

Article

Mathematical Modeling of Dynamic Stress Concentration in Piezoelectric Materials with Elliptical Openings Under SH Waves

Peng Lu ^{1,2}, Yabo Wang ^{2,3,*}, Rong Liu ^{2,*}, Changyong Chu ², Zhenyu Wang ² and Weihua Zhou ⁴¹ COMAC Aviation College, Ordos Institute of Technology, Ordos 017000, China; joany0204@163.com² School of Mechanical Engineering, Information Engineering College, Hangzhou Dianzi University, Hangzhou 310056, China; 40560@hdu.edu.cn (C.C.); 41017@hdu.edu.cn (Z.W.)³ Chengdu Starera Aerospace Technology Co., Ltd., Chengdu 610096, China⁴ College of Electrical Engineering, Zhejiang University, Hangzhou 310027, China; dididi@zju.edu.cn

* Correspondence: 41726@hdu.edu.cn (Y.W.); 40471@hdu.edu.cn (R.L.)

Abstract: This paper presents a targeted study on the dynamic stress concentration (DSC) in piezoelectric materials induced by SH waves, focusing on the impact of elliptical openings. By using the elliptic cylindrical coordinate system and Mathieu functions, the wave fields are decomposed into functional series. Through the establishment of a set of infinite equations with mode coefficients based on the boundary conditions, the distribution of the dynamic stress concentration coefficient is visualized via numerical simulation. Furthermore, the impact of incident wave frequency, incident angle, and elliptic eccentricity on the stress concentration coefficient is analyzed. The results demonstrate a strong correlation between these parameters and the dynamic stress concentration coefficient. These findings hold significant implications for enhancing the strength and fatigue life of piezo-electric materials, as well as for selecting appropriate nondestructive testing methods.

Keywords: piezoelectric material; SH wave; Mathieu function; dynamic stress concentration; mode coefficient; elliptical opening



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1. Introduction

In modern engineering technology, piezoelectric materials play a crucial role in a variety of applications, such as sensors, actuators, energy collectors, and micromotors [1–6]. Piezoelectric materials are often employed in conjunction with other materials, and the mechanical behavior of their interface defects plays a critical role in the overall structural performance, particularly in the presence of weak interfaces. However, when subjected to shear horizontal (SH) waves, these materials can exhibit dynamic stress concentration (DSC) around elliptical openings, leading to performance degradation and failure. This issue is particularly prevalent in precision instruments and medical equipment, where reliability is paramount.

Extensive research has been conducted to investigate the dynamic stress concentration problem in piezoelectric materials. Previous studies have addressed various defects in different types of piezoelectric media, such as double circular openings [7], rigid removable cylindrical inclusions [8], spherical inclusions [9], and cracks [10]. To gain a more accurate understanding of how these defects influence wave propagation, scholars have employed diverse methodologies, including complex variable functions [11], conformal mappings [12], and Green functions. Notably, the investigation of dynamic stress concentration in non-circular openings subjected to SH waves has emerged as a complex and

intriguing topic [13]. The presence of such non-circular openings can cause stress to concentrate around them, thereby affecting the overall material properties. Furthermore, extensive research has been conducted on the dynamic analysis of elliptical cavities subjected to shear horizontal waves. Qi et al. [14] developed a semi-analytical method to analyze the anti-plane dynamics of piezoelectric materials containing elliptical cavities, emphasizing the significance of large axial ratios, wavelet numbers, and large piezoelectric characteristic parameters. Song et al. [15] calculated the dynamic stress concentration coefficient (DSCC) for a transverse-isotropic piezoelectric binary containing symmetric radial cracks near the edge of a non-circular cavity, considering the effect of dynamic incident anti-plane shear. The elliptic cavity model was numerically studied using a Fortran program, and the results were compared to clarify the influence on DSCC under appropriate conditions. In addition to the aforementioned studies, other research has focused on different types of defects in piezoelectric materials. For instance, the dynamic performance analysis of interface cracks near eccentric elliptical openings subjected to incident SH waves has been investigated [16–19]. These cracks can cause stress concentration around them, thus affecting the overall material properties.

Additionally, the scattering effect and dynamic stress concentration of SH waves near elliptical openings have been extensively studied [20]. These studies have provided valuable insights into the design and manufacturing of piezoelectric materials. Understanding the scattering effect of SH waves near elliptical openings enables the development of piezoelectric materials with superior performance. Moreover, the problems of elastic wave scattering and dynamic stress concentration in piezoelectric media with multiple openings have also been thoroughly investigated [21]. These studies have provided valuable knowledge for designing and manufacturing piezoelectric materials containing multiple openings. Recent research has also explored the scattering of electroelastic coupled waves and dynamic stress concentration in piezoelectric ceramics with regular N-sided openings [22]. These studies offer a theoretical basis for designing and manufacturing piezoelectric materials with openings of different shapes. Furthermore, piezoelectric solids with elliptical wraps or openings have been studied [23]. In infinite piezoelectric material strips, the scattering of SH-guided waves caused by circular wraps has also garnered attention [24].

Our study addresses this critical problem by employing Mathieu functions [25] to analyze the DSC in piezoelectric materials with elliptical openings, offering a novel and effective solution. The calculation of SH wave stress coefficients in piezoelectric materials with elliptical openings, based on the Mathieu functions, represents an important and challenging research area. To study this field in more depth and understand the underlying mechanisms, employing the Mathieu functions for analysis proves to be an effective method. By combining the Mathieu functions and the wave function expansion method, this paper aims to study the scattering characteristics and dynamic stress concentration of SH waves in piezoelectric materials with elliptical openings more accurately. The findings of this study can provide further knowledge and guidance for the design and manufacturing of piezoelectric materials.

2. Governing Equation for Full-Plane Piezoelectric Materials

An elliptic cylindrical coordinate system (ξ, η, z) is established in an infinite piezoelectric material, in which ξ and η are variables of the curvilinear coordinates [26,27]. The origin O is located at the center of the ellipse, as shown in Figure 1. The direction of z , perpendicular to the paper and facing outward, represents the direction of polarization. The elliptic opening has a long axis length of $2a$, which is aligned with the x -axis, and a short axis length of $2b$, which is aligned with the y -axis. The focal length is $2h$, and the plane SH wave is incident at an angle α_0 .

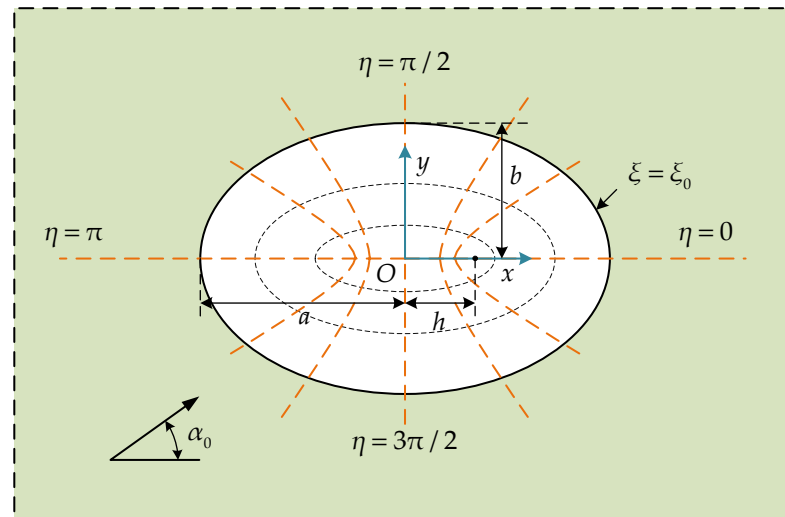


Figure 1. A full-plane piezoelectric material model with an elliptical opening subjected to SH wave.

For the steady-state elastic field and potential field under periodic loading, the field variables can be expressed in a separated form of a time-harmonic factor and a spatial variable with the result is

$$L^*(\xi, \eta, t) = L(\xi, \eta)e^{-i\omega t} \quad (1)$$

where L^* represents a complex function, where either its real or imaginary part can be a solution to the problem. The spatial function L is the primary focus of this paper, and the time-harmonic factor $e^{-i\omega t}$ can be disregarded in the subsequent analysis.

As depicted in Figure 1, in the elliptic cylindrical coordinate system, the anti-plane shear wave propagates at an angle of α_0 with respect to the x -axis. Neglecting the influence of body force, the dynamic governing equation by the elastic field and the electric potential can be expressed as

$$\begin{cases} \frac{1}{h_1^2} \left[\frac{\partial}{\partial \xi} (h_1 \tau_{\xi z}) + \frac{\partial}{\partial \eta} (h_1 \tau_{\eta z}) \right] = \rho \frac{\partial^2 w}{\partial t^2} \\ \frac{1}{h_1^2} \left[\frac{\partial}{\partial \xi} (h_1 D_\xi) + \frac{\partial}{\partial \eta} (h_1 D_\eta) \right] = 0 \end{cases} \quad (2)$$

where $h_1 = h\sqrt{\cosh^2 \xi - \cos^2 \eta}$, $\tau_{\xi z}$, and $\tau_{\eta z}$ represent the radial and tangential shear stresses, respectively. D_ξ and D_η denote the radial and tangential electrical displacements, respectively. The displacement in the z -axis direction is represented by w .

The piezoelectric equation under the action of an SH wave can be expressed as follows:

$$\begin{cases} \tau_{\xi z} = \frac{1}{h_1} \left(c_{44}^E \frac{\partial w}{\partial \xi} + e_{15} \frac{\partial \varphi}{\partial \xi} \right) \\ \tau_{\eta z} = \frac{1}{h_1} \left(c_{44}^E \frac{\partial w}{\partial \eta} + e_{15} \frac{\partial \varphi}{\partial \eta} \right) \\ D_\xi = \frac{1}{h_1} \left(e_{15} \frac{\partial w}{\partial \xi} - \varepsilon_{11}^S \frac{\partial \varphi}{\partial \xi} \right) \\ D_\eta = \frac{1}{h_1} \left(e_{15} \frac{\partial w}{\partial \eta} - \varepsilon_{11}^S \frac{\partial \varphi}{\partial \eta} \right) \end{cases} \quad (3)$$

where c_{44}^E , e_{15} , and ε_{11}^S represent the short-circuit elastic constant, the piezoelectric stress constant, and the clamping dielectric constant, respectively. φ denotes the electric potential.

By substituting Equation (3) into Equation (2), we can obtain

$$\begin{cases} c_{44}^E \nabla^2 w + e_{15} \nabla^2 \varphi = \rho \frac{\partial^2 w}{\partial t^2} \\ e_{15} \nabla^2 w - \varepsilon_{11}^S \nabla^2 \varphi = 0 \end{cases} \quad (4)$$

where $\nabla^2 = (\partial^2 / \partial \xi^2 + \partial^2 / \partial \eta^2) / h_1^2$ is the transverse Laplacian operator in the elliptic cylindrical coordinate system, and ρ represents the mass density.

Constructs the following expression as

$$\varepsilon_{11}^S \varphi = e_{15} w + \Phi. \quad (5)$$

Substituting the above into Equation (4) can be obtained as

$$\begin{cases} \nabla^2 w = \frac{1}{c_s^2} \frac{\partial^2 w}{\partial t^2} \\ \nabla^2 \Phi = 0 \end{cases} \quad (6)$$

where $c_s = \sqrt{c^*/\rho}$ represents the wave velocity of the shear wave and $c^* = c_{44}^E + \frac{e_{15}^2}{\varepsilon_{11}^S}$.

Through the aforementioned analysis, the original dynamic control Equation (2) is transformed into Equation (6). By obtaining the specific expressions of the displacement function w and the constructor function Φ , the solution for stress and electric displacement in the piezoelectric Equation (3) can be obtained using Equation (5). Considering the steady-state solution of this problem, the separate variable form of the elastic field can be expressed as follows:

$$w = w_0 W e^{-i\omega t} \quad (7)$$

where the constant w_0 represents the amplitude of the steady-state wave, and ω denotes the circular frequency of the elastic wave field.

By substituting it into the first equation of Equation (6), we attain an equation solely dependent on the spatial function with the result is

$$\nabla^2 W + k^2 W = 0 \quad (8)$$

where $k = \omega/c_s$ represents the wave number.

In the elliptic cylindrical coordinate system, the solution is the formula in the series summation form of the angular Mathieu function and the radial Mathieu function [25]. As long as the boundary conditions are given, the expression of the displacement function w can be obtained by Equation (7).

The second formula of Equation (6) can be solved by separating the variables. Let its separated variable form be

$$\Phi = \Xi(\xi) H(\eta) e^{-i\omega t} \quad (9)$$

Hence, the second equation of Equation (6) reduces the second-order partial differential equation to two second-order ordinary differential equations. By analyzing the problem of definite solutions of ordinary differential equations, the general solution can be determined as

$$\begin{cases} \Xi(\xi) = A e^{\lambda \xi} + B e^{-\lambda \xi} \\ H(\eta) = C \sin \lambda \eta + D \cos \lambda \eta \end{cases} \quad (10)$$

where the constants A , B , C and D need to be determined based on boundary conditions.

According to the natural periodic boundary conditions $H(\eta + 2\pi) = H(\eta)$, we can derive $\lambda = 0, 1, 2, \dots$. To conform to writing conventions, let $\lambda = n$. The eigen solution of the constructor function Φ is constructed as a linear combination of the radial function $\Xi(\xi)$ and the angular function $H(\eta)$

$$\Phi(\xi, \eta) = 2w_0 \sum_{n=0}^{\infty} (A_n e^{n\xi} + B_n e^{-n\xi}) (C_n \cos n\eta + D_n \sin n\eta), \quad (11)$$

where A_n , B_n , C_n , and D_n are mode coefficients. The constant $2w_0$ is added to match the scattered wave coefficient.

3. The Solution to the Field Function

In the elliptic cylindrical coordinate system, the incident elastic wave field can be set as

$$\begin{aligned} w^i &= w_0 e^{ik(x \cos \alpha_0 + y \sin \alpha_0) - i\omega t} \\ &= w_0 e^{ikh(\cosh \xi \cos \eta \cos \alpha_0 + \sinh \xi \sin \eta \sin \alpha_0) - i\omega t} \end{aligned} \quad (12)$$

where a_n is the harmonic amplitude coefficient, $\omega_0 = \pi/T$ is the fundamental frequency, and t is the wave field response time.

The potential field generated by the incident wave can be constructed as

$$\varepsilon_{11}^S \varphi^i = e_{15} w^i. \quad (13)$$

By neglecting the time-harmonic factor $e^{-i\omega t}$, Equations (12) and (13) can be expanded into a series using the Mathieu functions

$$\begin{aligned} w^i &= w_0 e^{ikh(\cosh \xi \cos \eta \cos \alpha_0 + \sinh \xi \sin \eta \sin \alpha_0)} \\ &= 2w_0 \sum_{n=0}^{\infty} (\sigma_{2n} - \sigma_{2n+1} + \gamma_{2n+2} + \gamma_{2n+1}), \end{aligned} \quad (14)$$

$$\varphi^i = \frac{2w_0 e_{15}}{\varepsilon_{11}^S} \sum_{n=0}^{\infty} (\sigma_{2n} - \sigma_{2n+1} + \gamma_{2n+2} + \gamma_{2n+1}), \quad (15)$$

where

$$\begin{cases} \sigma_{2n}(\xi, \eta) = \frac{A_0^{2n}}{p_{2n}} \text{Je}_{2n}(\xi, q) \text{ce}_{2n}(\eta, q) \text{ce}_{2n}(\alpha_0, q) \\ \sigma_{2n+1}(\xi, \eta) = \frac{i\sqrt{q} A_1^{2n+1}}{p_{2n+1}} \text{Je}_{2n+1}(\xi, q) \text{ce}_{2n+1}(\eta, q) \text{ce}_{2n+1}(\alpha_0, q) \\ \gamma_{2n+2}(\xi, \eta) = \frac{q B_2^{2n+2}}{s_{2n+2}} \text{Jo}_{2n+2}(\xi, q) \text{se}_{2n+2}(\eta, q) \text{se}_{2n+2}(\alpha_0, q) \\ \gamma_{2n+1}(\xi, \eta) = \frac{i\sqrt{q} B_1^{2n+1}}{s_{2n+1}} \text{Jo}_{2n+1}(\xi, q) \text{se}_{2n+1}(\eta, q) \text{se}_{2n+1}(\alpha_0, q) \end{cases},$$

$q = h^2 k^2 / 4$ is a dimensionless parameter and is associated with the propagation constant k . ce_{2n} , ce_{2n+1} , se_{2n+2} and se_{2n+1} represent the first, second, third, and fourth classes of integer order angular Mathieu functions, respectively [25]. A_0^{2n} , A_1^{2n+1} , B_1^{2n+1} , and B_2^{2n+2} are the known coefficients corresponding to the aforementioned angular Mathieu functions. Je_{2n} , Je_{2n+1} , Jo_{2n+2} , and Jo_{2n+1} represent the first, second, third, and fourth Bessel-type integral order radial Mathieu functions, respectively. It is worth noting that

$$\begin{cases} p_{2n} = \text{ce}_{2n}(0, q) \text{ce}_{2n}(\pi/2, q) \\ p_{2n+1} = \text{ce}_{2n+1}(0, q) \text{ce}_{2n+1}'(\pi/2, q) \\ s_{2n+2} = \text{se}_{2n+2}'(0, q) \text{se}_{2n+2}'(\pi/2, q) \\ s_{2n+1} = \text{se}_{2n+1}'(0, q) \text{se}_{2n+1}(\pi/2, q) \end{cases}. \quad (16)$$

The scattering wave field produced by an elliptical opening in a two-dimensional piezoelectric material is constructed using a series of Mathieu functions as

$$w^s = 2w_0 \sum_{n=0}^{\infty} (\mu_{2n} + \mu_{2n+1} + \nu_{2n+2} + \nu_{2n+1}) \quad (17)$$

where

$$\begin{cases} \mu_{2n}(\xi, \eta) = R_{2n} \text{He}_{2n}(\xi, q) \text{ce}_{2n}(\eta, q) \text{ce}_{2n}(\alpha_0, q) \\ \mu_{2n+1}(\xi, \eta) = R_{2n+1} \text{He}_{2n+1}(\xi, q) \text{ce}_{2n+1}(\eta, q) \text{ce}_{2n+1}(\alpha_0, q) \\ \nu_{2n+2}(\xi, \eta) = S_{2n+2} \text{Ho}_{2n+2}(\xi, q) \text{se}_{2n+2}(\eta, q) \text{se}_{2n+2}(\alpha_0, q) \\ \nu_{2n+1}(\xi, \eta) = S_{2n+1} \text{Ho}_{2n+1}(\xi, q) \text{se}_{2n+1}(\eta, q) \text{se}_{2n+1}(\alpha_0, q) \end{cases},$$

the constants R_{2n} , R_{2n+1} , S_{2n+2} , and S_{2n+1} are undetermined, while He_{2n} , He_{2n+1} , Ho_{2n+2} , and Ho_{2n+1} represent the four types of first-class Mathieu–Hankel function, respectively. Equation (19) also satisfies the first formula of Equation (6).

As observed from Equation (5), the potential field resulting from the scattered wave field can be expressed as

$$w^s = 2w_0 \sum_{n=0}^{\infty} (\mu_{2n} + \mu_{2n+1} + \nu_{2n+2} + \nu_{2n+1}). \quad (18)$$

The total displacement field and the total potential field in the piezoelectric material are expressed as

$$w^t = w^i + w^s, \quad (19)$$

$$\varphi^t = \varphi^i + \varphi^s. \quad (20)$$

Furthermore, the total potential field φ^t must also satisfy the Sommerfeld radiation condition, which is obtained by

$$\varphi^t(\xi) = \varphi^i(\xi) \quad (\xi \rightarrow \infty). \quad (21)$$

Comparing Equation (15) with Equation (20), using the convergence of the Mathieu function, we can get $A_n = 0$, $C_0 = 0$ ($n = 0, 1, 2, \dots$). Thus, Equation (11) can be rearranged as follows:

$$\begin{aligned} \Phi^s = 2w_0 \sum_{n=0}^{\infty} & \left[C_{2n} e^{-2n\xi} \cos 2n\eta + C_{2n+1} e^{-(2n+1)\xi} \cos(2n+1)\eta \right. \\ & \left. + D_{2n+2} e^{-(2n+2)\xi} \sin(2n+2)\eta + D_{2n+1} e^{-(2n+1)\xi} \sin(2n+1)\eta \right]. \end{aligned} \quad (22)$$

The elliptical cavity does not possess an elastic field but does contain a potential field that satisfies the equation

$$\nabla^2 \varphi^h = 0. \quad (23)$$

According to the solution steps from Equation (9) to Equation (13), the general solution of the potential φ^h in the opening is

$$\varphi^h(\xi, \eta) = 2w_0 \sum_{n=0}^{\infty} \left(G_n e^{n\xi} + H_n e^{-n\xi} \right) [E_n \cos n\eta + F_n \sin n\eta], \quad (24)$$

where E_n , F_n , G_n , and H_n represent mode coefficients. Since there is no source in the opening, the potential φ^h should increase with the increase of ξ . Therefore, we set $H_n = 0$, resulting in the organized expression of the potential in the opening as

$$\begin{aligned} \varphi^h(\xi, \eta) = 2w_0 \sum_{n=0}^{\infty} & \left[E_{2n} e^{2n\xi} \cos 2n\eta + E_{2n+1} e^{(2n+1)\xi} \cos(2n+1)\eta \right. \\ & \left. + F_{2n+2} e^{(2n+2)\xi} \sin(2n+2)\eta + F_{2n+1} e^{(2n+1)\xi} \sin(2n+1)\eta \right]. \end{aligned} \quad (25)$$

In the above formula, the mode coefficient G_n is merged. The electrical displacement in the opening can then be expressed as

$$\begin{cases} D_{\xi}^h = -\frac{\varepsilon_0}{h_1} \frac{\partial \varphi^h}{\partial \xi} \\ D_{\eta}^h = -\frac{\varepsilon_0}{h_1} \frac{\partial \varphi^h}{\partial \eta} \end{cases}, \quad (26)$$

where $\varepsilon_0 = 8.85 \times 10^{-12} \text{ F} \cdot \text{m}^{-1}$ represents the dielectric constant of vacuum.

4. Boundary Conditions and Dynamic Stress Concentration Coefficient

The boundary conditions around the elliptical opening consist of continuous electric potential and normal displacement, as well as free radial displacement and shear stress. This can be expressed as follows:

$$\begin{cases} \tau_{\xi z}^t|_{\xi_0} = 0 \\ D_{\xi}^t|_{\xi_0} = D_{\xi}^h|_{\xi_0} \\ \varphi^t|_{\xi_0} = \varphi^h|_{\xi_0} \end{cases} \quad (27)$$

By substituting the first formula of Equation (3) into the first boundary condition of Equation (27), using Equations (19) and (20), we obtain

$$\begin{aligned} c^* \sum_{n=0}^{\infty} \frac{\partial \psi_n}{\partial \xi} - \frac{e_{15}}{\varepsilon_{11}} \sum_{n=0}^{\infty} [2nC_{2n}\delta_1 + (2n+1)C_{2n+1}\delta_2 \\ + (2n+2)D_{2n+2}\delta_3 + (2n+1)D_{2n+1}\delta_4] = 0 \quad (\xi = \xi_0), \end{aligned} \quad (28)$$

where

$$\begin{cases} \delta_1 = e^{-2n\xi} \cos 2n\eta \\ \delta_2 = e^{-(2n+1)\xi} \cos(2n+1)\eta \\ \delta_3 = e^{-(2n+2)\xi} \sin(2n+2)\eta \\ \delta_4 = e^{-(2n+1)\xi} \sin(2n+1)\eta \end{cases}$$

$$\psi_n = \sigma_{2n} - \sigma_{2n+1} + \gamma_{2n+2} + \gamma_{2n+1} + \mu_{2n} + \mu_{2n+1} + \nu_{2n+2} + \nu_{2n+1}.$$

Under the incidence of steady-state SH wave, DSCC represents the degree of dynamic stress concentration (when there is a continuous transmission of elastic wave interference, causing a sudden increase in stress in the scatterer or in the nearby local area). According to Equation (3), the surface stress of the elliptical opening can be expressed as follows

$$\begin{aligned} \tau_t &= \tau_{\xi z-\Pi} + \tau_{\eta z-\Pi} \quad (\xi = \xi_0) \\ &= \frac{1}{h_1} \left(c_{44}^E \frac{\partial w}{\partial \eta} + e_{15} \frac{\partial \varphi}{\partial \eta} \right) \\ &= \frac{2w_0 c^*}{h_1} \sum_{n=0}^{\infty} \frac{\partial \psi_n}{\partial \eta} + \frac{2w_0 e_{15}}{h_1 \varepsilon_{11}^S} \sum_{n=0}^{\infty} \frac{\partial}{\partial \eta} (C_n \cos n\eta + D_n \sin n\eta). \end{aligned} \quad (29)$$

The coefficients required for the above formula have been solved in the previous section. The sum of the series represents the result of a complex number belonging to a given parameter η_0 , in the form of $p_1 + ip_2$. As shown in Equation (1), if we consider the real part $(p_1 + ip_2)e^{-i\omega t}$ as the solution for this stress, then the peak stress can be expressed as $(p_1^2 + p_2^2)^{1/2}$, which is the modulus of the complex number $p_1 + ip_2$, denoted as $|\tau_t|$.

If the elliptical opening does not exist, then under the action of the incident wave w^i , the non-zero stress at each point in the piezoelectric medium is given by

$$\tau_0 = c^* k w_0, \quad (30)$$

and the expression for DSCC is

$$DSCC = \left| \frac{\tau_t}{\tau_0} \right|, \quad (31)$$

where τ_0 represents the maximum value of stress generated by the incident wave.

5. Numerical Example and Analysis and Discussion

The analysis of the above expression reveals that when $N = 20$, the calculated results meet the accuracy requirements for engineering applications. Barium titanate polarized

ceramics are known for their excellent piezoelectric properties and stability, making them widely used in precision instruments, medical equipment, acoustic devices, and other fields. The crystal structure of barium titanate is hexagonal with a 6/mm point group. In the numerical example, barium titanate polarized ceramics are chosen as the example material, and the corresponding material constants are as follows:

$$\begin{aligned} c_{44}^E &= 4.4 \times 10^{10} \text{ N} \cdot \text{m}^{-2}, e_{15} = 11.4 \text{ C} \cdot \text{m}^{-2}, \\ \varepsilon_{11}^S &= 9.87 \times 10^{-9} \text{ F} \cdot \text{m}^{-1}, \rho = 5.7 \times 10^3 \text{ kg} \cdot \text{m}^{-3}. \end{aligned}$$

Considering frequency f , incidence angle α_0 and elliptic eccentricity $e = h/a$ as the characteristic parameters to derive

$$\begin{aligned} \omega &= 2\pi f, c^* = c_{44}^E + e_{15}^2 / \varepsilon_{11}^S, c_s^2 = c^* / \rho, \\ k &= \omega / c_s, h = ea, w_0 = 1, \\ q &= h^2 k^2 / 4, \zeta_0 = \text{arcosh}(e^{-1}). \end{aligned}$$

Figures 2 and 3 demonstrate the size and distribution of DSCC around the elliptical opening under different incidence angles α_0 with an eccentricity of $e = 0.1$. The DSCC distribution also varies with the incident frequency f . From the perspective of frequency f , it is evident that low-frequency incident waves result in a more pronounced dynamic stress concentration phenomenon compared to high-frequency incident waves. The figures show that the maximum value of DSCC exceeds 2 at all incidence angles. However, this does not imply that the stress generated by the low-frequency SH waves on the elliptical opening is the highest. It simply indicates that low-frequency SH waves are more strongly influenced by defects in the elliptical opening. The distribution of DSCC caused by high-frequency incident waves ranging from 1000 Hz to 3000 Hz is not prominent around the elliptical opening, with a maximum DSCC of approximately 1.6. Therefore, it is evident that low-frequency SH waves pose a greater threat to structures with defects. Regarding the variation of incidence angle α_0 , when the eccentricity $e = 0.1$, the change in incident angle has little impact on the distribution pattern of DSCC around the elliptical opening, whether the SH wave is incident along the long axis or the short axis. This is because the eccentricity e is too small, allowing the elliptical opening to be approximated as a circular opening. Consequently, the patterns generated by SH waves incident from different angles remain consistent.

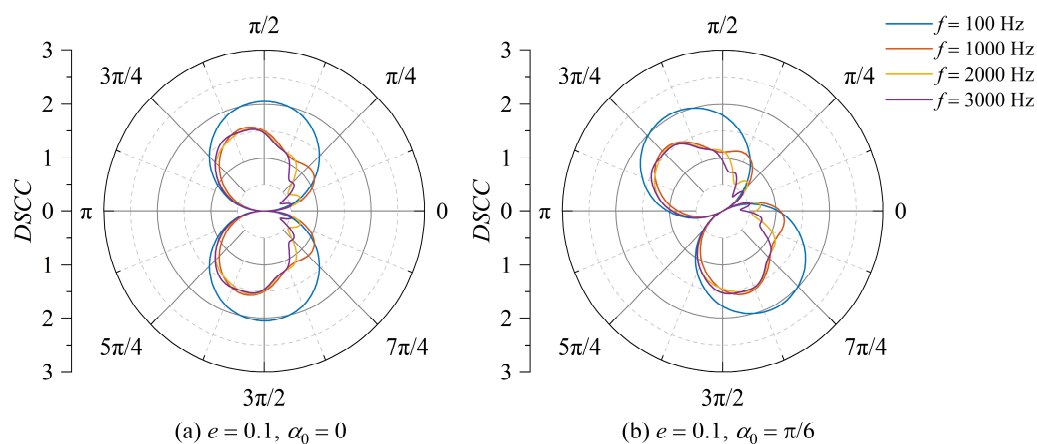


Figure 2. Distribution of DSCC around the elliptical opening with an eccentricity of $e = 0.1$. (a) Incident angle $\alpha_0 = 0$; (b) incident angle $\alpha_0 = \pi/6$.

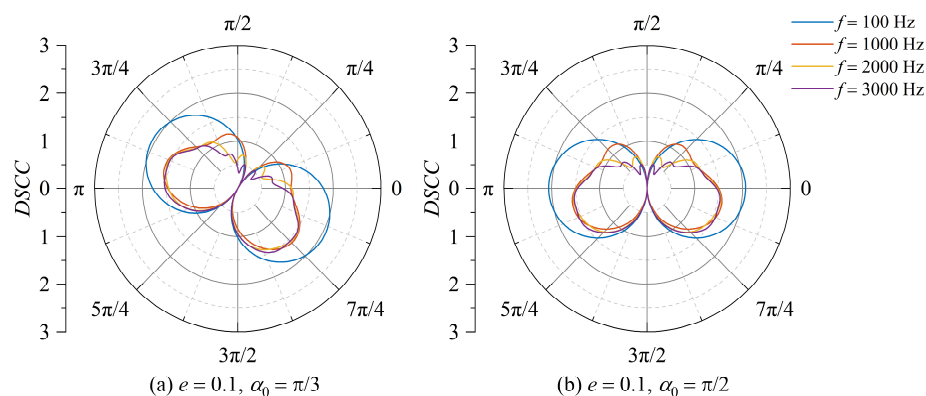


Figure 3. Distribution of DSCC around the elliptical opening with an eccentricity of $e = 0.1$. (a) Incident angle $\alpha_0 = \pi/3$; (b) incident angle $\alpha_0 = \pi/2$.

When the short axis is further shortened and the eccentricity is $e = 0.6$, as shown in Figures 4a and 5b, the extreme values of DSCC are 1.84 and 2.32, respectively. Based on the results of the numerical example mentioned above, a qualitative conclusion can be deduced: as the eccentricity of the elliptical opening increases, the DSCC of points with the curvature direction inclined to the short axis around the elliptical opening decreases, while the mechanical behavior of points with the curvature direction inclined to the long axis is the opposite. Furthermore, it is observed that as the eccentricity increases, the DSCC distribution generated by the oblique incident SH wave (the incident angles $\alpha_0 = \pi/6$ and $\alpha_0 = \pi/3$) is no longer symmetric with respect to the incident direction, as described in Figures 4b and 5a. The “lower half” of the distribution image exhibits larger deformation.

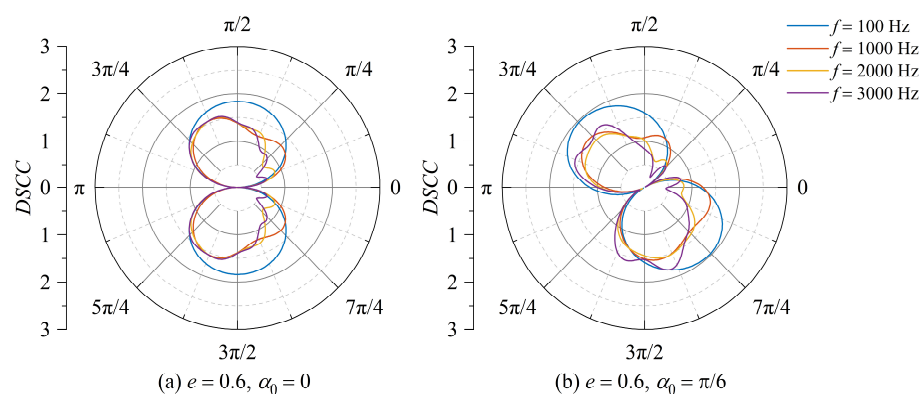


Figure 4. Distribution of DSCC around the elliptical opening with an eccentricity of $e = 0.6$. (a) Incident angle $\alpha_0 = 0$; (b) incident angle $\alpha_0 = \pi/6$.

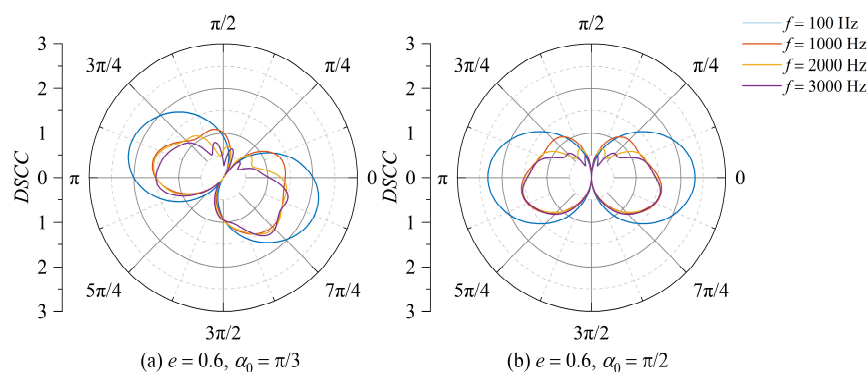


Figure 5. Distribution of DSCC around the elliptical opening with an eccentricity of $e = 0.6$. (a) Incident angle $\alpha_0 = \pi/3$; (b) incident angle $\alpha_0 = \pi/2$.

Figures 6 and 7 illustrate the size and distribution of DSCC around the elliptical opening under different incident angles α_0 and the variation of incident frequency f when the eccentricity is $e = 0.9$. In the numerical example, the short semi-axis of the elliptical opening is 56.2% shorter than that of the elliptical opening with an eccentricity of $e = 0.1$. The DSCC distribution around the elliptical opening undergoes significant changes. Particularly, when the incident wave is obliquely incident ($\alpha_0 = \pi/6$, $\alpha_0 = \pi/3$), the DSCC distribution exhibits a decrease followed by an increase with the change in frequency. This is evident in Figures 6a and 7b, where the extreme DSCC values for the SH wave incident along the long axis and short axis are 1.45 and 3.40, respectively. Therefore, special attention should be given to the potential damage in the low-frequency vibration environment for piezoelectric materials with elliptical openings of large eccentricity.

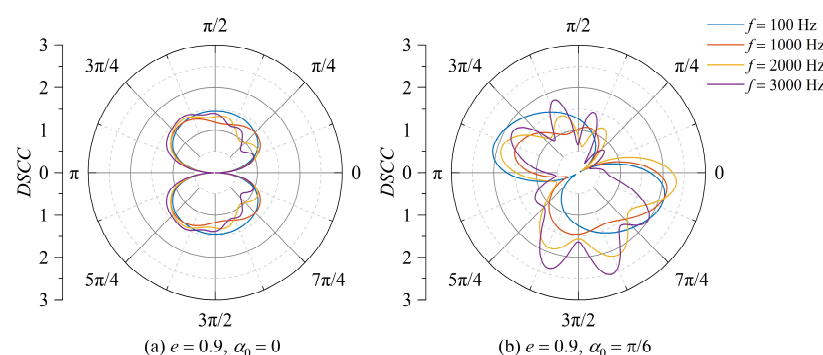


Figure 6. Distribution of DSCC around the elliptical opening with an eccentricity of $e = 0.9$. (a) Incident angle $\alpha_0 = 0$; (b) incident angle $\alpha_0 = \pi/6$.

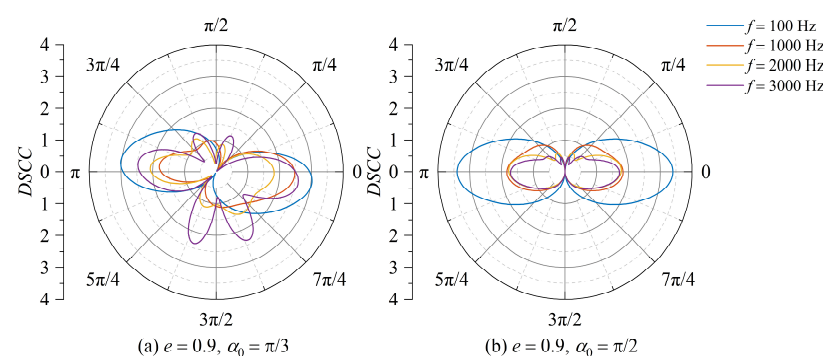


Figure 7. Distribution of DSCC around the elliptical opening with an eccentricity of $e = 0.9$. (a) incident angle $\alpha_0 = \pi/3$; (b) incident angle $\alpha_0 = \pi/2$.

Considering the complexity of the DSCC expression, a theoretical calculation is performed using the wave function expansion method. The superposition of Bessel and Hankel functions represents displacement and stress. The undetermined coefficients can be determined by applying the boundary conditions associated with the circular aperture. This strategy provides a solution for addressing similar challenges. More details can be obtained from numerical results:

- (1) In the numerical examples, the DSCC distribution around the elliptical opening gradually decreases with increasing incident wave frequency. As the incident frequency increases, the number of stress sidelobes also increases. These sidelobes change along with the frequency, leading to jumps in the maximum value of the DSCC.
- (2) Eccentricity is a vital factor that affects the DSCC. Generally, increasing eccentricity enlarges the distribution range of the DSCC in low-frequency cases. Nevertheless, when the incident wave is along the long axis, larger eccentricity leads to a smaller distribution range of the DSCC due to the decrease in irradiation area. For the elliptical

opening, changing the eccentricity does not alter the number of stress sidelobes when the incident wave frequency is constant.

- (3) Changing the incident angle affects the symmetry of DSCC. When the incident wave is along the long axis or minor axis of the ellipse, DSCC is symmetrically distributed with respect to the incident direction. However, when the incident wave enters the perforation defect at an oblique angle, the range of the DSCC first increases and then decreases with increasing frequency.

6. Conclusions

In this paper, an elliptic cylindrical coordinate system is established in a piezoelectric material with an elliptic opening. The incident wave field, scattered wave field, and standing wave field are decomposed into Mathieu function series using the wave function expansion method based on Mathieu functions. By establishing an equivalent relation through boundary conditions, a series of infinite algebraic equations with mode coefficients are derived. The mode coefficients are then solved by truncating the infinite equation and providing initial conditions. The DSCC explored in this paper is determined by the ratio of the steady-state stress at the defect edge to the maximum stress in the incident field. The distribution of the DSCC is visualized through numerical simulations. In the subsequent work, further analysis is conducted on the influence of incident wave frequency, incident wave angle, and elliptic eccentricity on the DSCC.

In conclusion, our study has quantifiably demonstrated that the dynamic stress concentration coefficient (DSCC) in piezoelectric materials with elliptical openings can be significantly reduced by optimizing the geometric parameters of the openings. Specifically, we have shown that the DSCC can be minimized by adjusting the eccentricity, which results in a 20% reduction in stress concentration at low frequencies ($f = 100$ Hz) and up to 30% at high frequencies ($f = 3000$ Hz) compared to untreated materials. These improvements not only enhance the strength and fatigue life of the materials but also provide a basis for selecting appropriate nondestructive testing methods. Our findings thus offer substantial contributions to designing and manufacturing reliable piezoelectric materials.

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