{2a_1} q^T q \\ &\quad +\tilde{\theta}^T\Gamma^{-1}\Big(\Gamma\phi_1 z_1 - \dot{\hat{\theta}}\Big) + \frac{1}{\lambda}\tilde{\vartheta}\Big(\lambda\hat{\psi}_1 z_1 - \dot{\hat{\vartheta}}\Big) \end{aligned} \tag{17}$$

Introduce the first virtual control α_1 as

$$\alpha_1 = \frac{1}{g_1}\Big(-k_1 z_1 - \hat{\theta}\phi_1 - \hat{\vartheta}\hat{\psi}_1 - \frac{a_1}{2\gamma} z_1 s_1^T s_1 + \dot{y}_d\Big) \tag{18}$$

with $k_1 > 0$ being a design parameter.

Invoking (18) into (17), $\dot{V}_1$ is presented as

$$\begin{aligned} \dot{V}_1 &< -k_1 z_1^2 + z_1 g_1 z_2 + \tilde{\theta}^T\Gamma^{-1}\Big(\Gamma\phi_1 z_1 - \dot{\hat{\theta}}\Big) + \frac{1}{\lambda}\vartheta\Big(\lambda\hat{\psi}_1 z_1 - \dot{\hat{\vartheta}}\Big) + \vartheta^*\sigma_1 + \frac{\gamma}{2a_1} q^T q \\ &= -k_1 z_1^2 + z_1 g_1 z_2 + \tilde{\theta}^T\Gamma^{-1}\Big(\tau_1 - \dot{\hat{\theta}}\Big) + \frac{1}{\lambda}\tilde{\vartheta}\Big(\varpi_1 - \dot{\hat{\vartheta}}\Big) + \frac{\gamma}{2a_1} q^T q + D_1 \end{aligned} \tag{19}$$

where $\tau_1 = \Gamma\phi_1 z_1$, $\varpi_1 = \lambda\hat{\psi}_1 z_1$, $D_1 = \vartheta^*\sigma_1(t)$.

Step j ($j = 2, \cdots, n-1$): The derivative of α_{j-1} can be expressed as

$$\begin{aligned} \dot{\alpha}_{j-1} &= \sum_{i=1}^{j-1}\frac{\partial\alpha_{j-1}}{\partial x_i} x_{i+1} + \sum_{i=1}^{j-1}\frac{\partial\alpha_{j-1}}{\partial x_i}\theta^T\phi_i + \sum_{i=1}^{j-1}\frac{\partial\alpha_{j-1}}{\partial x_i}\Delta_i + \sum_{i=1}^{j-1}\frac{\partial\alpha_{j-1}}{\partial x_i} s_i^T q \\ &\quad + \sum_{i=0}^{j-1}\frac{\partial\alpha_{j-1}}{\partial y_d^{(i)}} y_d^{(i+1)} + \frac{\partial\alpha_{j-1}}{\partial t} + \frac{\partial\alpha_{j-1}}{\partial\hat{\theta}}\dot{\hat{\theta}} + \frac{\partial\alpha_{j-1}}{\partial\hat{\vartheta}}\dot{\hat{\vartheta}} \end{aligned} \tag{20}$$

According to the coordinate transformation (8) and (20), $\dot{z}_j$ can be presented as

$$\begin{aligned}\dot{z}_j &= \dot{x}_j - \dot{\alpha}_{j-1} \\ &= g_j z_{j+1} + g_j \alpha_j + \theta^T \phi_j + \Delta_j + s_j^T q - \dot{\alpha}_{j-1} \\ &= g_j z_{j+1} + g_j \alpha_j + \theta^T \Phi_j + \bar{\Delta}_j + S_j^T q + F_j - \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \dot{\hat{\theta}} - \frac{\partial \alpha_{j-1}}{\partial \hat{\vartheta}} \dot{\hat{\vartheta}}\end{aligned} \tag{21}$$

where

$$\begin{aligned}\bar{\Delta}_j &= \Delta_j - \sum_{i=1}^{j-1} \frac{\partial \alpha_{j-1}}{\partial x_i} \Delta_i \\ \Phi_j &= \phi_j - \sum_{i=1}^{j-1} \frac{\partial \alpha_{j-1}}{\partial x_i} \theta^T \phi_i \\ S_j &= s_j - \sum_{i=1}^{j-1} \frac{\partial \alpha_{j-1}}{\partial x_i} s_i \\ F_j &= -\sum_{i=1}^{j-1} \frac{\partial \alpha_{j-1}}{\partial x_i} x_{i+1} - \sum_{i=0}^{j-1} \frac{\partial \alpha_{j-1}}{\partial y_d^{(i)}} y_d^{(i+1)} - \frac{\partial \alpha_{j-1}}{\partial t}\end{aligned}$$

Select the jth Lyapunov function candidate

$$V_j = V_{j-1} + \frac{1}{2} z_j^2$$

and $\dot{V}_j$ is given by

$$\begin{aligned}\dot{V}_j < & -\sum_{i=1}^{j-1} k_i z_i^2 + \tilde{\theta}^T \Gamma^{-1} \left(\tau_{j-1} - \dot{\hat{\theta}} \right) + \frac{1}{\lambda} \tilde{\vartheta} \left(\varpi_{j-1} - \dot{\hat{\vartheta}} \right) + D_{j-1} + z_j g_{j-1} z_{j-1} \\ & + \sum_{i=2}^{j-1} z_i \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \left(\tau_{j-1} - \dot{\hat{\theta}} \right) + \sum_{i=2}^{j-1} z_i \frac{\partial \alpha_{i-1}}{\partial \hat{\vartheta}} \left(\varpi_{j-1} - \dot{\hat{\vartheta}} \right) \\ & + z_j \left(g_j z_{j+1} + g_j \alpha_j + \theta^T \Phi_j + \bar{\Delta}_j + S_j^T q + F_j \right) - z_j \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \dot{\hat{\theta}} \\ & - z_j \frac{\partial \alpha_{j-1}}{\partial \hat{\vartheta}} \dot{\hat{\vartheta}} + \sum_{i=1}^{j-1} \frac{\gamma}{2 a_i} q^T q\end{aligned} \tag{22}$$

By utilizing Assumption 1, it can be obtained that

$$\begin{aligned}|z_j \bar{\Delta}_j| &\leq |z_j||\Delta_j| + |z_j| \sum_{i=1}^{j-1} \left| \frac{\partial \alpha_{j-1}}{\partial x_i} \right| |\Delta_i| \\ &\leq |z_j| \left(\vartheta_{j1}^* \psi_{j1}(|\bar{x}_j|) + \vartheta_{j2}^* \psi_{j2}(|\varrho|) + \sum_{i=1}^{j-1} \left| \frac{\partial \alpha_{j-1}}{\partial x_i} \right| \vartheta_{i1}^* \psi_{i1}(|\bar{x}_i|) \right. \\ &\quad \left. + \sum_{i=1}^{j-1} \left| \frac{\partial \alpha_{j-1}}{\partial x_i} \right| \vartheta_{i2}^* \psi_{i2}(|\varrho|) \right)\end{aligned}$$

With Assumption 2 and (12), it can be deduced that

$$\begin{aligned}|z_j \bar{\Delta}_j| &\leq |z_j| \vartheta^* \left(\psi_{j1}(|\bar{x}_j|) + \psi_{j2}\left[\alpha_1^{-1}(D(t_0, t)) \right] + \sum_{i=1}^{j-1} \left| \frac{\partial \alpha_{j-1}}{\partial x_i} \right| \psi_{i1}(|\bar{x}_i|) \right. \\ &\quad \left. + \sum_{i=1}^{j-1} \left| \frac{\partial \alpha_{j-1}}{\partial x_i} \right| \psi_{i2}\left[\alpha_1^{-1}(D(t_0, t)) \right] \right) \\ &= \vartheta^* |z_j| \bar{\psi}_j\end{aligned} \tag{23}$$

where

$$\begin{aligned}\bar{\psi}_j &= \psi_{j1}(|\bar{x}_j|) + \psi_{j2}\left[\alpha_1^{-1}(D(t_0,t))\right] + \sum_{i=1}^{j-1}\left|\frac{\partial \alpha_{j-1}}{\partial x_i}\right|\psi_{i1}(|\bar{x}_i|) \\ &\quad + \sum_{i=1}^{j-1}\left|\frac{\partial \alpha_{j-1}}{\partial x_i}\right|\psi_{i2}\left[\alpha_1^{-1}(D(t_0,t))\right]\end{aligned} \tag{24}$$

Based on Lemma 1, $|z_j\bar{\Delta}_j|$ can be further estimated by

$$|z_j\bar{\Delta}_j| < \vartheta^* z_j\hat{\psi}_j + \vartheta^*\sigma_j(t) \tag{25}$$

where $\hat{\psi}_j = \frac{z_j\bar{\psi}_j^2}{\sqrt{z_j^2\bar{\psi}_j^2+\sigma_j^2}}$ with $\sigma_j(t)$ selected as an arbitrary positive function that satisfies (6).

Utilizing Young's inequality, we have

$$z_j S_j^T q \le \frac{a_j}{2\gamma} z_j^2 S_j^T S_j + \frac{1}{2a_j}\gamma q^T q \tag{26}$$

where $a_j > 0$ is a constant.

By subtituting (23) and (26), $\dot{V}_j$ is reformulated as

$$\begin{aligned}\dot{V}_j &< -\sum_{i=1}^{j-1} k_i z_i^2 + \tilde{\theta}^T\Gamma^{-1}\left(\tau_{j-1} - \dot{\hat{\theta}}\right) + \frac{1}{\lambda}\tilde{\vartheta}\left(\varpi_{j-1} - \dot{\hat{\vartheta}}\right) + D_{j-1} + \sum_{i=1}^{j}\frac{\gamma}{2a_i}q^T q \\ &\quad + \sum_{i=2}^{j-1} z_i\frac{\partial \alpha_{i-1}}{\partial \hat{\theta}}\left(\tau_{j-1} - \dot{\hat{\theta}}\right) + \sum_{i=2}^{j-1} z_i\frac{\partial \alpha_{i-1}}{\partial \hat{\vartheta}}\left(\varpi_{j-1} - \dot{\hat{\vartheta}}\right) + \vartheta^*\sigma_j + z_j g_j z_{j+1} \\ &\quad + z_j\left(g_{j-1}z_{j-1} + g_j\alpha_j + \theta^T\Phi_j + \vartheta^*\hat{\psi}_j + F_j - z_j\frac{\partial \alpha_{j-1}}{\partial \hat{\theta}}\dot{\hat{\theta}} - z_j\frac{\partial \alpha_{j-1}}{\partial \hat{\vartheta}}\dot{\hat{\vartheta}} + \frac{a_j}{2\gamma}z_j S_j^T S_j\right) \\ &= -\sum_{i=1}^{j-1} k_i z_i^2 + \tilde{\theta}^T\Gamma^{-1}\left(\tau_j - \dot{\hat{\theta}}\right) + \frac{1}{\lambda}\tilde{\vartheta}\left(\varpi_{j-1} - \dot{\hat{\vartheta}}\right) + D_{j-1} + \sum_{i=2}^{j} z_i\frac{\partial \alpha_{i-1}}{\partial \hat{\theta}}\left(\tau_j - \dot{\hat{\theta}}\right) \\ &\quad + \sum_{i=2}^{j} z_i\frac{\partial \alpha_{i-1}}{\partial \hat{\vartheta}}\left(\varpi_j - \dot{\hat{\vartheta}}\right) + \vartheta^*\sigma_j + z_j g_j z_{j+1} + z_j(g_{j-1}z_{j-1} + g_j\alpha_j) \\ &\quad + z_j\left(\left(\hat{\theta}^T - \Gamma\sum_{i=2}^{j-1}\frac{\partial \alpha_{i-1}}{\partial \hat{\theta}}z_i\right)\Phi_j + \left(\hat{\vartheta} - \lambda\sum_{i=2}^{j-1}\frac{\partial \alpha_{i-1}}{\partial \hat{\vartheta}}z_i\right)\hat{\psi}_j + \frac{a_j}{2\gamma}z_j S_j^T S_j\right. \\ &\quad \left. + F_j - \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}}\tau_j - \frac{\partial \alpha_{j-1}}{\partial \hat{\vartheta}}\varpi_j\right) + \sum_{i=1}^{j}\frac{\gamma}{2a_i}q^T q\end{aligned} \tag{27}$$

where $\tau_j = \tau_{j-1} + \Gamma\Phi_j z_j$, $\varpi_j = \varpi_{j-1} + \lambda\hat{\psi}_j z_j$.

The virtual control law α_j are defined by

$$\begin{aligned}\alpha_j &= \frac{1}{g_i}\left(-k_j z_j - z_{j-1} - \left(\hat{\theta}^T - \Gamma\sum_{i=2}^{j-1}\frac{\partial \alpha_{i-1}}{\partial \hat{\theta}}z_i\right)\Phi_j - \left(\hat{\vartheta} - \lambda\sum_{i=2}^{j-1}\frac{\partial \alpha_{i-1}}{\partial \hat{\vartheta}}z_i\right)\hat{\psi}_j\right. \\ &\quad \left. - \frac{a_j}{2\gamma}z_j S_j^T S_j - F_j + \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}}\tau_j + \frac{\partial \alpha_{j-1}}{\partial \hat{\vartheta}}\varpi_j\right)\end{aligned} \tag{28}$$

Substituting (28) into (27) leads to

$$\begin{aligned}\dot{V}_j &< -\sum_{i=1}^{j} k_i z_i^2 + z_j g_i z_{j+1} + \tilde{\theta}^T\Gamma^{-1}\left(\tau_j - \dot{\hat{\theta}}\right) + D_j + \frac{1}{\lambda}\tilde{\vartheta}\left(\varpi_j - \dot{\hat{\vartheta}}\right) \\ &\quad + \sum_{i=2}^{j} z_i\frac{\partial \alpha_{i-1}}{\partial \hat{\theta}}\left(\tau_j - \dot{\hat{\theta}}\right) + \sum_{i=2}^{j} z_i\frac{\partial \alpha_{i-1}}{\partial \hat{\vartheta}}\left(\varpi_j - \dot{\hat{\vartheta}}\right) + \sum_{i=1}^{j}\frac{\gamma}{2a_i}q^T q\end{aligned}$$

where $D_j = D_{j-1} + \vartheta^*\sigma_j(t)$.

Step n: $\dot{z}_n$ can be expressed as

$$\begin{aligned}\dot{z}_n &= \dot{x}_n - \dot{\alpha}_{n-1} \\ &= g_n u + \theta^T \phi_n + \Delta_n + s_n^T q - \dot{\alpha}_{n-1} \\ &= g_n u + \theta^T \Phi_n + \bar{\Delta}_n + S_n^T q + F_n - \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}} \dot{\hat{\theta}} - \frac{\partial \alpha_{n-1}}{\partial \hat{\vartheta}} \dot{\hat{\vartheta}}\end{aligned} \tag{29}$$

where

$$\begin{aligned}\bar{\Delta}_n &= \Delta_n - \textstyle\sum_{i=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_i} \Delta_i \\ \Phi_n &= \phi_n - \textstyle\sum_{i=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_i} \phi_i \\ S_n &= s_n - \textstyle\sum_{i=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_i} s_i \\ F_n &= -\textstyle\sum_{i=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_i} x_{i+1} - \sum_{i=0}^{n-1} \frac{\partial \alpha_{n-1}}{\partial y_d^{(i)}} y_d^{(i+1)} - \frac{\partial \alpha_{n-1}}{\partial t}\end{aligned}$$

Define a Lyapunov function candidate V_n as

$$V_n = V_{n-1} + \frac{1}{2} z_n^2 \tag{30}$$

and $\dot{V}_n$ can be deduced that

$$\begin{aligned}\dot{V}_n < & -\textstyle\sum_{i=1}^{n-1} k_i z_i^2 + \tilde{\theta}^T \Gamma^{-1} \left(\tau_{n-1} - \dot{\hat{\theta}} \right) + \frac{1}{\lambda} \tilde{\vartheta} \left(\varpi_{n-1} - \dot{\hat{\vartheta}} \right) + D_{n-1} + z_n g_{n-1} z_{n-1} \\ & + \textstyle\sum_{i=2}^{n-1} z_i \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \left(\tau_{n-1} - \dot{\hat{\theta}} \right) + \sum_{i=2}^{n-1} z_i \frac{\partial \alpha_{i-1}}{\partial \hat{\vartheta}} \left(\varpi_{n-1} - \dot{\hat{\vartheta}} \right) \\ & + z_n \left(g_n u + \theta^T \Phi_n + \bar{\Delta}_n + S_n^T q - \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}} \dot{\hat{\theta}} - \frac{\partial \alpha_{n-1}}{\partial \hat{\vartheta}} \dot{\hat{\vartheta}} \right) + \sum_{i=1}^{n-1} \frac{\gamma}{2 a_i} q^T q\end{aligned} \tag{31}$$

Similar to the step j, $|z_n \bar{\Delta}_n|$ becomes

$$\begin{aligned}|z_n \bar{\Delta}_n| &\leq |z_n||\Delta_n| + |z_n| \textstyle\sum_{i=1}^{n-1} \left| \frac{\partial \alpha_{n-1}}{\partial x_i} \right| |\Delta_i| \\ &\leq |z_n| \Big(\vartheta_{n1}^* \psi_{n1}(|\bar{x}_n|) + \vartheta_{n2}^* \psi_{n2}(|\varrho|) + \textstyle\sum_{i=1}^{n-1} \left| \frac{\partial \alpha_{n-1}}{\partial x_i} \right| \vartheta_{i1}^* \psi_{i1}(|\bar{x}_i|) \\ &\quad + \textstyle\sum_{i=1}^{n-1} \left| \frac{\partial \alpha_{n-1}}{\partial x_i} \right| \vartheta_{i2}^* \psi_{i2}(|\varrho|) \Big) \\ &\leq |z_n| \vartheta^* \Big(\psi_{n1}(|\bar{x}_n|) + \psi_{n2} \left[\alpha_1^{-1}(D(t_0, t)) \right] + \textstyle\sum_{i=1}^{n-1} \left| \frac{\partial \alpha_{n-1}}{\partial x_i} \right| \psi_{i1}(|\bar{x}_i|) \\ &\quad + \textstyle\sum_{i=1}^{n-1} \left| \frac{\partial \alpha_{n-1}}{\partial x_i} \right| \psi_{i2} \left[\alpha_1^{-1}(D(t_0, t)) \right] \Big) \\ &= \vartheta^* z_n \bar{\psi}_n < \vartheta^* z_n \hat{\psi}_n + \vartheta^* \sigma_n(t)\end{aligned} \tag{32}$$

where

$$\begin{aligned}\bar{\psi}_n &= \psi_{n1}(|\bar{x}_n|) + \psi_{n2} \left[\alpha_1^{-1}(D(t_0, t)) \right] + \textstyle\sum_{i=1}^{n-1} \left| \frac{\partial \alpha_{n-1}}{\partial x_i} \right| \psi_{i1}(|\bar{x}_i|) \\ &\quad + \textstyle\sum_{i=1}^{n-1} \left| \frac{\partial \alpha_{n-1}}{\partial x_i} \right| \psi_{i2} \left[\alpha_1^{-1}(D(t_0, t)) \right]\end{aligned}$$

$\hat{\psi}_n = \frac{z_n \bar{\psi}_n^2}{\sqrt{z_n^2 \bar{\psi}_n^2 + \sigma_n^2}}$ with $\sigma_n(t)$ satisfying Lemma 1.

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Utilizing Young's inequality, we have

$$z_n S_n^T q \leq \frac{a_n}{2\gamma} z_n^2 S_n^T S_n + \frac{1}{2a_n}\gamma q^T q \tag{33}$$

where $a_j > 0$ is a constant.

Substituting (32) and (33) into (31), $\dot{V}_n$ is presented as

$$\begin{aligned}
\dot{V}_n < & -\sum_{i=1}^{n-1} k_i z_i^2 + \tilde{\theta}^T \Gamma^{-1}\left(\tau_{n-1} - \dot{\hat{\theta}}\right) + \frac{1}{\lambda}\tilde{\vartheta}\left(\varpi_{n-1} - \dot{\hat{\vartheta}}\right) + D_{n-1} + z_n g_{n-1} z_{n-1} \\
& + \sum_{i=2}^{n-1} z_i \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}}\left(\tau_{n-1} - \dot{\hat{\theta}}\right) + \sum_{i=2}^{n-1} z_i \frac{\partial \alpha_{i-1}}{\partial \hat{\vartheta}}\left(\varpi_{n-1} - \dot{\hat{\vartheta}}\right) + \vartheta^* \sigma_n \\
& + z_n\left(u + \theta^T \Phi_n + \vartheta^* \hat{\psi}_n + \frac{a_n}{2\gamma} z_n S_n^T S_n + F_n - \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}}\dot{\hat{\theta}} - \frac{\partial \alpha_{n-1}}{\partial \hat{\vartheta}}\dot{\hat{\vartheta}}\right) \\
& + \sum_{i=1}^{n} \frac{\gamma}{2a_i} q^T q \\
= & -\sum_{i=1}^{n-1} k_i z_i^2 + \tilde{\theta}^T \Gamma^{-1}\left(\tau_n - \dot{\hat{\theta}}\right) + \frac{1}{\lambda}\vartheta\left(\varpi_n - \dot{\vartheta}\right) + D_{n-1} + \sum_{i=2}^{n} z_i \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}}\left(\tau_n - \dot{\hat{\theta}}\right) \\
& + \vartheta^* \sigma_n + \sum_{i=2}^{n} z_i \frac{\partial \alpha_{i-1}}{\partial \hat{\vartheta}}\left(\varpi_n - \dot{\hat{\vartheta}}\right) + z_n(g_{n-1} z_{n-1} + u + \\
& \left(\hat{\theta}^T - \Gamma \sum_{i=1}^{n-1} \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} z_i\right)\Phi_n + \left(\hat{\vartheta} - \lambda \sum_{i=1}^{n-1} \frac{\partial \alpha_{i-1}}{\partial \vartheta} z_i\right)\hat{\psi}_n + \frac{a_n}{2\gamma} z_n S_n^T S_n \\
& + F_n - \frac{\partial \alpha_{n-1}}{\partial \hat{\vartheta}}\varpi_n - \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}}\tau_n\Big) + \sum_{i=1}^{n} \frac{\gamma}{2a_i} q^T q
\end{aligned} \tag{34}$$

where $\tau_n = \tau_{n-1} + \Gamma \Phi_n z_n$, $\varpi_n = \varpi_{n-1} + \lambda \hat{\psi}_n z_n$.

Define the control law u, and the adaptive laws $\dot{\hat{\theta}}$ and $\dot{\hat{\vartheta}}$ as

$$\begin{aligned}
u = & \frac{1}{g_n}\Bigg(-g_{n-1} z_{n-1} - k_n z_n - F_n + \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}}\tau_n + \frac{\partial \alpha_{n-1}}{\partial \hat{\vartheta}}\varpi_n \\
& - \left(\hat{\theta}^T - \Gamma \sum_{i=2}^{n-1} \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} z_i\right)\Phi_n - \frac{a_n}{2\gamma} z_n S_n^T S_n \\
& - \left(\hat{\vartheta} - \lambda \sum_{i=2}^{n-1} \frac{\partial \alpha_{i-1}}{\partial \hat{\vartheta}} z_i\right)\hat{\psi}_n\Bigg)
\end{aligned} \tag{35}$$

$$\dot{\hat{\theta}} = \tau_n \tag{36}$$

$$\dot{\hat{\vartheta}} = \varpi_n \tag{37}$$

From (35)–(37), $\dot{V}_n$ gives

$$\begin{aligned}
\dot{V}_n < & -\sum_{i=1}^{n} k_i z_i^2 + D_{n-1} + \vartheta^* \sigma_n + \sum_{i=1}^{n} \frac{\gamma}{2a_i} q^T q \\
= & -\sum_{i=1}^{n} k_i z_i^2 + D_n + k_1 \bar{\gamma} q^T q
\end{aligned} \tag{38}$$

where $D_n = D_{n-1} + \vartheta^* \sigma_n(t)$, $\bar{\gamma} = \frac{1}{k_1}\sum_{i=1}^{n} \frac{\gamma}{2a_i}$.

Theorem 1. *Consider system (1) under Assumptions 1–3. Let $T > 0$ be an arbitrary finite operation horizon (task duration). If the control law u (35) and the adaptive laws (36), and (37) are applied, then the following properties hold for all $t \in (0, T]$:*

(1) $y(t)$ and $x_i(t)$, $i = 2, 3, \ldots, n$, are bounded when $q(t) = 0$.

(2) If $q(t) \in \mathcal{L}_2$, under zero initial conditions, then the following inequality.

$$\int_0^t \|y(\tau) - y_d(\tau)\| d\tau \leq \bar{\gamma} \int_0^t \|q(\tau)\| d\tau + \varepsilon \tag{39}$$

holds for $0 \leq \varepsilon < \infty$.

Remark 1. *It is worth noting that the constant T in Theorem 1 specifies the range of time, during which the properties in Theorem 1 are true and this T is different from that for fixed-time control in [47,48].*

Proof. It can be easily verified that (38) can be rewritten as

$$\dot{V}_n < -k_1 z_1^2 + D_n(t) + k_1 \bar{\gamma} q^T q \tag{40}$$

Integrating (40) gives

$$V_n(z(t)) - V(z(0)) < -k_1 \int_0^t z_1^2(\tau) d\tau + k_1 \varepsilon + k_1 \bar{\gamma} \int_0^t q^T(\tau) q(\tau) d\tau \tag{41}$$

where $\varepsilon = \frac{1}{k_1} \int_0^t D_n(\tau) d\tau$.

The inequality (41) can be further simplified as

$$0 < V_n(z(t)) < -k_1 \int_0^t z_1^2(\tau) d\tau + k_1 \varepsilon + k_1 \bar{\gamma} \int_0^t q^T(\tau) q(\tau) d\tau \tag{42}$$

which means that (39) holds.

When $q(t) = 0$, integrating (38) gives

$$V_n(t) < V_n(0) - \sum_{i=1}^{n} \int_0^t k_i z_i^2(\tau) d\tau + \int_0^t D_n(\tau) d\tau < V_n(0) + \int_0^t D_n(\tau) d\tau < \infty \tag{43}$$

Based on (43), we have

$$\sum_{i=1}^{n} \int_0^t k_i z_i^2(\tau) d\tau < -V_n(t) + V_n(0) + \int_0^t D_n(\tau) d\tau \leq V_n(0) + \int_0^t D_n(\tau) d\tau < \infty \tag{44}$$

From (30), (43) and (44), the uniform boundedness of z_i, $\tilde{\theta}$ and $\tilde{\vartheta}$ can be guaranteed. With the aid of (18), the following inequality can be obtained

$$\begin{aligned} \alpha_1 &= -k_1 z_1 - \hat{\theta}\phi_1 - \hat{\vartheta}\hat{\psi}_1 - \frac{a_1}{2\gamma} z_1 s_1^T s_1 + \dot{y}_d \\ &= -k_1 z_1 - \hat{\theta}\phi_1 - \hat{\vartheta} \frac{z_1 \bar{\psi}_1^2}{\sqrt{z_1^2 \bar{\psi}_1^2 + \sigma_1^2}} - \frac{a_1}{2\gamma} z_1 s_1^T s_1 + \dot{y}_d \\ &\leq k_1 |z_1| + \left|\hat{\theta}\right| |\phi_1| + \left|\hat{\vartheta}\right| |\psi_{11}(|x_1|)| + \frac{a_1}{2\gamma} |z_1| s_1^T s_1 \\ &\quad + \left|\hat{\vartheta}\right| \psi_{12}\left[\alpha_1^{-1}(D(t_0, t))\right] + |\dot{y}_d| \end{aligned} \tag{45}$$

From Assumptions 1–2, ψ_{11}, $\psi_{12} \circ \alpha_1^{-1}(D(t_0, t))$ and s_1 are continuous functions, together with the boundedness of z_1 and $\dot{y}_d$, it can be verified that α_1 is bounded. With $|x_2| \leq |z_2| + |\alpha_1|$, the boundedness of x_2 can be obtained. From $\dot{z}_1 = x_2 + \theta^T \phi_1 + \Delta_1 + s_1^T q$, it can be proved that $\dot{z}_1$ is bounded. Similarly, the uniform continuity of x_j is ensured by the boundedness of α_j (28).

Thus, the WDD issue of a nonlinear system (1) can be handled by the actual controller u (35), adaptive laws (36) and (37). Simultaneously, the tracking performance of y is achieved. □

Algorithm 1 outlines the proposed adaptive backstepping WDD controller design procedure, which is as follows.

Algorithm 1. Algorithmic outline of the proposed adaptive backstepping WDD controller design procedure

Given: design parameters $k_i > 0$, $a_i > 0$, $\Gamma > 0$, $\lambda > 0$, $\gamma > 0$, $\sigma_i > 0$ $(i = 1, \ldots, n)$.
Measure: states $x_1, \ldots, x_n$; generate: y_d and derivatives $\dot{y}_d, \ldots, y_d^{(n)}$.
Initialize: $\hat{\theta}(0)$, $\hat{\vartheta}(0)$; set $\tau_0 = 0$, $\varpi_0 = 0$.
Step 1:
Compute $z_1 = x_1 - y_d$.
Compute $\bar{\psi}_1$ and $\hat{\psi}_1$ according to the definitions used in (14) and (15).
Estimate inequality term $s_1^T q$ in (16).
Compute the first virtual control α_1 (18), and set $z_2 = x_2 - \alpha_1$.
Update $\tau_1 = \Gamma \phi_1 z_1$ and $\varpi_1 = \lambda \hat{\psi}_1 z_1$.
Step j = 2 to n−1:
Compute $\bar{\psi}_j$ and $\hat{\psi}_j$ according to (24) and (25).
Estimate inequality term $s_j^T q$ in (26).
Compute α_j (28) and set $z_{j+1} = x_{j+1} - \alpha_j$.
Update $\tau_j = \tau_{j-1} + \Gamma \Phi_j z_j$ and $\varpi_j = \varpi_{j-1} + \lambda \hat{\psi}_j z_j$.
Step n:
Compute the actual control input u (35).
Estimate inequality term $s_n^T q$ in (33).
Update adaptive laws: $\dot{\hat{\theta}} = \tau_n$ (36) and $\dot{\hat{\vartheta}} = \varpi_n$ (37).

4. Simulation Results

Consider a nonlinear system with unmodeled dynamics and disturbances as follows

$$\begin{cases} \dot{\varrho} = -2\varrho \\ \dot{x}_1 = x_2 + \theta^T x_1^2 + \varrho x_1 \sin x_1 + q \\ \dot{x}_2 = u + \theta^T x_1 x_2 + \varrho x_1 x_2 + q \\ y = x_1 \end{cases} \tag{46}$$

where θ is the unknown parameter with the true value -0.15. To verify the effectiveness of the proposed WDD method, the disturbance $q = 0.2 \sin(t) e^{-0.1t}$ is added at $20s$. The control goal is to develop an adaptive backstepping controller, so that y can track the reference trajectory $y_d = 0.05 \sin(t)$ well.

To satisfy Assumption 2 for system (46), we choose $V(\varrho) = \varrho^2$. Then, it can be verified that $\dot{V}(\varrho) \leq -4\varrho^2$.

Based on Assumption 1, it can be verified that $\psi_{11}(x_1) = \frac{1}{4} x_1^2 \sin^2(x_1)$, $\psi_{21}(x_1, x_2) = \frac{1}{4} x_1^2 x_2^2$, $\psi_{12}(\varrho) = \psi_{22}(\varrho) = \varrho^2$. Then, we choose $\alpha_1(\varrho) = 0.5\varrho^2$, $\alpha_2(\varrho) = 2\varrho^2$.

In the simulation, choose $k_1 = k_2 = 10$, $a_1 = a_2 = 0.1$, $\Gamma = 100$, and $\lambda = 100$. In order to observe the effect of the controller on disturbance decoupling, γ is selected to 1, 0.5 and 0.05, respectively. Initialize the controlled plant as $\left[x_1(0), x_2(0), \hat{\theta}, \vartheta\right]^T = [0.1, 0.1, 0, 0]$.

Remark 2. *The gains (e.g., λ) are tuning parameters of the controller/adaptation dynamics. Larger values typically yield faster transients and stronger attenuation, but overly large gains may increase*

control effort and noise sensitivity, and may trigger actuator saturation or excite unmodeled dynamics. Hence, these gains should be selected by balancing performance and practical constraints (actuator bounds, sampling rate, and noise level).

Figures 1–5 demonstrate the simulation results. Figure 1 shows the system output x_1 and the reference signal y_d. It can be verified that the system output x_1 tracks the reference signal y_d well even though the disturbance q is added at $t = 20s$. Figure 2 shows the tracking error of x_1. It can be seen that the tracking error converge to zero when $q = 0$. Figure 3 shows the trajectory of state x_2. The control input u is displayed in Figure 4. When the parameter γ decreases, the disturbance decoupling effect of the output x_1 becomes better, the initial value of the control input u goes to larger and the amplitude of u tends to smaller. From Figures 5 and 6, it can be observed that the curve $\hat{\theta}$ and ϑ are bounded. It is verified that the WDD controller succeeds in attenuating the influence of the disturbance on the output for the nonlinear with unmodeled dynamics. Simultaneously, the WDD controller guarantees that the tracking performance of system output under zero disturbance.

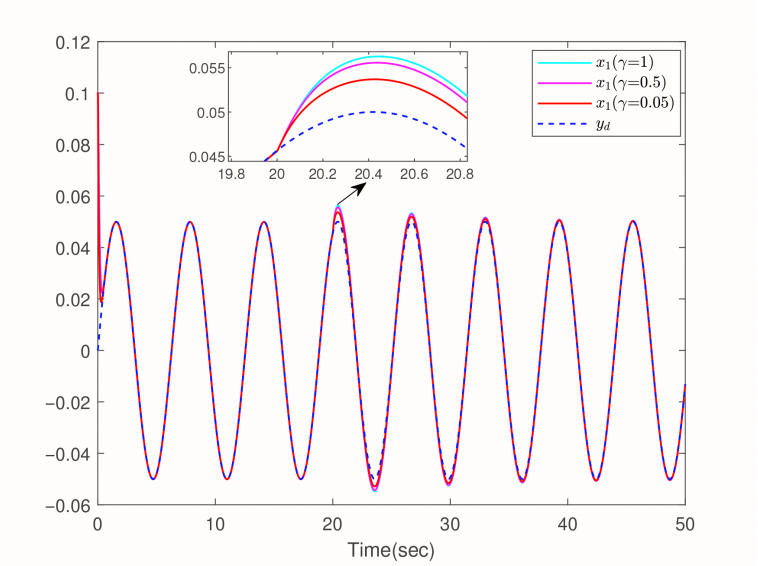


Figure 1. Output signals x_1 and reference signal y_d.

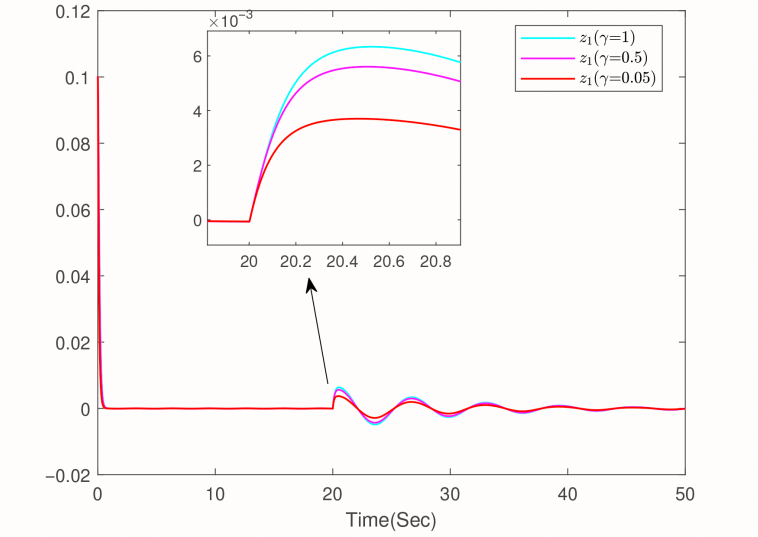


Figure 2. Tracking errors z_1.

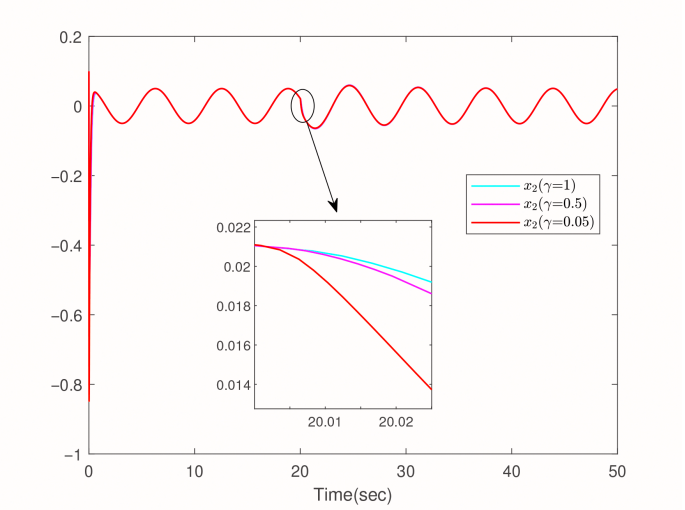


Figure 3. State variables x_2.

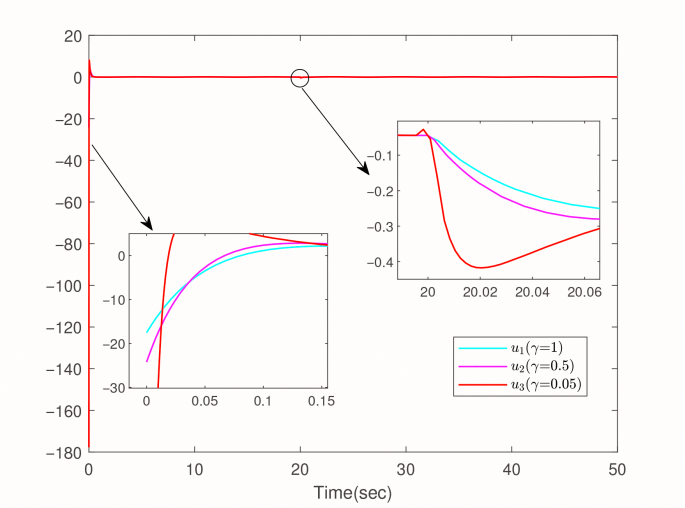


Figure 4. Control inputs u.

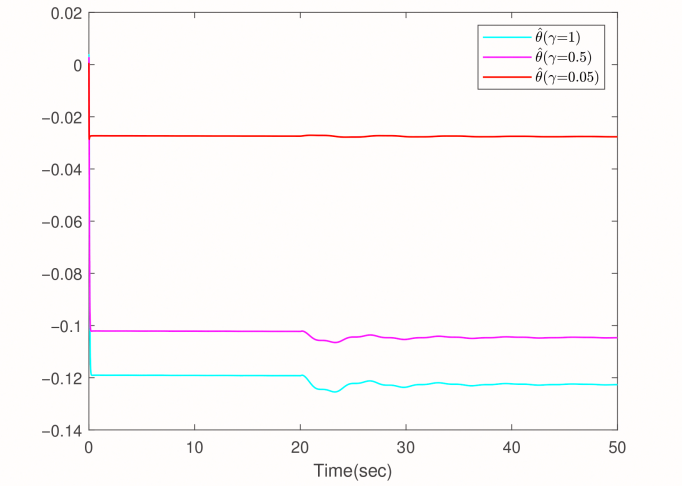


Figure 5. Adaptive parameters $\hat{\theta}$.

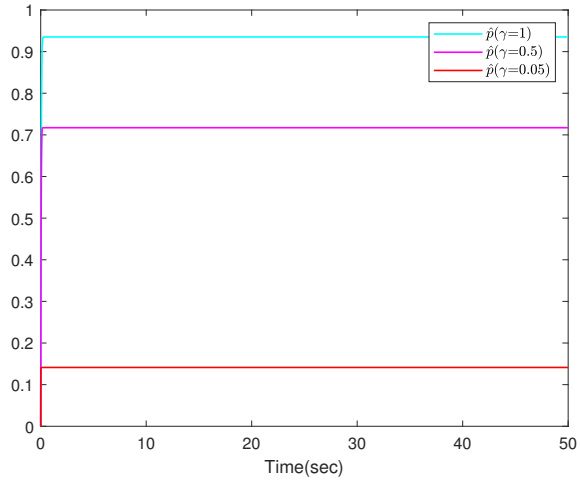


Figure 6. Adaptive parameters $\hat{p}$.

5. Experiments Results on Manipulator System

In this subsection, the proposed method is applied to the 2-DOF manipulator (Nanjing Qiaodong Technology Co., Ltd., Nanjing, China), which is actually an overhead crane with payload swing angles and their derivatives being set to zero and rope length being considered as a constant, as shown in Figure 7. In this setup, the actuator torque is taken as the control input u, whereas the trolley dynamics driven by F_η is treated as unmodeled internal dynamics ϱ.

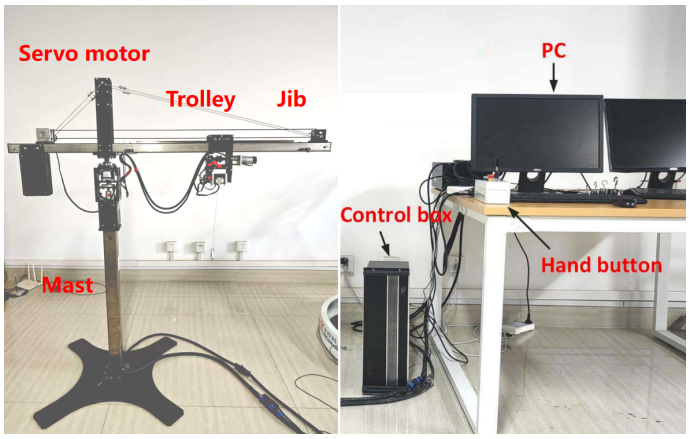


Figure 7. Experiment platform.

The trolley displacement and the jib slew angle are measured by incremental encoders. Specifically, the trolley position is measured using an encoder with a resolution of 550 pulses per revolution (PPR), while the jib slew angle is measured using a 1024 PPR encoder (K14-abz-1024ppr, Taiwan Karamay Technology Co., Ltd., 402 Shiji Road, Xitun District, Taichung, Taiwan, China). The power interface unit amplifies the command signal from the PC to the DC motor and converts the encoder pulse outputs into 16-bit digital data readable by the PC. The PC communicates with the power interface unit via an RT-DAC/PCI multifunction digital I/O board. The Xilinx device on the RT-DAC/PCI board (Nanjing Qiaodong Technology Co., Ltd., Nanjing, China) is programmed to implement encoder signal acquisition and to generate pulse-width modulation (PWM) waveforms for driving the DC motor. The control algorithm is executed in MATLAB/Simulink R2020b Real-Time Windows Target with a sampling period of $T_s = 10$ ms (100 Hz).

To this end, the following dynamic model for the 2 DOF manipulator can be easily obtained from the model of a 4 DOF overhead crane in [49] by setting payload swing angles (θ_1 and θ_2) and their derivatives ($\dot{\theta}_1$, $\dot{\theta}_2$, $\ddot{\theta}_1$ and $\ddot{\theta}_2$) to zero.

$$M\left(\ddot{\eta} - \eta\dot{\phi}^2\right) + d_1\dot{\eta} = F_\eta \tag{47}$$

$$\left(J + M\eta^2\right)\ddot{\phi} + (2M\eta\dot{\eta} + d_2)\dot{\phi} = \tau_\phi + q \tag{48}$$

where ϕ is the jib slew angle, η is the trolley translation displacement, q is the disturbance of the jib slew angle, $M = 1.852$ kg is the trolley mass, $J = 1.532\text{kg m}^2$ is the moment of inertia, and $d_1 = 0.1$ and $d_2 = 0.1$ are the friction coefficients of the trolley and jib.

The dynamics for the trolley with the nonlinear backstepping controller $F_\eta = -M\left((1+\xi_1\xi_2)\varrho_1 + (\xi_1+\xi_2)\varrho_2 + \eta\dot{\phi}^2 - \frac{d_1}{M}\dot{\eta} - \ddot{y}_{d_\eta}\right)$ is considered as the unmodeled dynamics $\varrho = [\varrho_1, \varrho_2]^T$ in (1), where $\beta = A\varrho$, $\varrho_1 = \eta - y_{d_\eta}$, $\varrho_2 = \dot{\eta} - \dot{y}_{d_\eta}$, $\xi_1 = 2$ and $\xi_2 = 10$ with $A = \begin{bmatrix} 0 & 1 \\ -1-\xi_2\xi_1 & -(\xi_1+\xi_2) \end{bmatrix}$. By defining $V = \frac{1}{2}\varrho_1^2 + \frac{1}{2}(\varrho_2 + \xi_1\varrho_1)^2$, it is straightforward to verify that $\dot{V} \le -\xi_1\varrho_1^2 - \xi_2(\varrho_2 + \xi_1\varrho_1)^2 \le -cV$ with $c = 2\min\{\xi_1, \xi_2\}$, which implies that Assumption 2 is true.

It can be seen from (48) that $\ddot{\phi} = \frac{1}{J+M\eta^2}\tau_\phi - \frac{2M\eta\dot{\eta}+d_2}{J+M\eta^2}\dot{\phi} + \frac{1}{J+M\eta^2}q$, from which one can tell that the control coefficient $\frac{1}{J+M\eta^2}$ depends on the displacement $\eta = \varrho_1 + y_{d_\eta}$ of the trolley. Therefore, in order to make the control coefficient independent of ϱ_1, (48) is linearized at the point $\eta = \eta_e = -0.485$ m, $\dot{\eta} = 0$ and $\ddot{\eta} = 0$ as

$$\ddot{\phi} = \frac{1}{J + M\eta_e^2}\tau_\phi - \frac{d_2}{J + M\eta_e^2}\dot{\phi} + \frac{1}{J + M\eta_e^2}q \tag{49}$$

Set $x_1 = \phi - y_{d_\phi}$ and $x_2 = \dot{\phi} - \dot{y}_{d_\phi}$. It can be easily checked that $x = [x_1, x_2]^T$ satisfies (1) with $g_1 = 1$, $g_2 = \frac{1}{J+M\eta_e^2}$, $\phi_1 = 0$, $\phi_2 = -\frac{1}{J+M\eta_e^2}x_2$, $\theta = d_2$, $\Delta_1 = \Delta_2 = 0$, $s_1 = 0$, and $s_2 = \frac{1}{J+M\eta_e^2}$, which implies that Assumption 1 is satisfied.

The reference trajectories y_{d_ϕ} and y_{d_η} are selected as

$$y_{d_\eta} = \bar{A}_\eta \sin\left(2\pi f_\eta (t - t_1)\right) + \eta_0 \tag{50}$$

$$y_{d_\phi} = \bar{A}_\phi \sin\left(2\pi f_\phi (t - t_1)\right) \tag{51}$$

with $\bar{A}_\eta = -0.2$, $\bar{A}_\phi = 25$, $f_\eta = \frac{1}{30}$, $f_\phi = \frac{1}{40}$, and $\eta_0 = -0.47$.

The controller parameters for the proposed controller are chosen as $k_1 = k_2 = 1$. The adaptive parameter $\Gamma = 0.1$. In addition, $a_2 = 1$, $\gamma_2 = 0.2$. The disturbance

$$q = \begin{cases} e^{(-0.5(t-10))}, & 10 < t < 15 \\ -1, & 25 < t < 27 \end{cases} \tag{52}$$

is added to the control input. To demonstrate the effectiveness of the proposed UD-WDD method, it is compared with UD and PD controllers.

UD controllers for (47) and (48) by using the design method in [50] are designed as follows:

$$
\begin{aligned}
F_\eta &= -M\Big((\xi_1+\xi_2)\varrho_2+(1+\xi_1\xi_2)\varrho_1+\eta\dot{\phi}^2-\frac{d_1}{M}\dot{\eta}-\ddot{y}_{d_\eta}\Big)\\
\alpha_1 &= -k_1z_1+\dot{y}_d\\
\tau_\phi &= \frac{1}{g_2}\Big(-g_1z_1-k_2z_2-F_2-\hat{\theta}^T\Phi_2\Big)\\
\dot{\hat{\theta}} &= \Gamma\Phi_2z_2
\end{aligned}
$$

where $F_2=-\frac{\partial\alpha_1}{\partial x_1}x_2-\frac{\partial\alpha_1}{\partial y_d}\dot{y}_d-\frac{\partial\alpha_1}{\partial\dot{y}_d}y_d^{(2)}$ and $\Phi_2=\phi_2$.

PD controllers for (47) and (48) by using the design method in [51] are developed as follows:

$$F_\eta = M\Big(K_{p\eta}\varrho_1+K_{d\eta}\varrho_2\Big), \tag{53}$$

$$\tau_\phi = \Big(J+M\eta_e^2\Big)\Big(K_{p\phi}\varphi_1+K_{d\phi}\dot{\varphi}_1\Big) \tag{54}$$

where $K_{p\eta1}=-(1+\xi_1\xi_2)$, $K_{d\eta1}=-(\xi_1+\xi_2)$, $K_{p\phi}=1+k_1k_2$ and $K_{d\phi}=k_1+k_2$.

The output signals ϕ and the tracking errors $\phi-y_{d_\phi}$ are shown in Figures 8 and 9. It can be seen that the tracking error with the proposed UD-WDD method subject to unmodeled dynamics is smaller than other methods. In addition, the proposed control method shows superior disturbance attenuation ability compared to that of the UD control method and the PD control method. Figure 10 shows the control signals τ_ϕ of the three methods, all of which are within a reasonable range.

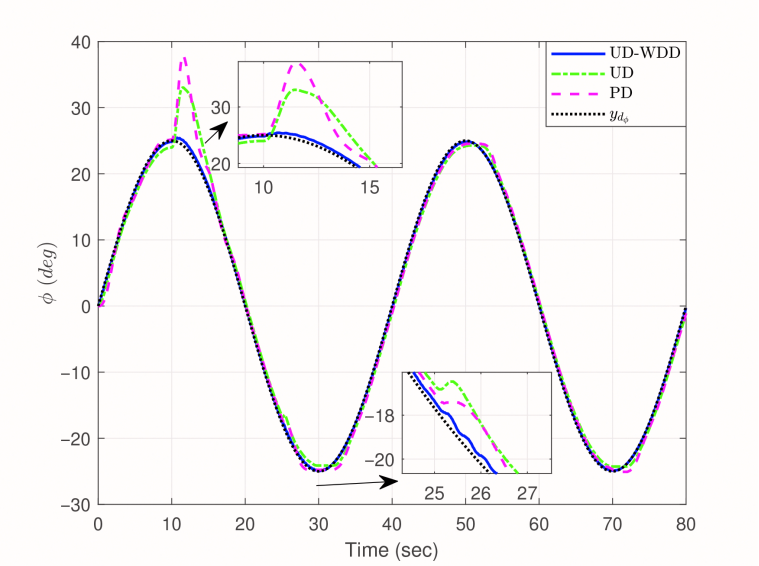


Figure 8. The state ϕ.

To objectively evaluate the tracking performance of different controllers, we compute three standard performance indices for the tracking error $e(t)=\phi(t)-y_{d_\phi}(t)$, including the integral of absolute error (IAE), the integral of squared error (ISE), and the maximum absolute tracking error $e_{\max}$. The calculated indices based on the experimental data are summarized in Table 2, which provides an objective comparison among the proposed UD-WDD controller, the UD controller, and the PD controller. To quantify this phenomenon, we further report the variance of the tracking error, Var(e) (see Table 2).

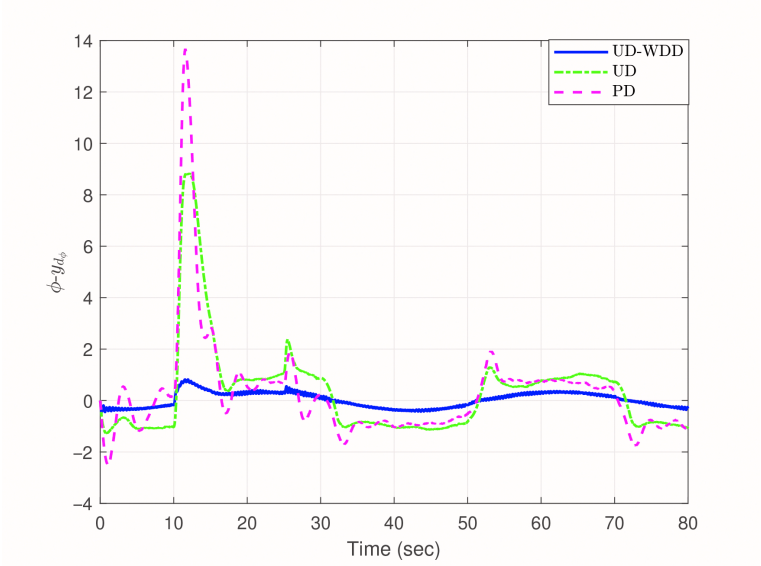


Figure 9. The tracking error $\phi - y_{d_\phi}$.

The results in Table 2 show that the proposed UD-WDD controller achieves the smallest ISE, IAE, and maximum absolute tracking error, which objectively demonstrates its superiority in tracking and disturbance attenuation compared to UD and PD controllers.

The performance indices are defined as

$$\text{IAE} = \int_0^T |e(t)|\,dt, \text{ISE} = \int_0^T e^2(t)\,dt, e_{\max} = \max_{t\in[0,T]} |e(t)|. \tag{55}$$

For sampled experimental data with sampling period T_s, the integrals are approximated by

$$\text{IAE} \approx T_s \sum_{k=1}^{N} |e[k]|, \text{ISE} \approx T_s \sum_{k=1}^{N} e^2[k], \text{Var}(e) = \frac{1}{N-1} \sum_{k=1}^{N} (e[k] - \bar{e})^2 \tag{56}$$

where $\bar{e}$ denotes the sample means.

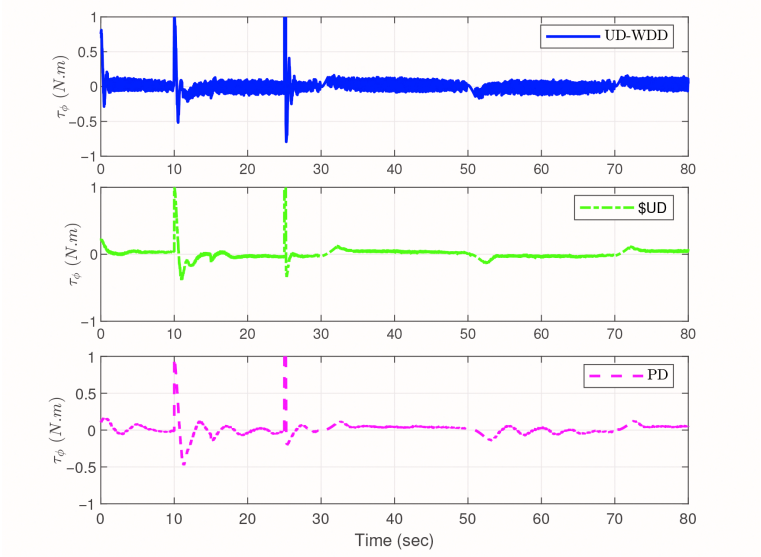


Figure 10. The controller τ_ϕ.

Table 2. Quantitative evaluation of tracking performance and noise sensitivity in the manipulator experiment.

Controller	IAE (deg·s)	ISE (deg^2·s)	e_{max} (deg)	Var(e_ϕ) (deg^2)
UD-WDD	38.3181	6.9259	0.8155	0.0858
UD	97.8486	280.1124	8.8416	3.3943
PD	92.9997	361.5718	13.5740	4.4392

6. Conclusions

An adaptive backstepping WDD control problem was studied for uncertain strict-feedback systems with unmodeled dynamics. By constructing Lyapunov functions, a WDD controller has been designed and the WDD control tracking performance can be guaranteed. The influence of the disturbances on the system output can be attenuated to a certain degree when disturbance belong to L_2 norm. Furthermore, the tracking performance of system output can be ensured in the case of zero disturbance. The stability analysis, simulation and experiment example show the effectiveness of the proposed WDD controller. Future work will extend the proposed WDD tracking framework to finite-time and fixed-time designs.

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