

Article

Robust and Non-Fragile Path Tracking Control for Autonomous Vehicles

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Abstract

Path tracking is a fundamental function for autonomous vehicles, but its performance often degrades under parameter variations and controller fragility—an issue seldom addressed together in prior studies. This paper develops a robust non-fragile Linear Quadratic Regulator (LQR) using linear matrix inequality (LMI) optimization, explicitly considering uncertainties in vehicle speed, mass, and cornering stiffness as well as gain perturbations from implementation. A two-degrees-of-freedom bicycle model is employed for controller design, and a weighted least-squares allocation method integrates multiple actuators, including front steering, rear steering, four-wheel independent drive, and braking. A double lane-change maneuver in CarSim evaluates the proposed design. The robust and non-fragile LQR maintains lateral offset within 0.02 m and overshoot below 1% under $\pm 20\%$ parameter variation, offering improved stability margins compared with the baseline LQR. The results highlight context-dependent actuator effects and clarify the trade-off between control complexity, robustness, and real-world applicability.

Keywords: path tracking control; robust non-fragile control; Linear Quadratic Regulator (LQR); control allocation; autonomous vehicles



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1. Introduction

Since the last decade, autonomous driving has attracted considerable attention as a transformative technology in both academia and the automotive industry [1]. The motivation stems not only from the potential to reduce collisions caused by human errors but also from the promise of improving energy efficiency and alleviating traffic congestion [2]. For instance, aggressive driving behaviors, such as abrupt acceleration or braking, are closely related to unnecessary fuel consumption and pollutant emissions, while heterogeneous human driving patterns often prevent road networks from being utilized at their maximum throughput. A typical autonomous driving framework consists of several functional modules, including perception, localization, prediction, planning, and control. More recently, an alternative paradigm known as end-to-end driving has been explored, where perception and decision-making are jointly optimized. Within this broad spectrum of research, path tracking control (PTC) has emerged as a particularly critical component, and extensive studies have been devoted to this topic over the past decade [3–15]. In addition to control accuracy, recent studies have emphasized actuator smoothness and implementation feasibility in real driving conditions, considering factors such as delay and sampling limitations [16].

Nevertheless, the design of path tracking controllers is not straightforward because real vehicle dynamics inevitably involve parameter uncertainties. Typical examples are variations in vehicle mass and speed. Another factor that strongly influences control performance is the tire–road friction coefficient, which is difficult to measure or estimate in real time. Consequently, vehicle systems should be regarded as time-varying and uncertain. In most cases, however, controllers are designed based on models with fixed or nominal parameter values, and such mismatches often degrade control performance [17–19]. The same issue also arises in the domain of PTC [7–9,11–13,15]. Recently, some approaches have attempted to integrate path planning and robust control under real-world uncertainties to enhance overall vehicle safety and consistency [20]. To cope with these challenges, a variety of robust design methods have been introduced for PTC since 2015 [21–39].

In view of controller design methodology, the quadratic stabilization with linear matrix inequality (LMI) with H_2 /LQR and H_∞ control, sliding mode control (SMC), model predictive control (MPC), gain scheduling, and adaptive control have been applied to PTC to date. The quadratic stabilization with H_2 /LQR and H_∞ control adopts full-state or output feedback with an LQ objective function, and it is solved with LMI optimization [21,22,29,30]. This method tries to control an uncertain system with a single gain. As a result, this method has worse robust performance than the others. SMC is known to be robust against parameter uncertainty and external disturbance. For this reason, SMC has been adopted as a robust controller for PTC [36–38]. SMC has not been used alone but in combination with others, such as adaptive control [36–38]. In recent years, robust MPC approaches with LMI-based constraint enforcement have been proposed for autonomous steering systems, addressing parameter uncertainty and control constraints simultaneously [40]. MPC has been adopted as a robust controller for PTC in combination with H_∞ control, adaptive control, or gain scheduling since 2019 [23,27,33]. The advantage of MPC is the inclusion of other methods, such as H_∞ control, adaptive control, and gain scheduling. For example, H_∞ performance can be integrated with the objective function of MPC [33]. An adaptive control or parameter adaptation scheme can be used to estimate uncertain parameters or disturbances of the model used in MPC [27].

Gain scheduling has been widely investigated for PTC as a means of handling parameter variations [24,33,35,39]. The idea is to pre-design controllers at several representative operating conditions and then interpolate among them parameters such as speed or mass change. Although effective in covering broad operating regions, this approach requires multiple controller designs and is sensitive to the choice of scheduling variables, which are often vehicle states or their inverses. In contrast, adaptive control adjusts controller gains online in response to parameter variations [25–28,31,32,34–38]. Estimation techniques, such as recursive least squares (RLS) or neural networks [27,31,32,37,38], are typically employed to identify uncertain parameters, which are then fed back into the controller. While adaptive methods can offer improved robustness, their effectiveness depends heavily on the speed and accuracy of the adaptation mechanism. In many studies, gain scheduling and adaptive control have also been combined with H_∞ control, MPC, or SMC, often formulated as optimization problems using LMI [21,23,24,29,33,35,39]. Motivated by this trend, the present work also employs an LMI-based design framework.

With regard to vehicle modeling, the majority of PTC studies adopt a state-space formulation based on the reference trajectory and the two-degrees-of-freedom (2-DOF) bicycle model [21–39]. This model inherently contains parameters that are uncertain or time-varying, such as mass, yaw inertia, forward velocity, and tire cornering stiffness. In particular, variations in velocity and cornering stiffness have been shown to strongly influence tracking performance. Therefore, following common practice in robust control design, this study considers these two parameters as uncertain. Unlike approaches that rely

on linear parameter-varying (LPV) gain scheduling, we focus on developing a controller that maintains robustness against unknown parameter fluctuations.

From the perspective of controller gains, parameter uncertainties naturally lead to gain variations. When the uncertainty is significant, the corresponding change in the controller gain can also be large. Therefore, a controller that is robust to gain fluctuations can also be regarded as robust against parameter uncertainties. Historically, gain variation—often referred to as controller fragility—was mainly considered an implementation issue arising from fixed-point arithmetic in microprocessors. A controller designed to withstand such fragility is called a non-fragile controller [41,42]. In this study, fragility is interpreted more broadly as a manifestation of parameter uncertainty, and a non-fragile controller is developed accordingly. To this end, an LQR formulation solved via LMI optimization is adopted [42], and the resulting design is further integrated with a robust LQR framework.

Regarding actuators for PTC, most existing studies employ front-wheel steering (FS, δ_f) as the primary control input. Some works also consider four-wheel steering (4S, δ_f, δ_r), which combines FS with rear-wheel steering (RS, δ_r) [21,22,26,34]. In addition, a few studies incorporate yaw moment control (ΔM_z) alongside FS [23,32,35,39]. The yaw moment can be generated by tire forces through four-wheel independent driving (4ID) or a torque vectoring device (TVD) and alternatively by four-wheel independent braking (4IB) [43,44]. Despite these efforts, the combined influence of multiple actuators—such as FS, RS, 4ID, and 4IB—on robustness and tracking performance has not been fully explored. Motivated by this gap, the present study investigates the effectiveness of RS, 4ID, and 4IB as actuators for robust PTC.

This study proposes a design framework for a robust and non-fragile controller in the context of path tracking. Specifically, a Linear Quadratic Regulator (LQR) is employed, formulated on a state-space representation that includes lateral offset, heading error, side-slip angle, and yaw rate as state variables [21,22,24,31,35]. A preview term is incorporated into the model to enhance anticipative behavior [23,24,29,31,37,39]. Three input configurations are investigated, $[\delta_f]$, $[\delta_f, \delta_r]$, and $[\delta_f, \Delta M_z]$, which involve front-wheel steering (δ_f), rear-wheel steering (δ_r), and yaw moment control (ΔM_z) [43]. The system is assumed to contain norm-bounded uncertainties derived from polytopic descriptions, including variations in cornering stiffness, vehicle mass, velocity, and tire–road friction. Robust, non-fragile, and robust non-fragile LQR controllers are then designed for each input configuration using LMI optimization, and their path tracking performance is compared. When ΔM_z is chosen as a control input, control allocation is applied to distribute it into tire forces generated by multiple actuators such as RS, 4ID, and 4IB [43,44]. The proposed method is validated in CarSim through a double lane-change maneuver scenario for collision avoidance [25,35], demonstrating its effectiveness under parameter variations. Unlike conventional robust control frameworks that primarily address model uncertainty or rely on extensive gain scheduling, the present study emphasizes controller fragility as an equally critical factor. By integrating robustness and non-fragility within a unified LMI-based LQR design, this work offers a distinct contribution that bridges theoretical stability analysis and practical implementation reliability.

The main contributions of this work can be summarized as follows:

1. A robust and non-fragile path tracking controller is newly formulated using an LMI-based LQR framework. The proposed approach explicitly integrates the non-fragile property—an aspect rarely addressed in previous path tracking studies—into the controller design, thereby improving both robustness and implementation reliability under parameter perturbations.
2. A systematic comparative analysis of multiple actuators (RS, 4ID, and 4IB) for yaw-moment generation is conducted. While most existing robust PTC studies have relied

solely on FS or 4S, this work provides the first structured evaluation of how different actuator combinations affect path tracking robustness and overall vehicle stability.

3. The comprehensive simulation results identify optimal controller–actuator configurations for practical implementation. The study demonstrates that when performance differences are minor, a simpler LQR design remains the most effective choice—offering clear design guidance for real-world autonomous vehicle control.

The remainder of this paper is organized as follows. Section 2 describes the vehicle model and outlines the controller design methodology. Section 3 introduces new evaluation metrics for path tracking. Section 4 presents the simulation settings and results, followed by a discussion. Finally, Section 5 concludes the paper.

2. Design of a Robust Non-Fragile Path Tracking Controller

Section 2 presents the research methodology, including controller design and modeling procedures. In this study, the path tracking task is addressed using an LQR-based control framework. A full-state feedback structure is considered, and the corresponding LQ cost function is employed to balance tracking accuracy and control effort. Robust and non-fragile variants of the LQR are formulated within an LMI optimization setting. In addition, when yaw moment control (ΔM_z) is introduced, a control allocation scheme is applied so that the commanded moment is distributed to physically realizable inputs such as rear steering angle, wheel braking torques, or driving torques generated by RS, 4ID, and 4IB. These actuator signals, computed through allocation, are then utilized to generate the necessary tire forces for vehicle maneuvering. Figure 1 shows the overall structure of the proposed robust and non-fragile path tracking controller.

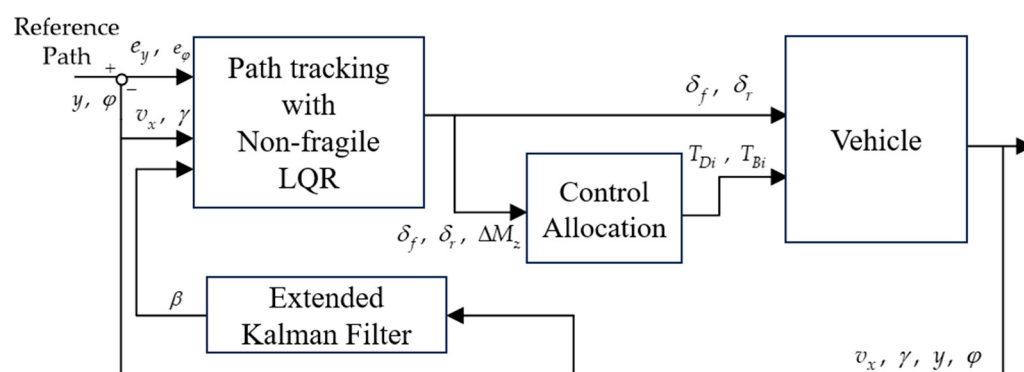


Figure 1. Overall structure of the proposed robust and non-fragile path tracking controller.

2.1. State-Space Model Formulation

For controller design, a two-degrees-of-freedom (2-DOF) bicycle representation is adopted, which has been widely used in path tracking studies [43–45]. This modeling approach has also been validated through various experimental studies on real vehicles under different operating conditions [46,47]. The coordinate system and relevant variables are illustrated in Figure 2 together with the reference path. This simplified model captures lateral dynamics and yaw motion while assuming a constant longitudinal velocity (v_x). The state vector is defined by yaw rate γ and side-slip angle β , leading to the equations of motion shown in (1) [21–23,31,35]. The front and rear tire slip angles, α_f and α_r , are introduced in (2), and the associated lateral tire forces, F_{yf} and F_{yr} , are expressed in linearized form in (3). By combining (1)–(3), the state-space equations of the bicycle model are obtained, as summarized in (4).

$$\begin{cases} m v_x (\dot{\beta} + \gamma) = F_{yf} + F_{yr} \\ I_z \dot{\gamma} = l_f F_{yf} - l_r F_{yr} + \Delta M_z \end{cases} \quad (1)$$

$$\mathbf{A} = \begin{bmatrix} 0 & v_x & -v_x & -L_p \\ 0 & 0 & 0 & -1 \\ 0 & 0 & \frac{\zeta_1}{mv_x} & \frac{\zeta_2}{mv_x^2} - 1 \\ 0 & 0 & \frac{\zeta_2}{I_z} & \frac{\zeta_3}{I_z v_x} \end{bmatrix}, \mathbf{B}_1 = \begin{bmatrix} 0 \\ v_x \\ 0 \\ 0 \end{bmatrix}, \mathbf{B}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{2C_f}{mv_x} & \frac{2C_r}{mv_x} & 0 \\ \frac{2l_f C_f}{I_z} & -\frac{2l_r C_r}{I_z} & \frac{1}{I_z} \end{bmatrix} \quad (9)$$

The control input, denoted as $\mathbf{u}$ in Equation (7), comprises three elements: δ_f , δ_r , and ΔM_z . These elements are used to define three distinct control inputs, named $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$, corresponding to input configurations *CONFIG-1*, *CONFIG-2*, and *CONFIG-3*, as outlined in reference [41]. Derived from Equation (9), the input matrices $\mathbf{B}_{21}$, $\mathbf{B}_{22}$, and $\mathbf{B}_{23}$ are determined in Equation (10), with $\mathbf{B}_2(k)$ representing the k -th column of matrix $\mathbf{B}_2$. Notably, δ_f is available in all input configurations, while δ_r is exclusively employed in *CONFIG-2* and *CONFIG-3*. In the case of *CONFIG-3*, involving ΔM_z , the generation of ΔM_z involves the availability of 4IB and 4ID, which are instrumental in converting ΔM_z into δ_r and determining the braking and traction torques (T_{Bi} and T_{Di}) for each wheel. This paper adopts the assumption that T_{Bi} and T_{Di} are generated through the use of 4IB and 4ID, respectively.

$$\begin{cases} \mathbf{u}_1 = \delta_f & \mathbf{B}_{21} = [\mathbf{B}_2(1)] & \text{for } \text{CONFIG} - 1 \\ \mathbf{u}_2 = \begin{bmatrix} \delta_f & \delta_r \end{bmatrix}^T & \mathbf{B}_{22} = \begin{bmatrix} \mathbf{B}_2(1) & \mathbf{B}_2(2) \end{bmatrix} & \text{for } \text{CONFIG} - 2 \\ \mathbf{u}_3 = \begin{bmatrix} \delta_f & \Delta M_z \end{bmatrix}^T & \mathbf{B}_{23} = \begin{bmatrix} \mathbf{B}_2(1) & \mathbf{B}_2(3) \end{bmatrix} & \text{for } \text{CONFIG} - 3 \end{cases} \quad (10)$$

2.2. Design of LQR

For each of the three input configurations (*CONFIG-1*, *CONFIG-2*, and *CONFIG-3*), a quadratic performance index is defined, denoted by J_1 , J_2 , and J_3 , respectively, as expressed in (11). These cost functions are then rewritten in vector–matrix form, as shown in (12). The weighting matrices $\mathbf{Q}$ and $\mathbf{R}_i$ are specified in (13). The selection of the weights ρ_i follows Bryson's rule [43,48], in which the parameter ξ_i indicates the maximum permissible value for each state or input. By adjusting ξ_i , the balance between tracking performance and control effort can be tuned.

Within the LQR framework, the control input $\mathbf{u}_i$ corresponding to *CONFIG- i* is obtained according to (15). Here, $\mathbf{P}_i$, which is the solution of the associated Riccati equation, determines the optimal feedback gain and thus plays a central role in the computation.

$$\begin{aligned} J_0 &= \int_0^\infty \left\{ \rho_1 e_y^2 + \rho_2 e_\varphi^2 + \rho_3 \beta^2 + \rho_4 \gamma^2 \right\} dt \\ \begin{cases} J_1 = J_0 + \int_0^\infty \left(\rho_5 \delta_f^2 \right) dt & \text{for } \text{CONFIG} - 1 \\ J_2 = J_0 + \int_0^\infty \left(\rho_5 \delta_f^2 + \rho_6 \delta_r^2 \right) dt & \text{for } \text{CONFIG} - 2 \\ J_3 = J_0 + \int_0^\infty \left(\rho_5 \delta_f^2 + \rho_7 \Delta M_z^2 \right) dt & \text{for } \text{CONFIG} - 3 \end{cases} \end{aligned} \quad (11)$$

$$J_i = \int_0^\infty \begin{bmatrix} \mathbf{x} \\ \mathbf{u}_i \end{bmatrix}^T \begin{bmatrix} \mathbf{Q} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_i \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{u}_i \end{bmatrix} dt, \quad i = 1, 2, 3 \quad (12)$$

$$\mathbf{Q} = \text{diag}(\rho_1, \rho_2, \rho_3, \rho_4), \quad \begin{cases} \mathbf{R}_1 = \rho_5 & \text{for } \text{CONFIG} - 1 \\ \mathbf{R}_2 = \text{diag}(\rho_5, \rho_6) & \text{for } \text{CONFIG} - 2 \\ \mathbf{R}_3 = \text{diag}(\rho_5, \rho_7) & \text{for } \text{CONFIG} - 3 \end{cases} \quad (13)$$

$$\rho_i = \frac{1}{\xi_i^2} \quad (14)$$

$$\mathbf{u}_i = -\mathbf{K}_i \mathbf{x} = -\mathbf{R}_i^{-1} \mathbf{B}_{2i}^T \mathbf{P}_i \mathbf{x}, \quad i = 1, 2, 3 \quad (15)$$

2.3. Design of Robust Non-Fragile LQR

In Section 2.2, the fundamental formulation of the LQR problem was presented, focusing on the nominal system without uncertainty. In this section, the LQR framework is extended to explicitly address parameter uncertainty and controller fragility. A system subject to both effects can be represented as in (16), where the state-space matrices and controller gains include uncertain components. The uncertain components, denoted as $\Delta\mathbf{A}$, $\Delta\mathbf{B}_2$, and $\Delta\mathbf{K}$, are outlined in (17), with the assumption that $\Phi(t)$ and $\Lambda(t)$ are diagonal and bound within a norm. Within (17), matrices $\mathbf{H}$, $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{F}$, and $\mathbf{G}$ can be readily derived from a polytopic uncertainty model or within a specified range centered around a nominal operating point. In (16), $\mathbf{B}_2$ takes one of three forms, namely, $\mathbf{B}_{21}$, $\mathbf{B}_{22}$, and $\mathbf{B}_{23}$, as defined in (10), contingent on the specific input configuration. Given the control input outlined in (16), designing a robust controller is challenging due to the inherent uncertainty in the control gains. To address this challenge, it is assumed that the controller gains are not uncertain, as described in (16). Leveraging the state-space model and control input from (16), in conjunction with the LQ objective function from (12), a robust non-fragile controller is formulated through LMI optimization [49].

$$\begin{cases} \dot{\mathbf{x}} = (\mathbf{A} + \Delta\mathbf{A})\mathbf{x} + \mathbf{B}_1\mathbf{w} + (\mathbf{B}_2 + \Delta\mathbf{B}_2)\mathbf{u} \\ \mathbf{u} = (\mathbf{K} + \Delta\mathbf{K})\mathbf{x} \end{cases} \quad (16)$$

$$\begin{cases} [\Delta\mathbf{A} \ \Delta\mathbf{B}_2] = \mathbf{H}\Phi(t) \begin{bmatrix} \mathbf{E}_1 & \mathbf{E}_2 \end{bmatrix} & \Phi^T(t)\Phi(t) \leq 1 \\ \Delta\mathbf{K} = \mathbf{F}\Lambda(t)\mathbf{G} & \Lambda^T(t)\Lambda(t) \leq 1 \end{cases} \quad (17)$$

To develop LQR, the first step is to ensure that the controller gain remains stable without fragilities, meaning that $\Delta\mathbf{K}$ in Equation (16) is set to zero. To achieve robust stability, it is necessary to meet the quadratic stabilization condition, expressed by the Riccati inequality in Equation (18). Throughout the subsequent equations, the subscript i refers to the input configuration specified in Equation (10). In Equation (18), $\text{sym}\{\mathbf{X}\}$ denotes the symmetric part of $\mathbf{X}$, and $\mathbf{P}_i$ represents a positive definite symmetric matrix, which is the solution to the Riccati Equation (15). The key challenge lies in eliminating the uncertain block, $\Phi(t)$, from Equation (18). This problem is addressed by employing the quadratic inequality presented in Equation (19), with ε as a non-zero tuning parameter through LMI optimization. Equation (19) is applied to the terms containing the uncertain block $\Phi(t)$ in (18). This step effectively removes that source of uncertainty. Consequently, Equation (20) is derived, wherein the uncertainty block, $\Phi(t)$, has been successfully eliminated. Subsequently, Equation (18) is transformed into Equation (21). By applying the Schur complement on Equation (21), the LMI condition for quadratic stabilization is derived, as indicated in Equation (22). The application of variable changes, as defined in Equation (23), and the linear congruent transform results in the final form of the LMI, presented as Equation (24). Notably, Equation (24) introduces the term denoted as Π_i on the left-hand side, which serves as the LMI utilized in the design of a robust LQR.

$$\text{sym}\{\mathbf{P}_i[\mathbf{A} + \mathbf{B}_2\mathbf{K} + \mathbf{H}\Phi(t)(\mathbf{E}_1 + \mathbf{E}_2\mathbf{K})]\} + \mathbf{Q} + \mathbf{K}^T\mathbf{R}_i\mathbf{K} < 0 \quad (18)$$

$$\mathbf{H}\Phi(t)\mathbf{E}^T + \mathbf{E}\Phi(t)\mathbf{H}^T \leq \varepsilon\mathbf{H}\Phi^T(t)\Phi(t)\mathbf{H}^T + \frac{1}{\varepsilon}\mathbf{E}^T\mathbf{E} \leq \varepsilon\mathbf{H}\mathbf{H}^T + \frac{1}{\varepsilon}\mathbf{E}^T\mathbf{E} \quad \text{if} \quad \Phi^T(t)\Phi(t) \leq 1 \quad (19)$$

$$\text{sym}\{\mathbf{P}_i\mathbf{H}\Phi(t)(\mathbf{E}_1 + \mathbf{E}_2\mathbf{K})\} \leq \varepsilon_1\mathbf{P}_i\mathbf{H}\mathbf{H}^T\mathbf{P}_i + \frac{1}{\varepsilon_1}(\mathbf{E}_1 + \mathbf{E}_2\mathbf{K})^T(\mathbf{E}_1 + \mathbf{E}_2\mathbf{K}) \quad (20)$$

$$\text{sym}\{\mathbf{P}_i(\mathbf{A} + \mathbf{B}_2\mathbf{K})\} + \mathbf{Q} + \mathbf{K}^T\mathbf{R}_i\mathbf{K} + \varepsilon_1\mathbf{P}_i\mathbf{H}\mathbf{H}^T\mathbf{P}_i + \frac{1}{\varepsilon_1}(\mathbf{E}_1 + \mathbf{E}_2\mathbf{K})^T(\mathbf{E}_1 + \mathbf{E}_2\mathbf{K}) < 0 \quad (21)$$

$$\begin{bmatrix} \text{sym}\{\mathbf{P}_i(\mathbf{A} + \mathbf{B}_2\mathbf{K})\} + \varepsilon_1\mathbf{P}_i\mathbf{H}\mathbf{H}^T\mathbf{P}_i & * & * & * \\ \mathbf{E}_1 + \mathbf{E}_2\mathbf{K} & -\varepsilon_1\mathbf{I} & * & * \\ \mathbf{I} & \mathbf{0} & -\mathbf{Q}^{-1} & * \\ \mathbf{K} & \mathbf{0} & \mathbf{0} & -\mathbf{R}_i^{-1} \end{bmatrix} < 0 \quad (22)$$

$$\mathbf{Y}_i = \mathbf{P}_i^{-1}, \mathbf{L}_i = \mathbf{K}\mathbf{Y}_i \quad (23)$$

$$\mathbf{\Pi}_i = \begin{bmatrix} \text{sym}\{(\mathbf{A}\mathbf{Y}_i + \mathbf{B}_2\mathbf{L}_i)\} + \varepsilon_1\mathbf{H}\mathbf{H}^T & * & * & * \\ \mathbf{E}_1\mathbf{Y}_i + \mathbf{E}_2\mathbf{L}_i & -\varepsilon_1\mathbf{I} & * & * \\ \mathbf{I} & \mathbf{0} & -\mathbf{Q}^{-1} & * \\ \mathbf{L}_i & \mathbf{0} & \mathbf{0} & -\mathbf{R}_i^{-1} \end{bmatrix} < 0 \quad (24)$$

The next phase involves the development of a non-fragile LQR controller, with the assumption that there are no parameter uncertainties in matrices $\mathbf{A}$ and $\mathbf{B}_2$ (i.e., $\Delta\mathbf{A} = \Delta\mathbf{B}_2 = 0$). It starts with the nominal system represented by $(\mathbf{A}, \mathbf{B}_2)$ and the fragile controller $\mathbf{K} + \Delta\mathbf{K}$. The Riccati inequality (25) is defined within the nominal system $(\mathbf{A}, \mathbf{B}_2)$ and the fragile controller $\mathbf{K} + \Delta\mathbf{K}$. Applying the Schur complement to (25) yields (26). Equations (26) and (27) segregate LMIs with and without $\Delta\mathbf{K}$. Applying the quadratic inequality to the second LMI on the left-hand side of (27) leads to (28). By further applying the Schur complement to (28) and combining it with the first LMI on the left-hand side of (27), we obtain (29). The final form of the LMI (30) is achieved through the application of the change in variables defined in (23) and the linear congruent transform. In (30), the left-hand term is denoted as Ω_i , and this LMI is used for designing a non-fragile LQR.

$$\text{sym}\{\mathbf{P}_i[\mathbf{A} + \mathbf{B}_2(\mathbf{K} + \Delta\mathbf{K})]\} + \mathbf{Q} + (\mathbf{K} + \Delta\mathbf{K})^T\mathbf{R}_i(\mathbf{K} + \Delta\mathbf{K}) < 0 \quad (25)$$

$$\begin{bmatrix} \text{sym}\{\mathbf{P}_i(\mathbf{A} + \mathbf{B}_2\mathbf{K} + \mathbf{B}_2\Delta\mathbf{K})\} & * & * \\ \mathbf{I} & -\mathbf{Q}^{-1} & * \\ \mathbf{K} + \Delta\mathbf{K} & \mathbf{0} & -\mathbf{R}_i^{-1} \end{bmatrix} < 0 \quad (26)$$

$$\begin{bmatrix} \text{sym}\{\mathbf{P}_i(\mathbf{A} + \mathbf{B}_2\mathbf{K})\} & * & * \\ \mathbf{I} & -\mathbf{Q}^{-1} & * \\ \mathbf{K} & \mathbf{0} & -\mathbf{R}_i^{-1} \end{bmatrix} + \begin{bmatrix} \text{sym}\{\mathbf{P}_i(\mathbf{B}_2\mathbf{F}\Delta(t)\mathbf{G})\} & * & * \\ \mathbf{0} & \mathbf{0} & * \\ \mathbf{F}\Delta(t)\mathbf{G} & \mathbf{0} & \mathbf{0} \end{bmatrix} < 0 \quad (27)$$

$$\text{sym}\left(\begin{bmatrix} \mathbf{P}_i\mathbf{B}_2\mathbf{F} \\ \mathbf{0} \\ \mathbf{F} \end{bmatrix} \Lambda(t) \begin{bmatrix} \mathbf{G} & \mathbf{0} & \mathbf{0} \end{bmatrix}\right) \leq \varepsilon_2 \begin{bmatrix} \mathbf{P}_i\mathbf{B}_2\mathbf{F} \\ \mathbf{0} \\ \mathbf{F} \end{bmatrix} \begin{bmatrix} \mathbf{P}_i\mathbf{B}_2\mathbf{F} \\ \mathbf{0} \\ \mathbf{F} \end{bmatrix}^T + \frac{1}{\varepsilon_2} \begin{bmatrix} \mathbf{G}^T \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{G} & \mathbf{0} & \mathbf{0} \end{bmatrix} < 0 \quad (28)$$

$$\begin{bmatrix} \text{sym}\{\mathbf{P}_i(\mathbf{A} + \mathbf{B}_2\mathbf{K})\} & * & * & * & * \\ \mathbf{I} & -\mathbf{Q}^{-1} & * & * & * \\ \mathbf{K} & \mathbf{0} & -\mathbf{R}_i^{-1} & * & * \\ \varepsilon_2(\mathbf{P}_i\mathbf{B}_2\mathbf{F})^T & \mathbf{0} & \varepsilon_2\mathbf{F}^T & -\varepsilon_2\mathbf{I} & * \\ \mathbf{G} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\varepsilon_2\mathbf{I} \end{bmatrix} < 0 \quad (29)$$

$$\Omega_i = \begin{bmatrix} \text{sym}\{(\mathbf{A}\mathbf{Y}_i + \mathbf{B}_2\mathbf{L}_i)\} & * & * & * & * \\ \mathbf{Y}_i & -\mathbf{Q}^{-1} & * & * & * \\ \mathbf{L}_i & \mathbf{0} & -\mathbf{R}_i^{-1} & * & * \\ \varepsilon_2(\mathbf{B}_2\mathbf{F})^T & \mathbf{0} & \varepsilon_2\mathbf{F}^T & -\varepsilon_2\mathbf{I} & * \\ \mathbf{G}\mathbf{Y}_i & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\varepsilon_2\mathbf{I} \end{bmatrix} < 0 \quad (30)$$

A robust non-fragile LQR is designed in the third step. Instead of using $\mathbf{K}$, $\mathbf{K} + \Delta\mathbf{K}$ is substituted in (22). The terms containing the uncertainty block, $\Delta\mathbf{K}$, are separated from those not in (22), resulting in (31). $\Delta\mathbf{K}$ is replaced with $\mathbf{F}\Delta(t)\mathbf{G}$ in the second term of the left-hand ones in (31), giving us (32). The quadratic inequality is applied to this term,

eliminating the uncertain block $\Lambda(t)$, as shown in (33). The second term of the left-hand ones in (31) is replaced with (33), and the Schur complement is applied to it, resulting in the LMI presented in (34). By employing the change in variables defined in (23) and the linear congruent transform, the final form of the LMI is obtained as (35). Let the left-hand side of (35) be denoted as Θ .

$$\Pi_i + \begin{bmatrix} \text{sym}\{\mathbf{P}_i(\mathbf{B}_2\Delta\mathbf{K})\} & * & * & * \\ \mathbf{E}_2\Delta\mathbf{K} & \mathbf{0} & * & * \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & * \\ \Delta\mathbf{K} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} < 0 \quad (31)$$

$$\begin{bmatrix} \text{sym}\{\mathbf{P}_i(\mathbf{B}_2\Delta\mathbf{K})\} & * & * & * \\ \mathbf{E}_2\Delta\mathbf{K} & \mathbf{0} & * & * \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & * \\ \Delta\mathbf{K} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} = \text{sym} \left(\begin{bmatrix} \mathbf{P}_i\mathbf{B}_2\mathbf{H}_K \\ \mathbf{E}_2\mathbf{F} \\ \mathbf{0} \\ \mathbf{F} \end{bmatrix} \Lambda(t) \begin{bmatrix} \mathbf{G} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \right) \quad (32)$$

$$\text{sym} \left(\begin{bmatrix} \mathbf{P}_i\mathbf{B}_2\mathbf{H}_K \\ \mathbf{E}_2\mathbf{F} \\ \mathbf{0} \\ \mathbf{F} \end{bmatrix} \Lambda(t) \begin{bmatrix} \mathbf{G} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \right) \leq \varepsilon_2 \begin{bmatrix} \mathbf{P}_i\mathbf{B}_2\mathbf{H}_K \\ \mathbf{E}_2\mathbf{F} \\ \mathbf{0} \\ \mathbf{F} \end{bmatrix} \begin{bmatrix} \mathbf{P}_i\mathbf{B}_2\mathbf{H}_K \\ \mathbf{E}_2\mathbf{F} \\ \mathbf{0} \\ \mathbf{F} \end{bmatrix}^T + \frac{1}{\varepsilon_2} \begin{bmatrix} \mathbf{G}^T \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{G} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} < 0 \quad (33)$$

$$\begin{bmatrix} (\mathbf{A} + \mathbf{B}_2\mathbf{K})^T\mathbf{P}_i + \mathbf{P}_i(\mathbf{A} + \mathbf{B}_2\mathbf{K}) + \varepsilon_1\mathbf{P}_i\mathbf{H}\mathbf{H}^T\mathbf{P}_i & * & * & * & * & * \\ \mathbf{E}_1 + \mathbf{E}_2\mathbf{K} & -\varepsilon_1\mathbf{I} & * & * & * & * \\ \mathbf{I} & \mathbf{0} & -\mathbf{Q}^{-1} & * & * & * \\ \mathbf{K} & \mathbf{0} & \mathbf{0} & -\mathbf{R}_i^{-1} & * & * \\ \varepsilon_2(\mathbf{P}_i\mathbf{B}_2\mathbf{F})^T & \varepsilon_2(\mathbf{E}_2\mathbf{F})^T & \mathbf{0} & \varepsilon_2\mathbf{F}^T & -\varepsilon_2\mathbf{I} & * \\ \mathbf{G} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\varepsilon_2\mathbf{I} \end{bmatrix} < 0 \quad (34)$$

$$\Theta_i = \begin{bmatrix} (\mathbf{A}\mathbf{Y}_i + \mathbf{B}_2\mathbf{L})^T + (\mathbf{A}\mathbf{Y}_i + \mathbf{B}_2\mathbf{L}_i) + \varepsilon_1\mathbf{H}\mathbf{H}^T & * & * & * & * & * \\ \mathbf{E}_1\mathbf{Y}_i + \mathbf{E}_2\mathbf{L}_i & -\varepsilon_1\mathbf{I} & * & * & * & * \\ \mathbf{Y}_i & \mathbf{0} & -\mathbf{Q}^{-1} & * & * & * \\ \mathbf{L}_i & \mathbf{0} & \mathbf{0} & -\mathbf{R}_i^{-1} & * & * \\ \varepsilon_2(\mathbf{B}_2\mathbf{F})^T & \varepsilon_2(\mathbf{E}_2\mathbf{F})^T & \mathbf{0} & \varepsilon_2\mathbf{F}^T & -\varepsilon_2\mathbf{I} & * \\ \mathbf{G}\mathbf{Y}_i & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\varepsilon_2\mathbf{I} \end{bmatrix} < 0 \quad (35)$$

The LMIs for robust non-fragile controllers have been derived as (24), (30), and (35), respectively. Equations (35) and (35) comprise a combination of (24) and (30), a well-known characteristic of LMIs. Utilizing these LMIs, LMI optimization problems are formulated for robust, non-fragile, and robust non-fragile control, denoted as (36). In (36), three distinct LMI optimization problems emerge, LMI.R, LMI.NF, and LMI.RNF, signifying the robust, non-fragile, and robust non-fragile controllers, respectively. As depicted in (36), LMI.R, LMI.NF, and LMI.RNF involve four, four, and five optimization variables, respectively. The controller gain, $\mathbf{K}_i$, for the input configuration i is obtained from the solutions $\mathbf{Y}_i$ and $\mathbf{L}_i$ presented in (36), resulting in (37).

$$\begin{array}{ccc} \text{LMI.R} & \text{LMI.NF} & \text{LMI.RNF} \\ \left\{ \begin{array}{l} \min_{\mathbf{Y}_i, \mathbf{L}_i, \mathbf{Z}, \varepsilon_1} \text{trace}(\mathbf{Z}) \\ \text{s.t.} \quad \mathbf{Y}_i = \mathbf{Y}_i^T > 0 \\ \begin{bmatrix} \mathbf{Z} & \mathbf{I} \\ \mathbf{I} & \mathbf{Y}_i \end{bmatrix} > 0 \\ \Pi_i < 0 \end{array} \right. & \left\{ \begin{array}{l} \min_{\mathbf{Y}_i, \mathbf{L}_i, \mathbf{Z}, \varepsilon_2} \text{trace}(\mathbf{Z}) \\ \text{s.t.} \quad \mathbf{Y}_i = \mathbf{Y}_i^T > 0 \\ \begin{bmatrix} \mathbf{Z} & \mathbf{I} \\ \mathbf{I} & \mathbf{Y}_i \end{bmatrix} > 0 \\ \Omega_i < 0 \end{array} \right. & \left\{ \begin{array}{l} \min_{\mathbf{Y}_i, \mathbf{L}_i, \mathbf{Z}, \varepsilon_1, \varepsilon_2} \text{trace}(\mathbf{Z}) \\ \text{s.t.} \quad \mathbf{Y}_i = \mathbf{Y}_i^T > 0 \\ \begin{bmatrix} \mathbf{Z} & \mathbf{I} \\ \mathbf{I} & \mathbf{Y}_i \end{bmatrix} > 0 \\ \Theta_i < 0 \end{array} \right. \end{array} \quad (36)$$

$$\mathbf{K}_i = \mathbf{L}_i \mathbf{Y}_i^{-1} \quad (37)$$

2.4. Constraints Related to Tire Slip Angle

Previous studies have introduced slip-angle constraints in order to prevent lateral tire force (F_y) saturation [50]. Excessive steering can cause the front and rear lateral forces (F_{yf} and F_{yr}) to reach their limits when the slip angle α exceeds a critical value α_m , which corresponds to the angle that yields the peak force. Once these forces are saturated, path tracking accuracy is noticeably degraded. To avoid this effect, the steering inputs δ_f and δ_r are restricted so that α remains below α_m .

It is assumed that the maximum lateral force, $F_{y,max}$, occurs at α_m . Beyond this point, F_y decreases with increasing α , meaning that the controller cannot fully exploit the available tire capacity [51,52]. For this reason, the slip angle is constrained to remain below a prescribed threshold, α_{max} , as expressed in (38). Unlike α_m , which is determined by tire characteristics or experimental data, such as carpet plots, α_{max} is selected by the designer. Combining (38) with the slip angle definitions in (2) leads to (39), and from (39), the constraints on δ_f and δ_r are derived, as shown in (40). These constraints are subsequently imposed on the steering commands generated by LQR, LMI.R, LMI.NF, and LMI.RNF controllers. In earlier studies, α_{max} was typically set to about 8° .

$$|\alpha| \leq \alpha_{max} \quad (38)$$

$$|\alpha_f| = \left| \delta_f - \beta - \frac{l_f \gamma}{v_x} \right| \leq \alpha_{max}, \quad |\alpha_r| = \left| \delta_r - \beta + \frac{l_r \gamma}{v_x} \right| \leq \alpha_{max} \quad (39)$$

$$\begin{cases} -\alpha_{max} + \beta + \frac{l_f \gamma}{v_x} \leq \delta_f \leq \alpha_{max} + \beta + \frac{l_f \gamma}{v_x} \\ -\alpha_{max} + \beta - \frac{l_r \gamma}{v_x} \leq \delta_r \leq \alpha_{max} + \beta - \frac{l_r \gamma}{v_x} \end{cases} \quad (40)$$

2.5. Control Allocation for LQR with CONFIG-3

According to (10), input configuration CONFIG-3 produces a control yaw moment, ΔM_z . Once this quantity is obtained, it must be distributed into longitudinal and lateral forces at the wheels. Such forces can be realized through different actuator combinations, including RS, 4IB, and 4ID. The transformation process is referred to as control allocation. In this study, a weighted least-squares (WLS) approach is adopted, following methods reported in earlier works [23,32,35,39]. The allocation problem is formulated as a quadratic program with equality constraints. Figure 3 depicts the mapping between ΔM_z and the corresponding tire forces for the case of a positive yaw moment [43,44,53,54].

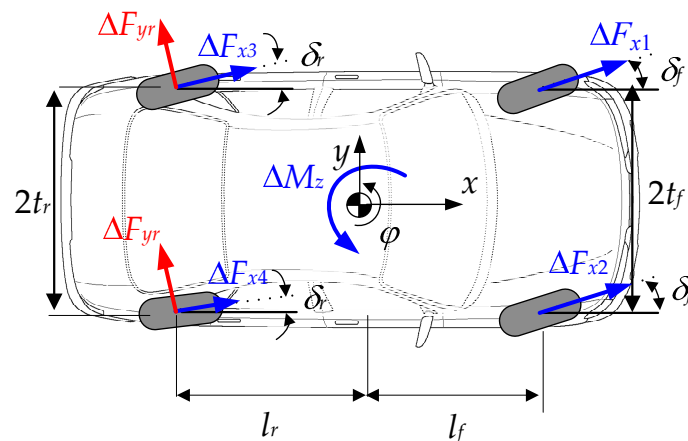


Figure 3. Coordinate system on vehicle geometry with tire forces and control yaw moment.

In Figure 3, ΔF_{yr} denotes the additional lateral force produced by the rear wheels via RS, while ΔF_{x1} , ΔF_{x2} , ΔF_{x3} , and ΔF_{x4} represent the longitudinal forces generated by 4ID or 4IB. When ΔF_{xi} is positive, it corresponds to traction torque T_{Di} from 4ID; if it is negative, it corresponds to braking torque T_{Bi} from 4IB. These five force components collectively realize the commanded yaw moment, and their values are computed using the WLS-based allocation scheme.

The equilibrium condition relating ΔM_z and tire forces is derived from the geometric relationship depicted in Figure 2, as expressed in (41) [43,44,53,54]. The components of vector $\mathbf{p}$ are defined in (42).

$$\underbrace{\begin{bmatrix} p_1 & p_2 & p_3 & p_4 & p_5 \end{bmatrix}}_{\mathbf{p}} \underbrace{\begin{bmatrix} \Delta F_{yr} \\ \Delta F_{x1} \\ \Delta F_{x2} \\ \Delta F_{x3} \\ \Delta F_{x4} \end{bmatrix}}_{\mathbf{q}} = \Delta M_z \quad (41)$$

$$\begin{cases} p_1 = 2l_r \cos \delta_r, \\ p_2 = -l_f \sin \delta_f + t_f \cos \delta_f, & p_3 = -l_f \sin \delta_f - t_f \cos \delta_f, \\ p_4 = l_r \sin \delta_r + t_r \cos \delta_r, & p_5 = l_r \sin \delta_r - t_r \cos \delta_r \end{cases} \quad (42)$$

The quadratic objective function for WLS is given in (43). In this formulation, μF_{zi} denotes the friction circle radius at wheel i . On a flat road, this value can be estimated from sensor-based acceleration measurements [55]. A vector of virtual weights, κ , is also introduced in (43) to specify the contribution of each actuator combination [44,53,54,56]. The equality constraint in (41) is reformulated into quadratic form, as shown in (44). Earlier studies emphasized that satisfying (41) was critical for generating ΔM_B . By combining the cost function (40) with the constraint (41) using a Lagrange multiplier ζ , a unified expression (45) is obtained. To guarantee feasibility, ζ must be set to 1 or greater; otherwise, the constraint in (41) cannot be enforced. The closed-form solution of (45) is then derived as (46) by differentiating with respect to $\mathbf{q}$ [43].

$$\begin{aligned} J_E &= \frac{\kappa_2 \Delta F_{x1}^2}{(\mu F_{z1})^2} + \frac{\kappa_3 \Delta F_{x2}^2}{(\mu F_{z2})^2} + \frac{\kappa_1 \Delta F_{yr}^2 + \kappa_4 \Delta F_{x3}^2}{(\mu F_{z3})^2} + \frac{\kappa_1 \Delta F_{yr}^2 + \kappa_5 \Delta F_{x4}^2}{(\mu F_{z4})^2} = \mathbf{p}^T \mathbf{W} \mathbf{p} \\ \mathbf{W} &= \text{diag} \left[\frac{1}{(\mu F_{z3})^2} + \frac{1}{(\mu F_{z4})^2}, \frac{1}{(\mu F_{z1})^2}, \frac{1}{(\mu F_{z2})^2}, \frac{1}{(\mu F_{z3})^2}, \frac{1}{(\mu F_{z4})^2} \right] \kappa \\ \kappa &= \text{diag} \begin{bmatrix} \kappa_1 & \kappa_2 & \kappa_3 & \kappa_4 & \kappa_5 \end{bmatrix} \end{aligned} \quad (43)$$

$$J_{EC} = (\mathbf{p} \mathbf{q} - \Delta M_z)^T (\mathbf{p} \mathbf{q} - \Delta M_z) \quad (44)$$

$$J_{CA} = J_E + \zeta \cdot J_{EC} = \mathbf{q}^T \mathbf{W} \mathbf{q} + \zeta (\mathbf{p} \mathbf{q} - \Delta M_z)^T (\mathbf{p} \mathbf{q} - \Delta M_z) \quad (45)$$

$$\mathbf{q}_{opt} = \zeta \left(\mathbf{W} + \eta \mathbf{p}^T \mathbf{p} \right)^{-1} \mathbf{p}^T \Delta M_z \quad (46)$$

When WLS is applied for distributing ΔM_z , different actuator sets—RS, 4IB, and 4ID—can be flexibly considered, depending on the input configuration. To reflect these choices, the virtual weights κ_i are adjusted according to the selected actuator combination [44,53,54]. The weight vector for RS is shown in (47). Initially, all weight components are set to unity. In (47), ε is 10^{-4} , and the entries marked with a “•” indicate negligible influence on the steering actuator. The first term of κ corresponds to ΔF_{yr} , which is converted into δ_r . When

this weight is replaced with ε , ΔF_{yr} is generated through the WLS process, as indicated in the first row of (47).

$$\left\{ \begin{array}{l} \text{RS: } \kappa = \text{diag} \begin{bmatrix} \varepsilon & \bullet & \bullet & \bullet & \bullet \end{bmatrix} \\ \text{No RS: } \kappa = \text{diag} \begin{bmatrix} 1 & \bullet & \bullet & \bullet & \bullet \end{bmatrix} \end{array} \right. \quad (47)$$

The vectors corresponding to 4IB and 4ID are summarized in (48). As illustrated in Figure 3, when only 4IB is available and ΔM_z is negative, tire forces ΔF_{x2} and ΔF_{x4} must be produced; this situation is described in the second row of (48). In contrast, when both 4IB and 4ID are available, no restrictions are placed on the longitudinal force distribution, as shown in the last row of (48). Thus, the virtual weight vectors in (47) and (48) allow flexible configuration of actuator combinations for generating ΔM_z .

$$\left\{ \begin{array}{l} 4\text{ID: } \left\{ \begin{array}{l} \kappa = \text{diag} \begin{bmatrix} \bullet & 1 & \varepsilon & 1 & \varepsilon \end{bmatrix} \text{ if } \Delta M_z > 0 \\ \kappa = \text{diag} \begin{bmatrix} \bullet & \varepsilon & 1 & \varepsilon & 1 \end{bmatrix} \text{ if } \Delta M_z < 0 \end{array} \right. \\ 4\text{IB: } \left\{ \begin{array}{l} \kappa = \text{diag} \begin{bmatrix} \bullet & \varepsilon & 1 & \varepsilon & 1 \end{bmatrix} \text{ if } \Delta M_z > 0 \\ \kappa = \text{diag} \begin{bmatrix} \bullet & 1 & \varepsilon & 1 & \varepsilon \end{bmatrix} \text{ if } \Delta M_z < 0 \end{array} \right. \\ 4\text{ID} + 4\text{IB: } \kappa = \text{diag} \begin{bmatrix} \bullet & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{bmatrix} \end{array} \right. \quad (48)$$

Table 1 summarizes the virtual weight vectors used for different actuator combinations in CONFIG-3 [57]. By appropriately configuring these weights, seven distinct actuator combinations—SET-3 through SET-9—are defined within this input configuration.

Table 1. Virtual weight settings for the control allocation scheme.

Actuator Combinations			Vector of Virtual Weights
CONFIG-1	SET-1	FS	
CONFIG-2	SET-2	4S	
CONFIG-3 FS	SET-3	+RS	$\kappa = \text{diag}[\varepsilon \quad 1 \quad 1 \quad 1 \quad 1]$
	SET-4	+RS + 4ID	$\left\{ \begin{array}{l} \kappa = \text{diag} \begin{bmatrix} \varepsilon & 1 & \varepsilon & 1 & \varepsilon \end{bmatrix} \text{ if } \Delta M_z > 0 \\ \kappa = \text{diag} \begin{bmatrix} \varepsilon & \varepsilon & 1 & \varepsilon & 1 \end{bmatrix} \text{ if } \Delta M_z < 0 \end{array} \right.$
	SET-5	+RS + 4IB	$\left\{ \begin{array}{l} \kappa = \text{diag} \begin{bmatrix} \varepsilon & \varepsilon & 1 & \varepsilon & 1 \end{bmatrix} \text{ if } \Delta M_z > 0 \\ \kappa = \text{diag} \begin{bmatrix} \varepsilon & 1 & \varepsilon & 1 & \varepsilon \end{bmatrix} \text{ if } \Delta M_z < 0 \end{array} \right.$
	SET-6	+RS + 4ID + 4IB	$\kappa = \text{diag}[\varepsilon \quad \varepsilon \quad \varepsilon \quad \varepsilon \quad \varepsilon]$
	SET-7	+4ID	$\left\{ \begin{array}{l} \kappa = \text{diag} \begin{bmatrix} 1 & 1 & \varepsilon & 1 & \varepsilon \end{bmatrix} \text{ if } \Delta M_z > 0 \\ \kappa = \text{diag} \begin{bmatrix} 1 & \varepsilon & 1 & \varepsilon & 1 \end{bmatrix} \text{ if } \Delta M_z < 0 \end{array} \right.$
	SET-8	+4IB	$\left\{ \begin{array}{l} \kappa = \text{diag} \begin{bmatrix} 1 & \varepsilon & 1 & \varepsilon & 1 \end{bmatrix} \text{ if } \Delta M_z > 0 \\ \kappa = \text{diag} \begin{bmatrix} 1 & 1 & \varepsilon & 1 & \varepsilon \end{bmatrix} \text{ if } \Delta M_z < 0 \end{array} \right.$
	SET-9	+4ID + 4IB	$\kappa = \text{diag}[1 \quad \varepsilon \quad \varepsilon \quad \varepsilon \quad \varepsilon]$

In this study, the traction and braking torques (T_{Bi} and T_{Di}) at wheel i are derived from the longitudinal forces ΔF_{x1} , ΔF_{x2} , ΔF_{x3} , and ΔF_{x4} , which are obtained via the WLS method using the optimal solution $\mathbf{q}_{opt}$ in (46). The sign of ΔF_{xi} determines whether the value corresponds to T_{Bi} and T_{Di} , and the computation is carried out as given in (49) [44,55,56,58].

$$\left\{ \begin{array}{l} T_{Bi} \text{ if } \Delta F_{xi} < 0 \\ T_{Di} \text{ if } \Delta F_{xi} > 0 \end{array} \right\} = h \left(\frac{r_{wi} \Delta F_{xi}}{v_i}, \omega_i \right), \quad i = 1, \dots, 4 \quad (49)$$

The rear steering angle δ_r is computed from ΔF_{yr} , and it is also obtained from $\mathbf{q}_{opt}$, in combination with the definitions in (2) and (3). Specifically, ΔF_{yr} is first converted into the rear slip angle α_r using the second term of (3), and then δ_r is determined according to (2), as shown in (50) [58]. Since this procedure requires knowledge of the side-slip angle β , an extended Kalman filter is applied in this work to provide its estimate [59].

$$\delta_r = -\alpha_r + \beta - \frac{l_r \gamma}{v_x} \approx -\frac{\Delta F_{yr}}{2C_r} + \beta - \frac{l_r \gamma}{v_x} \quad (50)$$

3. Performance Measures for Path Tracking Control

In path tracking control (PTC), the lateral displacement error (e_y) and heading error (e_ϕ) are conventionally used as performance indicators. In this work, however, evaluation is carried out using a double lane-change maneuver as the reference trajectory, representing a typical collision-avoidance scenario [25,35,43,51,57]. This maneuver is particularly suitable for assessing the agility and reachability of a path tracking controller [43,45]. Figure 4 illustrates the reference path together with the vehicle trajectory. Based on these trajectories, five performance metrics are defined in (51): peak center offset (ΔX), peak lateral offset (ΔY), overshoot percentage ($OS\%$), response delay (ΔDX), and settling delay (ΔSX). These metrics are also depicted in Figure 3, where the subscripts X and Y indicate the respective coordinates of the marked point. In general, smaller magnitudes of these quantities correspond to improved tracking performance.

$$\begin{cases} \Delta X = D_X - A_X = D_X - 73.20 \\ \Delta Y = D_Y - A_Y = D_Y - 3.53 \\ \Delta DX = E_X - B_X = E_X - 91.50 \\ \Delta SX = G_X - C_X = G_X - 102.50 \\ OS\% = \frac{|F_Y| - 1.45}{1.65 + 3.53} \times 100 \end{cases} \quad (51)$$

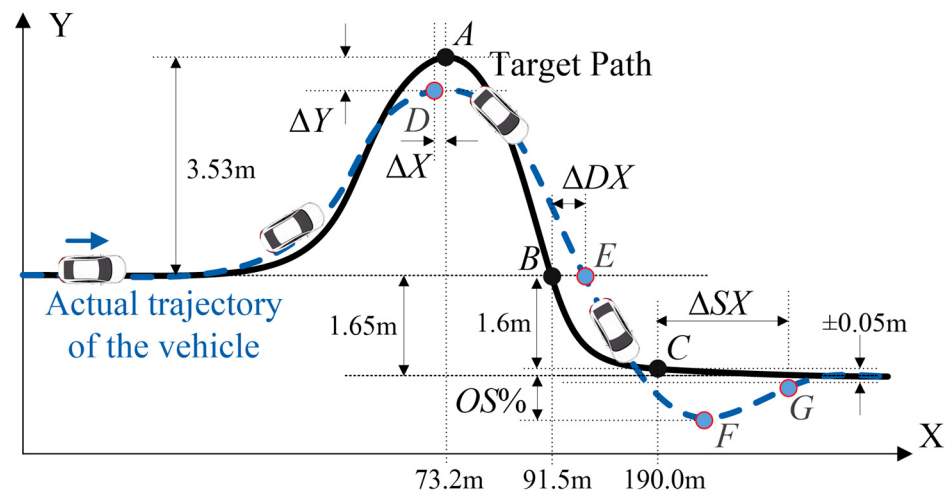


Figure 4. Measures for path tracking performance.

In Figure 4, agility is represented by ΔX and reachability by ΔY . A performance criterion is satisfied if ΔY remains greater than -0.05 m. The overshoot percentage $OS\%$ reflects lateral damping and agility; acceptable performance is achieved if $OS\%$ stays below 16%, which corresponds to a lateral overshoot of 0.85 m. The metric ΔDX denotes response delay, indicating the promptness of longitudinal motion, while ΔSX corresponds to settling time, reflecting how quickly the vehicle converges to the target lateral position. In this study, performance is considered satisfactory when $\Delta SX < 16$ m. The best results are generally

obtained when $OS\%$ is around 1% while ensuring $\Delta Y > -0.05$ m. For controller tuning, the weights and k_v in J_1 , J_2 , and J_3 are adjusted so that ΔY is maintained close to -0.02 m, which consistently yields $OS\%$ values below 1%. Additional details on these metrics are available in [43,45,51,57]. Conventional metrics such as steady-state error and settling time mainly assess steady-state tracking accuracy, whereas the proposed measures (ΔX , ΔY , $OS\%$, ΔDX , ΔSX) capture both lateral and longitudinal agility. In particular, ΔDX and ΔSX quantify transient delay more directly, offering clearer physical interpretation for double lane-change maneuvers.

4. Simulation and Validation

The performance measures defined in Section 3 are now applied to evaluate the proposed controllers through simulation. Numerical simulations were carried out to compare the performance of four controllers—LQR, LMI.R, LMI.NF, and LMI.RNF—under low-friction road conditions. The controllers were implemented in MATLAB/Simulink and coupled with the CarSim vehicle simulation platform. The double lane-change maneuver shown in Figure 4 served as the test scenario. For the simulations, the built-in F-segment sedan model available in CarSim was used [60], and its parameters were mapped to the bicycle model as listed in Table 2. All parameters and input signals were derived from the validated CarSim vehicle database and literature-based tire data, ensuring consistency with experimentally verified vehicle dynamics.

Table 2. Parameter of the F-segment sedan in CarSim.

Parameter	Value	Unit
m_s	1823	kg
l_f	1.27	m
l_r	1.90	m
I_z	6286	kg-m ²
t_r	0.80	m
t_f	0.80	m
C_f	42,000	N/rad
C_r	62,000	N/rad

The steering actuators for FS and RS were modeled as first-order systems with a time constant of 0.05 s. Similarly, 4IB and 4ID were modeled as first-order systems with a time constant of 0.1 s. During controller design, the road friction coefficient was fixed at $\mu = 0.6$, and the vehicle speed was maintained at 50 km/h. A built-in speed controller in CarSim was used to hold this constant velocity. The maximum steering ranges were set to 30° for FS and 5° for RS.

4.1. Controller Tuning

With nine actuator configurations, SET-1~SET-9, given in Table 1, the controllers LQR, LMI.R, LMI.NF, and LMI.RNF were tuned such that ΔY is near -0.02 m, and ΔX and ΔSX were as small as possible. The tuning parameters are the maximum allowable values ξ_i in J 's and k_v .

In this paper, the parameters m , I_z , C_f , C_r , v_x , and k_v were assumed to be uncertain. When designing LMI.R and LMI.RNF, the percentage variations of the uncertain parameters [m I_z C_f C_r v_x k_v] were set as [5% 5% 20% 20% 10% 10%] of the nominal values given in Table 2. The variations of C_f and C_r are proportional to μ . For this reason, the variations of C_f and C_r were given by multiplying μ by the nominal one in Table 2. The variations of controller gains in LMI.NF and LMI.RNF were set to 1% for the gains of LQR.

Tables 3–6 show the tuning results on LQR, LMI.R, LMI.NF, and LMI.RNF with the nominal parameters given in Table 2. The set of the maximum allowable values in J 's and k_v , used to calculate these controllers, are given in the tables in Appendix A. As shown in those tables, the controllers show nearly the same performance as each other. As mentioned earlier, $OS\%$ is less than 1 if ΔY is near -0.02 m. Moreover, most of ΔDX , ΔSX , and β are less than 1, 0, and 2, respectively. This is caused by the fact that those controllers were tuned such that ΔY is near -0.02 m and the other measures are maintained as small as possible.

Table 3. Simulation results for the LQR designed with the nominal parameters.

	Actuator Combinations		ΔX (m)	ΔY (m)	$OS\%$ (%)	ΔDX (m)	ΔSX (m)	$\max \beta $ (deg)	$\max \dot{\beta} $ (deg/s)
CONFIG-1	SET-1	FS	0.39	-0.020	0.5	0.69	-2.29	1.5	5.1
CONFIG-2	SET-2	4S	0.39	-0.020	0.5	0.67	-2.32	2.0	6.4
CONFIG-3 FS	SET-3	+RS	0.31	-0.021	0.4	0.54	-2.53	1.9	6.2
	SET-4	+RS + 4ID	0.34	-0.020	0.6	0.59	-3.08	1.8	6.2
	SET-5	+RS + 4IB	0.36	-0.020	0.6	0.57	-2.76	2.0	6.4
	SET-6	+RS + 4ID + 4IB	0.35	-0.020	0.6	0.54	-3.13	2.0	6.4
	SET-7	+4ID	0.33	-0.020	0.7	0.88	-4.19	1.2	6.0
	SET-8	+4IB	0.29	-0.021	0.4	0.47	-2.51	1.7	5.2
	SET-9	+4ID + 4IB	0.28	-0.020	0.3	0.48	-2.79	1.5	5.6

Table 4. Simulation results for LMI.R designed with the nominal parameters.

	Actuator Combinations		ΔX (m)	ΔY (m)	$OS\%$ (%)	ΔDX (m)	ΔSX (m)	$\max \beta $ (deg)	$\max \dot{\beta} $ (deg/s)
CONFIG-1	SET-1	FS	0.59	-0.020	0.9	0.95	-3.04	1.5	4.9
CONFIG-2	SET-2	4S	0.57	-0.021	0.9	0.89	-2.83	2.5	7.7
CONFIG-3 FS	SET-3	+RS	0.39	-0.021	0.6	0.63	-3.07	1.9	6.1
	SET-4	+RS + 4ID	0.41	-0.021	0.9	0.75	-3.67	1.7	6.0
	SET-5	+RS + 4IB	0.49	-0.020	0.9	0.75	-3.41	2.0	6.2
	SET-6	+RS + 4ID + 4IB	0.49	-0.021	0.9	0.74	-3.54	1.9	6.2
	SET-7	+4ID	0.34	-0.021	0.8	0.98	-4.52	1.1	7.7
	SET-8	+4IB	0.31	-0.021	0.3	0.52	-2.79	1.7	5.5
	SET-9	+4ID + 4IB	0.28	-0.021	0.2	0.52	-2.76	1.5	5.6

The controllers, i.e., LMI.R and LMI.RNF, gave a smaller ΔX than the others for SET-3~SET-9. As shown in those tables, the controllers with RS, i.e., SET-2~SET-6, have a larger β than the other actuator configurations. Especially, SET-2 is the largest among the actuator configurations. On the contrary, the controllers without RS, i.e., SET-7~SET-9, have a smaller β . This is caused by the fact that the latter was assisted by 4ID or /and 4IB. The tuning parameters used for the controller design are summarized in Tables A1–A4.

Table 5. Simulation results for LMI.NF designed with the nominal parameters.

	Actuator Combinations		ΔX (m)	ΔY (m)	OS% (%)	ΔDX (m)	ΔSX (m)	$\max \beta $ (deg)	$\max \dot{\beta} $ (deg/s)
CONFIG-1	SET-1	FS	0.32	−0.021	0.4	0.59	−2.03	1.5	5.2
CONFIG-2	SET-2	4S	0.12	−0.020	0.1	0.30	−0.94	2.2	7.4
CONFIG-3 FS	SET-3	+RS	0.39	−0.020	0.5	0.67	−1.79	1.9	6.5
	SET-4	+RS + 4ID	0.43	−0.021	0.7	0.72	−2.64	1.9	6.3
	SET-5	+RS + 4IB	0.23	−0.021	0.3	0.41	−1.96	2.0	6.7
	SET-6	+RS + 4ID + 4IB	0.44	−0.020	0.8	0.68	−2.76	2.0	6.3
	SET-7	+4ID	0.43	−0.020	0.9	1.04	−3.52	1.2	5.5
	SET-8	+4IB	0.35	−0.020	0.5	0.59	−2.41	1.7	5.3
	SET-9	+4ID + 4IB	0.35	−0.020	0.5	0.61	−2.65	1.6	5.6

Table 6. Simulation results for LMI.RNF designed with the nominal parameters.

	Actuator Combinations		ΔX (m)	ΔY (m)	OS% (%)	ΔDX (m)	ΔSX (m)	$\max \beta $ (deg)	$\max \dot{\beta} $ (deg/s)
CONFIG-1	SET-1	FS	0.57	−0.021	0.9	0.95	−3.01	1.5	4.9
CONFIG-2	SET-2	4S	0.59	−0.020	0.9	0.92	−2.75	2.5	7.7
CONFIG-3 FS	SET-3	+RS	0.39	−0.020	0.5	0.67	−1.79	1.9	6.5
	SET-4	+RS + 4ID	0.14	−0.021	0.3	0.53	−1.62	1.7	19.3
	SET-5	+RS + 4IB	0.13	−0.021	0.1	0.31	−0.53	2.0	7.1
	SET-6	+RS + 4ID + 4IB	0.12	−0.021	0.0	0.34	0.14	1.9	7.2
	SET-7	+4ID	0.30	−0.020	0.5	1.16	−2.45	1.1	21.7
	SET-8	+4IB	0.27	−0.020	0.1	0.55	−0.89	1.7	6.3
	SET-9	+4ID + 4IB	0.28	−0.021	0.1	0.65	−0.31	1.5	6.0

4.2. Simulation on CarSim for the Controllers with Actuator Configurations

In this subsection, the robust performance of four controllers, LQR, LMI.R, LMI.NF, and LMI.RNF, designed in the previous subsection is tested on CarSim. From the simulation results, the robustness of the controllers and the effects of actuator configurations on the robustness are analyzed.

When simulating the controllers on CarSim, the percentage variations of the uncertain parameters $[m \ I_z \ C_f \ C_r \ v_x \ k_v]$ were set as [5% 5% 20% 20% 10% 10%] of the nominal values given in CarSim. The variations of C_f and C_r are proportional to μ . For this reason, the variations of C_f and C_r were given as [0.5, 0.7] of μ in CarSim. The variation of v_x was also set as [40, 60] km/h in CarSim. The variation of k_v was set by multiplying 0.9 and 1.1 by the value obtained in Section 4.1. The pair of m and I_z varies in the same way because those variables are proportional to each other. The pair of C_f and C_r changes in the same way as μ varies in CarSim. For this reason, four parameters, (m, I_z) , (C_f, C_r) , v_x , and k_v , are uncertain. As a result, the number of sets of the uncertain parameters is 16. For 36 pairs of the controllers and the actuator configuration given in Tables 3–6, a simulation was performed with those sets of uncertain parameters. As a result, 576 runs of simulation were performed.

Tables 7–10 show the simulation results of LQR, LMI.R, LMI.NF, and LMI.RNF on 16 sets of uncertain parameters in CarSim. Table 11 shows the simulation results for

four controllers on various actuator configurations and uncertain parameters. Table 12 shows the simulation results of each actuator configuration for four controllers on various uncertain parameters. The values in those tables are the means, and there are 95% confidence intervals for the simulation results.

Table 7. Simulation results for LQR under various conditions on CarSim.

	AC's	ΔX (m)	ΔY (m)	OS% (%)	ΔDX (m)	ΔSX (m)	$\max \beta $ (deg)	$\max \dot{\beta} $ (deg/s)
CONFIG-1	SET-1	0.56 ± 0.40	-0.023 ± 0.021	0.9 ± 0.6	1.79 ± 1.17	8.44 ± 4.91	1.5 ± 0.4	7.7 ± 1.9
CONFIG-2	SET-2	0.56 ± 0.38	-0.022 ± 0.021	1.5 ± 1.1	1.75 ± 1.15	9.57 ± 5.97	2.0 ± 0.3	8.3 ± 2.0
CONFIG-3 FS	SET-3	0.65 ± 0.40	-0.006 ± 0.025	2.3 ± 1.7	1.82 ± 1.16	9.18 ± 5.62	1.5 ± 0.3	7.5 ± 1.8
	SET-4	0.65 ± 0.40	-0.006 ± 0.025	2.3 ± 1.7	1.82 ± 1.16	9.18 ± 5.62	1.5 ± 0.3	7.5 ± 1.8
	SET-5	0.65 ± 0.40	-0.006 ± 0.025	2.3 ± 1.7	1.82 ± 1.16	9.18 ± 5.62	1.5 ± 0.3	7.5 ± 1.8
	SET-6	0.69 ± 0.41	-0.001 ± 0.026	2.4 ± 1.8	1.85 ± 1.17	9.27 ± 5.74	1.5 ± 0.3	7.5 ± 1.8
	SET-7	0.36 ± 0.33	-0.037 ± 0.023	3.4 ± 3.0	1.48 ± 1.01	7.81 ± 5.73	1.5 ± 0.3	7.7 ± 1.9
	SET-8	0.47 ± 0.36	-0.019 ± 0.026	3.7 ± 3.2	1.57 ± 1.07	7.85 ± 6.09	1.5 ± 0.3	7.8 ± 1.9
	SET-9	0.48 ± 0.37	-0.017 ± 0.026	3.7 ± 3.2	1.58 ± 1.08	7.84 ± 6.12	1.5 ± 0.3	7.8 ± 1.9

Table 8. Simulation results for LMI.R under various conditions on CarSim.

	AC's	ΔX (m)	ΔY (m)	OS% (%)	ΔDX (m)	ΔSX (m)	$\max \beta $ (deg)	$\max \dot{\beta} $ (deg/s)
CONFIG-1	SET-1	0.74 ± 0.46	-0.028 ± 0.022	1.1 ± 0.6	2.06 ± 1.25	10.71 ± 5.45	1.5 ± 0.4	8.1 ± 2.5
CONFIG-2	SET-2	0.72 ± 0.43	-0.027 ± 0.020	1.8 ± 1.1	1.97 ± 1.20	11.89 ± 6.47	2.5 ± 0.2	9.4 ± 2.5
CONFIG-3 FS	SET-3	0.78 ± 0.45	-0.003 ± 0.028	3.4 ± 2.5	1.96 ± 1.20	9.63 ± 5.88	1.5 ± 0.4	8.0 ± 2.3
	SET-4	0.71 ± 0.43	-0.010 ± 0.027	3.5 ± 2.7	1.87 ± 1.16	9.60 ± 5.71	1.5 ± 0.4	8.1 ± 2.3
	SET-5	0.87 ± 0.46	-0.002 ± 0.026	2.6 ± 1.6	2.09 ± 1.22	9.48 ± 5.94	1.5 ± 0.4	7.8 ± 2.2
	SET-6	0.87 ± 0.46	-0.002 ± 0.026	2.6 ± 1.6	2.09 ± 1.22	9.48 ± 5.94	1.5 ± 0.4	7.8 ± 2.2
	SET-7	0.33 ± 0.35	-0.045 ± 0.025	4.3 ± 3.8	1.44 ± 1.01	8.78 ± 5.96	1.5 ± 0.4	8.5 ± 2.4
	SET-8	0.57 ± 0.41	-0.016 ± 0.029	4.7 ± 3.9	1.69 ± 1.11	9.49 ± 5.66	1.5 ± 0.4	8.4 ± 2.4
	SET-9	0.51 ± 0.39	-0.022 ± 0.028	4.8 ± 4.1	1.63 ± 1.09	9.29 ± 5.66	1.5 ± 0.4	8.4 ± 2.4

Tables 7–10 show the means and their 95% confidence intervals of the results of 16 runs of simulation. For this reason, a t -distribution is used to calculate the 95% confidence interval because the number of samples is 16. On the other hand, Tables 11 and 12 show the means and their 95% confidence intervals of the results of 144 and 64 runs of simulation, respectively. For this reason, a normal distribution is used to calculate the 95% confidence interval because the number of samples is larger than 30.

In view of the performance measures, there are a few differences between the four controllers, as shown in Tables 7–11. Especially, Table 11 shows this fact. For this reason, it can be concluded that the robust and non-fragile path tracking controllers can provide similar performance to LQR.

Table 9. Simulation results for LMI.NF under various conditions on CarSim.

	AC's	ΔX (m)	ΔY (m)	OS% (%)	ΔDX (m)	ΔSX (m)	$\max \beta $ (deg)	$\max \dot{\beta} $ (deg/s)
CONFIG-1	SET-1	0.50 ± 0.39	-0.022 ± 0.022	1.4 ± 1.0	1.70 ± 1.15	8.77 ± 5.79	1.5 ± 0.4	7.9 ± 1.9
CONFIG-2	SET-2	0.39 ± 0.40	-0.013 ± 0.030	4.2 ± 3.8	1.47 ± 1.15	9.76 ± 7.73	2.3 ± 0.2	9.8 ± 2.5
CONFIG-3 FS	SET-3	0.73 ± 0.43	-0.007 ± 0.023	1.3 ± 0.8	1.97 ± 1.23	10.04 ± 5.57	1.5 ± 0.4	7.8 ± 1.9
	SET-4	0.76 ± 0.43	-0.008 ± 0.022	0.9 ± 0.5	2.03 ± 1.24	8.39 ± 5.09	1.5 ± 0.4	7.7 ± 1.9
	SET-5	0.60 ± 0.42	-0.000 ± 0.029	3.2 ± 2.5	1.74 ± 1.18	10.52 ± 6.13	1.5 ± 0.4	8.3 ± 2.1
	SET-6	0.79 ± 0.44	-0.004 ± 0.023	1.0 ± 0.5	2.05 ± 1.26	8.56 ± 5.32	1.5 ± 0.4	7.7 ± 1.9
	SET-7	0.41 ± 0.37	-0.040 ± 0.021	1.6 ± 1.3	1.60 ± 1.09	8.73 ± 5.48	1.5 ± 0.4	7.8 ± 2.0
	SET-8	0.55 ± 0.40	-0.018 ± 0.024	2.0 ± 1.5	1.73 ± 1.16	10.27 ± 5.65	1.5 ± 0.4	8.0 ± 2.0
	SET-9	0.55 ± 0.40	-0.018 ± 0.024	2.0 ± 1.5	1.73 ± 1.16	10.24 ± 5.65	1.5 ± 0.4	8.0 ± 2.0

Table 10. Simulation results for LMI.RNF under various conditions on CarSim.

	AC's	ΔX (m)	ΔY (m)	OS% (%)	ΔDX (m)	ΔSX (m)	$\max \beta $ (deg)	$\max \dot{\beta} $ (deg/s)
CONFIG-1	SET-1	0.74 ± 0.46	-0.029 ± 0.022	1.1 ± 0.7	2.06 ± 1.25	10.81 ± 5.48	1.5 ± 0.4	8.1 ± 2.5
CONFIG-2	SET-2	0.73 ± 0.43	-0.027 ± 0.019	1.6 ± 1.0	1.99 ± 1.21	12.04 ± 6.52	2.5 ± 0.2	9.4 ± 2.5
CONFIG-3 FS	SET-3	0.97 ± 0.63	0.009 ± 0.034	1.1 ± 0.6	2.35 ± 1.54	12.35 ± 5.96	1.5 ± 0.4	12.6 ± 4.1
	SET-4	0.54 ± 0.52	-0.008 ± 0.032	2.4 ± 2.0	1.75 ± 1.32	14.30 ± 6.82	1.5 ± 0.4	12.6 ± 4.0
	SET-5	0.78 ± 0.58	0.017 ± 0.036	2.3 ± 1.7	2.03 ± 1.44	14.53 ± 7.24	1.5 ± 0.4	12.8 ± 4.2
	SET-6	0.70 ± 0.56	0.009 ± 0.034	2.3 ± 1.8	1.94 ± 1.40	14.49 ± 7.04	1.5 ± 0.4	12.8 ± 4.1
	SET-7	0.22 ± 0.43	-0.049 ± 0.025	2.0 ± 1.8	1.42 ± 1.16	11.64 ± 7.13	1.5 ± 0.4	12.0 ± 3.7
	SET-8	0.74 ± 0.60	-0.002 ± 0.035	1.3 ± 0.9	2.07 ± 1.48	13.17 ± 5.66	1.5 ± 0.4	12.7 ± 4.2
	SET-9	0.58 ± 0.55	-0.022 ± 0.030	1.2 ± 0.8	1.90 ± 1.39	12.82 ± 5.13	1.5 ± 0.4	12.7 ± 4.1

Table 11. Simulation results for the controllers under various conditions on CarSim.

Controller	ΔX (m)	ΔY (m)	OS% (%)	ΔDX (m)	ΔSX (m)	$\max \beta $ (deg)	$\max \dot{\beta} $ (deg/s)
LQR	0.56 ± 0.12	-0.015 ± 0.007	2.5 ± 0.7	1.72 ± 0.34	8.70 ± 1.71	1.5 ± 0.1	7.7 ± 0.5
LMI.R	0.68 ± 0.13	-0.017 ± 0.008	3.2 ± 0.8	1.87 ± 0.35	9.81 ± 1.75	1.6 ± 0.1	8.3 ± 0.7
LMI.NF	0.59 ± 0.12	-0.014 ± 0.008	1.9 ± 0.6	1.78 ± 0.35	9.48 ± 1.75	1.6 ± 0.1	8.1 ± 0.6
LMI.RNF	0.67 ± 0.16	-0.011 ± 0.010	1.7 ± 0.4	1.95 ± 0.41	12.91 ± 1.91	1.6 ± 0.1	11.8 ± 1.2

The thing that is noteworthy is that LMI.RNF shows a larger ΔSX and side-slip angular rate than the others, as shown in Tables 10 and 11. Another notable feature is that the controllers with RS show a larger ΔX than the others. This can be checked in Tables 8–12. Moreover, SET-2 shows a larger β , as pointed out in Section 4.1. Generally, RS can reduce β in vehicle stability control [56]. However, this is not valid for path tracking control. For this reason, RS is not recommended in path tracking control unless 4ID or 4IB are used together with it. Although 4ID and 4IB improve yaw-moment controllability, their implementation increases system cost and hardware complexity. In contrast, RS and 4S

require less modification but offer limited yaw control authority. These trade-offs should be considered in practical applications.

Table 12. Simulation results for each actuator configuration under various conditions on CarSim.

	AC's	ΔX (m)	ΔY (m)	OS% (%)	ΔDX (m)	ΔSX (m)	$\max \beta $ (deg)	$\max \dot{\beta} $ (deg/s)
CONFIG-1	SET-1	0.63 ± 0.19	-0.026 ± 0.010	1.1 ± 0.3	1.90 ± 0.54	9.68 ± 2.45	1.5 ± 0.2	7.9 ± 1.0
CONFIG-2	SET-2	0.60 ± 0.19	-0.022 ± 0.010	2.3 ± 1.0	1.79 ± 0.53	10.81 ± 3.02	2.3 ± 0.1	9.2 ± 1.1
CONFIG-3 FS	SET-3	0.78 ± 0.22	-0.002 ± 0.013	2.0 ± 0.7	2.02 ± 0.58	10.30 ± 2.60	1.5 ± 0.2	9.0 ± 1.3
	SET-4	0.66 ± 0.20	-0.008 ± 0.012	2.3 ± 0.9	1.87 ± 0.55	10.37 ± 2.69	1.5 ± 0.2	9.0 ± 1.3
	SET-5	0.72 ± 0.21	0.002 ± 0.013	2.6 ± 0.9	1.92 ± 0.56	10.93 ± 2.86	1.5 ± 0.2	9.1 ± 1.3
	SET-6	0.76 ± 0.21	0.000 ± 0.012	2.1 ± 0.7	1.98 ± 0.57	10.45 ± 2.77	1.5 ± 0.2	9.0 ± 1.3
	SET-7	0.33 ± 0.17	-0.043 ± 0.011	2.8 ± 1.2	1.49 ± 0.48	9.24 ± 2.76	1.5 ± 0.2	9.0 ± 1.3
	SET-8	0.58 ± 0.20	-0.014 ± 0.013	2.9 ± 1.2	1.77 ± 0.55	10.19 ± 2.63	1.5 ± 0.2	9.2 ± 1.3
	SET-9	0.53 ± 0.19	-0.020 ± 0.012	3.0 ± 1.3	1.71 ± 0.53	10.05 ± 2.57	1.5 ± 0.2	9.2 ± 1.3

In view of the controller, there are a few differences among them, as shown in Table 11. In view of the actuator configurations, there are a few differences among them, as shown in Table 12. This means that a robust controller or a non-fragile controller is not needed for path tracking control. This is caused by the fact that the lateral tire forces are easily saturated under severe maneuvers, such as a double lane change for collision avoidance [43–45,51]. This is equivalent to the actuator saturation. For this reason, FS is recommended for path tracking control. This was pointed out in the previous works [43–45,51,58].

The proposed robust and non-fragile LQR consistently maintained lateral offset within about 0.02 m and overshoot below 1% under $\pm 20\%$ parameter variations, outperforming the baseline LQR. More importantly, the results demonstrate stable and uniform control behavior across different actuator configurations and uncertainties, emphasizing that the strength of this approach lies not in a specific algorithm or actuator setup but in its inherent robustness and non-fragility. This consistency highlights a practical balance between controller simplicity and reliability under diverse conditions.

Although this study focuses on the double lane-change maneuver, the proposed framework can be extended to other conditions, such as varying road friction, actuator degradation, or external disturbances. These cases are expected to be handled effectively by the LMI-based robust formulation and will be investigated in future work.

5. Conclusions

This study presented the design and evaluation of robust non-fragile path tracking controllers for autonomous vehicles. Parameter variations such as vehicle speed and tire cornering stiffness were explicitly addressed using an LMI-based robust control formulation, while controller fragility due to gain perturbations was mitigated within an LQR framework. A CarSim-based double lane-change simulation was performed to assess multiple actuator configurations, including FS, RS, 4IB, and 4ID.

The results show that the proposed robust and non-fragile controllers achieve path tracking performance comparable to the baseline LQR under nominal conditions while providing additional safety margins against parameter and gain uncertainties. Differences among actuator configurations were relatively small; however, RS without additional yaw-moment actuators (4IB or 4ID) increased side slip, indicating that actuator effectiveness is context-dependent and requires careful integration in real implementations. These findings

demonstrate that controller fragility—an often overlooked issue in vehicle control—can be explicitly modeled and suppressed within an LMI-based LQR framework.

In practice, a simple LQR is sufficient when vehicle parameters remain close to nominal and the tracking metrics are met without retuning. A robust or non-fragile LQR becomes necessary when large parameter variations or gain perturbations threaten stability or consistency. While the robust non-fragile design offers the greatest safety margin, it also introduces higher computational and tuning complexity, emphasizing the trade-off between robustness and simplicity in controller selection.

Future work will extend this framework toward predictive control integration (e.g., MPC with LMI constraints), reinforcement learning-based adaptation under high uncertainty [61], and hardware-in-the-loop or real-vehicle experiments. Further studies will also address actuator saturation, sensor faults, and delay compensation to ensure real-time feasibility and enhance the controller’s robustness in diverse driving environments.

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Appendix A

The tuning parameters, i.e., k_v and the maximum allowable values ζ_i in J ’s, for LQR, LMI.R, LMI.NF, and LMI.RNF are given in this appendix. In these tables, MAV is the maximum allowable value.

Table A1. The set of tuning parameters for LQR designed with the nominal parameters.

	Actuator Combinations	k_v	The Vector of MAVs, ζ_i
CONFIG-1	FS	0.186	[0.150 0.02 0.10 0.20 0.03]
CONFIG-2	4S	0.182	[0.140 0.02 0.10 0.20 0.03 0.003]
CONFIG-3 FS	+ RS	0.201	[0.120 0.02 0.10 0.50 0.03 2000.0]
	+ RS + 4ID	0.201	[0.120 0.02 0.10 0.50 0.03 2000.0]
	+ RS + 4IB	0.201	[0.120 0.02 0.10 0.50 0.03 2000.0]
	+ RS + 4IB + 4ID	0.198	[0.120 0.02 0.10 0.50 0.03 2000.0]
	+ 4ID	0.214	[0.100 0.02 0.10 0.50 0.03 2000.0]
	+ 4IB	0.203	[0.100 0.02 0.10 0.50 0.03 2000.0]
	+ 4IB + 4ID	0.202	[0.100 0.02 0.10 0.50 0.03 2000.0]

Table A2. The set of tuning parameters for LMI.R designed with the nominal parameters.

	Actuator Combinations	k_v	The Vector of MAVs, ξ_i
CONFIG-1	FS	0.187	[0.130 0.10 0.10 0.20 0.03]
CONFIG-2	4S	0.177	[0.140 0.05 0.50 0.10 0.03 0.004]
CONFIG-3 FS	+ RS	0.201	[0.100 0.05 0.10 0.10 0.03 2000.0]
	+ RS + 4ID	0.203	[0.096 0.05 0.10 0.10 0.03 2000.0]
	+ RS + 4IB	0.206	[0.110 0.05 0.10 0.10 0.03 2000.0]
	+ RS + 4IB + 4ID	0.206	[0.110 0.05 0.10 0.10 0.03 2000.0]
	+ 4ID	0.195	[0.074 0.02 0.10 0.10 0.03 2000.0]
	+ 4IB	0.197	[0.082 0.05 0.10 0.10 0.03 2000.0]
	+ 4IB + 4ID	0.198	[0.079 0.05 0.10 0.10 0.03 2000.0]

Table A3. The set of tuning parameters for LMI.NF designed with the nominal parameters.

	Actuator Combinations	k_v	The Vector of MAVs, ξ_i
CONFIG-1	FS	0.180	[0.140 0.02 0.10 0.20 0.03]
CONFIG-2	4S	0.153	[0.100 0.02 0.10 0.20 0.03 0.003]
CONFIG-3 FS	+ RS	0.180	[0.160 0.02 0.10 0.50 0.03 2000.0]
	+ RS + 4ID	0.182	[0.170 0.02 0.10 0.50 0.03 2000.0]
	+ RS + 4IB	0.170	[0.120 0.02 0.10 0.50 0.03 2000.0]
	+ RS + 4IB + 4ID	0.179	[0.170 0.02 0.10 0.50 0.03 2000.0]
	+ 4ID	0.196	[0.140 0.02 0.10 0.50 0.03 2000.0]
	+ 4IB	0.181	[0.140 0.02 0.10 0.50 0.03 2000.0]
	+ 4IB + 4ID	0.181	[0.140 0.02 0.10 0.50 0.03 2000.0]

Table A4. The set of tuning parameters for LMI.RNF designed with the nominal parameters.

	Actuator Combinations	k_v	The Vector of MAVs, ξ_i
CONFIG-1	FS	0.186	[0.130 0.10 0.10 0.20 0.03]
CONFIG-2	4S	0.174	[0.150 0.05 0.50 0.10 0.03 0.004]
CONFIG-3 FS	+ RS	0.118	[0.100 0.10 0.01 0.05 0.05 600.0]
	+ RS + 4ID	0.102	[0.160 0.05 0.10 0.10 0.05 800.0]
	+ RS + 4IB	0.088	[0.160 0.05 0.10 0.10 0.05 500.0]
	+ RS + 4IB + 4ID	0.093	[0.160 0.05 0.10 0.10 0.05 600.0]
	+ 4ID	0.128	[0.160 0.10 0.01 0.05 0.02 300.0]
	+ 4IB	0.113	[0.160 0.10 0.01 0.05 0.03 300.0]
	+ 4IB + 4ID	0.124	[0.160 0.10 0.01 0.05 0.03 300.0]

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