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Depth Control of Variable Buoyancy Systems: A Low Energy Approach Using a VSC with a Variable-Amplitude Law

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Abstract

Underwater exploration relies heavily on autonomous underwater vehicles and sensor platforms for sustained monitoring of marine environments, yet their operational duration is limited by energy constraints. To enhance energy efficiency, various control strategies have been proposed, including robust, optimal, and disturbance-aware approaches. Recent work introduced a variable structure controller (VSC) with a constant-amplitude control action for depth control of a platform equipped with a variable buoyancy module, achieving an average 22% reduction in energy use in comparison with conventional PID-based controllers. In a separate paper, the conditions for its closed-loop stability were proven. This study extends these works by proposing a controller with a variable-amplitude control action designed to minimize energy consumption. A formal proof of stability is provided to guarantee safe operation even under conservative assumptions. The controller is applied to a previously developed depth-regulated sensor platform using a validated physical model. Additionally, this study analyzes how the controller parameters and mission requirements affect stability regions, offering practical guidelines for parameter tuning. A method to estimate oscillation amplitude during hovering tasks is also introduced. Simulation trials validate the proposed approach, showing energy savings of up to 16% when compared to the controller using a constant-amplitude control action.

Keywords: variable structure controllers; autonomous underwater vehicles; variable buoyancy; energy savings; depth control



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1. Introduction

Energy is one of the most important topics in today's world, given the scarcity of resources humankind temporarily faces. Besides looking for alternative energy sources for buildings [1], industry [2], and transportation [3], research is also focused on how to reduce energy consumption in many different areas such as traffic [4], vehicle consumption [5] and industry [6].

In this energy shortage context, humans are devoting increasing attention to the oceans. Oceans cover over 70% of the Earth's surface yet remain largely unexplored, making underwater exploration essential for advancing knowledge of marine ecosystems, biodiversity, and deep-sea environments. These efforts provide critical insights into how oceans regulate climate, sustain life, and offer resources such as food and energy. In this

context, maintaining a continuous underwater presence is vital for long-term research, conservation, and resource management. Persistent monitoring enables the collection of real-time data on environmental dynamics, marine behavior, and ecosystem health, as well as the detection of subtle or rapid changes driven by climate change, pollution, or anthropogenic impacts such as overfishing [7–11]. Autonomous platforms—including underwater vehicles [12] and sensor-equipped buoys [13]—play a key role in this effort, supporting effective marine resource management and providing early warning systems for natural hazards like tsunamis and harmful algal blooms [14,15]. However, their operational range is often limited by energy autonomy. To address this constraint, energy-efficient control strategies are critical. By optimizing actuator usage, such control laws reduce energy consumption, thereby extending mission durations and expanding the reach of underwater exploration.

In order to minimize energy consumption, several strategies can be found in the literature. In [16], an event triggered sliding mode controller was developed for the depth control of an AUV. This controller updates control inputs only when a triggering condition is met, reducing control update frequency significantly and thereby saving computing and actuation energy. Energy savings of approximately 5% in simulation and approximately 3% in field experiments are reported. A similar event-triggered approach was followed in [17], where dynamic event-triggered fixed-time controllers were designed for multi-AUV systems to achieve coordinated formation or trajectory tracking while minimizing unnecessary control updates.

In [18] a combined depth control strategy was proposed to enable both hovering and efficient low-speed, long-range cruising for AUVs. This strategy alternates between: (1) a water ballast type VBS with a hierarchical architecture-based on-off flowrate controller for stable hovering and large depth changes, and (2) bow and stern fins with propulsion to maintain cruising depth efficiently. Numerical and basin tests demonstrated satisfactory control performance, achieving neglectable steady-state depth error even under model uncertainties. However, steady-state errors emerged in the presence of external disturbances, and no energy consumption data was reported. Regarding energy efficient depth control, in [19] a high-accuracy buoyancy-actuated system (HA-BA) was developed for multimodal underwater vehicles (MUVs), capable of operating as either Argo floats or a Gliders. The HA-BA features a hydraulic variable buoyancy system that uses two drive strategies for floating motions: pumping oil once for rapid ascent and pumping oil multiple times to adjust buoyancy based on floating speed. Simulations suggested a 37% reduction in energy consumption during a deep water to surface ascent using the latter strategy. However, these findings were not experimentally validated as the sea trials focused on evaluating the HA-BA control performance. Energy efficiency was also a focus in [20], where a model predictive control strategy incorporating a quadratic energy term into the cost function (MPC-QE) was developed based on a state-space model for AUVs. The quadratic energy term aimed to reduce energy consumption while minimizing deviations from the reference trajectory and changes in control input. Simulation trials were conducted using step, sinusoidal, and triangular depth references, comparing MPC-QE to conventional MPC. Results showed that MPC-QE reduced energy consumption by 34% for step trajectories and by over 20% for tracking trajectories, even in the presence of significant model uncertainty. However, the approach was not validated through experimental trials. In [21], the depth control of an AUV was simulated using four approaches: propeller-driven control (with a PID controller), hydraulic VBS-driven control (also with PID), and two hybrid strategies combining both. In the first hybrid method, depth is initially stabilized using the propellers, after which the integral term of the PID controller is transferred to the VBS. In the second, both systems operate simultaneously: the VBS with a PID controller and the propeller with

a PD controller. A step depth reference from 0.75 m to 1.36 m was used for the simulation trials. Results showed the propeller-only method was the fastest and most energy-efficient. However, the study did not clarify how energy consumption was calculated, and the hydraulic pump efficiency was not disclosed. Energy consumption was also a concern in [22] where a robust controller for an underwater vehicle operating at depths of up to 150 m was developed. This method combined sliding mode control (SMC) for robustness against modeling errors and environmental disturbances with classical optimal control techniques to minimize control effort. During helicoidal trajectory tracking, the controller achieved up to 30% energy savings, albeit with some compromise in tracking accuracy.

In this context, the authors of this paper recently proposed a variable structure depth controller with a constant-amplitude control action. This controller was designed to reduce energy consumption in an underwater sensor platform with a variable buoyancy module (VBM) [23]. The controller activates only when system dynamics drive the vehicle away from the desired depth and deactivates when they naturally guide it back, minimizing unnecessary actuation. Results in [8] show that this approach reduces energy use by an average of 22% compared to the most efficient PID-based alternative, albeit with a trade-off in control performance. The formal proof of stability of the controller was presented in [24] by demonstrating that if the control action remains within specified upper and lower bounds, the system state remains bounded regardless of its initial rest conditions. The study in [24] also highlights how stability regions vary with parameter choices and mission requirements, offering practical guidance for controller tuning. The proposed control law in [24] was shown to ensure formal stability, even under conservative assumptions. Stability analysis is critical for ensuring safe and reliable operation of underwater vehicles in complex marine environments. Without it, control laws may lead to unpredictable behavior, such as instability, oscillations, or mission failure. Stability proofs such as those based on Lyapunov-based methods [25–27] provide assurance of safe autonomous operation, enhancing mission success and reducing risks.

It is important to highlight that the control law proposed in [24] is of the switched type, thereby characterizing the overall system as a switched control system. In such systems, the stability analysis is inherently more complex than in non-switched counterparts. This is due not only to the potential need to establish the stability of each individual subsystem, but also to the requirement that the switching between subsystems does not induce instability [28]. A common methodology employed in this context involves the use of multiple Lyapunov functions [28–30]. However, this approach presents a significant challenge, as it demands the construction of appropriate Lyapunov functions for each subsystem—an inherently difficult and non-trivial task.

In previous papers [23,24] the controller uses a fixed amplitude control action in the operating regions where it is on. For stability this constant value must be above the one determined by the stability conditions. Since these conditions vary according to the system state, the constant value may be often oversized. To avoid this potential energy waste, the controller proposed in this study uses a variable value that matches the instantaneous stability conditions. Potential energy savings might therefore be obtained. Since the formal proof developed in [24] assumes a constant control action when the controller is on, it is no longer applicable to the one developed in this study. Furthermore, the work developed in this paper also examines how controller parameters and mission requirements influence stability regions and how these regions can be used to estimate the order of magnitude of oscillation amplitude during hovering. It is shown through simulation trials that the proposed controller can further reduce energy consumption compared to two previously developed controllers in [23,24]. From these findings it is also discussed how parameter choices affect energy consumption.

The main contributions of this paper are therefore the following:

- (i) a new variable structure controller that further reduces the energy consumption obtained with the controllers in [24]. The proposed controller achieves energy savings of up to 16% when compared to the most efficient controller presented in [24];
- (ii) a formal proof of stability for the new controller;
- (iii) the newly proposed formal stability proof differs from conventional approaches found in the literature on switched systems, which are typically based on the use of multiple Lyapunov functions [28];
- (iv) a procedure to estimate the order of magnitude of the depth error band that can be obtained with the new controller.

The paper is structured as follows: Section 2 introduces the underwater device model and Section 3 describes the proposed controller. Section 4 presents the formal stability proof, with Section 4.1 outlining the proof and Sections 4.2–4.4 detailing its key components. Section 5 starts by quantifying stability bounds using a previously developed prototype as a case study and ends by presenting simulation trials. Finally, Section 6 draws the key conclusions of this work.

2. Underwater Device Closed Loop Model

The prototype model used in this work integrates the VBM model with the vertical motion model, as developed in [31]. Given that actuator dynamics are significantly faster than the rest of the system, a reduced-order model, represented in the Laplace domain, is shown in Figure 1.

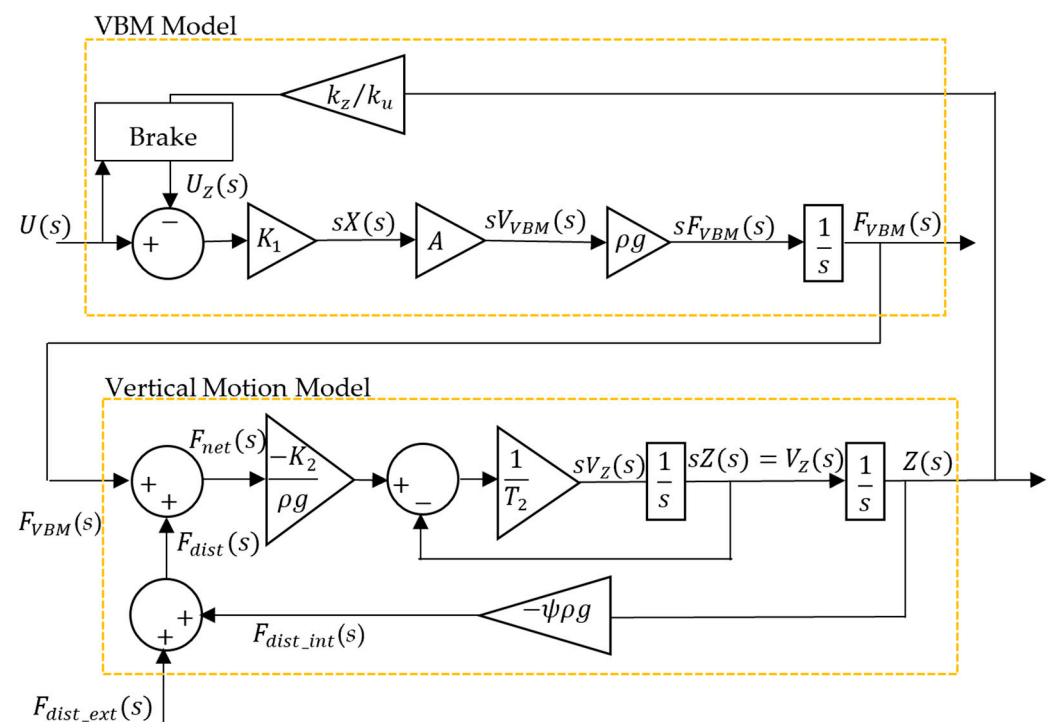


Figure 1. Prototype reduced-order model.

Figure 1 resumes the model that has already been used in [24]. In this figure, K_1 represents the steady-state gain of the actuator's velocity dynamics, experimentally identified in [32], and A , the area of the piston in contact with seawater. The piston position is represented by $X(s)$ and piston velocity by $sX(s)$. The mechanical brake's effect on the actuator is represented in the 'brake' block of Figure 1. When $U = 0$, the brake is engaged, resulting in an equivalent depth voltage $U_Z = 0$. When $U \neq 0$, the brake is disengaged,

and $U_Z = (k_z/k_u) Z$, reflecting the influence of depth-induced pressure on actuator motion. Z is the vehicle depth and the ratio k_z/k_u , determined in [33], captures the torque exerted on the motor by external pressure forces and the motor's stall torque at the given voltage.

The net force F_{net} is the difference between the actual VBM buoyancy force F_{VBM} and the total disturbance force F_{dist} , which includes both internal ($F_{dist_{int}} = -\psi\rho gZ$) and external ($F_{dist_{ext}}$) components.

$$F_{net} = F_{VBM} + F_{dist_{ext}} - \psi\rho gZ \quad (1)$$

Parameters K_2 and T_2 , which characterize the depth dynamics, were identified in [31]. The parameters ρ and g represent the water density at 1 atm and the gravitational acceleration, respectively. Parameter ψ accounts for the variation in water density with depth and the volume loss per meter due to hull deformation with pressure. A method for estimating this volumetric loss was presented in [23].

The model is governed by the following differential equations:

$$\begin{cases} \dot{V}_Z = \frac{K_2}{T_2\rho g}\psi\rho gZ - \frac{1}{T_2}V_Z - \frac{K_2}{T_2\rho g}F_{VBM} - \frac{K_2}{T_2\rho g}F_{dist_{ext}} \\ \dot{F}_{VBM} = -\rho gK_1A(k_z/k_u)Z + \rho gK_1AU \end{cases} \quad (2)$$

where $V_Z = \dot{Z}$ is the vertical speed of the prototype. It will be considered, without loss of generality, that the control purpose is to drive the depth error E_Z to $E_Z = 0$. If control action U is switched between two states $U_i, i = 1, 2$, where

$$U_i = \begin{cases} 0 & \text{if } i = 1 \\ -U_{switch}\text{sign}(E_Z) & \text{if } i = 2 \end{cases} \quad (3)$$

with U_{switch} being a variable control action to be described later in Theorem 2 of Section 4.2.

Then the closed loop dynamics are described by:

$$\dot{\mathbf{x}} = \mathbf{f}_i(\mathbf{x}(t), F_{dist_{ext}}) \quad (4)$$

where the state $\mathbf{x}(t)$ is defined as $\mathbf{x}(t) = [Z \ V_Z \ F_{VBM}]^T$ and $\mathbf{f}_1(\mathbf{x}(t), F_{dist_{ext}})$ and $\mathbf{f}_2(\mathbf{x}(t), F_{dist_{ext}})$ defined as:

$$\mathbf{f}_1(\mathbf{x}(t), F_{dist_{ext}}) = \begin{cases} \dot{Z} = V_Z \\ \dot{V}_Z = \frac{K_2}{T_2\rho g}\psi\rho gZ - \frac{1}{T_2}V_Z - \frac{K_2}{T_2\rho g}F_{VBM} - \frac{K_2}{T_2\rho g}F_{dist_{ext}} \\ \dot{F}_{VBM} = 0 \end{cases} \quad (5)$$

$$\mathbf{f}_2(\mathbf{x}(t), F_{dist_{ext}}) = \begin{cases} \dot{Z} = V_Z \\ \dot{V}_Z = \frac{K_2}{T_2\rho g}\psi\rho gZ - \frac{1}{T_2}V_Z - \frac{K_2}{T_2\rho g}F_{VBM} - \frac{K_2}{T_2\rho g}F_{dist_{ext}} \\ \dot{F}_{VBM} = -\rho gK_1A k_z/k_u Z - \rho gK_1AU_{switch}\text{sign}(E_Z) \end{cases} \quad (6)$$

3. Controller Description

The controller presented in this work has a similar structure to the one presented in [23,24] and will therefore only be briefly presented in the following. Both controllers attempt to minimize consumption of the VBM by leveraging the system's inherent dynamics to favor low energy depth control. This approach activates the VBM whenever the underwater device deviates from its target depth. Conversely, if the device moves in a way that reduces the depth error, the VBM is deactivated. As a result, the VBM is expected to remain off for a substantial portion of the mission, leading to significant energy savings. Figure 2 presents a block

diagram of the controller structure. It should be noticed that while F_{VBM} can be obtained by measuring the piston position X , the disturbances are estimated using a disturbance observer, as presented in [23]. Further details about the controller and the observer can be found in [23,24].

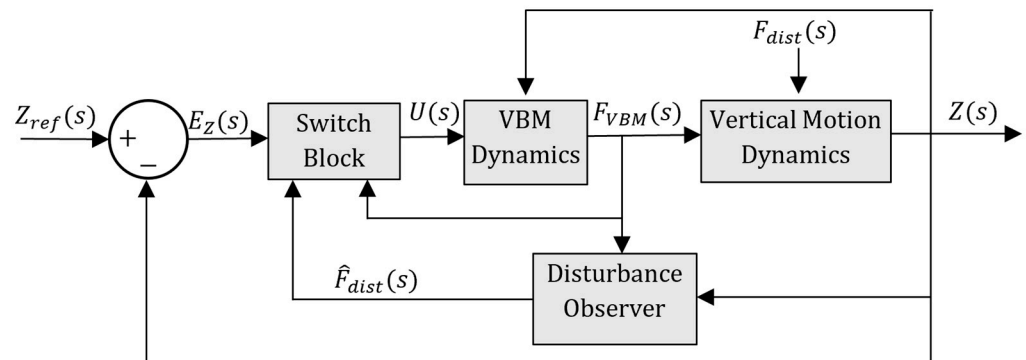


Figure 2. VSC system block diagram.

In Figure 2, Z_{ref} is the target depth reference. The depth control error and its derivative can be written as (7) and (8), respectively, since Z_{ref} is assumed to be constant. Figure 3 presents the system model notation.

$$E_Z = Z_{ref} - Z \quad (7)$$

$$\dot{E}_Z = -V_Z \quad (8)$$

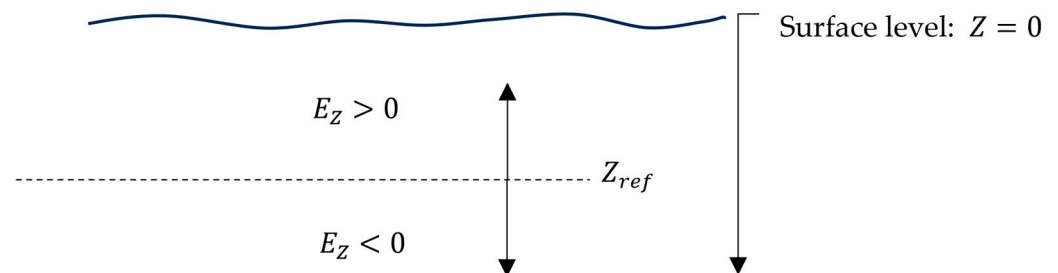


Figure 3. System model notation.

The switch block, described below, determines whether the VBM should be turned OFF—by engaging the brake and applying control law (3) with $i = 1$ —or turned ON—by disengaging the brake and using control law (3) with $i = 2$. Figure 4 illustrates the decision-making process of the switch block.

In the shaded areas of the schematic in Figure 4, the VBM is switched OFF (corresponding to $i = 1$ in Equation (3)), while in the unshaded regions, it is switched ON ($i = 2$). Specifically, the VBM is OFF in regions 1a, 1b, and 1c, and ON in regions 2a and 2b. To account for disturbances, the net force F_{net} ($F_{net} = F_{VBM} + F_{dist}$) is represented on the vertical axis of the schematic.

Table 1 presents the different switch block decisions and control action U according to each region in the schematic of Figure 4.

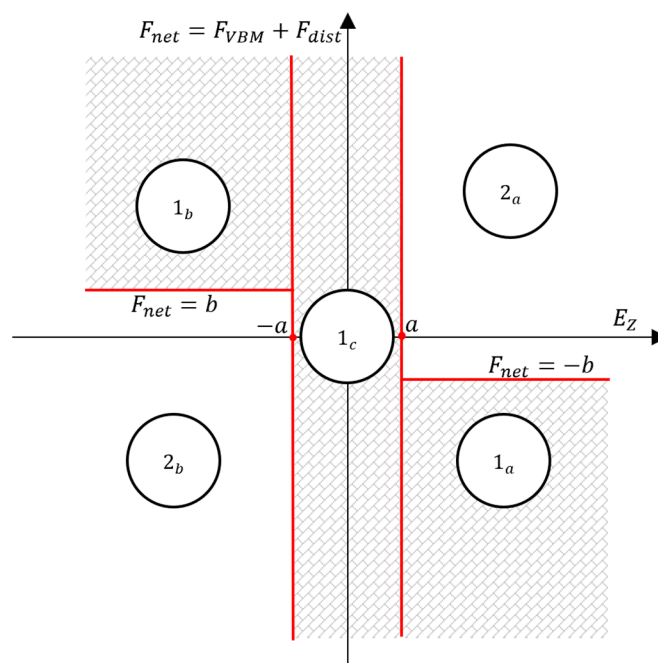


Figure 4. Schematic of the switch block decision process.

Table 1. Switch block decision table.

Regions	E_Z	F_{net}	Decision	i	Control Action U
1_c	$-a < E_Z < a$	F_{net}	OFF	1	0
2_a	$E_Z > a$	$F_{net} \geq -b$	ON	2	$-U_{switch}$
1_a	$E_Z > a$	$F_{net} < -b$	OFF	1	0
1_b	$E_Z < -a$	$F_{net} > b$	OFF	1	0
2_b	$E_Z < -a$	$F_{net} \leq b$	ON	2	U_{switch}

The controller parameters a and b in Table 1 are strictly positive constants that are used by both the controller developed in [24] and by the one presented in this paper. The two controllers differ on the control action U_{switch} used. The controller presented in [23,24] uses a constant-amplitude control action, $U_{switch} = const.$, and therefore can be referred to as a variable structure controller with a constant-amplitude control action (VSC_{Uconst}). The controller presented in this work is similar but uses a variable-amplitude control action ($U_{switch} = U_{min}$), designed to be as low as possible while keeping the system stable. As will be shown, this control strategy further reduces energy consumption. This controller will be referred to as VSC_{Umin}, and the formal proof of its stability will be presented in the following sections.

4. Proof of Stability of the Closed-Loop System

4.1. Outline of the Proof of Stability

Section 4 will demonstrate that system (4), using control action (3) and the switching strategy from Table 1 is stable in the Lyapunov sense [34] for constant depth references Z_{ref} and constant external disturbances F_{dist_ext} .

The proof of stability of the controller rests upon three stability conditions:

1. For system (4): if F_{net} is bounded, then $\dot{E}_Z = -V_Z$ are also bounded;
2. For system (4) with control action (3) and the switching decision presented in Table 1: when the system (4) is in region 2a, $\dot{F}_{net} = -\eta$, $\eta > 0$ and when the system is in region 2b, $\dot{F}_{net} = \eta$, $\eta > 0$;

3. Using the control law presented in Section 3, and as long as condition 2 is satisfied: starting from an arbitrary state ξ such that $F_{net\xi} = 0$, $\dot{E}_{Z\xi} = 0$, $E_{Z\xi} < E_{Zmax}$, where E_{Zmax} is an arbitrary maximum depth (typically the rated depth of the vehicle), the system performs a cyclical sequence S , traversing points $\alpha, \beta, \gamma_1, \gamma_2, \delta, \varepsilon_1, \varepsilon_2$ of Figure 5, such that the following conditions are met:
 - a. $F_{net\alpha} = F_{net\varepsilon_2}$ and $E_{Z\varepsilon_2} \leq E_{Z\alpha}$;
 - b. $|E_{Z\beta}| = |E_{Z\xi}|$ and $|F_{net\xi}| \leq |F_{net\beta}|$

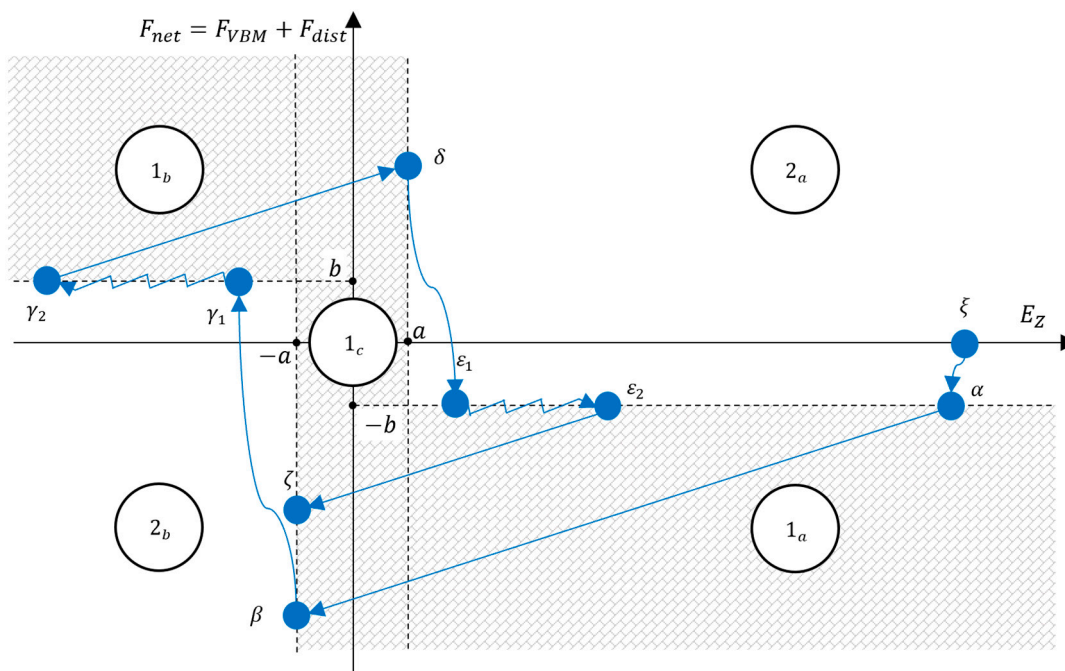


Figure 5. Sequence of states in the $E_Z - F_{net}$ plane.

By satisfying conditions 3a and 3b above, it follows that the absolute values in both F_{net} and E_Z are bounded. From condition 1, the boundedness of F_{net} implies that V_Z is also bounded. From Equation (7), if E_Z is bounded, Z is also bounded. Finally, from (1) and assuming constant external disturbances, the boundedness of F_{net} and Z ensures F_{VBM} is also bounded. In this work, it is assumed that F_{dist_ext} remains constant, given that, under typical operating conditions, disturbances such as fouling or salinity variations occur at sufficiently slow rates to be approximated as constant. Wave action is assumed to primarily affect depth measurement accuracy rather than buoyancy, as depth is commonly sensed via pressure sensors. Vertical currents may influence the buoyancy force, but, except in highly dynamic shallow waters, they are not expected to exhibit rapid variations. Furthermore, if these disturbances exhibit abrupt transitions at infrequent time instants, the corresponding time-derivatives are non-zero only at those particular time instants. Consequently, such behavior does not compromise system stability. It is therefore shown that satisfying conditions 3a and 3b ensures that the state $\mathbf{x}$ of Equation (4) remains bounded in the Lyapunov sense [34]:

$$\forall R > 0, \exists r > 0, \|\mathbf{x}(0)\| < r \implies \forall t \geq 0, \|\mathbf{x}(t)\| < R \quad (9)$$

4.2. Proof of Stability Condition 1

The proof of stability condition 1 rests on Theorem 1. The proof of Theorem 1 has been detailed in [24], and the following sections present the proof of the two remaining stability conditions. Theorem 1 is presented below for convenience.

Theorem 1. For the system defined by Equation (4) and by the block diagram in Figure 1, if F_{net} is bounded, $\dot{E}_Z$ is also bounded.

4.3. Proof of Stability Condition 2

Theorem 2. For the system represented by (4), the variable structure control action of Equation (3) with the switching decision block described in Table 1 ensures that when the system (4) is in region 2_a, $\dot{F}_{net} = -\eta$ and when the system (4) is in region 2_b, $\dot{F}_{net} = \eta$, with $\eta > 0$ as long as Equation (10) is satisfied.

$$U_{switch} = \frac{\eta}{\rho g K_1 A} + \text{sign}(E_Z) \left(-(k_z/k_u)Z - \left(\frac{\psi}{K_1 A} \right) V_Z \right) \quad (10)$$

Proof of Theorem 2. Consider region 2_a, where $E_Z > a \wedge F_{net} \geq -b$. In this case, to ensure that the system moves out of this region, it is necessary to impose $\dot{F}_{net} < 0$. In this work it will be ensured that $\dot{F}_{net} = -\eta$ so that a minimum F_{net} time rate is guaranteed. From the block diagram of Figure 1, the value of F_{net} can be written, in the Laplace domain, as:

$$F_{net}(s) = \frac{1}{s} (\rho g ((K_1 A)(U(s) - (k_z/k_u)Z(s)))) + F_{dist_{ext}}(s) - \psi \rho g Z(s) \quad (11)$$

So, the Laplace transform of F_{net} time derivative can be written as

$$F_{net}(s)s = \rho g K_1 A U(s) - \rho g K_1 A (k_z/k_u)Z(s) + s F_{dist_{ext}}(s) - \psi \rho g V_Z(s) \quad (12)$$

To ensure that $\dot{F}_{net} = -\eta$, the condition presented in Equation (13), written in the time domain, must be met:

$$\dot{F}_{net} = -\eta \rightarrow U = -\frac{\eta}{\rho g K_1 A} + (k_z/k_u)Z + \left(\frac{\psi}{K_1 A} \right) V_Z - \frac{1}{\rho g K_1 A} \dot{F}_{dist_{ext}} \quad (13)$$

Since at region 2_a, according to (3), $U = -\text{sign}(E_Z)U_{switch}$, and since in region 2_a, $E_Z > 0$, it follows that $U = -U_{switch}$. Replacing the value of U in Equation (13), and recalling that external disturbances are considered to be constant, Equation (14) can be achieved.

$$U_{switch} = \frac{\eta}{\rho g K_1 A} - (k_z/k_u)Z - \left(\frac{\psi}{K_1 A} \right) V_Z \quad (14)$$

Consider now region 2_b, where $E_Z < -a \wedge F_{net} \leq b$. In this case, to ensure that the system moves out of this region, $\dot{F}_{net} > 0$. For the same reason of region 2_a, it will be ensured that $\dot{F}_{net} = \eta$. In order to ensure that $\dot{F}_{net} = \eta$, Equation (12) can be used to reach Equation (15):

$$\dot{F}_{net} = \eta \rightarrow U = \frac{\eta}{\rho g K_1 A} + (k_z/k_u)Z + \left(\frac{\psi}{K_1 A} \right) V_Z - \frac{1}{\rho g K_1 A} \dot{F}_{dist_{ext}} \quad (15)$$

Since at region 2_b, according to (3), $U = -\text{sign}(E_Z)U_{switch}$, and since in region 2_b, $E_Z < 0$, then $U = U_{switch}$. Replacing U in Equation (15), and recalling that external disturbances are considered to be constant, leads to Equation (16).

$$U_{switch} = \frac{\eta}{\rho g K_1 A} + (k_z/k_u)Z + \left(\frac{\psi}{K_1 A} \right) V_Z \quad (16)$$

Conditions (14) and (16) can be written, in a compact form, as in Equation (10). Therefore, it is shown that control law (3) with U_{switch} given by (10) and using the switching decision law

presented in Table 1 ensures that when the system (4) is in region 2_a , $\dot{F}_{net} = -\eta$ and when the system (4) is in region 2_b , $\dot{F}_{net} = \eta$. Condition 2 of Section 4.1 is therefore satisfied. $\square$

4.4. Proof of Stability Condition 3

4.4.1. System Trajectory Description

Let S be a switching sequence associated with the switched system (4), the set of switching times be defined by $\{t_j\}$ and the interval of completion $\sqsupset(S|i)$ be defined as the set of time intervals during which subsystem i is active:

$$\sqsupset(S|i) = \bigcup_k \bar{t}_{i,k}, t_{i,k} \quad (17)$$

where

$$\{\bar{t}_{i,k}\} = \{t_j \mid \text{when subsystem } i \text{ is switched on}\} \quad (18)$$

and

$$\{t_{i,k}\} = \{t_j \mid \text{when subsystem } i \text{ is switched off}\} \quad (19)$$

Consider now Figure 6, showing a sequence of switching actions and the corresponding states in the $E_Z - F_{net}$ plane.

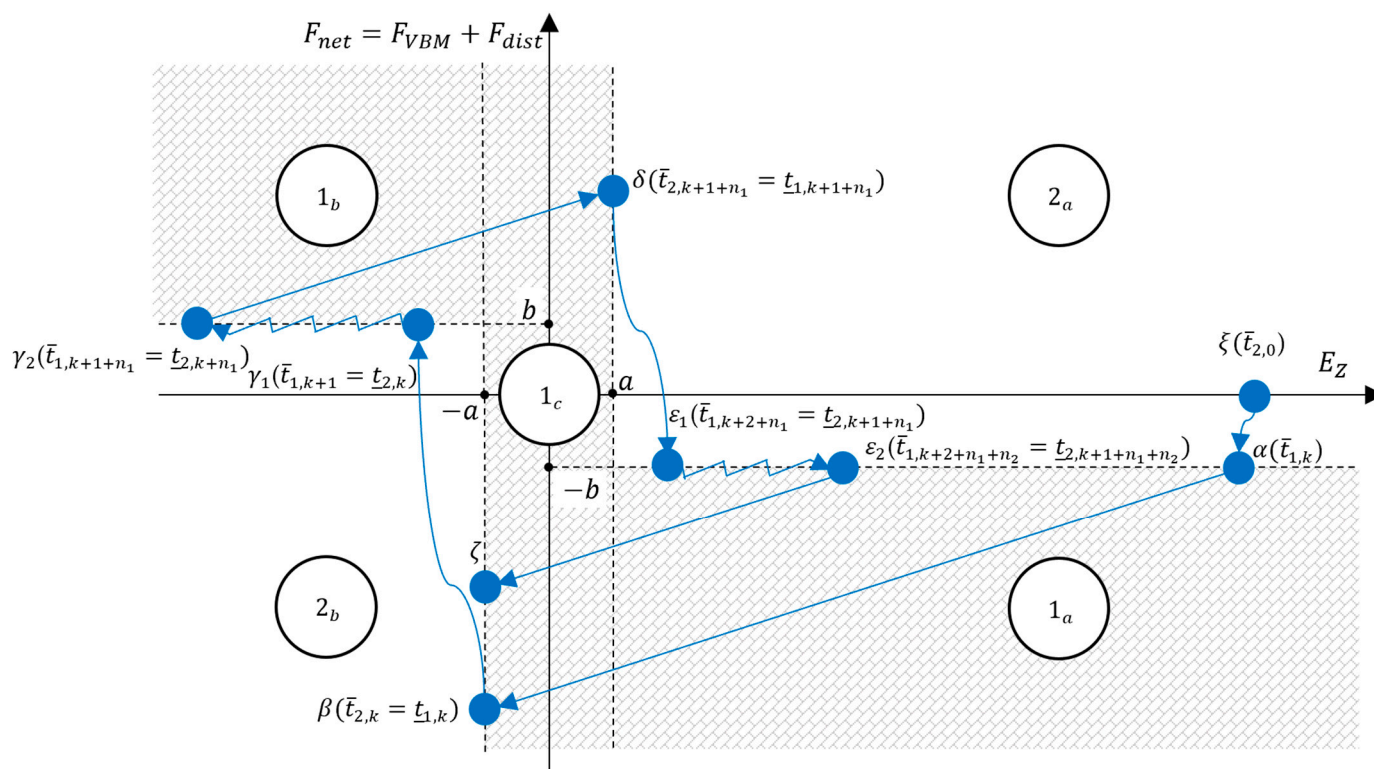


Figure 6. A sequence of switching actions and the corresponding states in the $E_Z - F_{net}$ plane.

As detailed in Section 4.1, it will be shown in the following that starting from point $\xi(\bar{t}_{2,0})$, the system will evolve to point $\varepsilon_2(t)$ crossing $\alpha(t)$, $\beta(t)$, $\gamma_1(t)$, $\gamma_2(t)$, $\delta(t)$ and $\varepsilon_1(t)$ in this order. The system then enters a cyclic sequence $S = \{\alpha(t), \beta(t), \gamma_1(t), \gamma_2(t), \delta(t), \varepsilon_1(t), \varepsilon_2(t)\}$, such that conditions 3 of Section 4.1 are met.

For that purpose, the following results derived from the block diagram of Figure 1 will be used:

$$F_{net} = F_{VBM} - \psi \rho g Z + F_{dist_ext} \quad (20)$$

And, since F_{dist_ext} is considered to be constant,

$$\dot{F}_{net} = \dot{F}_{VBM} - \psi \rho g V_Z \quad (21)$$

Also, from Figure 1 and Equation (3), the value of F_{VBM} and its derivative are given by:

$$F_{VBM} = \begin{cases} \text{constant}, i = 1 \\ \int K_1 A \rho g \left(-U_{switch} \text{sign}(E_Z) - \frac{k_Z}{k_u} Z \right) dt, i = 2 \end{cases} \quad (22)$$

$$\dot{F}_{VBM} = \begin{cases} 0, i = 1 \\ K_1 A \rho g \left(-U_{switch} \text{sign}(E_Z) - \frac{k_Z}{k_u} Z \right), i = 2 \end{cases} \quad (23)$$

At the initial point $\xi(\bar{t}_{2,0})$, it is considered that the system has zero velocity, zero net buoyancy and an initial arbitrary error $E_{Z\xi} > \mu$. The initial point can be defined as:

$$\xi(\bar{t}_{2,0}) : \begin{cases} E_Z(\bar{t}_{2,0}) = E_{Z\xi}; \dot{E}_Z(\bar{t}_{2,0}) = \dot{E}_{Z\xi} = 0 \\ Z(\bar{t}_{2,0}) = Z_\xi; V_Z(\bar{t}_{2,0}) = V_{Z\xi} = -\dot{E}_{Z\xi} = 0 \\ F_{net}(\bar{t}_{2,0}) = F_{net\xi} = 0 \end{cases} \quad (24)$$

The change in F_{net} between points $\xi(F_{net\xi})$ and $\alpha(F_{net\alpha})$, can be determined integrating Equation (21) with $i = 2$:

$$F_{net\alpha} - F_{net\xi} = \int_{\bar{t}_{2,0}}^{\bar{t}_{1,k}} (\dot{F}_{VBM} - \psi \rho g V_Z) dt = - \int_{\bar{t}_{2,0}}^{\bar{t}_{1,k}} \eta dt \quad (25)$$

Developing Equation (25):

$$F_{net\alpha} = F_{net\xi} - \eta(\bar{t}_{1,k} - \bar{t}_{2,0}) \quad (26)$$

Since $F_{net\alpha} = -b$, and $F_{net\xi} = 0$

$$(\bar{t}_{1,k} - \bar{t}_{2,0}) = \frac{b}{\eta} \quad (27)$$

Given that $\dot{E}_{Z\xi} = 0$ and as has been shown in [24], the sign of $\dot{E}_Z$ and F_{net} are the same, it follows that point $\alpha(\bar{t}_{1,k})$ can be defined as:

$$\alpha(\bar{t}_{1,k}) : \begin{cases} E_Z(\bar{t}_{1,k}) = E_{Z\alpha}; \dot{E}_Z(\bar{t}_{1,k}) = \dot{E}_{Z\alpha} \leq 0 \\ Z(\bar{t}_{1,k}) = Z_\alpha; V_Z(\bar{t}_{1,k}) = V_{Z\alpha} = -\dot{E}_{Z\alpha} \geq 0 \\ F_{net}(\bar{t}_{1,k}) = F_{net\alpha} = -b \end{cases} \quad (28)$$

Please notice that in the first cycle $V_{Z\alpha} = -\dot{E}_{Z\alpha} > 0$ but in subsequent cycles $V_{Z\alpha} = -\dot{E}_{Z\alpha} = 0$. This is because point ε_2 of one cycle is point α of the following cycle and $\dot{E}_{Z\varepsilon_2} = 0$, as will later be shown. Since $F_{net}(\bar{t}_{1,k}) = -b$ and using Equation (20),

$$F_{VBM}(\bar{t}_{1,k}) = \psi \rho g Z_\alpha - b - F_{dist_ext} = F_{VBM\alpha} \quad (29)$$

The change in F_{VBM} between points α , $F_{VBM\alpha}$, and β , $F_{VBM\beta}$, can be determined integrating Equation (23) with $i = 1$:

$$F_{VBM\beta} - F_{VBM\alpha} = 0 \rightarrow F_{VBM\beta} = F_{VBM\alpha} \quad (30)$$

The change in F_{net} between points α , $F_{net\alpha}$, and β , $F_{net\beta}$, can be determined integrating Equation (21) with $i = 1$.

$$F_{net\beta} - F_{net\alpha} = \int_{\bar{t}_{1,k}}^{\bar{t}_{1,k}} -\psi\rho g V_Z dt \quad (31)$$

Since $F_{net\alpha} = -b$ and $\bar{t}_{1,k} E_{Z\beta} = -a$

$$F_{net\beta} = -b + \psi\rho g \int_{E_{Z\alpha}}^{E_{Z\beta}} dE_Z = -b + \psi\rho g (E_{Z\beta} - E_{Z\alpha}) = -b - \psi\rho g (a + E_{Z\alpha}) \quad (32)$$

Since $\dot{E}_{Z\alpha} < 0$ and between α and β $F_{net} < 0$, the point $\beta(\bar{t}_{1,k})$ can therefore be defined as:

$$\beta(\bar{t}_{1,k}) : \begin{cases} E_Z(\bar{t}_{1,k}) = E_{Z\beta}; \dot{E}_Z(\bar{t}_{1,k}) = \dot{E}_{Z\beta} < 0 \\ Z(\bar{t}_{1,k}) = Z_{\beta}; V_Z(\bar{t}_{1,k}) = V_{Z\beta} = -\dot{E}_{Z\beta} > 0 \\ F_{net}(\bar{t}_{1,k}) = F_{net\beta} = -b - \psi\rho g (a + E_{Z\alpha}) \end{cases} \quad (33)$$

The change in F_{net} between points $\beta(F_{net\beta})$ and $\gamma_1(F_{net\gamma_1})$, can be determined integrating Equation (21) with $i = 2$:

$$F_{net\gamma_1} - F_{net\beta} = \int_{\bar{t}_{2,k}}^{\bar{t}_{2,k}} (\dot{F}_{VBM} - \psi\rho g V_Z) dt = \int_{\bar{t}_{2,k}}^{\bar{t}_{2,k}} \dot{F}_{net} dt = \int_{\bar{t}_{2,k}}^{\bar{t}_{2,k}} \eta dt \quad (34)$$

Developing Equation (34) leads to:

$$F_{net\gamma_1} = F_{net\beta} + \eta(\bar{t}_{2,k} - \bar{t}_{2,k}) \quad (35)$$

Since $F_{net\gamma_1} = b$ and considering (32):

$$(\bar{t}_{2,k} - \bar{t}_{2,k}) = \frac{2b + \psi\rho g (a + E_{Z\alpha})}{\eta} \quad (36)$$

Equation (36) expresses the necessary time for the system to move from β to γ_1 . From the block diagram of Figure 1:

$$\dot{V}_Z + \frac{1}{T_2} V_Z = -\frac{K_2}{T_2 \rho g} F_{net}, \quad \bar{t}_{2,k} \leq t \leq \bar{t}_{2,k} \quad (37)$$

where $V_Z(\bar{t}_{2,k}) = V_{Z\beta}$, and $F_{net}(\bar{t}_{2,k}) = F_{net\beta}$. Since in this region $\dot{F}_{net} = \eta$

$$\dot{F}_{net} = \eta \implies F_{net}(t) = F_{net\beta} + \eta(t - \bar{t}_{2,k}), \quad \bar{t}_{2,k} \leq t \leq \bar{t}_{2,k} \quad (38)$$

The expression for the depth velocity between β and γ_1 is thus obtained by solving Equation (37):

$$V_Z(t) = \left[V_{Z\beta} + \frac{K_2}{\rho g} (F_{net\beta} - \eta T_2) \right] e^{-(t - \bar{t}_{2,k})/T_2} - \frac{K_2}{\rho g} [F_{net\beta} + \eta(t - \bar{t}_{2,k}) - \eta T_2], \quad \bar{t}_{2,k} \leq t \leq \bar{t}_{2,k} \quad (39)$$

To find the depth expression, (39) must be integrated.

$$Z(t) = \int \left(\left[V_{Z\beta} + \frac{K_2}{\rho g} (F_{net\beta} - \eta T_2) \right] e^{-(t - \bar{t}_{2,k})/T_2} - \frac{K_2}{\rho g} [F_{net\beta} + \eta(t - \bar{t}_{2,k}) - \eta T_2] \right) dt, \quad \bar{t}_{2,k} \leq t \leq \bar{t}_{2,k} \quad (40)$$

After integration, (40) leads to:

$$\begin{aligned} Z(t) = & -T_2 \left[V_{Z\beta} + \frac{K_2}{\rho g} (F_{net\beta} - \eta T_2) \right] e^{-(t - \bar{t}_{2,k})/T_2} - \frac{K_2}{\rho g} \left[F_{net\beta} t + \eta \frac{(t - \bar{t}_{2,k})^2}{2} - \eta T_2 t \right] + Z_{\beta} \\ & + T_2 \left[V_{Z\beta} + \frac{K_2}{\rho g} (F_{net\beta} - \eta T_2) \right] + \frac{K_2}{\rho g} [F_{net\beta} \bar{t}_{2,k} - \eta T_2 \bar{t}_{2,k}], \quad \bar{t}_{2,k} \leq t \leq \bar{t}_{2,k} \end{aligned} \quad (41)$$

Since at $t = t_{2,k}$, $Z = Z_{\gamma_1}$

$$Z_{\gamma_1} = -T_2 \left(V_{Z\beta} + \frac{K_2}{\rho g} (F_{net\beta} - \eta T_2) \right) e^{-(t_{2,k} - \bar{t}_{2,k})/T_2} - \frac{K_2}{\rho g} \left(F_{net\beta} t_{2,k} + \frac{\eta(t_{2,k} - \bar{t}_{2,k})^2}{2} - \eta T_2 t_{2,k} \right) + Z_{\beta} \\ + T_2 \left(V_{Z\beta} + \frac{K_2}{\rho g} (F_{net\beta} - \eta T_2) \right) + \frac{K_2}{\rho g} (F_{net\beta} \bar{t}_{2,k} - \eta T_2 \bar{t}_{2,k}) \quad (42)$$

Rearranging Equation (42) leads to:

$$Z_{\gamma_1} = Z_{\beta} + T_2 \left(V_{Z\beta} + \frac{K_2}{\rho g} (F_{net\beta} - \eta T_2) \right) \left(1 - e^{-(t_{2,k} - \bar{t}_{2,k})/T_2} \right) \\ - \frac{K_2}{\rho g} \frac{\eta(t_{2,k} - \bar{t}_{2,k})^2}{2} + \frac{K_2}{\rho g} (t_{2,k} - \bar{t}_{2,k}) (\eta T_2 - F_{net\beta}) \quad (43)$$

Equation (43) defines the depth at γ_1 as a function of the depth at β and the vehicle depth dynamics. To find the depth error, Equation (7) should be considered:

$$E_{\gamma_1} = Z_{ref} - \left(Z_{\beta} + T_2 \left(V_{Z\beta} + \frac{K_2}{\rho g} (F_{net\beta} - \eta T_2) \right) \left(1 - e^{-(t_{2,k} - \bar{t}_{2,k})/T_2} \right) \right. \\ \left. - \frac{K_2}{\rho g} \frac{\eta(t_{2,k} - \bar{t}_{2,k})^2}{2} + \frac{K_2}{\rho g} (t_{2,k} - \bar{t}_{2,k}) (\eta T_2 - F_{net\beta}) \right) \quad (44)$$

Rearranging Equation (44)

$$E_{\gamma_1} = E_{Z\beta} - T_2 \left(V_{Z\beta} + \frac{K_2}{\rho g} (F_{net\beta} - \eta T_2) \right) \left(1 - e^{-(t_{2,k} - \bar{t}_{2,k})/T_2} \right) \\ + \frac{K_2}{\rho g} \frac{\eta}{2} (t_{2,k} - \bar{t}_{2,k})^2 - \frac{K_2}{\rho g} (t_{2,k} - \bar{t}_{2,k}) (\eta T_2 - F_{net\beta}) \quad (45)$$

Considering $F_{net\beta}$ from (32) and $(t_{2,k} - \bar{t}_{2,k})$ from (36)

$$E_{\gamma_1} = E_{Z\beta} - T_2 \left(V_{Z\beta} + \frac{K_2}{\rho g} ((-b - \psi \rho g(a + E_{Z\alpha})) - \eta T_2) \right) \left(1 - e^{-\left(\frac{2b + \psi \rho g(a + E_{Z\alpha})}{\eta}\right)/T_2} \right) \\ + \frac{K_2}{\rho g} \frac{\eta}{2} \left(\frac{2b + \psi \rho g(a + E_{Z\alpha})}{\eta} \right)^2 - \frac{K_2}{\rho g} \left(\frac{2b + \psi \rho g(a + E_{Z\alpha})}{\eta} \right) (\eta T_2 - (-b - \psi \rho g(a + E_{Z\alpha}))) \quad (46)$$

The depth error at γ_1 can therefore be given by:

$$E_{\gamma_1} = E_{Z\beta} - T_2 \left(V_{Z\beta} - \frac{K_2}{\rho g} b - K_2 \psi(a + E_{Z\alpha}) - \frac{K_2}{\rho g} \eta T_2 \right) \left(1 - e^{-\left(\frac{2b + \psi \rho g(a + E_{Z\alpha})}{\eta}\right)/T_2} \right) - \frac{K_2}{2\eta \rho g} (\psi \rho g(a + E_{Z\alpha}))^2 \\ - 2 \frac{K_2}{\rho g} b T_2 - K_2 \psi(a + E_{Z\alpha}) T_2 - \frac{K_2}{\eta} b \psi(a + E_{Z\alpha}) \quad (47)$$

Equation (47) can be rewritten as:

$$E_{\gamma_1} = E_{Z\beta} + \Delta E_{Z\beta\gamma} \quad (48)$$

with

$$\Delta E_{Z\beta\gamma} = -T_2 \left(V_{Z\beta} - \frac{K_2}{\rho g} b - K_2 \psi(a + E_{Z\alpha}) - \frac{K_2}{\rho g} \eta T_2 \right) \left(1 - e^{-\left(\frac{2b + \psi \rho g(a + E_{Z\alpha})}{\eta}\right)/T_2} \right) \\ - \frac{K_2}{2\eta \rho g} (\psi \rho g(a + E_{Z\alpha}))^2 - 2 \frac{K_2}{\rho g} b T_2 - K_2 \psi(a + E_{Z\alpha}) T_2 - \frac{K_2}{\eta} b \psi(a + E_{Z\alpha}) \quad (49)$$

Between β and γ_1 , the system is accelerated when $F_{net} < 0$ and is decelerated when $F_{net} > 0$. The velocity when $F_{net} < 0$ is positive and may or may not become negative when $0 < F_{net} < b$. The point $\gamma_1(\bar{t}_{1,k+1})$ can therefore be defined as:

$$\gamma_1(\bar{t}_{1,k+1}) : \begin{cases} E_Z(\bar{t}_{1,k+1}) = E_{Z\gamma_1}; \dot{E}_Z(\bar{t}_{1,k+1}) = \dot{E}_{Z\gamma_1} \\ Z(\bar{t}_{1,k+1}) = Z_{\gamma_1}; V_Z(\bar{t}_{1,k+1}) = V_{Z\gamma_1} \\ F_{net}(\bar{t}_{1,k+1}) = F_{net\gamma_1} = b \end{cases} \quad (50)$$

If at γ_1 the velocity $V_{Z\gamma_1}$ is positive, the system will undergo a particular behavior between points γ_1 and γ_2 . When the system enters infinitesimally at $i = 1$, the control action is $U = 0$, which makes $F_{vbm} = \text{constant}$ according to (20). Since $\dot{E}_Z$ is negative, Z will increase and so does the force magnitude factor due to $-\psi\rho gZ$. This leads to a decrease in the net force, and the system is pushed towards $i = 2$, causing the control action to turn on. This will increase the net force, causing the system to be moved again towards $i = 1$. So, in this situation, the system undergoes a switching phase with n_γ switchings between $i = 1$ and $i = 2$. This behavior will occur until the vehicle is fully decelerated, making the velocity $\dot{Z}$ to change sign from positive to negative. Also, since $\dot{E}_Z = -V_Z$, at this point, the sign of $\dot{E}_Z$ changes from negative to positive. When this happens, point $\gamma_2(\bar{t}_{1,k+1+n_\gamma})$ is reached. Since between γ_1 and γ_2 the average value of F_{net} is constant, the relation between $Z(\bar{t}_{1,k+1+n_\gamma}) = Z_{\gamma_2}$ and Z_{γ_1} can be determined, from the block diagram of Figure 1, by:

$$\dot{V}_Z + \frac{1}{T_2} V_Z = -\frac{K_2}{T_2 \rho g} b, V_{Z\gamma_1} > 0 \wedge t_{2,k} < t < t_{2,k+n_\gamma} \quad (51)$$

The solution to the differential Equation (51) is:

$$V_Z(t) = \left(V_{Z\gamma_1} + \frac{K_2}{\rho g} b \right) \cdot e^{-(t-t_{2,k})/T_2} - \frac{K_2}{\rho g} b, V_{Z\gamma_1} > 0 \wedge t_{2,k} < t < t_{2,k+n_\gamma} \quad (52)$$

where $V_{Z\gamma_1}$ is the depth velocity at γ_1 . To obtain the expression for depth when time varies between $t_{2,k}$ and $t_{2,k+n}$, expression (52) is integrated, leading to:

$$Z(t) = Z_{\gamma_1} + \left(V_{Z\gamma_1} + \frac{K_2}{\rho g} b \right) T_2 \cdot \left(1 - e^{-(t-t_{2,k})/T_2} \right) - \frac{K_2}{\rho g} b(t - t_{2,k}), V_{Z\gamma_1} > 0 \wedge t_{2,k} < t < t_{2,k+n_\gamma} \quad (53)$$

In (53) Z_{γ_1} is the depth at point γ_1 . The point γ_2 occurs when there is change in the sign of the velocity in (52):

$$0 = \left(V_{Z\gamma_1} + \frac{K_2}{\rho g} b \right) \cdot e^{-(t_{\gamma_2}-t_{2,k})/T_2} - \frac{K_2}{\rho g} b \quad (54)$$

The time instant t_{γ_2} is therefore given by:

$$t_{\gamma_2} = t_{2,k} - T_2 \ln \left(\frac{1}{\frac{V_{Z\gamma_1} \rho g}{b K_2} + 1} \right) \quad (55)$$

Replacing (55) in (53) leads to:

$$Z_{\gamma_2} = Z_{\gamma_1} + \left(V_{Z\gamma_1} + \frac{K_2}{\rho g} b \right) T_2 \cdot \left(1 - \frac{1}{\frac{V_{Z\gamma_1} \rho g}{b K_2} + 1} \right) + \frac{K_2}{\rho g} b T_2 \ln \left(\frac{1}{\frac{V_{Z\gamma_1} \rho g}{b K_2} + 1} \right) \quad (56)$$

Considering Equation (7) leads to:

$$E_{Z\gamma_2} = E_{Z\gamma_1} - \left(V_{Z\gamma_1} + \frac{K_2}{\rho g} b \right) T_2 \cdot \left(1 - \frac{1}{\frac{V_{Z\gamma_1} \rho g}{b K_2} + 1} \right) - \frac{K_2}{\rho g} b T_2 \ln \left(\frac{1}{\frac{V_{Z\gamma_1} \rho g}{b K_2} + 1} \right) \quad (57)$$

If at γ_1 the velocity $V_{Z\gamma_1}$ is zero, then according to (57), $E_{Z\gamma_2} = E_{Z\gamma_1}$, which means points γ_1 and γ_2 are coincident. If $V_{Z\gamma_1}$ is negative, then (57) is no longer valid since no switching occurs. The depth error in expression (57) can also be expressed as:

$$E_{Z\gamma_2} = E_{Z\gamma_1} + \Delta E_{Z\gamma} \quad (58)$$

where $\Delta E_{Z\gamma}$ is the change in depth error between γ_1 and γ_2 , given by:

$$\Delta E_{Z\gamma} = -\left(V_{Z\gamma_1} + \frac{K_2}{\rho g}b\right)T_2 \cdot \left(1 - \frac{1}{\frac{V_{Z\gamma_1}\rho g}{bK_2} + 1}\right) - \frac{K_2}{\rho g}bT_2 \ln\left(\frac{1}{\frac{V_{Z\gamma_1}\rho g}{bK_2} + 1}\right) \quad (59)$$

Expression (58) is valid when $V_{Z\gamma_1} \geq 0$ and switching occurs. On the other hand, since points γ_1 and γ_2 are coincident when $V_{Z\gamma_1} < 0$, Equation (58) can take a more general form:

$$E_{Z\gamma_2} = E_{Z\gamma_1} + \lambda_\gamma \Delta E_{Z\gamma} \quad (60)$$

where λ_γ is a parameter that expresses the occurrence of switching at state γ_1 :

$$\begin{cases} \lambda_\gamma = 1, & V_{Z\gamma_1} \geq 0 \\ \lambda_\gamma = 0, & V_{Z\gamma_1} < 0 \end{cases} \quad (61)$$

Point γ_2 can therefore be defined as:

$$\gamma_2(\bar{t}_{1,k+1+\lambda_\gamma n_\gamma}) : \begin{cases} E_Z(\bar{t}_{1,k+1+\lambda_\gamma n_\gamma}) = E_{Z\gamma_2} = E_{Z\gamma_1} + \lambda_\gamma \Delta E_{Z\gamma}; \dot{E}_{Z\gamma_2} = (1 - \lambda_\gamma)\dot{E}_{Z\gamma_1} \geq 0 \\ Z(\bar{t}_{1,k+1+\lambda_\gamma n_\gamma}) = Z_{\gamma_2}; V_{Z\gamma_2} = (1 - \lambda_\gamma)V_{Z\gamma_1} \leq 0 \\ F_{net}(\bar{t}_{1,k+1+\lambda_\gamma n_\gamma}) = F_{net\gamma_2} = F_{net\gamma_1} = b \end{cases} \quad (62)$$

From point γ_2 until point δ , $\dot{E}_Z$ is positive, so V_Z is negative (Equation (8)). Thus, the force due to the vehicle deformation decreases, since Z becomes smaller, and since F_{VBM} is constant, the value of F_{net} increases (Equation (20)) until the system reaches point δ .

The change in F_{net} between points γ_2 , $F_{net\gamma_2}$, and δ , $F_{net\delta}$, can be determined integrating Equation (21) with $i = 1$:

$$F_{net\delta} - F_{net\gamma_2} = \int_{\bar{t}_{1,k+1+\lambda_\gamma n_\gamma}}^{\bar{t}_{1,k+1+\lambda_\gamma n_\gamma}} -\psi\rho g V_Z dt \rightarrow F_{net\delta} = b + \psi\rho g \int_{\bar{t}_{1,k+1+\lambda_\gamma n_\gamma}}^{\bar{t}_{1,k+1+\lambda_\gamma n_\gamma}} \frac{dE_Z}{dt} dt \quad (63)$$

Since $E_{Z\delta} = a$, Equation (63) can be rewritten as:

$$F_{net\delta} = b + \psi\rho g(a - E_{Z\gamma_2}) \quad (64)$$

The point $\delta(\bar{t}_{1,k+1+\lambda_\gamma n_\gamma})$ can therefore be defined as:

$$\delta(\bar{t}_{1,k+1+\lambda_\gamma n_\gamma}) : \begin{cases} E_Z(\bar{t}_{1,k+1+\lambda_\gamma n_\gamma}) = E_{Z\delta} = a; \dot{E}_Z(\bar{t}_{1,k+1+\lambda_\gamma n_\gamma}) = \dot{E}_{Z\delta} > 0 \\ Z(\bar{t}_{1,k+1+\lambda_\gamma n_\gamma}) = Z_\delta; V_Z(\bar{t}_{1,k+1+\lambda_\gamma n_\gamma}) = V_{Z\delta} < 0 \\ F_{net}(\bar{t}_{1,k+1+\lambda_\gamma n_\gamma}) = F_{net\delta} = b + \psi\rho g(a - E_{Z\gamma_2}) \end{cases} \quad (65)$$

The change in F_{net} between points $\delta(F_{net\delta})$ and $\varepsilon_1(F_{net\varepsilon_1})$, can be determined integrating Equation (21) with $i = 2$:

$$F_{net\varepsilon_1} - F_{net\delta} = \int_{\bar{t}_{2,k+1+\lambda_\gamma n_\gamma}}^{\bar{t}_{2,k+1+\lambda_\gamma n_\gamma}} (\dot{F}_{VBM} - \psi\rho g V_Z) dt = \int_{\bar{t}_{2,k+1+\lambda_\gamma n_\gamma}}^{\bar{t}_{2,k+1+\lambda_\gamma n_\gamma}} (-\eta) dt \quad (66)$$

Developing (66) leads to:

$$F_{net\varepsilon_1} = F_{net\delta} - \eta(\bar{t}_{2,k+1+\lambda_\gamma n_\gamma} - \bar{t}_{2,k+1+\lambda_\gamma n_\gamma}) \quad (67)$$

Since $F_{net\varepsilon_1} = -b$ and given (64):

$$\left(\underline{t}_{2,k+1+\lambda_\gamma n_\gamma} - \bar{t}_{2,k+1+\lambda_\gamma n_\gamma}\right) = \frac{2b + \psi \rho g(a - E_{Z\gamma 2})}{\eta} \quad (68)$$

Equation (68) expresses the necessary time for the system to move from δ to ε_1 . From the block diagram of Figure 1:

$$\dot{V}_Z + \frac{1}{T_2} V_Z = -\frac{K_2}{T_2 \rho g} F_{net}, \quad \bar{t}_{2,k+1+\lambda_\gamma n_\gamma} \leq t \leq \underline{t}_{2,k+1+\lambda_\gamma n_\gamma} \quad (69)$$

where $V_Z(\bar{t}_{2,k+1+\lambda_\gamma n_\gamma}) = V_{Z\delta}$, and $F_{net}(\bar{t}_{2,k+1+\lambda_\gamma n_\gamma}) = F_{net\delta}$. Since in this region $\dot{F}_{net} = -\eta$

$$\dot{F}_{net} = -\eta \implies F_{net}(t) = F_{net\delta} - \eta(t - \bar{t}_{2,k+1+\lambda_\gamma n_\gamma}), \quad \bar{t}_{2,k+1+\lambda_\gamma n_\gamma} \leq t \leq \underline{t}_{2,k+1+\lambda_\gamma n_\gamma} \quad (70)$$

Integrating differential Equation (69), the expression for the depth velocity between δ and ε_1 is thus:

$$V_Z(t) = \left[V_{Z\delta} + \frac{K_2}{\rho g} (F_{net\delta} + \eta T_2) \right] \cdot e^{-(t - \bar{t}_{2,k+1+\lambda_\gamma n_\gamma})/T_2} - \frac{K_2}{\rho g} \left(F_{net\delta} - \eta \left[(t - \bar{t}_{2,k+1+\lambda_\gamma n_\gamma}) - T_2 \right] \right), \quad \bar{t}_{2,k+1+\lambda_\gamma n_\gamma} \leq t \leq \underline{t}_{2,k+1+\lambda_\gamma n_\gamma} \quad (71)$$

To find the depth expression, (71) must be integrated.

$$Z(t) = \int \left(\left[V_{Z\delta} + \frac{K_2}{\rho g} (F_{net\delta} + \eta T_2) \right] \cdot e^{-(t - \bar{t}_{2,k+1+\lambda_\gamma n_\gamma})/T_2} - \frac{K_2}{\rho g} \left(F_{net\delta} - \eta \left[(t - \bar{t}_{2,k+1+\lambda_\gamma n_\gamma}) - T_2 \right] \right) \right) dt, \quad \bar{t}_{2,k+1+\lambda_\gamma n_\gamma} \leq t \leq \underline{t}_{2,k+1+\lambda_\gamma n_\gamma} \quad (72)$$

The solution of Equation (72) is given by:

$$Z(t) = -T_2 \left[V_{Z\delta} + \frac{K_2}{\rho g} (F_{net\delta} + \eta T_2) \right] e^{-(t - \bar{t}_{2,k+1+\lambda_\gamma n_\gamma})/T_2} - \frac{K_2}{\rho g} \left(F_{net\delta} t - \frac{\eta}{2} (t - \bar{t}_{2,k+1+\lambda_\gamma n_\gamma})^2 + \eta T_2 t \right) + Z_\delta + T_2 \left[V_{Z\delta} + \frac{K_2}{\rho g} (F_{net\delta} + \eta T_2) \right] + \frac{K_2}{\rho g} \left(F_{net\delta} \bar{t}_{2,k+1+\lambda_\gamma n_\gamma} + \eta T_2 \bar{t}_{2,k+1+\lambda_\gamma n_\gamma} \right), \quad \bar{t}_{2,k+1+\lambda_\gamma n_\gamma} \leq t \leq \underline{t}_{2,k+1+\lambda_\gamma n_\gamma} \quad (73)$$

Since at $t = \underline{t}_{2,k+1+\lambda_\gamma n_\gamma}$, $Z = Z_{\varepsilon 1}$

$$Z_{\varepsilon 1} = -T_2 \left[V_{Z\delta} + \frac{K_2}{\rho g} (F_{net\delta} + \eta T_2) \right] e^{-(\underline{t}_{2,k+1+\lambda_\gamma n_\gamma} - \bar{t}_{2,k+1+\lambda_\gamma n_\gamma})/T_2} - \frac{K_2}{\rho g} \left(F_{net\delta} \underline{t}_{2,k+1+\lambda_\gamma n_\gamma} - \frac{\eta}{2} (\underline{t}_{2,k+1+\lambda_\gamma n_\gamma} - \bar{t}_{2,k+1+\lambda_\gamma n_\gamma})^2 + \eta T_2 \underline{t}_{2,k+1+\lambda_\gamma n_\gamma} \right) + Z_\delta + T_2 \left[V_{Z\delta} + \frac{K_2}{\rho g} (F_{net\delta} + \eta T_2) \right] + \frac{K_2}{\rho g} \left(F_{net\delta} \bar{t}_{2,k+1+\lambda_\gamma n_\gamma} + \eta T_2 \bar{t}_{2,k+1+\lambda_\gamma n_\gamma} \right) \quad (74)$$

Rearranging Equation (74),

$$Z_{\varepsilon 1} = Z_\delta + T_2 \left[V_{Z\delta} + \frac{K_2}{\rho g} (F_{net\delta} + \eta T_2) \right] \left(1 - e^{-(\underline{t}_{2,k+1+\lambda_\gamma n_\gamma} - \bar{t}_{2,k+1+\lambda_\gamma n_\gamma})/T_2} \right) + \frac{K_2 \eta}{\rho g 2} (\underline{t}_{2,k+1+\lambda_\gamma n_\gamma} - \bar{t}_{2,k+1+\lambda_\gamma n_\gamma})^2 - \frac{K_2}{\rho g} (\underline{t}_{2,k+1+\lambda_\gamma n_\gamma} - \bar{t}_{2,k+1+\lambda_\gamma n_\gamma}) (\eta T_2 + F_{net\delta}) \quad (75)$$

Equation (75) defines the depth at ε_1 as a function of the depth at δ and the vehicle depth dynamics. To find the depth error, Equation (7) should be considered:

$$E_{Z\epsilon_1} = Z_{ref} - \left(Z_\delta + T_2 \left[V_{Z\delta} + \frac{K_2}{\rho g} (F_{net\delta} + \eta T_2) \right] \left(1 - e^{-\frac{(t_{2,k+1+\lambda_\gamma n_\gamma} - \bar{t}_{2,k+1+\lambda_\gamma n_\gamma})}{T_2}} \right) + \frac{K_2}{\rho g} \frac{\eta}{2} \left(t_{2,k+1+\lambda_\gamma n_\gamma} - \bar{t}_{2,k+1+\lambda_\gamma n_\gamma} \right)^2 - \frac{K_2}{\rho g} \left(t_{2,k+1+\lambda_\gamma n_\gamma} - \bar{t}_{2,k+1+\lambda_\gamma n_\gamma} \right) (\eta T_2 + F_{net\delta}) \right) \quad (76)$$

Rearranging Equation (76),

$$E_{Z\epsilon_1} = E_{Z\delta} - T_2 \left[V_{Z\delta} + \frac{K_2}{\rho g} (F_{net\delta} + \eta T_2) \right] \left(1 - e^{-\frac{(t_{2,k+1+\lambda_\gamma n_\gamma} - \bar{t}_{2,k+1+\lambda_\gamma n_\gamma})}{T_2}} \right) - \frac{K_2}{\rho g} \frac{\eta}{2} \left(t_{2,k+1+\lambda_\gamma n_\gamma} - \bar{t}_{2,k+1+\lambda_\gamma n_\gamma} \right)^2 + \frac{K_2}{\rho g} \left(t_{2,k+1+\lambda_\gamma n_\gamma} - \bar{t}_{2,k+1+\lambda_\gamma n_\gamma} \right) (\eta T_2 + F_{net\delta}) \quad (77)$$

Considering $F_{net\delta}$ from (64) and $\left(t_{2,k+1+\lambda_\gamma n_\gamma} - \bar{t}_{2,k+1+\lambda_\gamma n_\gamma} \right)$ from (68)

$$E_{Z\epsilon_1} = E_{Z\delta} - T_2 \left[V_{Z\delta} + \frac{K_2}{\rho g} \left((b + \psi \rho g (a - E_{Z\gamma 2})) + \eta T_2 \right) \right] \left(1 - e^{-\frac{2b + \psi \rho g (a - E_{Z\gamma 2})}{\eta} / T_2} \right) - \frac{K_2}{\rho g} \frac{\eta}{2} \left(\frac{2b + \psi \rho g (a - E_{Z\gamma 2})}{\eta} \right)^2 + \frac{K_2}{\rho g} \frac{2b + \psi \rho g (a - E_{Z\gamma 2})}{\eta} (\eta T_2 + (b + \psi \rho g (a - E_{Z\gamma 2}))) \quad (78)$$

The depth error at ϵ_1 can therefore be given by:

$$E_{Z\epsilon_1} = E_{Z\delta} - T_2 \left[V_{Z\delta} + \frac{K_2}{\rho g} b + K_2 \psi (a - E_{Z\gamma 2}) + \frac{K_2}{\rho g} \eta T_2 \right] \left(1 - e^{-\frac{2b + \psi \rho g (a - E_{Z\gamma 2})}{\eta} / T_2} \right) + \frac{K_2}{\rho g} \frac{(\psi \rho g (a - E_{Z\gamma 2}))^2}{2\eta} + 2 \frac{K_2}{\rho g} b T_2 + K_2 \psi (a - E_{Z\gamma 2}) T_2 + \frac{K_2}{\eta} b \psi (a - E_{Z\gamma 2}) \quad (79)$$

Equation (80) can be rewritten as:

$$E_{Z\epsilon_1} = E_{Z\delta} + \Delta E_{Z\delta\epsilon} \quad (80)$$

With

$$\Delta E_{Z\delta\epsilon} = -T_2 \left[V_{Z\delta} + \frac{K_2}{\rho g} b + K_2 \psi (a - E_{Z\gamma 2}) + \frac{K_2}{\rho g} \eta T_2 \right] \left(1 - e^{-\frac{2b + \psi \rho g (a - E_{Z\gamma 2})}{\eta} / T_2} \right) + \frac{K_2}{\rho g} \frac{(\psi \rho g (a - E_{Z\gamma 2}))^2}{2\eta} + 2 \frac{K_2}{\rho g} b T_2 + K_2 \psi (a - E_{Z\gamma 2}) T_2 + \frac{K_2}{\eta} b \psi (a - E_{Z\gamma 2}) \quad (81)$$

Between δ and ϵ_1 , the system is accelerated when $F_{net} > 0$ and is decelerated when $F_{net} < 0$. The velocity V_Z in region 2a when $F_{net} > 0$ is negative and may or may not become positive when $0 > F_{net} > -b$. The point $\epsilon_1 \left(\bar{t}_{1,k+2+\lambda_\gamma n_\gamma} \right)$ can therefore be defined as:

$$\epsilon_1 \left(\bar{t}_{1,k+2+\lambda_\gamma n_\gamma} \right) : \begin{cases} E_Z \left(\bar{t}_{1,k+2+\lambda_\gamma n_\gamma} \right) = E_{Z\epsilon_1}; \dot{E}_Z \left(\bar{t}_{1,k+2+\lambda_\gamma n_\gamma} \right) = \dot{E}_{Z\epsilon_1} \\ Z \left(\bar{t}_{1,k+2+\lambda_\gamma n_\gamma} \right) = Z_{\gamma 1}; V_Z \left(\bar{t}_{1,k+2+\lambda_\gamma n_\gamma} \right) = V_{Z\epsilon_1} \\ F_{net} \left(\bar{t}_{1,k+2+\lambda_\gamma n_\gamma} \right) = F_{net\epsilon_1} = -b \end{cases} \quad (82)$$

If at ϵ_1 the velocity $V_{Z\epsilon_1}$ is negative, the system will undergo a particular behavior between points ϵ_1 and ϵ_2 . When the system enters infinitesimally at $i = 1$, the control action is $U = 0$, which makes $F_{vbm} = \text{constant}$ according to (20). Since $\dot{E}_Z$ is positive, Z will decrease and so does the force magnitude factor due to $-\psi \rho g Z$. This leads to an increase in the net force, and the system is pushed towards $i = 2$, causing the control action to turn on. This will increase the net force, causing the system to be moved again towards $i = 1$. So, in this situation, the system undergoes a switching phase with n_ϵ switchings between $i = 1$ and $i = 2$. This behavior will occur until the vehicle is fully decelerated,

making the velocity $\dot{Z}$ to change sign from negative to positive. Also, since $\dot{E}_Z = -V_Z$, at this point, the sign of $\dot{E}_Z$ changes from positive to negative. When this happens, point $\varepsilon_2(\bar{t}_{1,k+2+\lambda_\gamma n_\gamma+n_\varepsilon})$ is reached. Since between ε_1 and ε_2 the average value of F_{net} is constant and equal to $-b$, the relation between $Z(\bar{t}_{2,k+1+\lambda_\gamma n_\gamma}) = Z_{\varepsilon_2}$ and Z_{ε_1} can be determined, from the block diagram of Figure 1, as:

$$\dot{V}_Z + \frac{1}{T_2} V_Z = \frac{1}{T_2} \frac{K_2}{\rho g} b, V_{Z\varepsilon_1} < 0 \wedge \bar{t}_{2,k+1+\lambda_\gamma n_\gamma} < t < \bar{t}_{2,k+1+\lambda_\gamma n_\gamma+n_\varepsilon} \quad (83)$$

The solution to the differential Equation (83) is:

$$V_Z = \left(V_{Z\varepsilon_1} - \frac{K_2}{\rho g} b \right) \cdot e^{-(t-\bar{t}_{2,k+1+\lambda_\gamma n_\gamma})/T_2} + \frac{K_2}{\rho g} b, V_{Z\varepsilon_1} < 0 \wedge \bar{t}_{2,k+1+\lambda_\gamma n_\gamma} < t < \bar{t}_{2,k+1+\lambda_\gamma n_\gamma+n_\varepsilon} \quad (84)$$

Integrating (84):

$$Z = Z_{\varepsilon_1} + \left(V_{Z\varepsilon_1} - \frac{K_2}{\rho g} b \right) T_2 \left(1 - e^{-(t-\bar{t}_{2,k+1+\lambda_\gamma n_\gamma})/T_2} \right) + \frac{K_2}{\rho g} b \left(t - \bar{t}_{2,k+1+\lambda_\gamma n_\gamma} \right), V_{Z\varepsilon_1} < 0 \wedge \bar{t}_{2,k+1+\lambda_\gamma n_\gamma} < t < \bar{t}_{2,k+1+\lambda_\gamma n_\gamma+n_\varepsilon} \quad (85)$$

In (85) Z_{ε_1} is the depth at point ε_1 . The point ε_2 occurs when the velocity in (84) changes its sign:

$$0 = \left(V_{Z\varepsilon_1} - \frac{K_2}{\rho g} b \right) \cdot e^{-(t_{\varepsilon_2}-\bar{t}_{2,k+1+\lambda_\gamma n_\gamma})/T_2} + \frac{K_2}{\rho g} b \quad (86)$$

Rewriting (86):

$$t_{\varepsilon_2} = \bar{t}_{2,k+1+\lambda_\gamma n_\gamma} - T_2 \ln \left(\frac{1}{-\frac{V_{Z\varepsilon_1} \rho g}{K_2 b} + 1} \right) \quad (87)$$

Plugging (87) into (85):

$$Z_{\varepsilon_2} = Z_{\varepsilon_1} + \left(V_{Z\varepsilon_1} - \frac{K_2}{\rho g} b \right) T_2 \left(1 - e^{-(\bar{t}_{2,k+1+\lambda_\gamma n_\gamma} - T_2 \ln \left(\frac{1}{-\frac{V_{Z\varepsilon_1} \rho g}{K_2 b} + 1} \right) - \bar{t}_{2,k+1+\lambda_\gamma n_\gamma})/T_2} \right) + \frac{K_2}{\rho g} b \left(\bar{t}_{2,k+1+\lambda_\gamma n_\gamma} - T_2 \ln \left(\frac{1}{-\frac{V_{Z\varepsilon_1} \rho g}{K_2 b} + 1} \right) - \bar{t}_{2,k+1+\lambda_\gamma n_\gamma} \right) \quad (88)$$

Rearranging Equation (88) leads to:

$$Z_{\varepsilon_2} = Z_{\varepsilon_1} + \left(V_{Z\varepsilon_1} - \frac{K_2}{\rho g} b \right) T_2 \left(1 - \frac{1}{-\frac{V_{Z\varepsilon_1} \rho g}{K_2 b} + 1} \right) - \frac{K_2}{\rho g} b T_2 \ln \left(\frac{1}{-\frac{V_{Z\varepsilon_1} \rho g}{K_2 b} + 1} \right) \quad (89)$$

Considering Equation (7), the depth error at point ε_2 is given by:

$$E_{Z\varepsilon_2} = E_{Z\varepsilon_1} - \left(V_{Z\varepsilon_1} - \frac{K_2}{\rho g} b \right) T_2 \left(1 - \frac{1}{-\frac{V_{Z\varepsilon_1} \rho g}{K_2 b} + 1} \right) + \frac{K_2}{\rho g} b T_2 \ln \left(\frac{1}{-\frac{V_{Z\varepsilon_1} \rho g}{K_2 b} + 1} \right) \quad (90)$$

If at ε_1 the velocity $V_{Z\varepsilon_1}$ is zero, then according to (90), $E_{Z\varepsilon_2} = E_{Z\varepsilon_1}$, which means points ε_1 and ε_2 are coincident. If $V_{Z\varepsilon_1}$ is positive, then (90) is no longer valid since no switching occurs. The depth error in (90) can also be expressed as:

$$E_{Z\varepsilon_2} = E_{Z\varepsilon_1} + \Delta E_{Z\varepsilon} \quad (91)$$

where $\Delta E_{Z\varepsilon}$ is the change in depth error between ε_1 and ε_2 , is given by:

$$\Delta E_{Z\varepsilon} = -\left(V_{Z\varepsilon_1} - \frac{K_2}{\rho g}b\right)T_2\left(1 - \frac{1}{-\frac{V_{Z\varepsilon_1}\rho g}{K_2b} + 1}\right) + \frac{K_2}{\rho g}bT_2\ln\left(\frac{1}{-\frac{V_{Z\varepsilon_1}\rho g}{K_2b} + 1}\right) \quad (92)$$

Expression (91) is valid when $V_{Z\varepsilon_1} \leq 0$ and switching occurs. On the other hand, since points ε_1 and ε_2 are coincident when $V_{Z\varepsilon_1} > 0$, Equation (91) can take a more general form:

$$E_{Z\varepsilon_2} = E_{Z\varepsilon_1} + \lambda_\varepsilon \Delta E_{Z\varepsilon} \quad (93)$$

where λ_ε is a parameter that expresses the occurrence of switching at state ε_1 :

$$\begin{cases} \lambda_\varepsilon = 1, & V_{Z\varepsilon_1} \leq 0 \\ \lambda_\varepsilon = 0, & V_{Z\varepsilon_1} > 0 \end{cases} \quad (94)$$

Point ε_2 can therefore be defined as:

$$\varepsilon_2\left(\bar{t}_{1,k+2+\lambda_\gamma n_\gamma + \lambda_\varepsilon n_\varepsilon}\right) : \begin{cases} E_Z\left(\bar{t}_{1,k+2+\lambda_\gamma n_\gamma + \lambda_\varepsilon n_\varepsilon}\right) = E_{Z\varepsilon_2} = E_{Z\varepsilon_1} + \lambda_\varepsilon \Delta E_{Z\varepsilon}; \dot{E}_{Z\varepsilon_2} = (1 - \lambda_\varepsilon)\dot{E}_{Z\varepsilon_1} \leq 0 \\ Z\left(\bar{t}_{1,k+2+\lambda_\gamma n_\gamma + \lambda_\varepsilon n_\varepsilon}\right) = Z_{\varepsilon_2}; V_{Z\varepsilon_2} = (1 - \lambda_\varepsilon)V_{Z\varepsilon_1} \geq 0 \\ F_{net}\left(\bar{t}_{1,k+2+\lambda_\gamma n_\gamma + \lambda_\varepsilon n_\varepsilon}\right) = F_{net\varepsilon_2} = F_{net\varepsilon_1} = -b \end{cases} \quad (95)$$

The change in F_{net} between points ε_2 , $F_{VBM\varepsilon_2}$, and ζ , $F_{VBM\zeta}$, can be determined by integrating Equation (21) with $i = 1$.

$$F_{net\zeta} - F_{net\varepsilon_2} = \int_{\bar{t}_{1,k+2+\lambda_\gamma n_\gamma + \lambda_\varepsilon n_\varepsilon}}^{\bar{t}_{1,k+2+\lambda_\gamma n_\gamma + \lambda_\varepsilon n_\varepsilon}} -\psi\rho g V_Z dt \quad (96)$$

Since $F_{net\varepsilon_2} = -b$

$$F_{net\zeta} = -b + \psi\rho g \int_{\bar{t}_{1,k+2+\lambda_\gamma n_\gamma + \lambda_\varepsilon n_\varepsilon}}^{\bar{t}_{1,k+2+\lambda_\gamma n_\gamma + \lambda_\varepsilon n_\varepsilon}} \frac{dE_Z}{dt} dt \quad (97)$$

Since $E_{Z\zeta} = -a$, and replacing this value in (97), leads to:

$$F_{net\zeta} = -b - \psi\rho g(a + E_{Z\varepsilon_2}) \quad (98)$$

Point ζ can therefore be defined as:

$$\zeta\left(\bar{t}_{1,k+2+\lambda_\gamma n_\gamma + \lambda_\varepsilon n_\varepsilon}\right) : \begin{cases} E_Z\left(\bar{t}_{1,k+2+\lambda_\gamma n_\gamma + \lambda_\varepsilon n_\varepsilon}\right) = E_{Z\zeta}; \dot{E}_Z\left(\bar{t}_{1,k+2+\lambda_\gamma n_\gamma + \lambda_\varepsilon n_\varepsilon}\right) = \dot{E}_{Z\zeta} < 0 \\ Z\left(\bar{t}_{1,k+2+\lambda_\gamma n_\gamma + \lambda_\varepsilon n_\varepsilon}\right) = Z_\zeta; V_Z\left(\bar{t}_{1,k+2+\lambda_\gamma n_\gamma + \lambda_\varepsilon n_\varepsilon}\right) = V_{Z\zeta} = -\dot{E}_{Z\zeta} > 0 \\ F_{net}\left(\bar{t}_{1,k+2+\lambda_\gamma n_\gamma + \lambda_\varepsilon n_\varepsilon}\right) = F_{net\zeta} = -b - \psi\rho g(a + E_{Z\varepsilon_2}) \end{cases} \quad (99)$$

With all the points in the diagram defined, conditions 3 of Section 4.1 can finally be demonstrated to exist. Condition 3a requires that $F_{net\alpha} = F_{net\varepsilon_2}$ and $E_{Z\varepsilon_2} \leq E_{Z\alpha}$. From Figure 6, $F_{net\alpha} = F_{net\varepsilon_2}$ by definition. The demonstration that $E_{Z\varepsilon_2} \leq E_{Z\alpha}$ will be divided into two parts:

- (i) To demonstrate that $|E_{Z\gamma_2}| \leq E_{Z\alpha}$;
- (ii) To demonstrate that $E_{Z\varepsilon_2} \leq |E_{Z\gamma_2}|$;

By demonstrating that $E_{Z\varepsilon_2} \leq |E_{Z\gamma_2}|$ and $|E_{Z\gamma_2}| \leq E_{Z\alpha}$, it logically follows that $E_{Z\varepsilon_2} \leq E_{Z\alpha}$. The conditions for each part will be detailed in the following sections.

4.4.2. Detailing of Condition 3a: Part i

The condition for part i is:

$$|E_{Z\gamma_2}| \leq |E_{Z\alpha}| \quad (100)$$

Which can also be expressed as:

$$-E_{Z\gamma_2} \leq E_{Z\alpha} \quad (101)$$

From (48)

$$-E_{Z\beta} - \Delta E_{Z\beta\gamma} - \lambda_\gamma \Delta E_{Z\gamma} \leq E_{Z\alpha} \quad (102)$$

Since $E_{Z\beta} = -a$

$$E_{Z\alpha} \geq a - \Delta E_{Z\beta\gamma} - \lambda_\gamma \Delta E_{Z\gamma} \quad (103)$$

Considering $\lambda_\gamma = 1$, (49) and (59)

$$\begin{aligned} E_{Z\alpha} \geq a + T_2 \left(V_{Z\beta} - \frac{K_2}{\rho_g} b - K_2 \psi(a + E_{Z\alpha}) - \frac{K_2}{\rho_g} \eta T_2 \right) & \left(1 - e^{-\left(\frac{2b + \psi \rho_g (a + E_{Z\alpha})}{\eta} \right) / T_2} \right) \\ & + \frac{K_2}{2\eta \rho_g} (\psi \rho_g (a + E_{Z\alpha}))^2 + 2 \frac{K_2}{\rho_g} b T_2 + K_2 \psi(a + E_{Z\alpha}) T_2 + \frac{K_2}{\eta} b \psi(a + E_{Z\alpha}) \\ & + \left(V_{Z\gamma_1} + \frac{K_2}{\rho_g} b \right) T_2 \cdot \left(1 - \frac{1}{\frac{V_{Z\gamma_1} \rho_g}{b K_2} + 1} \right) + \frac{K_2}{\rho_g} b T_2 \ln \left(\frac{1}{\frac{V_{Z\gamma_1} \rho_g}{b K_2} + 1} \right) \end{aligned} \quad (104)$$

Expression (104) defines the minimum value of $E_{Z\alpha}$ that satisfies (100). However, it is not easy to extract direct information from this expression. Therefore (104) will be quantified in a later section.

4.4.3. Detailing of Condition 3a: Part ii

The condition for part ii is:

$$|E_{Z\epsilon_2}| \leq |E_{Z\gamma_2}| \quad (105)$$

$$E_{Z\epsilon_2} \leq -E_{Z\gamma_2} \quad (106)$$

From (93) and (80) and considering $E_{Z\delta} = a$

$$-E_{Z\gamma_2} \geq a + \Delta E_{Z\delta\epsilon} + \lambda_\epsilon \Delta E_{Z\epsilon} \quad (107)$$

Considering $\lambda_\epsilon = 1$, (81) and (92)

$$\begin{aligned} -E_{Z\gamma_2} \geq a - T_2 \left[V_{Z\delta} + \frac{K_2}{\rho_g} b + K_2 \psi(a - E_{Z\gamma_2}) + \frac{K_2}{\rho_g} \eta T_2 \right] & \left(1 - e^{-\frac{2b + \psi \rho_g (a - E_{Z\gamma_2})}{\eta} / T_2} \right) \\ & + \frac{K_2}{\rho_g} \frac{(\psi \rho_g (a - E_{Z\gamma_2}))^2}{2\eta} + 2 \frac{K_2}{\rho_g} b T_2 + K_2 \psi(a - E_{Z\gamma_2}) T_2 + \frac{K_2}{\eta} b \psi(a - E_{Z\gamma_2}) \\ & - \left(V_{Z\epsilon_1} - \frac{K_2}{\rho_g} b \right) T_2 \left(1 - \frac{1}{-\frac{V_{Z\epsilon_1} \rho_g}{K_2 b} + 1} \right) + \frac{K_2}{\rho_g} b T_2 \ln \left(\frac{1}{-\frac{V_{Z\epsilon_1} \rho_g}{K_2 b} + 1} \right) \end{aligned} \quad (108)$$

Rewriting (108) with absolute values:

$$\begin{aligned} |E_{Z\gamma_2}| \geq a + T_2 \left[|V_{Z\delta}| - \frac{K_2}{\rho_g} b - K_2 \psi(a + |E_{Z\gamma_2}|) - \frac{K_2}{\rho_g} \eta T_2 \right] & \left(1 - e^{-\frac{2b + \psi \rho_g (a + |E_{Z\gamma_2}|)}{\eta} / T_2} \right) \\ & + \frac{K_2}{2\eta \rho_g} (\psi \rho_g (a + |E_{Z\gamma_2}|))^2 + 2 \frac{K_2}{\rho_g} b T_2 + K_2 \psi(a + |E_{Z\gamma_2}|) T_2 + \frac{K_2}{\eta} b \psi(a + |E_{Z\gamma_2}|) \\ & + \left(|V_{Z\epsilon_1}| + \frac{K_2}{\rho_g} b \right) T_2 \left(1 - \frac{1}{\frac{|V_{Z\epsilon_1}| \rho_g}{K_2 b} + 1} \right) + \frac{K_2}{\rho_g} b T_2 \ln \left(\frac{1}{\frac{|V_{Z\epsilon_1}| \rho_g}{K_2 b} + 1} \right) \end{aligned} \quad (109)$$

Expression (109) defines the minimum value of $E_{Z\gamma_2}$ that satisfies (105). However, it is not easy to extract direct information from this expression. Therefore (109) will be quantified in a later section.

4.4.4. Detailing of Condition 3b

For condition 3b to be satisfied, $|E_{Z\beta}| = |E_{Z\zeta}|$ and $|F_{net\zeta}| \leq |F_{net\beta}|$. The first condition is ensured by the definition of points β and ζ . For the second condition to be met and since $F_{net\zeta} < 0$ and $F_{net\beta} < 0$:

$$-F_{net\zeta} \leq -F_{net\beta} \quad (110)$$

From (32) and (98):

$$-(-b - \psi\rho g(a + E_{Z\epsilon_2})) \leq -(-b - \psi\rho g(a + E_{Z\alpha})) \quad (111)$$

which leads to

$$E_{Z\epsilon_2} \leq E_{Z\alpha} \quad (112)$$

The result from expression (112) means that as long (104) and (109) are satisfied, then $|F_{net\zeta}| \leq |F_{net\beta}|$ and F_{net} is therefore bounded, meeting the requirements of condition 3b.

4.5. Quantifying the Stability Conditions

Since (104) defines a lower bound for $E_{Z\alpha}$, it is interesting, from a conservative approach, to find the maximum value for its right side. To estimate this maximum value, a few assumptions will be made:

1. From α to β , $F_{net} = F_{net\beta} = \text{constant}$, for the purposes of estimating $V_{Z\beta}$. Since $|F_{net\beta}|$ is the maximum magnitude value F_{net} takes between α and β , this will contribute to an overestimation of $V_{Z\beta}$, leading to a conservative estimation of $E_{Z\alpha}$;
2. At γ_1 , it will be considered the velocity is $V_{Z\gamma_1} = V_{Z\beta}$. Between β and γ_1 , F_{net} increases from $F_{net\beta} < 0$ to $F_{net\gamma_1} = b$. From assumption 1, $V_{Z\beta}$ is the steady-state velocity at $F_{net\beta}$. As such, from β to γ_1 , the velocity will decrease. Therefore, considering $V_{Z\gamma_1} = V_{Z\beta}$ contributes to an overestimation of $V_{Z\gamma_1}$, leading to a conservative estimation of $E_{Z\alpha}$.

Considering the block diagram of Figure 1.

$$s \cdot Z(s) = \frac{-K_2}{\rho g} \frac{1}{T_2 s + 1} F_{net}(s) \quad (113)$$

This means that maximum depth velocity occurs at steady-state, in which:

$$V_Z = \frac{-K_2}{\rho g} F_{net} \quad (114)$$

From assumption 1,

$$V_{Z\beta} = V_{Z\beta_max} = \frac{-K_2}{\rho g} F_{net\beta} = V_{Z\gamma_1} \quad (115)$$

From (32):

$$V_{Z\beta_max} = \frac{-K_2}{\rho g} (-b - \psi\rho g(a + E_{Z\alpha})) = K_2\psi(a + E_{Z\alpha}) + \frac{K_2}{\rho g}b = K_2 \left[\psi(a + E_{Z\alpha}) + \frac{b}{\rho g} \right] \quad (116)$$

Expression (104) then becomes:

$$\begin{aligned}
E_{Z\alpha} \geq & a + T_2 \left(K_2 \psi(a + E_{Z\alpha}) + \frac{K_2}{\rho g} b - \frac{K_2}{\rho g} b - K_2 \psi(a + E_{Z\alpha}) - \frac{K_2}{\rho g} \eta T_2 \right) \left(1 - e^{-\left(\frac{2b + \psi \rho g (a + E_{Z\alpha})}{\eta} \right) / T_2} \right) \\
& + \frac{K_2}{2\eta \rho g} (\psi \rho g (a + E_{Z\alpha}))^2 + 2 \frac{K_2}{\rho g} b T_2 + K_2 \psi(a + E_{Z\alpha}) T_2 + \frac{K_2}{\eta} b \psi(a + E_{Z\alpha}) \\
& + \left(K_2 \psi(a + E_{Z\alpha}) + \frac{K_2}{\rho g} b + \frac{K_2}{\rho g} b \right) T_2 \cdot \left(1 - \frac{1}{\frac{K_2 [\psi(a + E_{Z\alpha}) + \frac{b}{\rho g}] \rho g}{b K_2} + 1} \right) \\
& + \frac{K_2}{\rho g} b T_2 \ln \left(\frac{1}{\frac{K_2 [\psi(a + E_{Z\alpha}) + \frac{b}{\rho g}] \rho g}{b K_2} + 1} \right)
\end{aligned} \tag{117}$$

Rearranging (117):

$$\begin{aligned}
E_{Z\alpha} \geq & a - \frac{K_2}{\rho g} \eta T_2^2 \left(1 - e^{-\left(\frac{2b + \psi \rho g (a + E_{Z\alpha})}{\eta} \right) / T_2} \right) + \frac{K_2 \psi^2 \rho g}{2\eta} (a + E_{Z\alpha})^2 + 3 \frac{K_2}{\rho g} b T_2 + 2 K_2 \psi(a + E_{Z\alpha}) T_2 \\
& + \frac{K_2}{\eta} b \psi(a + E_{Z\alpha}) + \frac{K_2}{\rho g} b T_2 \ln \left(\frac{1}{\frac{K_2 [\psi(a + E_{Z\alpha}) + \frac{b}{\rho g}] \rho g}{b K_2} + 1} \right)
\end{aligned} \tag{118}$$

Expression (118) constitutes an overestimation of (104) so that (100) holds.

Regarding (109), since it defines a lower bound for $|E_{Z\gamma 2}|$, it is interesting to find the maximum value for its right side. To estimate this maximum value, a few assumptions will be made:

1. From γ_2 to δ , $F_{net} = F_{net\delta} = \text{constant}$, for the purposes of estimating $V_{Z\delta}$. Since $[F_{net\delta}]$ is the maximum magnitude value F_{net} takes between γ_2 and δ , this will contribute to an overestimation of $|V_{Z\delta}|$, leading to a conservative estimation of $|E_{Z\gamma 2}|$;
2. At ε_1 , it will be considered the velocity is $V_{Z\varepsilon 1} = V_{Z\delta}$. Between δ and ε_1 , F_{net} decreases from $F_{net\delta} > 0$ to $F_{net\varepsilon 1} = -b$. From assumption 3, $V_{Z\delta}$ is the steady-state velocity at $F_{net\delta}$. As such, from δ to ε_1 , the velocity magnitude will decrease. Therefore, considering $V_{Z\varepsilon 1} = V_{Z\delta}$ contributes to an overestimation of $V_{Z\gamma 1}$, leading to a conservative estimation of $E_{Z\alpha}$.

Using the maximum depth velocity at steady-state determined in Equation (114) and considering assumption 3,

$$V_{Z\delta} = V_{Z\delta_max} = \frac{-K_2}{\rho g} F_{net\delta} = V_{Z\varepsilon 1} \tag{119}$$

From (64):

$$V_{Z\delta_max} = \frac{-K_2}{\rho g} (b + \psi \rho g (a - E_{Z\gamma 2})) = -K_2 \psi (a + |E_{Z\gamma 2}|) - \frac{K_2}{\rho g} b \tag{120}$$

In terms of absolute values

$$|V_{Z\delta}| = K_2 \psi (a + |E_{Z\gamma 2}|) + \frac{K_2}{\rho g} b \tag{121}$$

Expression (109) then becomes:

$$\begin{aligned}
|E_{Z\gamma 2}| \geq & a + T_2 \left[K_2 \psi(a + |E_{Z\gamma 2}|) + \frac{K_2}{\rho_g} b - \frac{K_2}{\rho_g} b - K_2 \psi(a + |E_{Z\gamma 2}|) - \frac{K_2}{\rho_g} \eta T_2 \right] \left(1 - e^{-\frac{2b + \psi \rho_g (a + |E_{Z\gamma 2}|)}{\eta} / T_2} \right) \\
& + \frac{K_2}{2\eta \rho_g} (\psi \rho_g (a + |E_{Z\gamma 2}|))^2 + 2 \frac{K_2}{\rho_g} b T_2 + K_2 \psi(a + |E_{Z\gamma 2}|) T_2 + \frac{K_2}{\eta} b \psi(a + |E_{Z\gamma 2}|) \\
& + \left(K_2 \psi(a + |E_{Z\gamma 2}|) + \frac{K_2}{\rho_g} b + \frac{K_2}{\rho_g} b \right) T_2 \left(1 - \frac{1}{\frac{(K_2 \psi(a + |E_{Z\gamma 2}|) + \frac{K_2}{\rho_g} b) \rho_g}{K_2 b} + 1} \right) \\
& + \frac{K_2}{\rho_g} b T_2 \ln \left(\frac{1}{\frac{(K_2 \psi(a + |E_{Z\gamma 2}|) + \frac{K_2}{\rho_g} b) \rho_g}{K_2 b} + 1} \right)
\end{aligned} \tag{122}$$

Rearranging (122)

$$\begin{aligned}
|E_{Z\gamma 2}| \geq & a - \frac{K_2}{\rho_g} \eta T_2^2 \left(1 - e^{-\frac{2b + \psi \rho_g (a + |E_{Z\gamma 2}|)}{\eta} / T_2} \right) + \frac{K_2 \psi^2 \rho_g}{2\eta} (a + |E_{Z\gamma 2}|)^2 + 3 \frac{K_2}{\rho_g} b T_2 \\
& + 2 K_2 \psi(a + |E_{Z\gamma 2}|) T_2 + \frac{K_2}{\eta} b \psi(a + |E_{Z\gamma 2}|) + \frac{K_2}{\rho_g} b T_2 \ln \left(\frac{1}{(a + |E_{Z\gamma 2}|) \frac{\psi \rho_g}{b} + 2} \right)
\end{aligned} \tag{123}$$

Since from (100), $|E_{Z\gamma 2}| \leq |E_{Z\alpha}|$, in a worst-case scenario, $|E_{Z\gamma 2}| = |E_{Z\alpha}| = E_{Z\alpha}$. Therefore, (123) becomes:

$$\begin{aligned}
E_{Z\alpha} \geq & a - \frac{K_2}{\rho_g} \eta T_2^2 \left(1 - e^{-\frac{2b + \psi \rho_g (a + E_{Z\alpha})}{\eta} / T_2} \right) + \frac{K_2 \psi^2 \rho_g}{2\eta} (a + E_{Z\alpha})^2 + 3 \frac{K_2}{\rho_g} b T_2 + 2 K_2 \psi(a + E_{Z\alpha}) T_2 \\
& + \frac{K_2}{\eta} b \psi(a + E_{Z\alpha}) + \frac{K_2}{\rho_g} b T_2 \ln \left(\frac{1}{(a + E_{Z\alpha}) \frac{\psi \rho_g}{b} + 2} \right)
\end{aligned} \tag{124}$$

Expression (124) constitutes an overestimation so that (105) holds. Notice that expression (124) is equal to (118), which ensures that (100) holds. This means that if (118) (or (124)) are satisfied, then (100) and (105) are also satisfied so that condition 3a is ensured. In other words,

$$|E_{Z\gamma 2}| \leq |E_{Z\alpha}| \implies |E_{Z\epsilon 2}| \leq |E_{Z\gamma 2}| \implies |E_{Z\epsilon 2}| \leq |E_{Z\alpha}| \tag{125}$$

Consequently, (118) and (124) can be written as:

$$E_{Z\alpha} \geq E_{Z\epsilon 2}^* \tag{126}$$

where $E_{Z\epsilon 2}^*$ is an overestimation of $E_{Z\epsilon 2}$ represented by the right side of (124). Equation (126) will therefore be the basis of the controller stability analysis performed in the next section.

5. Controller Stability: Case Study

5.1. Stability Regions

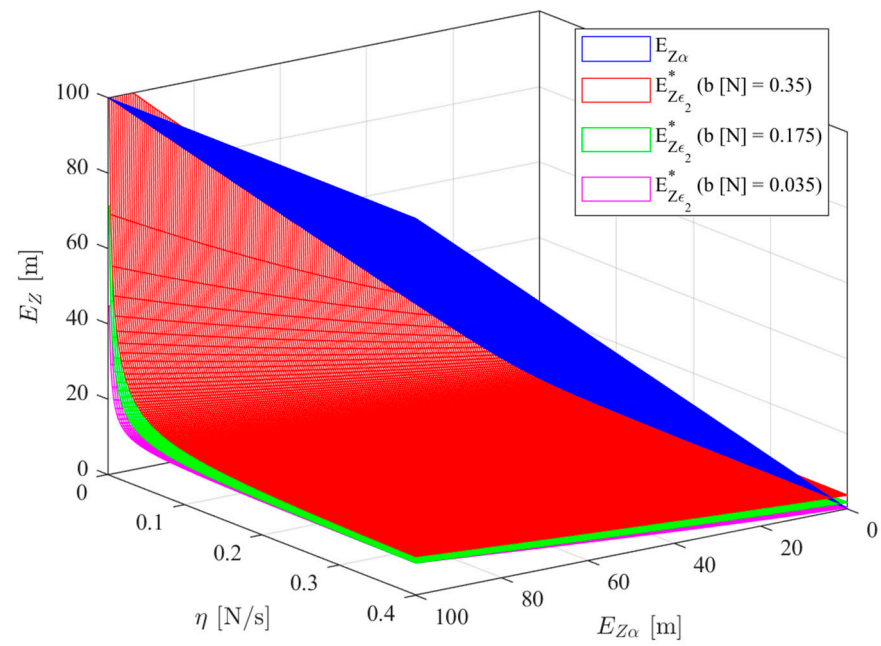
In order to present a case study of the application of the results obtained in this work to a prototype, in this section expression (124) is used to determine the stability regions according to the depth error at point α , $E_{Z\alpha}$. The system model parameters for these expressions are unique to the real system being modelled. For the purposes of this work, the prototype developed in [32] with the parameters fully determined in [23] was considered. The prototype, whose model can be described by the block diagram of Figure 1, uses a diaphragm sealed piston, driven by a DC motor, to change buoyancy (the reader is referred to [23,32] for further details).

The system model parameters are presented in Table 2. Figure 7 presents three 3D plots of $E_{Z\epsilon 2}^*$ and $E_{Z\alpha}$ as a function of $E_{Z\alpha}$ and η . By comparing the values of $E_{Z\epsilon 2}^*$ and $E_{Z\alpha}$, a direct assessment of expression (126) can be made. Different discrete values of the

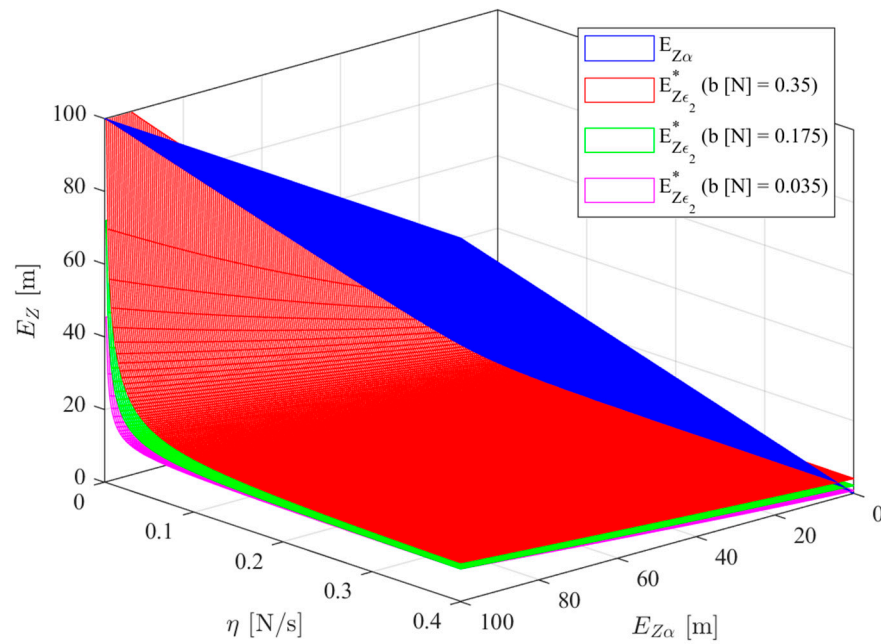
parameters a and b were used in the plots of Figure 7. Parameter a defines the depth band inside which the controller is turned off. Increasing it will tend to decrease energy consumption while tending to cause an increase in the depth control error. Parameter b defines a net force (F_{net}) layer inside which the controller remains turned ON although it could already be turned off. Increasing this parameter tends to increase energy consumption while preventing F_{net} determination errors to affect the controller behaviour. The chosen values of a correspond to 0.1%, 0.5%, 1% and 10% of the prototype maximum depth and the chosen values of b correspond to 1%, 5% and 10% of the maximum buoyancy. Parameter η defines how fast the net force is changed: increasing it will tend to increase energy consumption while tending to decrease the time the system takes to react. The range for η was chosen based on $\dot{F}_{VBM_max}$. From the block diagram of Figure 1, considering $k_z/k_u = 0$, $\dot{F}_{VBM} = K_1 A \rho g \cdot U$. Given the battery voltage of $U_{batt} = 14.8$ V, the maximum voltage considered was $U_{max} = 12$ V, to account for voltage drops in the electronics. As such, the considered range for η is $[0 \leq \eta \leq 0.4]$ N/s. In Figure 7, the regions where the pink, red and green surfaces ($E_{Z\epsilon_2}^*$) are below the blue plane ($E_{Z\alpha}$) are the regions where expression (126) is satisfied and, therefore, the system performs a convergent trajectory as explained below. When the pink, red and green surfaces are above the blue plane, the prototype will move away from the target depth reference and increase $E_{Z\alpha}$. This will occur until $E_{Z\alpha}$ is big enough so that (126) is again satisfied. This means the system is bounded and therefore Lyapunov stable as described in Section 4.1. From Figure 7, the impact of the parameters a , b and η on the system stability can be evaluated. Increasing parameter a leads to an increase in the minimum required value of $E_{Z\alpha}$ to satisfy (126). This result is expected since increasing a means expanding the region where the controller is turned OFF. Increasing b also leads to an increase in the minimum required value of $E_{Z\alpha}$ to satisfy (126), but for a different reason. Increasing b means that on the diagram of Figure 6 the system will reach point γ_2 with a higher net force magnitude, leading to an increase in velocity magnitude at point δ according to Equation (121). Therefore, the system reaches ϵ_1 with a higher velocity according to assumption 3, leading to a higher depth error at ϵ_2 . On the other hand, increasing η leads to a decrease in the minimum required value of $E_{Z\alpha}$ to satisfy (126). This is to be expected since it means it takes less time for the system to move from β to γ_1 (or from δ to ϵ_1), meaning the switching phase starts earlier, at a lower depth error magnitude at γ_1 (or ϵ_1). This is why, as can be seen in Figure 7, for very low values of η , the values of $E_{Z\epsilon_2}^*$ increase considerably. In Figure 8, sections of Figure 7b are presented: three 2D plots of $E_{Z\epsilon_2}^*$ and $E_{Z\alpha}$ as a function of $E_{Z\alpha}$ for an a of 0.5% of the maximum depth reference and for the same discrete values of parameters b and η . The chosen values for η correspond to 10%, 50% and 100% of $\dot{F}_{VBM_max}$.

Table 2. System model parameters.

Parameter	Value	Unit
A	7.7×10^{-3}	$[m^2]$
g	9.81	$[ms^{-2}]$
K_1	4.44×10^{-4}	$[ms^{-1}V^{-1}]$
K_2	7.9355×10^3	$[ms^{-1}m^{-3}]$
k_z/k_u	3.31×10^{-2}	$[Vm^{-1}]$
T_2	36.3	$[s]$
ρ	1×10^3	$[kgm^{-3}]$
ψ	3.4×10^{-7}	$[m^3m^{-1}]$
Z_{max}	100	$[m]$



(a)



(b)

Figure 7. Cont.

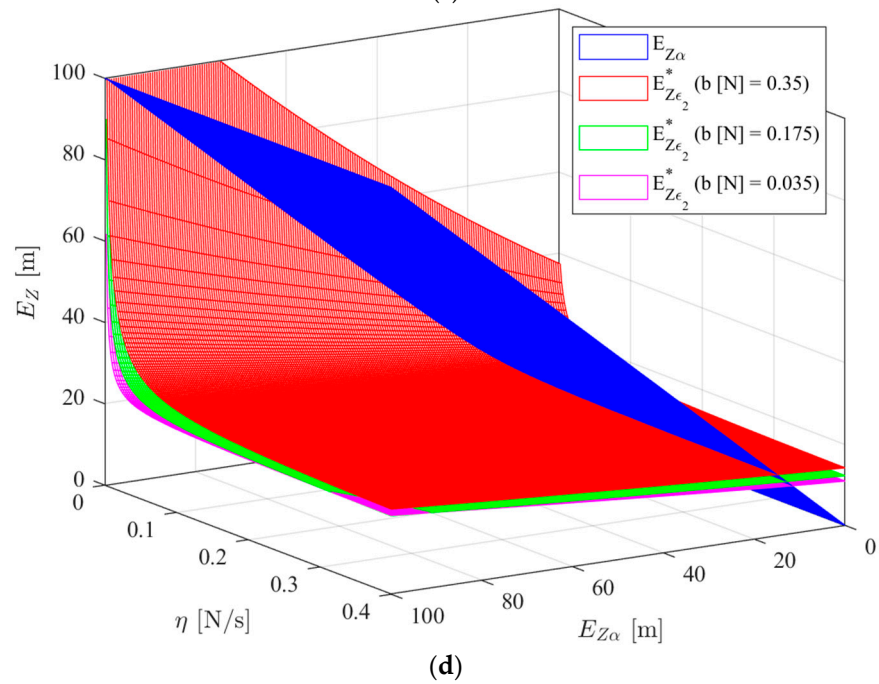
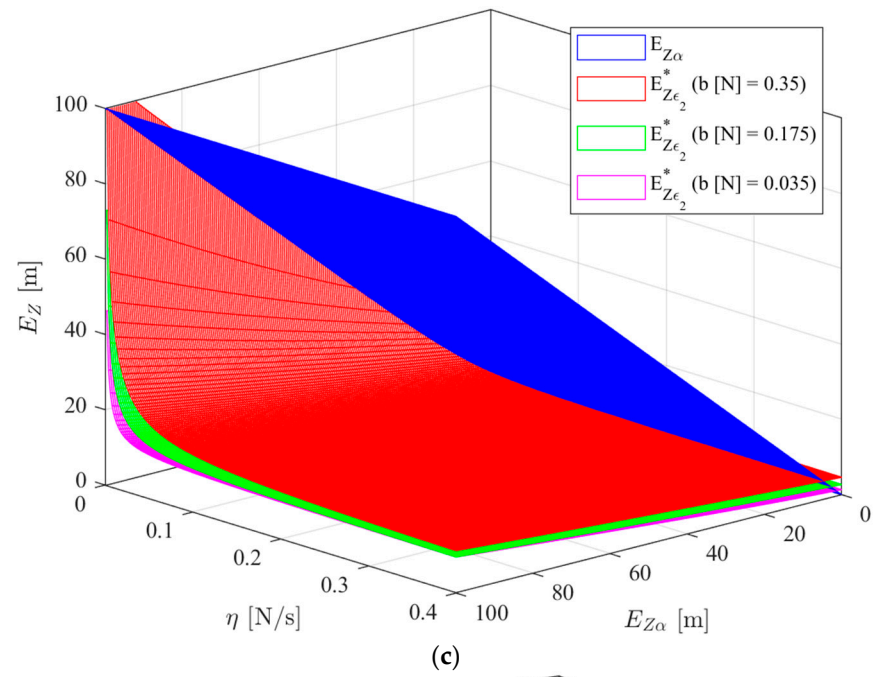
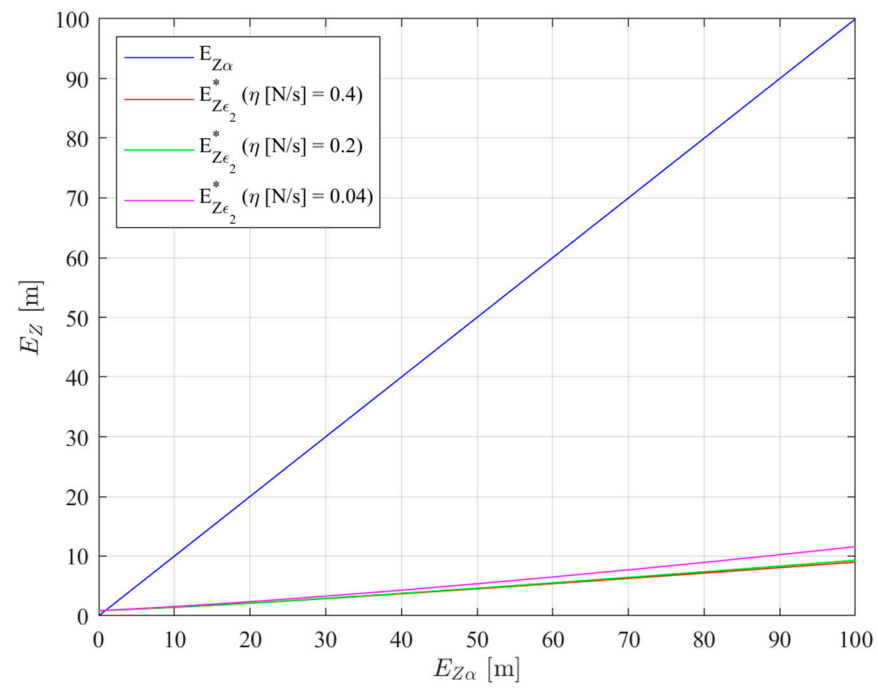
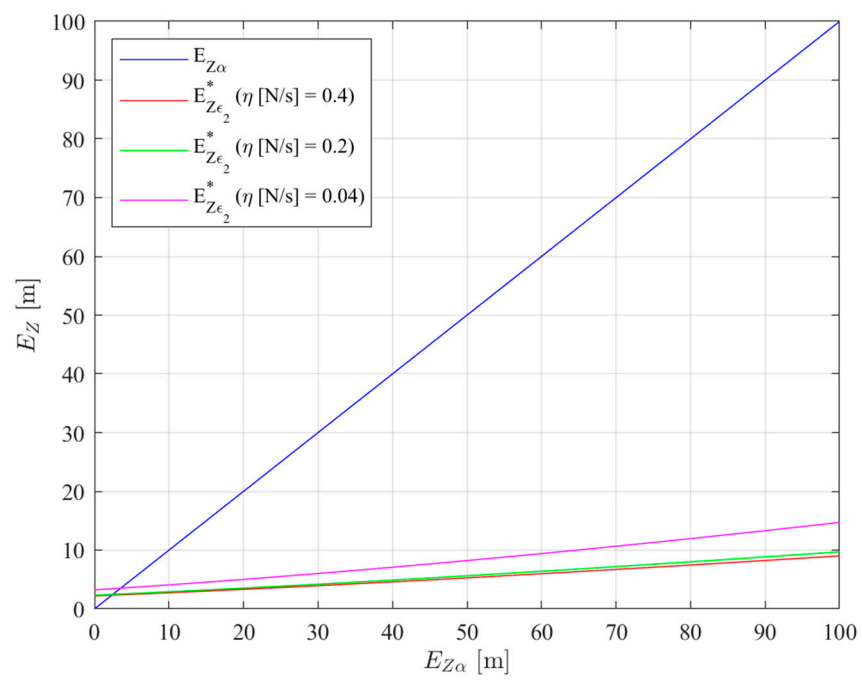


Figure 7. Plot of $E_{Z\alpha}$ and $E_{Z\epsilon_2}^*$ as functions of $E_{Z\alpha}$ and η for particular values of b : (a) $a = 0.1$ m; (b) $a = 0.5$ m; (c) $a = 1$ m; (d) $a = 10$ m.



(a)



(b)

Figure 8. Cont.

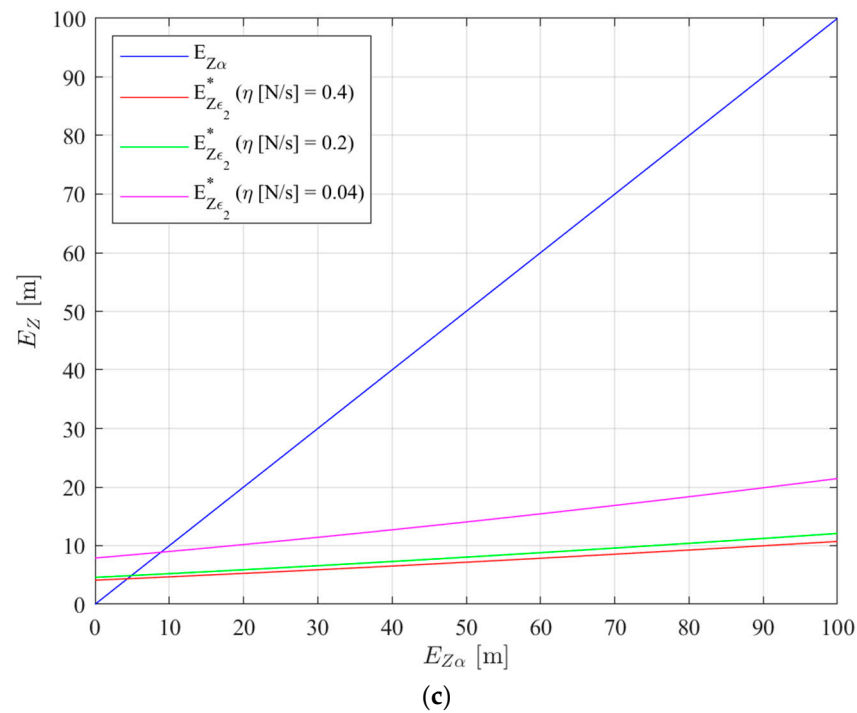


Figure 8. Plot of $E_{Z\alpha}$ and $E_{Z\epsilon_2}^*$ as functions of $E_{Z\alpha}$ for $a = 0.5$ m and for particular values of η : (a) $b = 0.035$ N; (b) $b = 0.175$ N; (c) $b = 0.35$ N.

In Figure 9, a close-up of Figure 8c for $\eta = 0.04$ N/s, $a = 0.5$ m and $b = 0.35$ N is presented. As an example, consider three consecutive switching sequences (S_1 , S_2 and S_3) the system might undergo. Assume that the estimation $E_{Z\epsilon_2}^*$ is perfect ($E_{Z\epsilon_2}^* = E_{Z\epsilon_2}$) and that switching sequence S_1 starts at point α with $E_{Z\alpha} = 18$ m, represented by the right-most triangular marker in Figure 9. In this situation, $E_{Z\epsilon_2}^* = 9.9$ m. This means that after performing the switching sequence S_1 , the depth error reduces from $E_{Z\alpha} = 18$ m to $E_{Z\epsilon_2} = 9.9$ m. Point ϵ_2 can now be seen as point α of switching sequence S_2 . In this situation, $E_{Z\epsilon_2}^* = 9$ m. This means that after performing switching sequence S_2 , the depth error reduces from $E_{Z\alpha} \cong 10$ m to $E_{Z\epsilon_2} = 9$ m. Point ϵ_2 can now be seen as point α of switching sequence S_3 . In this situation, $E_{Z\epsilon_2}^* = 8.9$ m, close to the intersection between the two curves of Figure 9. A similar process can be followed by starting another different sequence S_1 , at point α with $E_{Z\alpha} = 2$ m, represented by the left-most triangular marker in Figure 9. In this situation, $E_{Z\epsilon_2}^* = 8.1$ m. This means that after performing switching sequence S_1 , the depth error increases from $E_{Z\alpha} = 2$ m to $E_{Z\epsilon_2} = 8.1$ m. Point ϵ_2 can now be seen as point α of the new switching sequence S_2 . In this situation, $E_{Z\epsilon_2}^* = 8.8$ m. This means that after performing switching sequence S_2 , the depth error increases from $E_{Z\alpha} = 8.1$ m to $E_{Z\epsilon_2} = 8.8$ m. Point ϵ_2 can now be seen as point α of the new switching sequence S_3 . In this situation, $E_{Z\epsilon_2}^* = 8.9$ m. It is clear from these examples that whether the system starts to the left or to the right of the intersection between the curves of Figure 9, it always moves in the direction of the intersection. This means the depth error is bounded. Moreover, the information provided in Figures 7 and 8 allows the estimation of the order of magnitude of the depth error bounds around which the system oscillates. These bounds are given by the intersection between $E_{Z\alpha}$ and $E_{Z\epsilon_2}^*$. In the next section a numerical example will be presented.

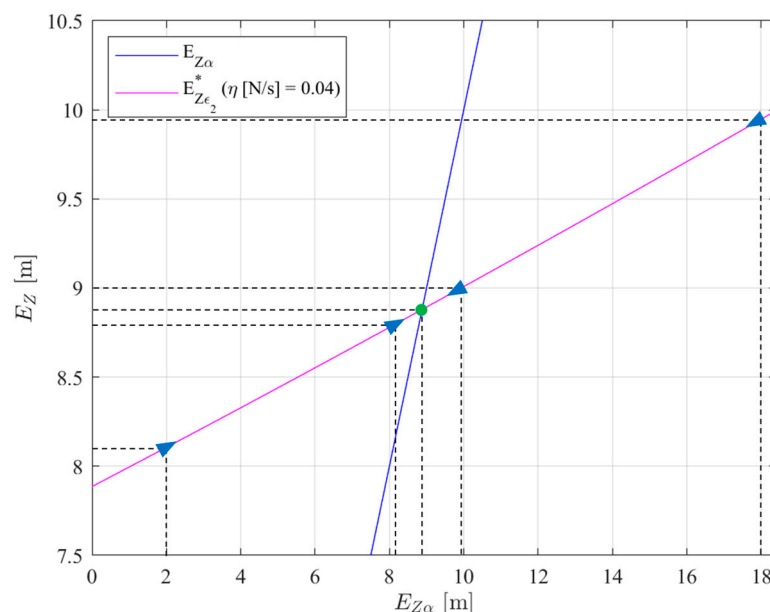


Figure 9. Close-up of Figure 8c.

5.2. Simulation Trial

In this section, the proposed controller is simulated and compared to controller 7b of [33] and to the controller proposed in [24]. In [33], the authors proposed a cascaded control strategy consisting of a depth controller and a volume controller for the VBM. The control structure 7b led to the smallest energy consumption and was therefore selected for datum comparison in this work. In [23,24] a variable structure controller with a constant control action was presented and its stability was proven. A particular version of this controller was simulated in [23] and its energy consumption was lower compared to the lowest energy consuming controllers from [33]. As such, the controllers used for the simulation trial are:

1. Controller 7b: a cascaded I-PD/PI control scheme, with the I-PD depth controller C_z in the outer loop and the PI volume controller C_{Vol} with a deadband DB_{Vol} in the inner loop;
2. Controller VSC_{Uconst} : a variable structure controller with a constant-amplitude control action $U_{switch} = const.$;
3. Controller VSC_{Umin} : the variable structure controller presented in this work, with a constant net force time rate $\left| \dot{F}_{net} \right| = \eta = const.$

Controllers VSC_{Uconst} and VSC_{Umin} require a disturbance observer, presented in [23], to provide an estimation of the net force acting on the prototype. The parameters of controller 7b are presented in Table 3, whereas the VSCs and observer parameters are shown in Table 4.

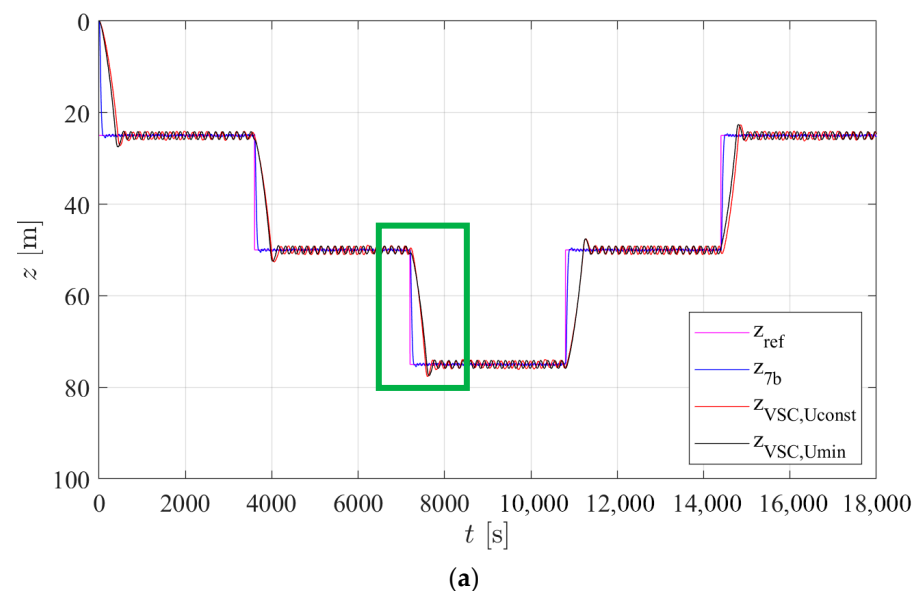
Table 3. Controller 7b parameters used in the simulation trial.

C_z			C_{Vol}		
K_{PZ} [m ³ /m]	K_{IZ} [m ³ /(ms)]	K_{DZ} [m ³ /(ms ^{−1})]	K_{PVol} [V/m ³]	K_{IVol} [V/(m ³ s)]	DB_{Vol} [m ³]
7.5×10^{-5}	-2×10^{-6}	1.1×10^{-3}	1×10^5	1×10^5	3.5×10^{-5}

Table 4. VSC and observer parameters used in the simulation trial.

Controller	a [m]	b [N]	U_{switch} [V]	η [N/s]	K_O [N/m]	K_{OD} [N/(ms ⁻¹)]
VSC _{Uconst}	0.5	0.035	12	variable	5	100
VSC _{Umin}	0.5	0.035	variable	0.28	5	100

Controller 7b tuning conditions are described in a previous paper by the authors [33], where several PID based structures and parameters were heuristically tuned to achieve low energy consumption. The controller parameters leading to the lowest energy consumption are the ones presented in Table 3. The choice of parameters for the VSC_{Uconst} was based on the charts presented in [24]. In particular, the controller voltage $U_{switch} = 12$ V was chosen so that inside the stability regions the range for the parameter η is $0.27 < \eta < 0.29$. The same values were used for parameters a and b on the VSC_{Umin}, and the average value of $\eta = 0.28$ was considered. In this manner, a fair comparison can be made between both controllers. The simulation trial conducted in Matlab Simulink corresponds to a type 1 test as described in [33]: a target depth consisting in a set of five equal amplitude steps, without external disturbances. The system model was implemented with quantization of the measured variables X and Z to account for sensor resolution as well as Zero Order Holds (ZOH) with a sampling time of 0.1 s (corresponding to the real system sampling time) to simulate the holding of control actions and measured variables calculated between sampling instants. The solver used was Matlab ode45, based on an explicit Runge–Kutta (4,5) formula, the Dormand–Prince pair. The simulations were run with a variable time-step. The solver uses a relative error tolerance of 10^{-3} . Further details of the implementation can be found in [33]. Figure 10 presents the prototype depth for the complete simulation trial and a close-up of the region marked with a green rectangle. In Figure 11, the control action is shown for the corresponding time intervals of Figure 10. Figure 12 exhibits the energy consumption.

**Figure 10.** Cont.

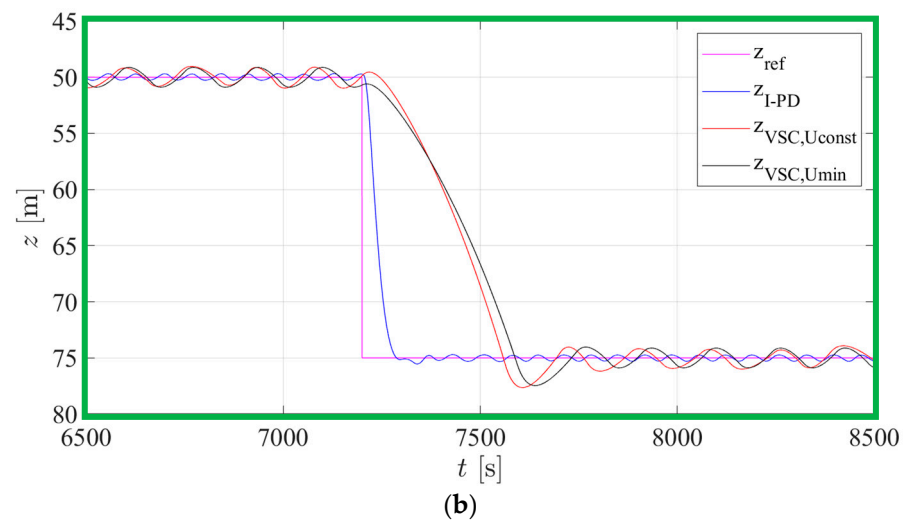


Figure 10. Depth in Simulation Trial: (a) full simulation; (b) close-up.

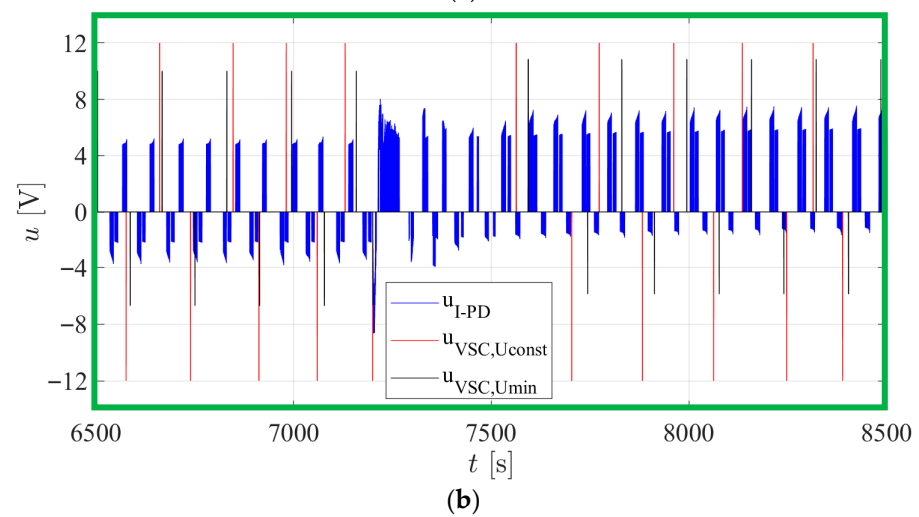
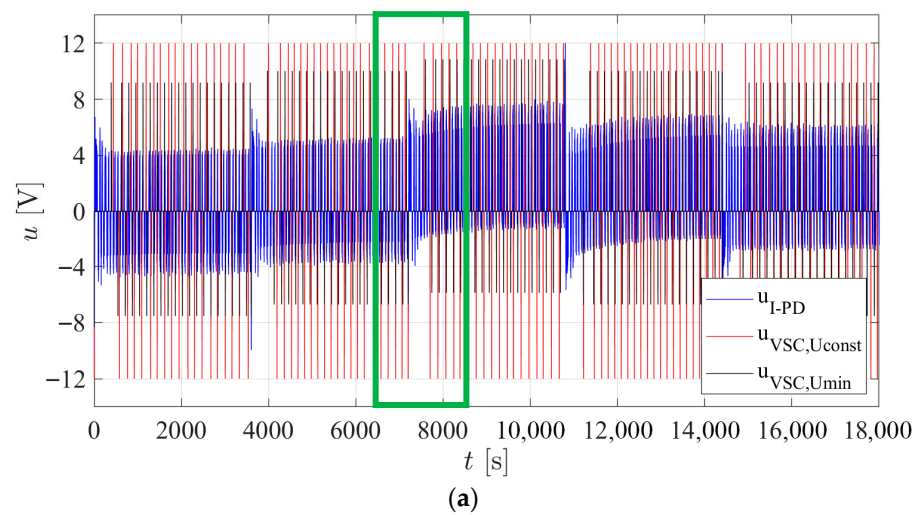


Figure 11. Control action in simulation trials: (a) full simulation; (b) close-up.

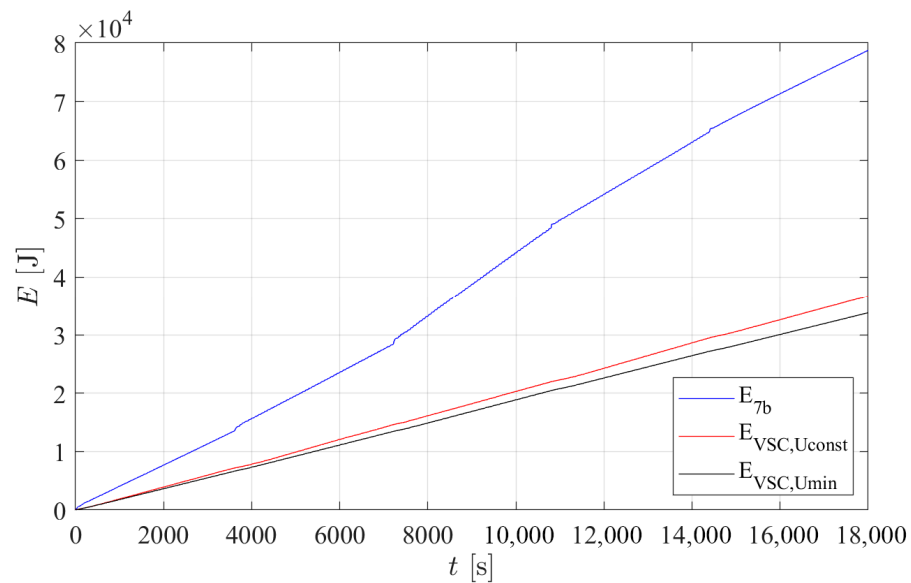


Figure 12. Simulation Trials: energy consumption.

From Figure 10 it is clear that controller 7b is faster than both VSCs. This comes at a price of a bigger energy consumption as can be seen in Figure 12. Comparing only the VSCs, they are similarly fast but the VSC_{Umin} is less energy demanding. The differences in energy consumption between the three controllers can be explained with the help of Figure 11. Controller 7b uses a control action with a smaller amplitude than both VSCs, but at a much higher frequency. In contrast, the VSCs operate at a similar frequency: VSC_{Uconst} maintains a fixed amplitude of 12 V, while VSC_{Umin} adjusts its amplitude to the minimum required by (10), leading to smaller amplitudes than VSC_{Uconst} . It should be noticed that the actual amplitude of the VSC_{Umin} control action is larger when increasing buoyancy since the external water pressure opposes piston movement. Conversely, when decreasing buoyancy, the surrounding water pressure aids the piston return motion, so a lower voltage is sufficient. As expected, Figure 11 shows this asymmetry becomes more pronounced with depth due to the higher pressure.

Considering the VSC_{Umin} with the parameters described in Table 3, the chart from Figure 8a is applicable. Figure 13 shows a close-up of Figures 10 and 8a.

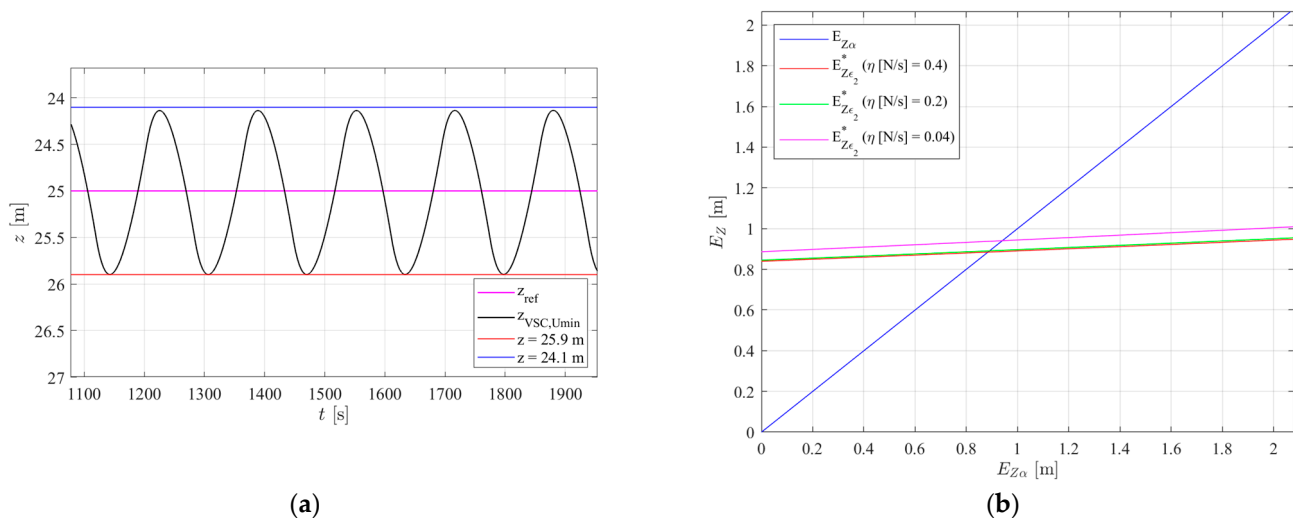


Figure 13. Close-up of: (a) Figure 10; (b) Figure 8a.

Given that for the simulation trial $\eta = 0.28$ N/s was used, the corresponding $E_{Z_{\varepsilon_2}}^*$ curve is located between the green and red curves of Figure 13b. The intersection between this curve of $E_{Z_{\varepsilon_2}}^*$ and the blue line $E_{Z_{\alpha}}$ gives an estimation of the maximum depth error band that can be obtained. As such, the system should tend to oscillate around the target depth with an amplitude of the same order of magnitude of the value given by this intersection. From Figure 13b, this value is $E_{Z_{\alpha}} = E_{Z_{\varepsilon_2}}^* \cong 0.9$ m, so when the target depth is 25 m, this corresponds to a minimum value of $z = 24.1$ m and a maximum value of $z = 25.9$ m, as represented by the red and blue lines of Figure 13a. This result matches the one presented in Figure 13a, where the oscillation has an amplitude of about 0.9 m. This shows how the charts from Figure 8 can be used to estimate the controlled system oscillation amplitude.

In order to assess the energy consumption of the controllers with different values of a and b , additional simulation trials were conducted. The energy consumption results are presented in Table 5.

Table 5. Energy consumption of the VSCs according to the chosen parameters.

η [N/s]	a [m]	b [N]	$E_{VSC_{Umin}}$ [kJ]	$E_{VSC_{Uconst}}$ [kJ]	Energy Savings [%]	Off-Time VSC _{Umin} [%]	Off-Time VSC _{Uconst} [%]
0.28	0.1	0.0175	33.92	38.07	10.9	67.2	55.5
		0.035	37.77	43.65	13.5	15.1	18
		0.07	41.46	49.37	16.0	11.7	11.6
	0.5	0.0175	31.37	34.87	10.0	99	95.4
		0.035	33.77	36.69	8.0	98.7	99.3
		0.07	36.09	40.48	10.8	98.1	81.1
	1	0.0175	30.58	32.77	6.7	99.6	99.4
		0.035	32.45	34.99	7.3	99.2	98.4
		0.07	34.31	38.04	9.8	98.1	98.4
	10	0.0175	29.85	30.34	1.6	99.8	99.7
		0.035	30.19	31.10	2.9	99.8	99
		0.07	31.05	32.28	3.8	99.1	98.9

The results from Table 5 show that absolute energy consumption increases with the parameter b . This is to be expected since it means that the controllers must perform larger buoyancy changes. On the other hand, it is also clear that increasing the value of parameter a leads to a reduction in absolute energy consumption at the cost of a larger depth error band. This is also not surprising since increasing a means increasing the admissible depth error and, as such, the amount of time the system is turned off also increases. Another important result is that for comparable parameters, the VSC_{Umin} always lead to less absolute energy consumption than the VSC_{Uconst}. This effect is particularly evident for small values of a , where up to 16% energy savings were achieved. The relative reduction in energy savings percentage for higher values of a can be explained by the fact that having a wider depth error band reduces the control action's ON time, thereby diminishing the relative benefit of the proposed controller. Finally, energy savings percentage also generally increases with the increase in parameter b . This may show that using a smaller control action for longer time (VSC_{Umin}) as opposed to using a larger control action for shorter times (VSC_{Uconst}) is beneficial as it increases the relative advantage of the proposed controller.

The performance of each controller in maintaining the sensor platform's suitability for underwater acoustic measurements was assessed. The results are also presented in the two last columns of Table 5. This assessment focused on measuring how long each controller remained inactive for periods of at least 60 s. The two last columns of Table 5 summarize the off-time for each trial and controller, as a percentage of the overall trial

durations. The results show that in most cases where the error band a is equal to 0.1 m, the controller is not able to be off more than 60 consecutive seconds in a significant part of the mission. This is expected since for tighter depth error bands, the controller must turn on more frequently. However, if the error band a is equal or higher than 0.5 m (a typically accepted error value in many missions), the controller is off most of the time. There also appears to be an increase in the 60 consecutive seconds period that the controller is off when b is decreased. This is consistent with the energy decrease noticed when b is reduced. Furthermore, there were no notable differences in inactive periods between using a constant-amplitude control action and a variable one. These findings highlight the strong suitability of the controller developed in this study for scenarios that demand minimal acoustic emissions.

Lastly, to assess the influence of parameter η on energy consumption, a few more simulations were conducted for different values of η . The results are presented in Table 6.

Table 6. Energy consumption of the VSC_{Umin} for different values of η .

a [m]	b [N]	η [N/s]	$E_{VSC_{Umin}}$ [kJ]
0.5	0.035	0.04	31.08
		0.2	31.93
		0.28	33.77
		0.4	36.92

The results presented in Table 6 show that parameter η contributes to increase energy consumption. This is expected since to achieve a higher value of η , a higher value of U_{switch} is required.

6. Conclusions

This paper presents a new variable structure controller with a variable control action designed to minimize energy consumption in underwater vehicles and platforms using variable buoyancy actuation. A formal proof of stability is provided, demonstrating that if the initial conditions are within specified depth error bounds, the system state remains bounded. These bounds can be quantified for a given system, offering practical guidance for selecting controller parameters. The proposed approach was applied to a previously developed depth-controlled sensor platform using a validated system model while accounting for physical constraints. Results confirm that the control law is implementable in such a prototype, despite the highly conservative approach considered.

This study further explores how stability regions are affected by controller parameters and mission requirements. It also shows that the stability conditions derived can be used to estimate the amplitude of oscillations during hovering tasks. Simulation trials were conducted to validate the proposed controller (VSC_{Umin}), comparing it with two previously developed controllers: a constant control action VSC (VSC_{Uconst}) and a controller from the PID family (a cascaded I-PD/PI). The variable structure controllers exhibit a slower response than the I-PD/PI but with the benefit of less energy consumption. Moreover, the VSC_{Umin} achieves up to a 16% energy reduction when compared to the VSC_{Uconst} . The analysis also highlighted how parameter choices impact the energy consumption of the VSC_{Umin} , particularly in comparison to VSC_{Uconst} . These findings provide a foundation for controller tuning with specific mission requirements. Future work will involve the experimental validation of the controller through pool and sea trials and extension of the stability proof to account for variable external disturbances. Also, the necessary changes regarding system parameters for electro-hydraulic and other actuators will be explored,

as well as a comparison with other controllers, such as Linear Quadratic Regulators and Model Predictive Control.

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Nomenclature

a	Depth deadband [m]
A	Area of the piston in contact with sea water [m ²]
b	F_{net} band [N]
C_z	Depth controller
C_{Vol}	Volume controller
DB_{Vol}	Volume dead band parameter for C_{Vol} [m ³]
$E_{VSC_{Uconst}}$	Energy spent by VSC_{Uconst} [J]
$E_{VSC_{Umin}}$	Energy spent by VSC_{Umin} [J]
E_Z	Depth control error [m]
E_{Zi}	Depth control error at state i [m]
E_{Zi}^*	Overestimation of E_{Zi}
$\mathbf{f}_1, \mathbf{f}_2$	System model state functions
F_{dist}	Total disturbance forces [N]
$F_{dist_{ext}}$	External disturbance forces [N]
$F_{dist_{int}}$	Internal disturbance forces [N]
F_{net}	Difference between F_{VBM} and F_{dist} [N]
F_{neti}	F_{net} at state i [N]
F_{VBM}	Variable buoyancy module force [N]
g	Acceleration of gravity [ms ⁻²]
K_1	Variable buoyancy module linear model steady-state gain [ms ⁻¹ V ⁻¹]
K_2	Vertical motion linear model steady-state gain [ms ⁻¹ m ⁻³]
K_{IV}	Integral gain of C_{VOL} [V/(m ³ s)]
K_{PV}	Proportional gain of C_{VOL} [V/m ³]
K_{DZ}	Derivative gain of C_Z [m ³ /(ms ⁻¹)]
K_{IZ}	Integral gain of C_Z [m ³ /(ms)]
K_{PZ}	Proportional gain of C_Z [m ³ /m]
K_O	Disturbance observer proportional gain [Nm ⁻¹]
K_{OD}	Disturbance observer derivative gain [Nsm ⁻¹]
k_z/k_u	Parameter relating the depth and the equivalent depth voltage [Vm ⁻¹]
p	Water pressure [Pa]
r, R	System state boundaries in the Lyapunov sense
S	Switching sequence
t	Time [s]
t_j	Switching time instant for subsystem j [s]
$\bar{t}_{i,k}$	Switching time instant when subsystem i is switched on for the k th time [s]
$\underline{t}_{i,k}$	Switching time instant when subsystem i is switched off for the k th time [s]
T_1	Variable buoyancy module linear model time constant [s]

T_2	Vertical motion linear model time constant [s]
U	Control action [V]
U_{min}	Variable-amplitude control action [V]
U_{switch}	Variable structure controller control action [V]
U_Z	Equivalent depth voltage [V]
V_Z	Depth velocity [ms^{-1}]
V_{Zi}	Depth velocity at state i [ms^{-1}]
VSC_{Uconst}	Constant-amplitude variable structure controller
VSC_{Umin}	Variable-amplitude variable structure controller
$\mathbf{x}$	System state vector
X	Piston position [m]
Z	Vehicle depth [m]
Z_i	Vehicle depth at state i [m]
Z_{max}	Maximum vehicle [m]
Z_{ref}	Depth reference [m]
$\alpha, \beta, \gamma_1, \gamma_2, \delta, \varepsilon_1, \varepsilon_2, \zeta, \xi$	System states
ΔE_{Zij}	Change in depth error between states i and j
ΔE_{Zi}	Change in depth error between i_1 and i_2
λ_γ	Parameter that expresses the occurrence of switching at state γ_1 [0, 1]
λ_ε	Parameter that expresses the occurrence of switching at state ε_1 [0, 1]
η	VSC_{Umin} controller parameter [Ns^{-1}]
ψ	Hull loss of volume per meter depth [m^3m^{-1}]
ρ	Water volumetric mass [kgm^{-3}]
$\mathcal{S}(i)$	Set of time intervals during which subsystem i is active
$(\bullet)$	Time derivative of $(\bullet)$
$(\bullet)_{max}$	Maximum value of $(\bullet)$
$ (\bullet) $	Absolute value of $(\bullet)$
$\ (\bullet)\ $	Norm of vector $(\bullet)$

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