

Article

Manufacture of a 4-Degree-of-Freedom Robot to Support an IRB 120 Robot

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Abstract: In this work, we present the construction and control of a four-degrees-of-freedom (DOF) manipulator aimed at addressing one of the key challenges faced by the Academy-Industry Cooperation Center (CCAI): the need for mechatronic equipment to support and facilitate the development of advanced robotic cells. We begin by designing the robot's structure and components using SolidWorks software for computer-aided design (CAD) modeling. This ensures that all the links and parts fit together properly without collisions. The robot links are then manufactured using 3D printing. Additionally, we performed kinematic modeling, dynamic analysis, and PI-V control, along with control using a trigonometric function (hyperbolic tangent). To evaluate the robot's movement, we simulate these processes using Matlab R2019a/Simulink software, focusing on key parameters such as position, velocity, and acceleration, which inform the design of PI-V control for each link. We also present the electrical and electronic designs, followed by system implementation. The kinematics of the robot play a crucial role in the dynamics and controller design. We validate the kinematics using Peter Corke's libraries based on the Denavit–Hartenberg parameters. The results show that the controller based on the trigonometric function improves the response time, particularly enhancing the performance of axes 2 and 3.

Keywords: robotic manipulator; direct kinematics; dynamics by Euler–Lagrange; PI-V control; PD tanh control; Lyapunov



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1. Introduction

One of the objectives of the development of this robot is to collaborate with the IRB 120 robot (ABB robot) to monitor paths for welding or cutting. This is because IRB 120 does not reach certain points. To solve this, a table is placed that rotates so that it can reach these points; however, there are times when the rotary table passes 15° or more, and the programming that is left loses the position where it ends, causing it to lack welding points or, in worse cases, welding is not performed. Therefore, the primary objective of this robot is to charge an artificial vision camera so that it provides the correct location where the last welding point is, and from there, it begins to weld the IRB 120.

There are several methods to obtain the dynamics of a robot, including the Newton–Euler equations, the multi-body serial chain method, and the Euler–Lagrange equations.

In [1], the development of a model and an optimal controller is presented, which uses a disturbance detector to stabilize a mobile robot with a three-link inverted pendulum. A key aspect of this work is the approach used to obtain the robot's mathematical model, which is based on the Euler-Lagrange equations.

In [2], they perform the dynamic performance of a hip rehabilitation robot with two degrees of freedom. Where they obtain the kinematics with the Denavit Hartenberg parameters, they subsequently obtain the dynamics with the Newton-Euler method. The contribution of [2] for this project is the guidance to find the kinematics and dynamics of the proposed robot.

One of the challenges in simulating the dynamics of a robot is singularities, which are divided into three types. The first occurs when the Jacobian matrix is equal to zero, the second when the determinant of the Jacobian matrix for the inverse model vanishes, and the third when both determinants vanish simultaneously [3]. In general, these types of singularities tend to become more frequent as the number of degrees of freedom increases.

In [4], a 6-degree-of-freedom manipulator robot is presented, which incorporates both rotational and prismatic joints. Additionally, graphical simulations are included, performed using software tools such as Matlab/Simulink. A key feature of this work is the free body diagrams that are shown, as well as the implementation of controllers on the 6-degree-of-freedom robot.

For the development and simulation of a robot, it is essential to understand both its kinematics and its dynamics. An interesting study is presented in [5], which highlights the importance of simulations to analyze and characterize the robot behavior in kinematic and dynamic terms, using the formulation of the Euler-Lagrange equations. The tools used in this work are Matlab/Guide, SolidWorks with Matlab/Simulink.

A key aspect when designing a controller is determining its gains. In [6], several methods for obtaining the gains of a PID controller are presented, among which "Particle Swarm Optimization" (PSO) and "Bee Colony" stand out. Additionally, an interesting contribution is the application of performance indices to evaluate the controllers, such as the Integral Squared Error (ISE), the Integral Time Squared Error (ITSE), the Integral Absolute Error (IAE), and the Integral Time Absolute Error (ITAE).

There are times when implementing a PID controller in a system is not enough to be able to take the system output to the desired value, therefore, there is the combination of the PID controller with some other strategy to improve the controller. In [7], an ankle exoskeleton is developed that stands out for using a PID controller based on neural networks (PIDNN) to adaptively manage the system and minimize the associated uncertainty.

To build a robot it is necessary to have knowledge of different disciplines, which range from design, programming, mathematics, electronics, among others. There is little work that makes known all these disciplines in a single document, this is seen in [8]. The contribution of [8] is the implementation of a fuzzy PID controller, where they find the gains through particle swarm optimization.

When the controller for a robot with multiple joints is made, it is important that the first link has enough strength and precision to move to the desired point, in addition to being fast and avoiding rattles. To achieve this, it is necessary to correctly calculate the controller gains and also add a function that limits the controller, these types of functions can be tanh, sat and sign. In [9] they develop a PID controller with a sign function which is implemented in a mobile vehicle. This control strategy is used to generate an additional moment and optimize the distribution of the longitudinal force of the tires.

In [10], we see again the development of a fuzzy PID controller applied to a 4-degree-of-freedom robot. This type of robot is frequently used in cable cars where the objective of the robot is to prevent strong winds from moving the trolley that goes through the cable or cables. The contribution of this work is the one that leaves explicitly the development of the controller implemented in its robot.

In the field of automatic control, there are various regulation strategies; however, one of the most used is the PID control. This stands out for its ability to correct previous errors, opti-

mize regulation in terms of response time, have a low computational processing consumption and be easily implemented in low-cost electronic cards. In addition, when combined with other control strategies, its characteristics can be significantly improved. There are numerous studies on the PID controller, among which some can be cited as [11–13].

In [14], the design of a new bipedal robot is presented, a lightweight and full-size robot that resembles a pair of human legs, with a mobility of 12 degrees of freedom, resulting in six degrees of freedom for each leg.

In [15], an interesting “review” of collaborative robot dynamics is seen, where an interesting phrase that the author makes known is “By formulating a control strategy that intentionally prioritizes sets of actuators near or at kinematic singularities and avoids those near “static” singularities, one can achieve increased load-bearing capabilities, such as lifting, with reduced joint deflections, improved accuracy, and superior overall performance. It is crucial to emphasize that employing singularity for load resistance introduces inherent risks and potential hazards. Therefore, careful consideration is advised for anyone contemplating this “approach”. One of the main problems that exists when performing simulations in Matlab is what this author mentions, which prevents the simulation from being completed or not being performed.

In this study [16], a cost-effective and efficient mobile robotic manipulator designed to sort objects in domestic and industrial environments is developed. To achieve this goal, the authors implemented a deep-learning approach using YOLO for accurate object detection. In addition, we incorporated reverse kinematics to facilitate precise positioning, placement, and movement of the robotic arms to the desired location of the object.

In [17], the authors discuss and present a multi-query resistive grid-based planning method applied to robotic manipulators with high degrees of freedom (DOF). The study presents a more comprehensive multi-query method capable of obtaining collision-free paths for high-DOF manipulators with multiple surrounding obstacles. Additionally, it presents potential solutions to problems that may arise in the project owing to collisions despite the robot having four degrees of freedom.

In [18], a highly advanced anthropomorphic control system that uses multi-sensor technologies to achieve precise control of a robotic arm based on an anthropomorphic robot is proposed, which not only improves the accuracy of the system but also improves the ability to adapt to dynamic environmental changes, thus ensuring the stable movement of the robotic arm.

In [19], a partial Lagrangian method is proposed that can handle the Lagrange equation efficiently, even for systems with multiple degrees of freedom, using a divide-and-conquer approach. The proposed method can easily handle system extensions and reconstructions, such as intermediate link changes, for serial link manipulators with multiple degrees of freedom.

This study [20] presents a three-joint exoskeleton located at the hip, knee, and ankle. In this study, the authors designed a model-based gravity compensator impedance controller for an exoskeleton system (SLE). A dynamic model describing the interaction between the user and the exoskeleton with remarkable accuracy was derived and validated. This model was achieved using the Euler–Lagrange formulation and following the Denavit–Hartenberg rules, and it was implemented and validated in Simscape and Simulink. The contribution of [20] to this project is the use of Simulink for the development of the dynamic model.

In this article [21], a fuzzy controller designed to monitor the displacement of a three-dimensional robotic mechanism of the lumbar spine driven by tendons is described. The implementation of this driver was performed using Matlab/Simulink R2023a software. When relating this study to the project of a robot with four degrees of freedom, where dynamics are carried out to simulate its behavior using the Euler–Lagrange mathematical model, a similarity in the methodology used is evident. Both cases use simulation tools and are based on advanced mathematical models (Lagrangian or Euler–Lagrange) to understand and control the dynamic movement of the mechanical systems in question.

Robotics has an increasing number of applications, one of which is seen in [22], where they carry out a methodology to develop an energy consumption model for a printed robot, developed and optimized specifically for this technology, where an innovative approach is incorporated to determine reduced efficiency maps, which allows the inclusion of drive efficiency parameters that are difficult to measure in the model. This results in a comprehensive model of the energy consumption of the printed robot, which reflects its operating characteristics in a real-world environment and is applied to an IRB 4600 and an experimental robot.

In [23], the integration of different technologies is carried out, such as the use of a PLC, an IRB 1200 robot, and an Intel Real Sense artificial vision camera, the objective of which is to verify images and colors, depending on which robot performs a specific routine.

In [24], a system analysis is conducted for the manipulation of the metric certification process for industrial robots. Certification is performed by correctly executing the relevant calculation procedures using the software products CoppeliaSim V4.8 and Matlab. Although this work is analogous to the proposed study, it has important implications for future projects.

In [25], a calibration method is performed for IRB 120 based on the kinematic separation of the manipulator. Finally, the IRB-1200 robot was successfully calibrated using the proposed method; the average position and angle errors of the robot after calibration decreased significantly. The position accuracy improved by 66.9%, and the attitude accuracy improved by 86.2%. Although this work is analogous to the proposed study, it has important implications for future projects.

A significant aspect in the field of automatic control is obtaining benefits from regulators, where there are various studies and approaches to determine such benefits; however, many of these methods are only applicable to linear systems, and the difficulty arises when the system dynamics is non-linear, even worse when this dynamic is unstable or presents oscillating responses. To solve this problem, a widely used method is Lyapunov stability. Some references that deal with this topic are [26–29].

2. Materials and Methods

The anthropomorphic robot is composed of four structures, which are connected to each other by a piece called a bride, which in turn is coupled to a GM 25-370 DC motor. Each of the links has the necessary spaces and holes for the installation of the motors.

The prototype is manufactured using the additive manufacturing technique (3D printing), and is made of PLA with 70% printing, which gives it the necessary rigidity to support both the weight of the links and the torque of the motors. This feature can be observed in Figure A8 that appears in Appendix A.1.

To adequately represent the kinematics of the 4 DOF robot, the libraries of Peter Corke and Matlab [30] are used. The objective is to generate a figure that reflects the kinematics of the proposed robot, which is shown in Figure A11 in Appendix A.2. This ensures that the dynamic calculations are correct.

The dynamics of the 4 DOF robot is developed using the Euler-Lagrange equations method [1,5,31]. To do this, an algorithm is created in Matlab in which the homogeneous transformation matrices of the robot are introduced.

The controller design takes as a reference the work of Takegaki and Arimoto, who address the non-linear dynamic model and the closed-loop equation generated by the controller [32].

The materials and equipment used are as follows:

1. A 3D printer;
2. A filament reel of PLA;
3. Four Arduino UNO boards;
4. Four L298N motor controllers;
5. Four Breadboards for Arduino UNO with 12 VDC 10 amp power supply;
6. Push buttons;

7. Potentiometers;
8. Multiple connectors;
9. $4 \times$ GM 25-370 DC motors with encoder.

3. Development and Simulations

3.1. PI-V Controller Design for Four-Degrees-of-Freedom Robot

This subsection examines the development of the PI-V controller for the robot. It is worth mentioning that the development of this controller does not apply to the entire robot, which is developed as a controller for each degree of freedom. In short, there are four PI-V controllers, and each controller has different gains because the force required by each motor is very different because some links carry more weight. The representation of the closed-loop system can be seen in Figure 1.

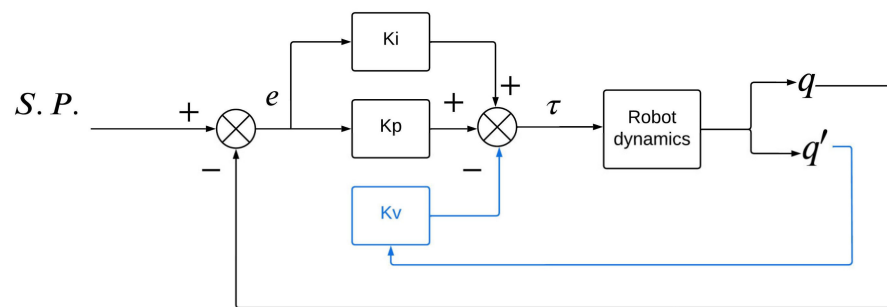


Figure 1. 4-DOF closed-loop robot system.

Remark 1. The derivative term has been avoided as it has been previously demonstrated that the derivative term may include instabilities to noise [33].

To develop the controller, it is necessary to consider the dynamics of the system; in this case, Equation (A20) from Appendix A, where $\tau = K_p e - K_v q' + K_i \int_0^t e(T) dT$.

The closed-loop equation in variables in state space can be defined as Equation (1):

$$\frac{d}{dt} \begin{pmatrix} v \\ e' \\ q' \end{pmatrix} = \begin{pmatrix} e \\ -q' \\ M(q)^{-1} [K_p e - K_v q' + K_i v - C q' - G - C_B] \end{pmatrix} \quad (1)$$

where e is the error, q is the position of the robot, q' is the speed of the robot, v is a variable to define the integral of the error, C_B is the viscosity coefficient of the motor of each axis, K_p is the proportional gain of the controller, K_v is the gain of the speed, K_i is the integrative gain, M is the inertia matrix, C is the Coriolis matrix, and G is the gravity matrix. The error is defined as $e = S.P. - q$, where $S.P.$ is the reference value.

3.2. Lyapunov Stability Criterion

An important point in the development of a controller is to calculate the appropriate gains for the plant (angular positions of the robot) to reach the desired reference values. However, calculating these profits using an analytical method is complicated. You can choose to find these gains through an optimization method, but the details are that the dynamics presented in this work are very extensive, and its computer calculation takes a long time. Therefore, we propose the use of Lyapunov's theory of stability [31,34], which guarantees that the stability of the closed-loop system of Equation (1) is satisfied.

Equation (2) is considered:

$$V(q', e) = \frac{1}{2} q'^T M(q) q' + U_p(k_p, e) \quad (2)$$

The first term of Equation (2) is positive because the inertia matrix $M(q)$ by definition is positive. The second term U_p is the potential energy, which is a quadratic function of the position error e , and by design, k_p is the matrix of the proportional gain, which is positive. Therefore, $V(q', e)$ is defined as positive.

If Equation (2) is derived, it provides the following as a result.

$$V'(q', e) = q'^T M(q) q'' + \frac{1}{2} q'^T M'(q) q' + \frac{\partial}{\partial e} U_p(k_p, e) e' \quad (3)$$

where $e' = -q'$ and $q'' = (M(q))^{-1} (\nabla U_p(k_p, e)) - Br_v(k_v, q') - C(q, q')q' - C_B q'$. Substituting this into Equation (3), V' can be expressed as in Equation (4):

$$V'(q', e) = -q'^T Br_v(k_v, q') - q'^T C(q, q')q' + \frac{1}{2} q'^T M'(q) q' - q'^T C_B q' \quad (4)$$

where $-q'^T C(q, q')q' + \frac{1}{2} q'^T M'(q) q'$ and $C(q, q')q' = M'(q)q' - \frac{\partial}{\partial q} \left(\frac{1}{2} q'^T M'(q) q' \right)$. Therefore, $-q'^T C(q, q')q' + \frac{1}{2} q'^T M'(q) q' = 0$. Taking this into consideration, Equation (4) can be rewritten and is represented in Equation (5):

$$V'(q', e) = -q'^T Br_v(k_v, q') - q'^T C_B q' \quad (5)$$

where Equation (5) is less than or equal to zero; in this way, it complies with Lyapunov's stability theorem, and in this way, the global stability of Equation (1) is proved.

Considering the above, the values of the gains are chosen, and they must be defined as positive. For the case of proportional gain, it is expressed in Equation (6):

$$K_p = \begin{bmatrix} 350 & 0 & 0 & 0 \\ 0 & 450 & 0 & 0 \\ 0 & 0 & 450 & 0 \\ 0 & 0 & 0 & 350 \end{bmatrix} \quad (6)$$

The values of the speed gain are observed in Equation (7).

$$K_v = \begin{bmatrix} 65 & 0 & 0 & 0 \\ 0 & 95 & 0 & 0 \\ 0 & 0 & 95 & 0 \\ 0 & 0 & 0 & 20 \end{bmatrix} \quad (7)$$

The values of the integrative gain are presented in Equation (8).

$$K_i = \begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & 135 & 0 & 0 \\ 0 & 0 & 135 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix} \quad (8)$$

It is worth mentioning that one of the advantages of Equation (8) is that it provides a small boost when the robot cannot climb for the effects of the weights of the other links. However, this gain causes the output response of this degree of freedom to rattle; q' is used to dampen the action that generates the proportional gain, broadly speaking, and what $K_v q'$ does is decrease the speed of the motor when it approaches the desired value ($S.P$).

The viscosity coefficient matrix is seen in Equation (9):

$$C_B = \begin{bmatrix} 0.2 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 0.2 \end{bmatrix} \quad (9)$$

3.3. Open-Loop Simulations

In this subsection, we present the simulations conducted on the 4-DOF robot. The software utilized for the development of this subsection is Matlab/Simulink 2019, with the ODE 45 equation solver employed. Simulations are executed for the dynamics of the 4-DOF robot and for the implementation of the PI-V controller for the 4-DOF robot.

For the development of this section, it is necessary to use Equation (A20) and provide the parameters of the robot of 4-DOF. These are presented in Table 1 and Figure 2.

Table 1. 4-DOF robot parameters.

Id	Symbol	Unit	Description
1	m_1	0.5 kg	Mass of the first link, the link coupling, and the motor
2	m_2	0.5 kg	Mass of the second link, the link coupling, and the motor
3	m_3	0.2 kg	Third link mass, link coupling, two bearings, and motor
4	m_4	0.1 kg	Flange mass
5	g	9.8 m/s ²	Gravity

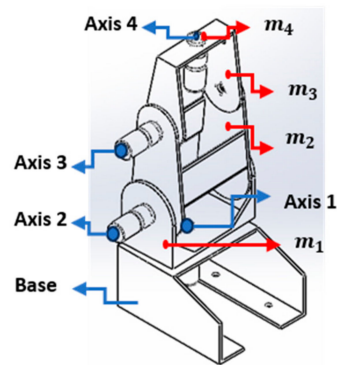


Figure 2. Robot sketch.

The distances between the robot links are listed in Table A1, which is found in Appendix A, Section 2. One of the first objectives is to determine how the simulation of the robot behaves when no force is applied to it and its robot positions are zero (initial conditions or steady state). This behavior is seen in Figure 3, where axis 1 remains at zero, axis 2 has an angular displacement of -0.6° due to the weight of the other links, and in a very similar way, occurs with axis number 3, which has a small angular displacement of approximately -0.2° . Axis 4 stays at approximately zero degrees.

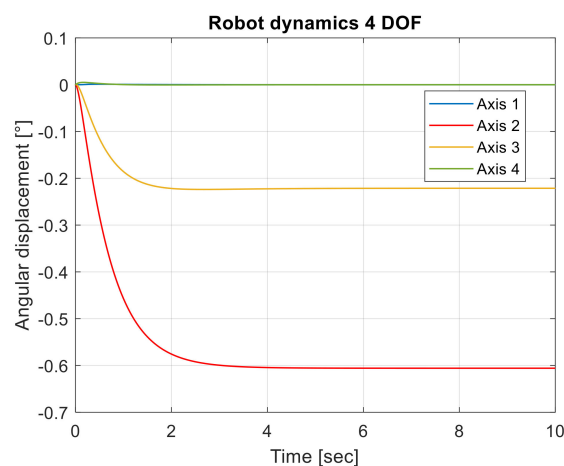


Figure 3. 4-DOF robot dynamics. No upfront conditions and zero torque.

This indicates that the dynamic model of the 4-DOF robot behaves appropriately, which may be because of the decimal points at the time the equation solver begins to solve the system. After the dynamics of the robot, initial conditions are provided, which are $\theta_1 = 0, \theta_2 = 90^\circ, \theta_3 = 90$, and $\theta_4 = 0$, and the behavior of it is seen in simulation. These initial conditions are used to exemplify a real situation of how the robot would be accommodated. This behavior is shown in Figure 4, where axes 2 and 3 start at 90° and fall. Axis 2 reaches -34° , and axis 3 reaches -12° because there are several operations with decimal numbers; however, the importance of this is that these axes should fall.

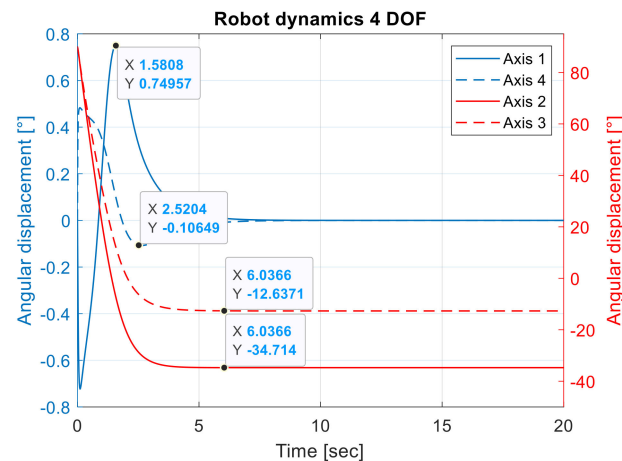


Figure 4. 4-DOF robot dynamics with initial conditions and zero torques.

Subsequently, the behavior of the robot's dynamics when a force is applied is observed; in this case, the force that is going to be applied is only a unitary step-type input for a time of two seconds, exemplifying a peak of energy in a very short time. In this case, the initial condition is zero, as shown in Figure 5.

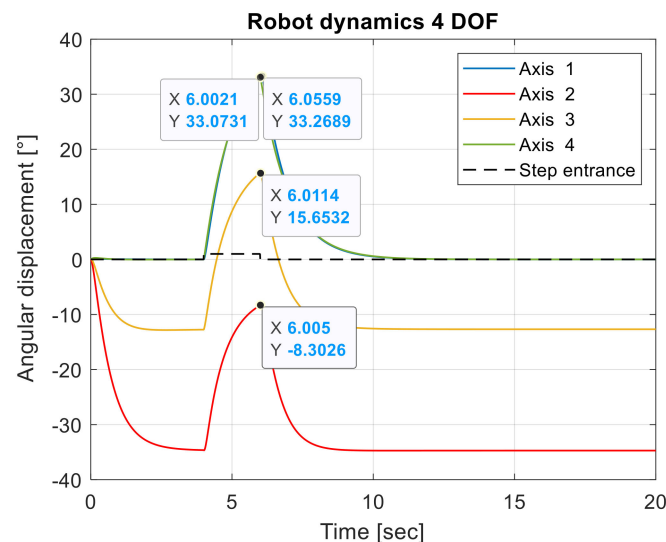


Figure 5. 4-DOF robot dynamics with a step entrance.

The analysis in Figure 5 is important because it reveals the behavior when the robot is energized for a certain time. The behaviors of axes 1 and 4 are similar because the amount of energy required by these axes is not high, resulting in an angular displacement of approximately 33° . Conversely, axis 2, under these temporal conditions and with this unit step input, is insufficient to elevate the additional linkages of the robotic system,

reaching a maximum of -8° approximately. For axis 3, the amount of force required was much less than that for axis 2. Therefore, it can break its starting point and reach 15° .

3.4. Closed-Loop Simulations

In this section, Equation (A20) from Appendix A, Equations (1) and (6)–(9) are presented, with the system parameters found in Tables 1 and A1. Additionally, the block diagram is illustrated in Figure 1.

The results obtained are shown in Figure 6, where the response time is approximately 1.7 s for a constant 45° input.

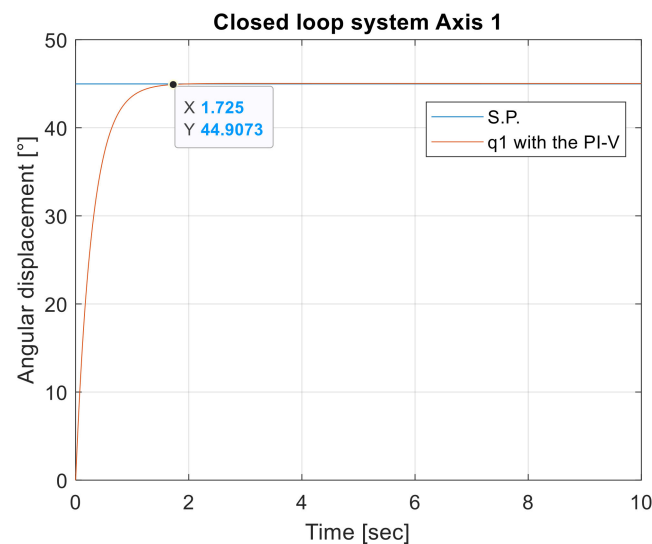


Figure 6. Response of the output signal q_1 in a feedback system with a PI-V controller.

For axis 2, the response of the system with the controller in Figure 7 shows that the reference angle we are looking for is 115° , which is approximately 2 s. It is worth mentioning that one of the most complicated axes to control is axis 2, which is due to all the weights falling on this axis.

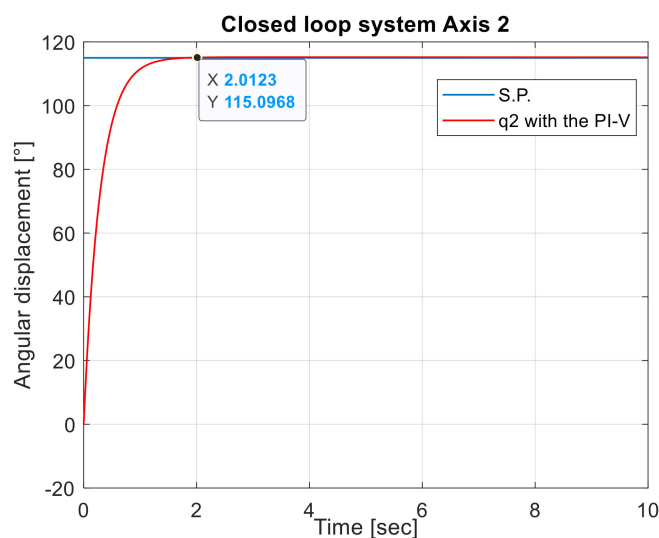


Figure 7. Response of the output signal q_2 in a feedback system with a PI-V controller.

Another important point is that axis 2 should not have over impulses since this would cause axis 2 to not reach the reference value and oscillate for a very long time; therefore, the integrative gain must be very small or non-existent. However, if integrative gain does

not exist and the robot's position exceeds the reference value (e.g., 120°), the robot may be left with a steady-state error.

The response of axis 3 is shown in Figure 8, where its response time is approximately 2 s with a reference value of 115° .

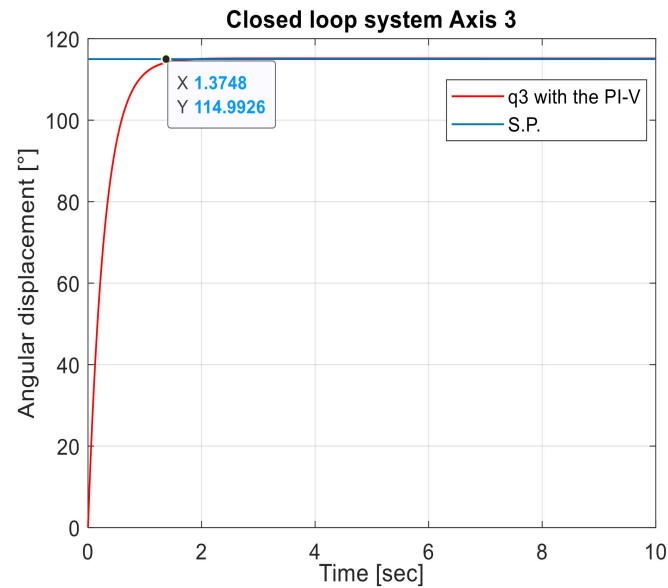


Figure 8. Response of the output signal q_3 in a feedback system with a PI-V controller.

Finally, the response of axis 4 is shown in Figure 9, where its response time is approximately 2 s with a reference value of 45° .

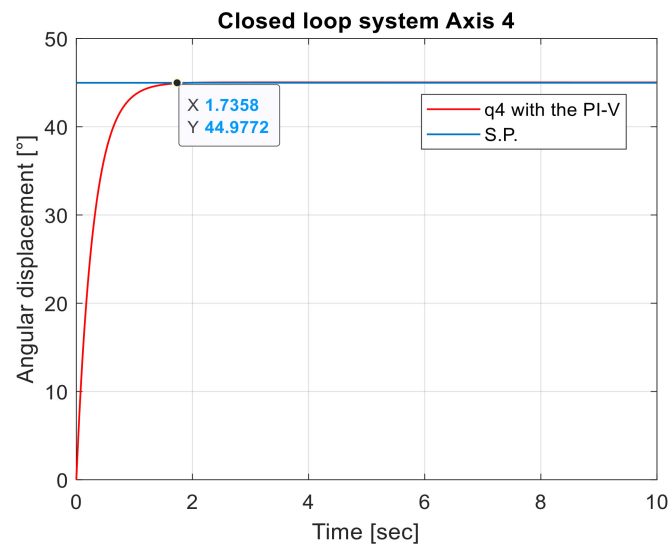


Figure 9. Response of the output signal q_4 in a feedback system with a PI-V controller.

It is worth mentioning that the reference values were placed randomly, even in extremely complicated positions for axes 2 and 3, where the maximum load was generated. An example of this is shown in Figure 10, which tracks the trajectory where there are two-step functions in very short periods of time. In the first step, because of the interval time of said function, it does not reach the reference value, which is 45° ; in the second step, it manages to reach the reference value with a response time of 1.7 s.

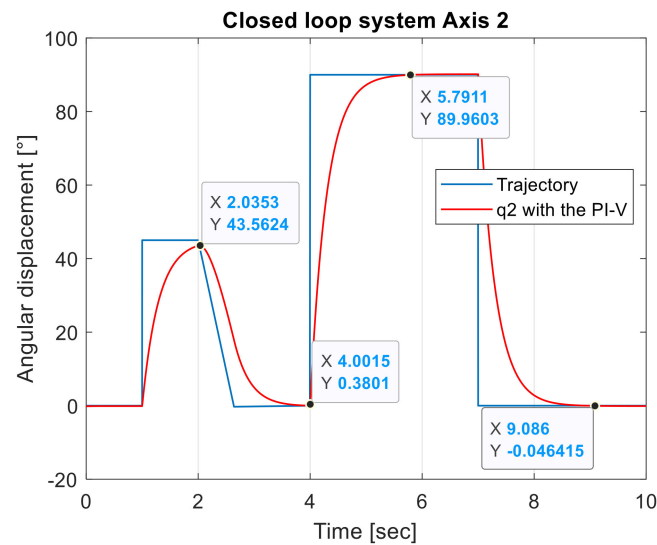


Figure 10. Response of the output signal q_2 in a feedback system with a PI-V controller.

It is worth mentioning that the changes in the trajectory of Figure 10 are drastic and in very short periods; therefore, the gain of the proportional part, which corresponds to axis 2, is increased. The results, which were significantly improved, are shown in Figures 10 and 11.

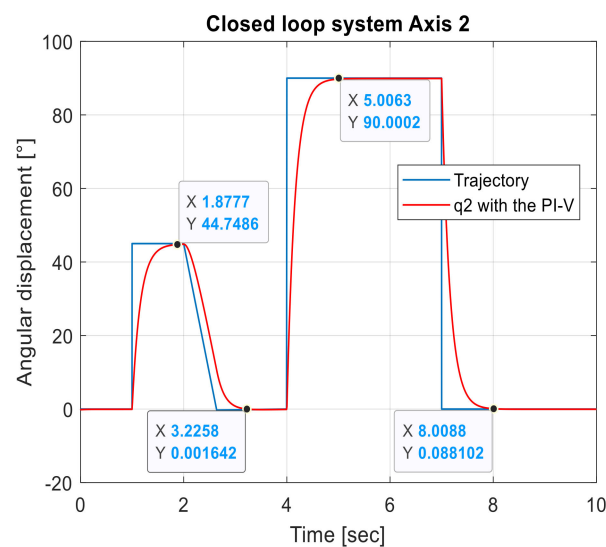


Figure 11. Response of the output signal q_2 in a feedback system with a version 2 PI-V controller.

One of the advantages of this controller is that, as the signal approaches the reference value, the speed decreases, which provides it the peculiarity of generating smooth responses.

3.5. Design of the Bounded PI-V Controller for the Four-Degrees-of-Freedom Robot

The design of this controller is used because the motor “torque” generated by the PI-V controller is too high, leading to overdrives, rattles, and thermal problems after a certain time in operation. It is worth mentioning that this occurs only when experimentation is carried out, and because of this situation in [31,34], it was found that one way to avoid these problems is by limiting the PI-V controller using functions such as the function $\text{sign}(\cdot)$, $\text{tanh}(\cdot)$, and $\text{atan}(\cdot)$. For this study, the function chosen is $\text{tanh}(\cdot)$, and the integrative part is eliminated because it generates a small overdrive.

The closed-loop system is presented in Equation (10):

$$\frac{d}{dt} \begin{pmatrix} e \\ q' \end{pmatrix} = \begin{pmatrix} -q' \\ M(q)^{-1} [K_p \tanh(Pe) - K_v \tanh(\Phi q') - Cq' - G - C_B] \end{pmatrix} \quad (10)$$

where $P = \text{diag}[\rho_1, \rho_2, \rho_3, \dots, \rho_n] \in \mathbb{R}^{n \times n}$ and $\Phi = \text{diag}[\phi_1, \phi_2, \phi_3, \dots, \phi_n] \in \mathbb{R}^{n \times n}$.

Lyapunov Stability Criterion for the Bounded PI-V

Similarly, the Lyapunov stability criterion of Equation (10) was applied to determine whether the closed-loop system is globally stable.

$$V(q', e) = \frac{1}{2} q'^T M(q) q' + \begin{bmatrix} \sqrt{\ln(\cosh(\rho_1 e_1))} \\ \sqrt{\ln(\cosh(\rho_2 e_2))} \\ \vdots \\ \sqrt{\ln(\cosh(\rho_n e_n))} \end{bmatrix}^T K_p \begin{bmatrix} \sqrt{\ln(\cosh(\rho_1 e_1))} \\ \sqrt{\ln(\cosh(\rho_2 e_2))} \\ \vdots \\ \sqrt{\ln(\cosh(\rho_n e_n))} \end{bmatrix} \quad (11)$$

Deriving Equation (11),

$$V'(q', e) = q'^T M(q) q'' + q'^T M'(q) q' + \frac{d}{dt} \begin{bmatrix} \sqrt{\ln(\cosh(e_1))} \\ \sqrt{\ln(\cosh(e_2))} \\ \vdots \\ \sqrt{\ln(\cosh(e_n))} \end{bmatrix}^T K_p P^{-1} \begin{bmatrix} \sqrt{\ln(\cosh(e_1))} \\ \sqrt{\ln(\cosh(e_2))} \\ \vdots \\ \sqrt{\ln(\cosh(e_n))} \end{bmatrix} \quad (12)$$

Taking into account $q'' = M(q)^{-1} [K_p \tanh(Pe) - K_v \tanh(\Phi q') - Cq' - G - C_B]$ and reducing terms, Equation (12) can be written as

$$V'(q', e) = -q'^T K_v \tanh(Pq') - q'^T Bq' \leq 0 \quad (13)$$

where $q'^T K_v \tanh(Pq') > 0$. In this way, it is shown that Equation (10) is globally stable. Considering the above, the values of the gains are chosen, and they must be defined as positive in the case of proportional gain, as demonstrated in Equation (14).

$$K_p = \begin{bmatrix} 350 & 0 & 0 & 0 \\ 0 & 450 & 0 & 0 \\ 0 & 0 & 450 & 0 \\ 0 & 0 & 0 & 350 \end{bmatrix} \quad (14)$$

The values of the speed gain are observed in Equation (15).

$$K_v = \begin{bmatrix} 150 & 0 & 0 & 0 \\ 0 & 250 & 0 & 0 \\ 0 & 0 & 150 & 0 \\ 0 & 0 & 0 & 150 \end{bmatrix} \quad (15)$$

The gain values for P and Φ are presented in Equations (16) and (17).

$$P = \begin{bmatrix} 40 & 0 & 0 & 0 \\ 0 & 20 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1.5 \end{bmatrix} \quad (16)$$

$$\Phi = \begin{bmatrix} 40 & 0 & 0 & 0 \\ 0 & 20 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1.5 \end{bmatrix} \quad (17)$$

The higher the values of P and Φ , the faster the system reaches the reference value; therefore, it is important to increase the gain values of K_v to mitigate the overdrive that is generated. This is observed in the simulation graphs.

3.6. Closed-Loop Simulations for Bounded PI-V

In this section, Equation (A20) of Appendix A, Equation (1), and Equations (14)–(17), plus the system parameters listed in Tables 1 and A1 are necessary for simulation.

The results obtained can be seen in Figure 12, where the response time is approximately 0.7 s for a constant 45° input. One drawback is that the trigonometric function generates small overpulses ranging from 1° to 3° ; however, the system becomes faster and safer when it comes to implementation.

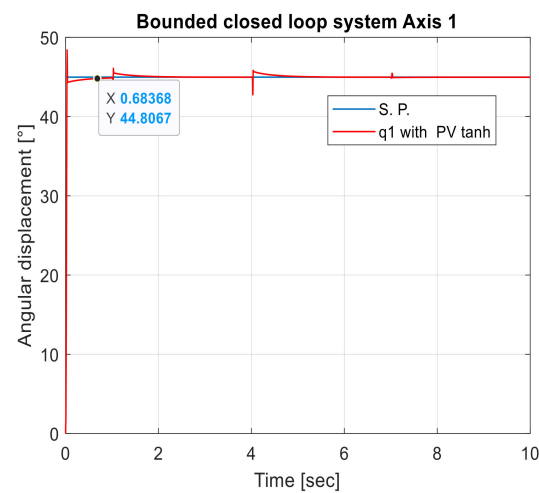


Figure 12. Output signal response in q_1 in a system feedback with a controller $PV \tanh$.

Axis number 2 is where the performance of the controller $PV \tanh$ can be seen since this axis is the one that carries the weight of links 2, 3, and 4; in addition to this, it is tested at angles where its highest load point is located, such as the angle of 45° , and then it has to be placed in a totally vertical way (90°) and then go to 0° in a very abrupt way, something that does not occur in industrial robots since in most cases they are ramp-type functions since the descent or ascent is always gradual.

Figure 13 shows the path tracking of axis 2 for this controller, where the path tracking is almost identical to that of the baseline.

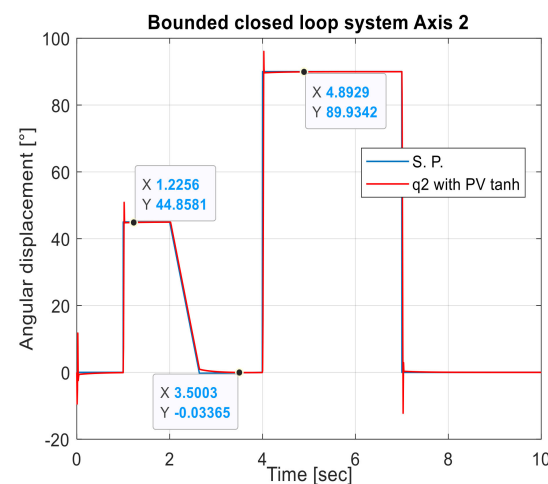


Figure 13. Output signal response in q_2 in a system feedback with a controller $PV \tanh$.

As illustrated in Figure 14, axis 3 exhibits an arrival time of approximately 0.1 s and an angle closely approximating the reference value (115°).

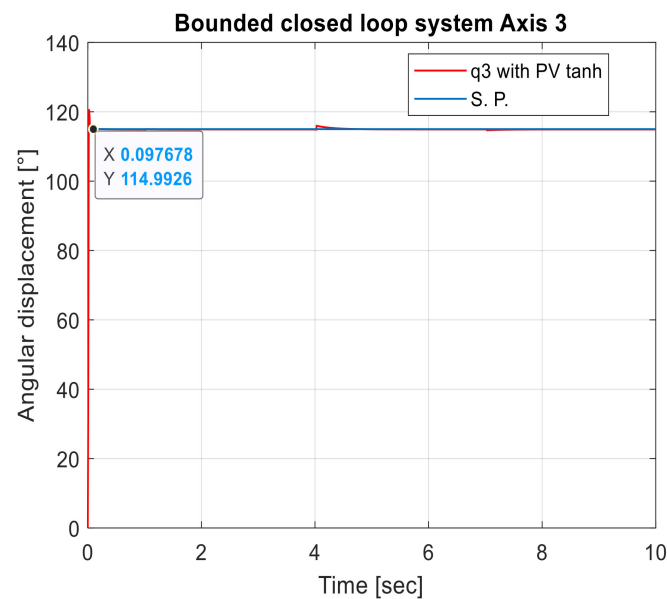


Figure 14. Output signal response in q_3 in a system feedback with a controller $PV \tanh$.

The behavior of axis 4 is shown in Figure 15, where the response time is approximately 1 s.

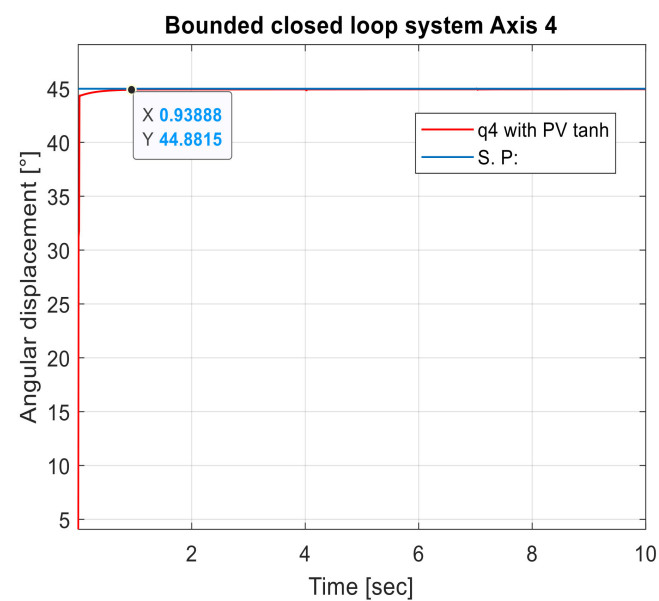


Figure 15. Output signal response in q_4 in a system feedback with a controller $PV \tanh$.

A better way to determine if there is trajectory tracking in a more appropriate way is by following a sine or cosine signal; for this work, the angular displacement goes from 0° to 180° (-90°). The result is shown in Figure 16.

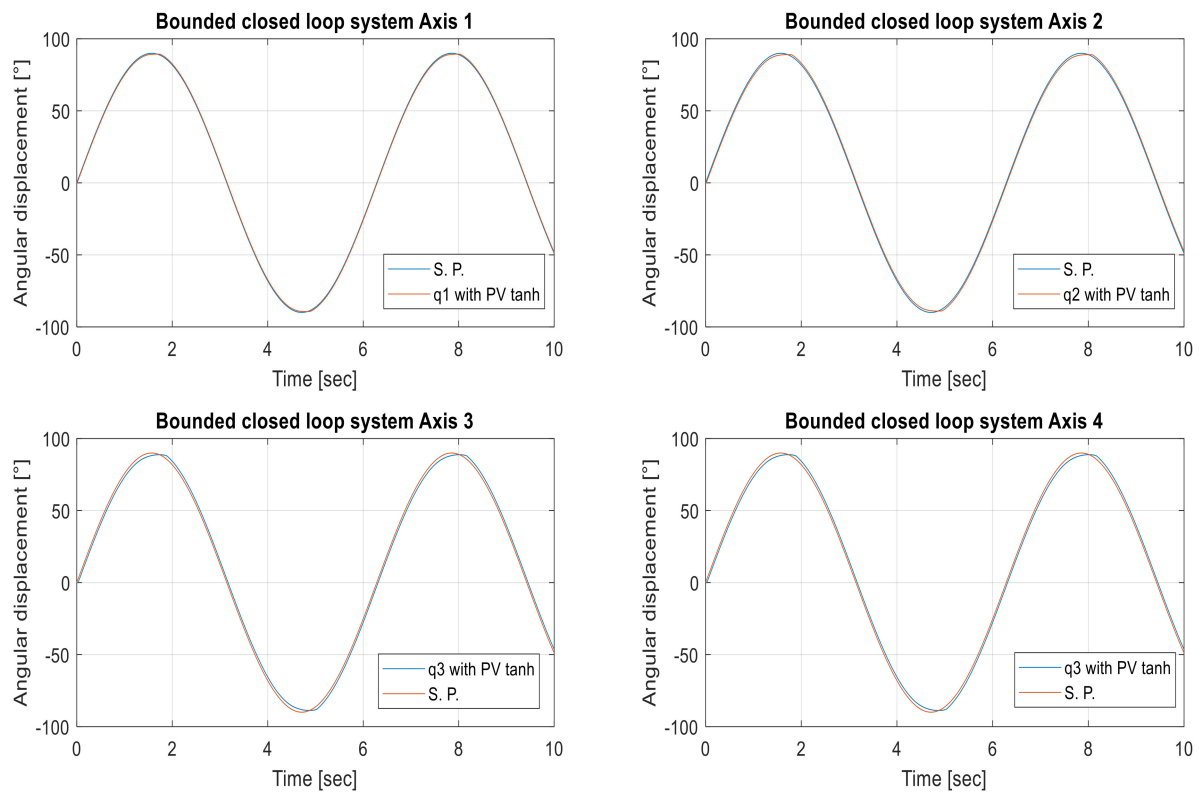


Figure 16. Responses of the output signals in a feedback system with a *PVtanh* controller with a sinusoidal reference signal.

3.7. Simulation Results of Robot Errors

In this section, an analysis of different errors of the controllers applied to the robot is carried out. Likewise, a simulation is presented where the two controllers presented in this article are compared.

In Figure 17, the output of each of the robot's axes is compared, applying the two controllers mentioned above. It can be seen that the *PV tanh* controller has a better performance than the *PI – V* controller.

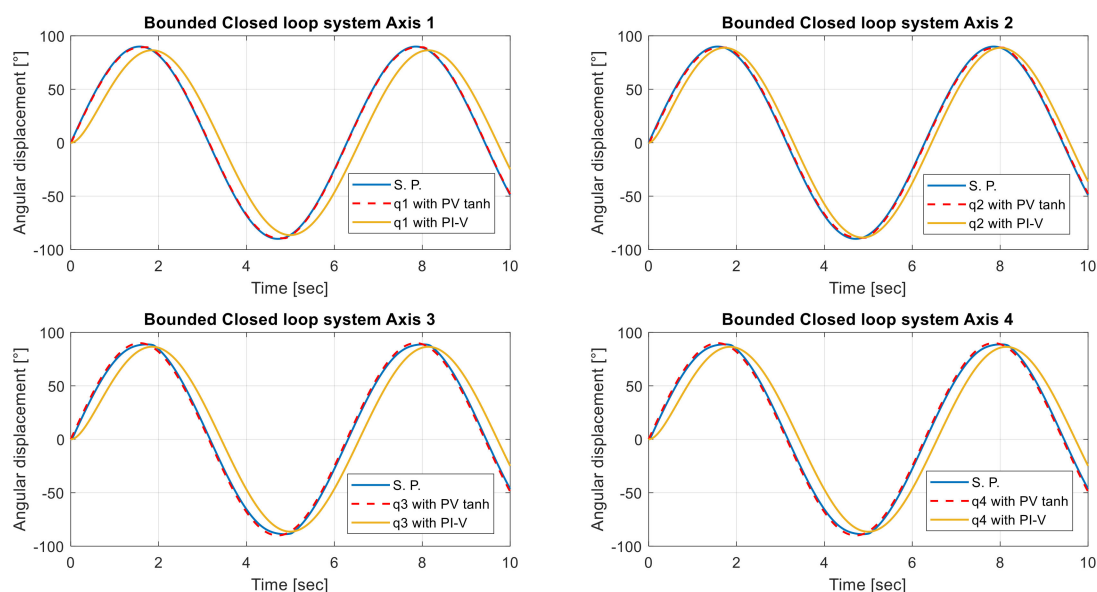


Figure 17. Comparison of *PVtanh* controller vs. *PI – V* controller.

In Figure 18, you can see the error graph of each controller, where the *PV tanh* controller has an error closer to zero.

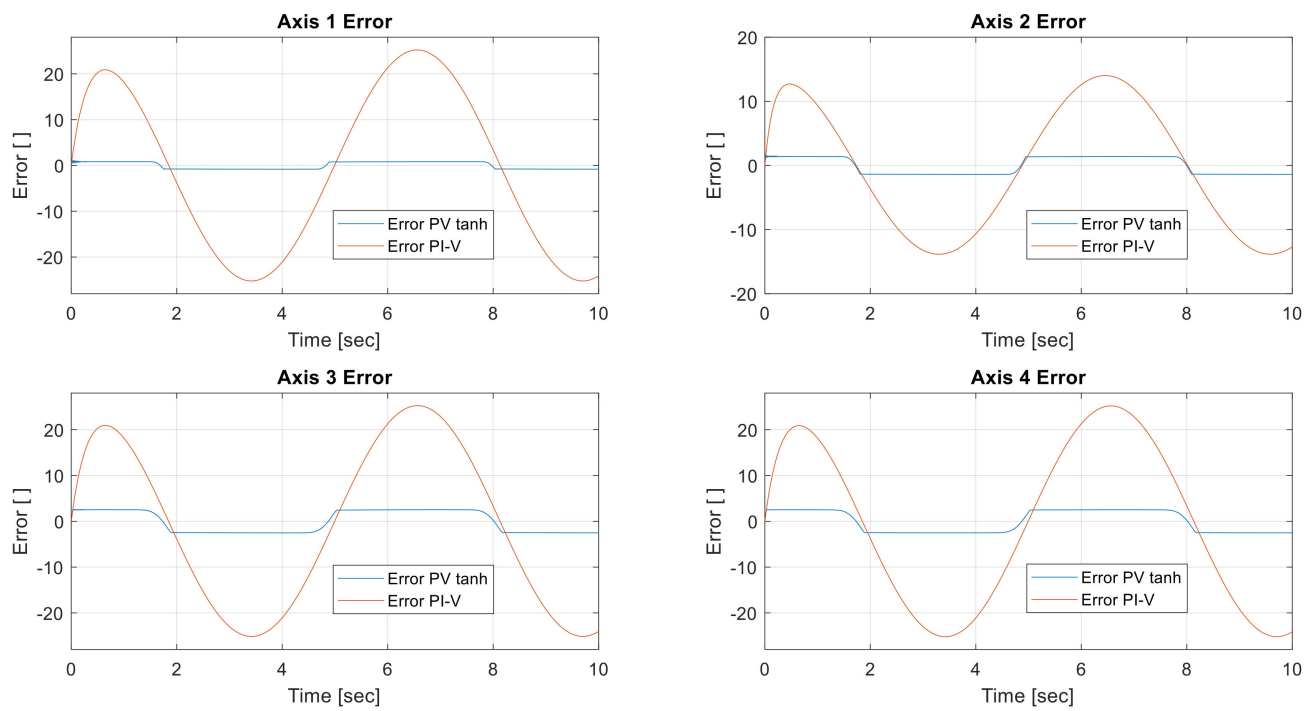


Figure 18. Comparison of *PV tanh* controller error vs. *PI – V* controller error.

To quantify what is seen in Figure 18, different performance indices are developed, as seen in Table 2. RMSE corresponds to the root mean square error, which is represented by Equation (18), IAE is the absolute integral of the error, as seen in Equation (19), ITAE is the integral of the absolute error multiplied by time, as represented by Equation (20), ITSE is the integral of the squared error multiplied by the time, as seen in Equation (21), and finally, ISE is the integral of the squared error, as seen in Equation (22).

$$RMSE = \sqrt{\frac{1}{T} \int_0^T e^2 dt} \quad (18)$$

$$IAE = \int_0^T |e| dt \quad (19)$$

$$ITAE = \int_0^T t |e| dt \quad (20)$$

$$ITSE = \int_0^T t e^2 dt \quad (21)$$

$$ISE = \int_0^T e^2 dt \quad (22)$$

Table 2. Performance indicators for controllers.

Eje 1					
Control	RMSE	IAE	ITAE	ISE	ITSE
PVTANH	0.742	2042	3702	1622	2928
PIV	10,434	32,490	55,664	506,001	809,115
Eje 2					
Control	RMSE	IAE	ITAE	ISE	ITSE
PVTANH	1202	3481	6422	4745	8822
PIV	5961	19,418	35,358	181,531	311,114
Eje 3					
Control	RMSE	IAE	ITAE	ISE	ITSE
PVTANH	1947	6062	11,471	14,680	28,281
PIV	10,443	32,549	55,890	508,019	814,458
Eje 4					
Control	RMSE	IAE	ITAE	ISE	ITSE
PVTANH	1945	6055	11,502	14,647	28,439
PIV	10,434	32,481	55,648	505,822	808,838

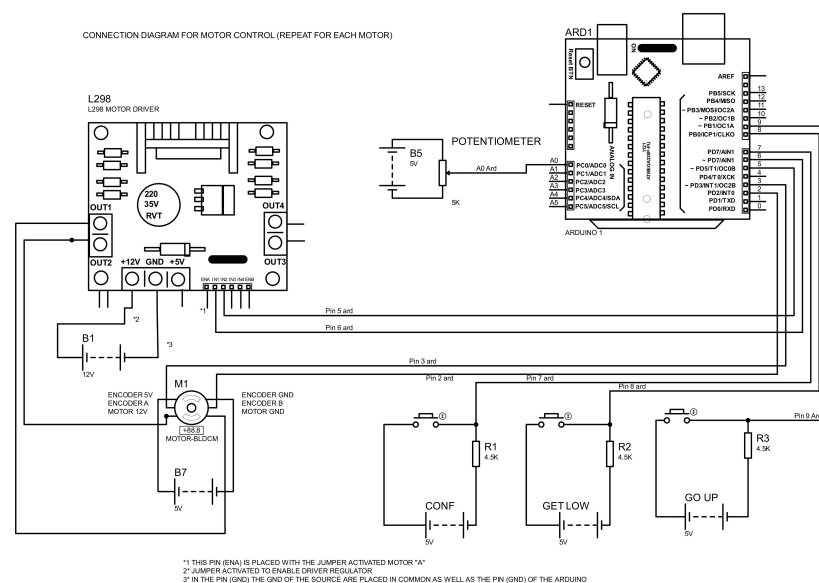
4. Prototype Development

The electrical diagrams are developed in PROTEUS version 8. The programming of the microcontroller is developed in Arduino IDE

4.1. Electrical Diagrams

In this section, the electrical and electronic diagrams are made where a direct-current motor controller board is integrated. This board integrates an L298N IC, which amplifies the PWM signal of the Arduino to a voltage of 12 VDC, which is the voltage of the motor “JGA25-370”. This motor already comes incorporated with its own encoder, which must make a relationship between the pulses provided by the encoder with respect to the revolutions provided by the motor shaft. This is known as the resolution of the encoder with respect to the motor.

For the development of this project, an Arduino board is used for each axis in the same way as a motor controller. Three buttons are added. The first button has the objective of moving the motor forward, and the second button moves the motor backward; it is worth mentioning that this is a manual movement. Finally, the third button activates the *PV tanh*. The diagram is shown in Figure 19.



The potentiometer sends the reference signal for the motor shaft where 0 volts is 0° and 5 volts is 360° (one turn).

An important part of this diagram is to make the common node of the encoder negative with the negative of the motor power source.

It is worth mentioning that one diagram is made for each motor (axis), providing a total of four similar diagrams, and the difference between them is found in the programming of each microcontroller.

4.2. Results

In this section, there are only photographs of the prototype development. This can be seen in Figures 20–23.

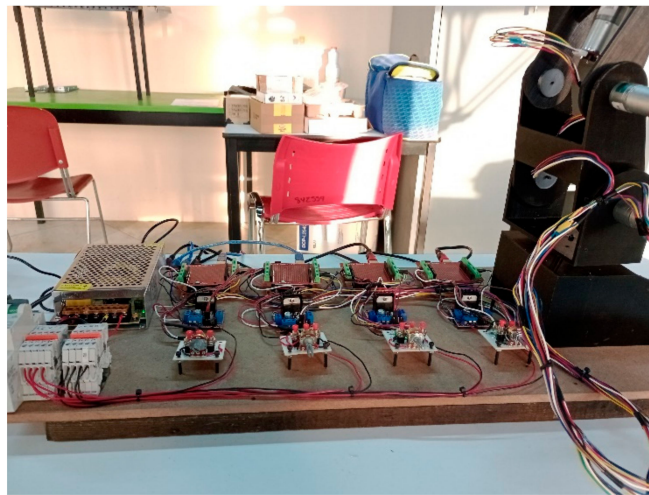


Figure 20. Physical hardware of the robot.

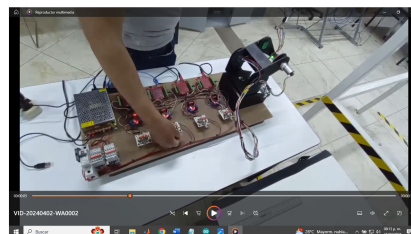


Figure 21. Physical hardware of the robot, test 1.

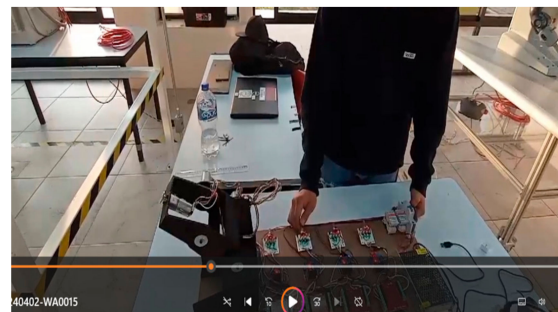


Figure 22. Physical hardware of the robot, test 2.

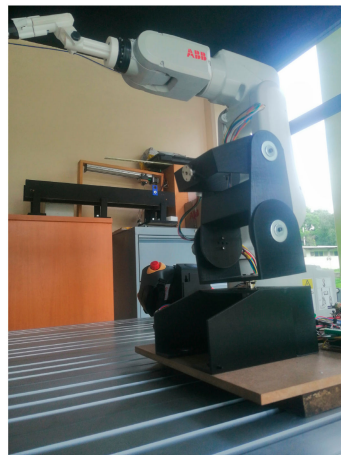


Figure 23. IRB 120 and prototype.

In Figure 20, the third axis is intended to rotate 65 degrees and remain in that position for a long time, verifying that this movement does not affect the position of the other axes, which does not happen.

Figures 21 and 22 provide evidence that there is a video of the movement of each of the robot's axes.

Figure 23 shows the 4-DOF robot in an angular configuration similar to that of the ABB IRB 120 robot, with the aim of the 4-DOF robot remaining in that position for an extended time.

4.3. Experimental Tests

Using the Arduino serial plotter, the behavior of the robot's axes can be seen, as shown in Table 3.

Table 3. Arduino serial plotter signal description.

Line Color		Description
	Blue	Reference value (S. P.)
	Red	Error
	Green	System output signal (angle given by the encoder)
	Orange	Control output value
	Purple	Engine speed

Figure 24 shows the four axes of the robot, with the first axis to be worked on being axis 1 until reaching axis 4. The idea is to vary each of the reference values using the potentiometer.

Axis 1 follows the trajectory appropriately, as seen in Figure 24. A very similar thing occurs with axis 2; this can be seen in Figures 24 and 25. Axis 3 takes a while to respond; however, it follows the trajectory a few milliseconds later, as seen in Figure 26. For axis 4, there is no problem when following the trajectory; subsequently, the reference value with respect to the output value (encoder) is the same, as seen in Figures 26 and 27.

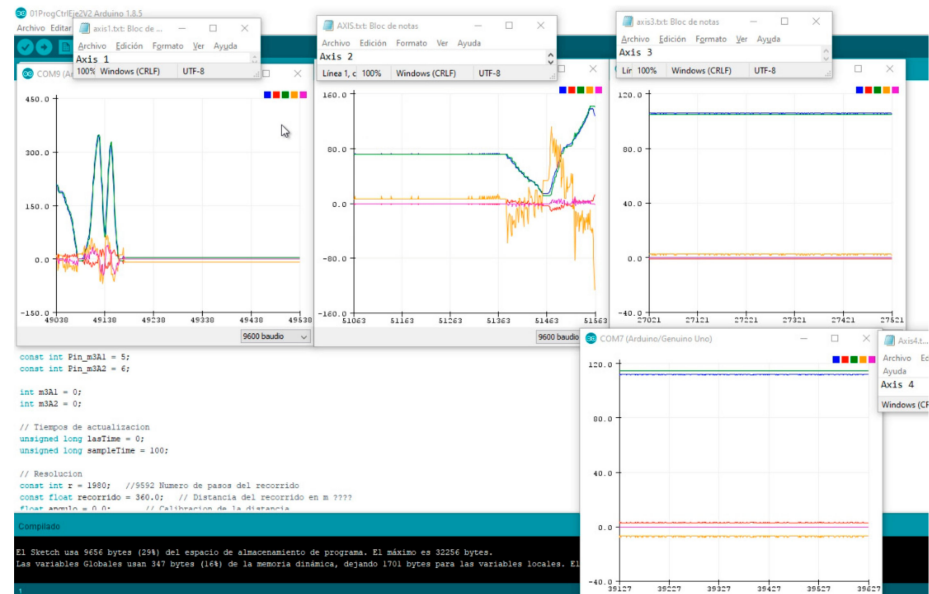


Figure 24. Experimental test part 1.

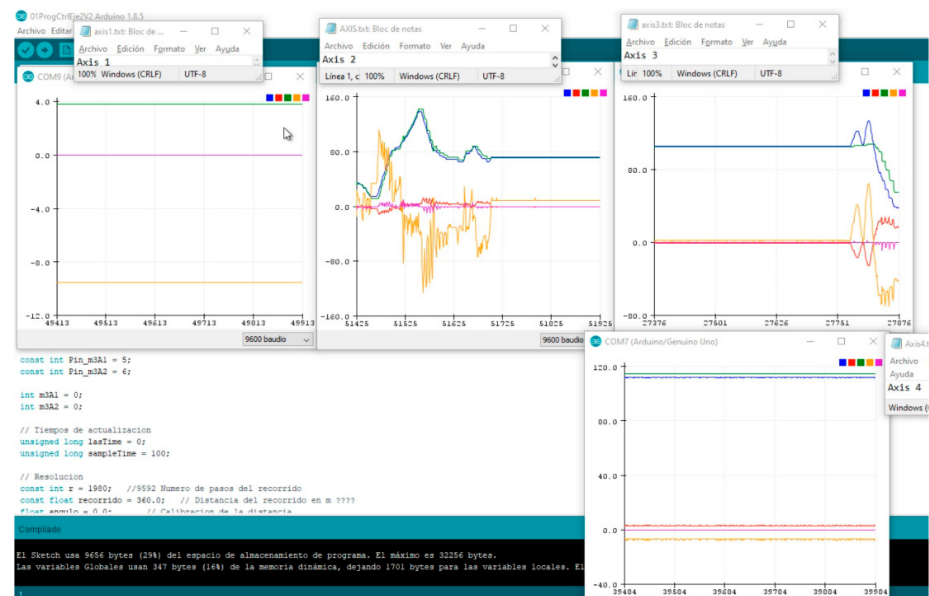


Figure 25. Experimental test part 2.

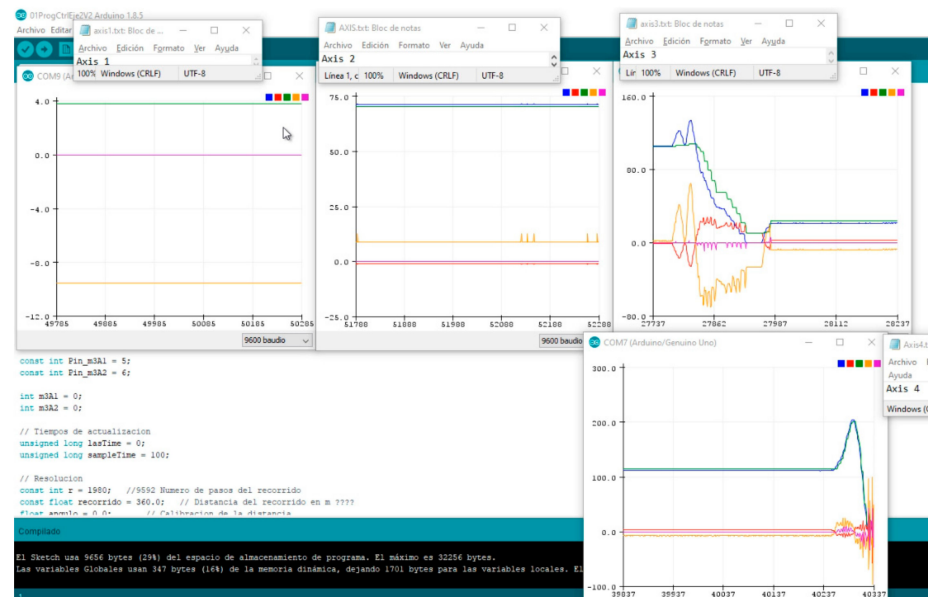


Figure 26. Experimental test part 3.



Figure 27. Experimental test part 4.

For Figure 25, on axis 1, the reference signal (blue) remained at 4 degrees and the output signal (green) remained at the same position as the reference signal. The motor speed signal (purple) does not need to increase or decrease its speed because the output value is at the desired value, therefore, its value is zero. A very similar case occurs for the error signal (red).

For the output control signal (orange) it will always have a small variation because all the axes affect axis 1.

For axis 2 in Figure 25, the reference value moves from 45° to 150° approximately and the output signal (green) follows the reference signal. The reference signal moves abruptly, this is in order to see the controller's response time. An important point is to leave axis 2 at an angle other than 90°, this with the objective of seeing the behavior of this axis when axis 3 and 4 are moved.

Regarding Figure 26, the reference value also changes abruptly from 90° to 0° , and the controller responds appropriately. At the beginning of this test, the output signal does not reach the reference value because its working range on axis three is from 0° to 130° .

For Figure 27, on axis 4, the output signal (green) follows the desired signal (blue) without any issues. There are not many complications in this test because axis 4 has no load. The working range for this axis is from 0° to 360° .

Axis 2 and axis 3 have a steady-state error, which is due to mechanical and electrical issues. The solution to the problem of axis 2 is to install a motor with a better gear system or to install two similar motors that work at the same time. Regarding axis 3, the solution is to install a motor with a better gear system. For the mechanical part, the structure must be improved by installing RIBS at certain points. This can be seen in Video S1 in Supplementary Materials

5. Discussion

The construction of a 4-degree-of-freedom robot is achieved using easily accessible materials. The programmable cards, motor controllers and electronics used are widely known in the field.

A key aspect in the development of this prototype is the decision not to use MG995 servomotors, since the objective is to demonstrate how to implement a PIV controller, which can be applied to different types of direct current motors, considering only the reading of the encoder and the power stage of the motor.

A programming language based on the Arduino platform is created, which facilitates the replication and improvement of the code. In addition, the use of external libraries is avoided in the development of the controller. A section of the code is dedicated exclusively to reading the encoder.

On the other hand, different controllers can be implemented, such as a fuzzy control and controller through other types of trigonometric functions, such as sine, cosine, sign, and tangent arc. In the same manner, the observers could be implemented.

The motor of axis 2 is changed to one that has the capacity to carry a greater weight such that the motor does not overheat or place two motors and that they are activated at the same time. Similarly, a better motor controller is used.

Finite element analysis was also required to determine how much stress the robot's structure could withstand.

On the other hand, it would be good to create a human—machine interface in which the linear displacement is seen through the coordinates (X, Y, Z) and angular displacement of each of the axes.

6. Conclusions

In general, all the objectives were met; in some cases, 100 percent of them were not achieved owing to unknown factors within the development of this project; however, the manufacture of the robot was achieved.

The development of the kinematics was fulfilled more than what was thought; this was thanks to Peter Corke's libraries, since in this way, it could be seen that these kinematics were very similar to what was being sought. It is worth mentioning that in this section, at least three parameter tables by Denavit–Hartenberg were made, where logically, these tables were not correct, as verified by Peter Corke's libraries. An advantage of this is that at the time of dynamic development, it is performed with fewer uncertainties.

According to the design, it was satisfactorily fulfilled, providing the support of seeing the points where the robot could have collisions or where the part will probably bend or fracture. The solution for this was to add supports to the sides or increase the thickness slightly without affecting the length of the motor shaft. On the other hand, the use achieved a design that did not have much weight and was functional, despite the fact that the printing was made in 3D.

For the development of the dynamics, it is necessary that the kinematics are correct because the use of the transformation matrices is required depending on the degree of freedom that is developed. To avoid calculating the entire dynamic algorithm, a code was developed in Matlab that provides the facility to find the mass, inertia, gravity, and Coriolis matrices. This allowed the amount of weight that axle two could support to be assessed.

In the development of the dynamics, despite not applying any electromotive force on the axes, the second and third axes exhibited angular displacements of 2° and 6° , respectively. Through the algorithm that was developed, we tried to find what generated this displacement; however, it was not found correctly, and it was concluded that they were probably details of the decimals that were not considered.

Another important point in the dynamics is to consider the coefficients of friction and viscosity; if this is not done, singularities are generated in the system, and the behavior of the axes cannot be seen as time passes.

At the time of the development of the PI-V controller, it was initially proposed to make a controller where the three gains were directly affected by the error; however, in several simulations, the answers that were obtained always generated oscillations, and after a while, these oscillations stabilized. It is worth mentioning that these oscillations in a real robot are not desirable because they lose precision. The aim was to improve the development of the controller by reading the motor speed and minimizing the speed when the motor shaft reached the desired point.

Another interesting point was the integrative gain because if this gain was zero, the robot was left with a steady-state error. However, if the integrative gain is very large, it generates small oscillations.

In general, the design and implementation of the PI-V controller provided the desired results, as can be seen in the figures of the simulations.

One of the details that the PI-V controller presented at the time of carrying out the experiments was the sustained oscillations, which stopped when a system was implemented, limiting a hyperbolic tangent function for this use case, which greatly improved the simulation and experimentation.

The manufacturing of the robot at the time had several drawbacks, one of which was that a support was not left on one of the sides; the solution for this was to place a piece that would provide the support and, at the same time, provide mobility.

It is worth mentioning that during the experimentation stage, it was found that after approximately an hour, the motor of axis 2 became very hot, causing it to no longer respond adequately. One of the reasons it is important to place an engine with a higher load capacity or place two similar engines that work at the same time.

Generally, at this stage, it is concluded that the results were satisfactory because the structure of the robot withstood very strong moments (torques). Similarly, in the same way, the robot has adequate mobility, its structure supports the weight and movement of the robot, and most importantly, the robot reaches the desired position.

Supplementary Materials: The following are available online at <https://www.mdpi.com/article/10.3390/act13120483/s1>, Video S1.

Author Contributions: Conceptualization, R.B., J.d.J.R., J.P., G.O. and G.S.G.; methodology, E.G., A.Z. and G.S.G.; software, R.B. and M.A.H.; validation, R.B., J.d.J.R., M.A.H., J.P. and A.-J.G.-C.; formal analysis, R.B., J.d.J.R. and M.A.H.; research, R.B. and J.d.J.R.; resources, G.O., A.Z. and G.S.G.; data curation, E.G. and G.O.; writing: preparation of the original draft, R.B.; writing: revision and editing, R.B., J.P. and A.-J.G.-C.; visualization, R.B., E.G. and A.-J.G.-C.; supervision, R.B. and A.Z.; project management, R.B. and M.A.H.; acquisition of funds, M.A.H., J.P., A.-J.G.-C., G.O. and G.S.G. All authors have read and agreed to the published version of the manuscript.

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Data Availability Statement: The data are contained in the article.

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Conflicts of Interest: The authors declare that they have no conflicts of interest.

Abbreviations

DOF	Degrees of Freedom
PID	Integral and Derivative Proportional
PI-V	Integral and Proportional—the gain of the speed
SILS	Single-Incision Laparoscopic Surgery
HIM	Lagrange–Euler
PIDNN	Integral and Derivative Proportional with Neural Networks
PSO	Particle Swarm Optimization
YOLO	You Only Look Once
SLE	Exoskeleton System
PLA	Polylactic acid, thermoplastic type
VCD	Direct-Current Voltage
DH	Denavit–Hartenberg
PWM	Pulse width modulation

Appendix A

Appendix A.1. Development of the 4-Degrees-of-Freedom Robot

In this chapter, we discuss the 3D design of the four-degrees-of-freedom robot to later make its 3D printing. Similarly, the development of electrical diagrams that connect each of the robot's axes is observed.

On the other hand, the development of the kinematics and dynamics of the four-degrees-of-freedom robot is seen, and then the design of the PID controller is developed, and the simulations of the robot in the open loop and closed loop are carried out.

CAD design of the 4-DOF robot

In order to be able to perform the kinematics of the robot, the first step is to create a free-body diagram, which is performed with the Solid Works software. The robot that will be used in this project is an anthropomorphic robot. This robot can be seen in Figure A1.

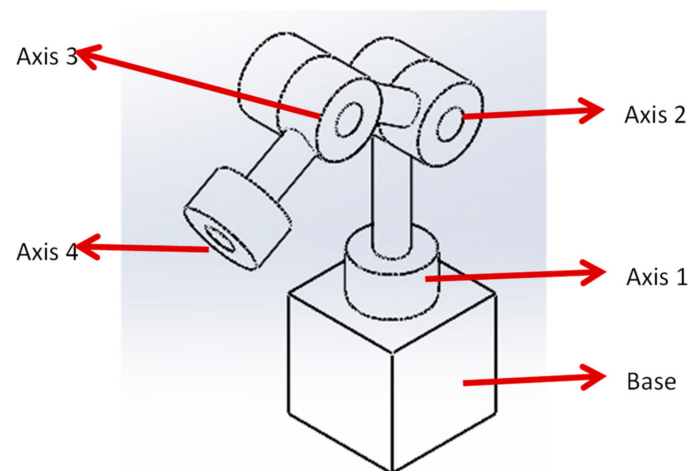


Figure A1. 4-DOF robot free-body diagram.

The four axes in Figure A1 represent the four degrees of freedom of the robot; these degrees of freedom are rotational in type. Taking this into account, each of the pieces that

will make up the robot is made from the base to the end effector. Similarly, flanges and couplings are made for the type of engine to be carried.

In Figure A2, the base of the robot is visible, and the bottom holes are left so that they can be screwed to a table. The holes above hold the flange and motor.

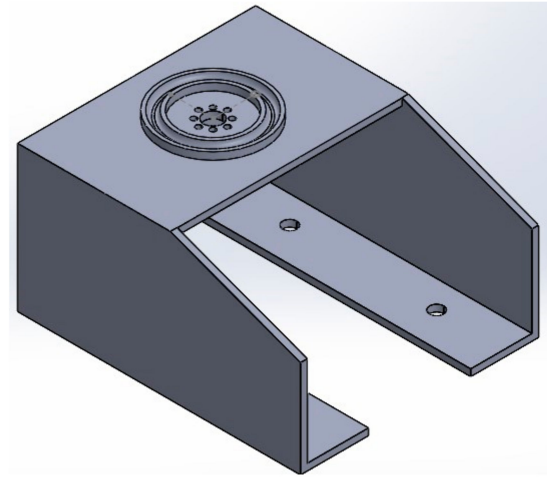


Figure A2. 4-DOF robot base.

In Figure A3, the motor is shown with the part that couples the motor with the base; this part is given the name of the flange. The motor is attached to the holes that are in the base, and the flange holds the next link of the robot.

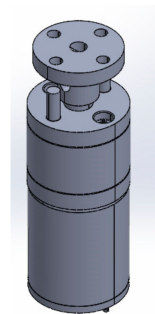


Figure A3. Flanged motor.

Subsequently, the link that corresponds to axis 2 is made; this link is shown in Figure A4. Four ribs are added to this link to prevent the piece from opening or breaking with the robot's movements. Figure A5 shows an example of the shaft assembly with a motor.

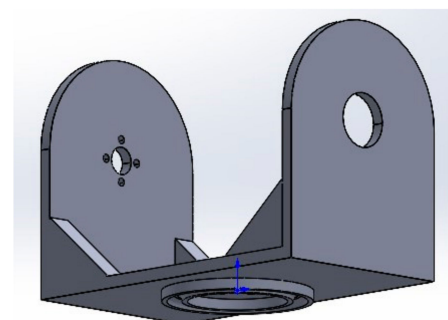


Figure A4. Axis link 2.

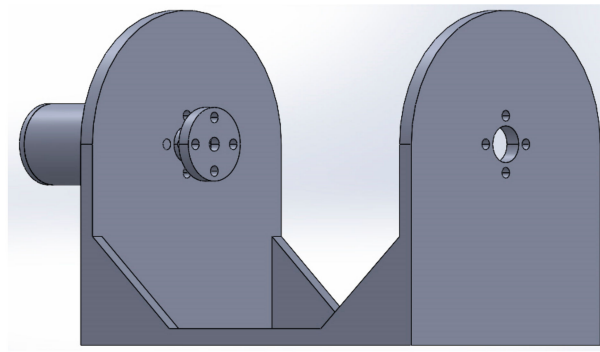


Figure A5. Shaft link 2 with motor and flange.

For the link of axis number 3, which is seen in Figure A6, the figure of an “H” is considered with the aim of joining the motor of axis 2 and the motor of axis 3. It is also considered that the upper part should be slenderer in order to reduce weight. As shown in Figure A6, they are not at the same distance to facilitate the coupling of the next link and the motor.

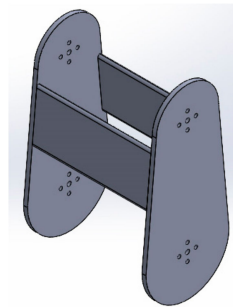


Figure A6. Axis link 3.

In Figure A7, the last link of the robot is shown, the idea of which is to join link 3, the motor of axis 3, and the motor of axis 4. In addition, it should not lose its symmetry with the previous link. In the same way, two ribs are placed in order to provide greater strength and support when placing the motors, and they are in operation.

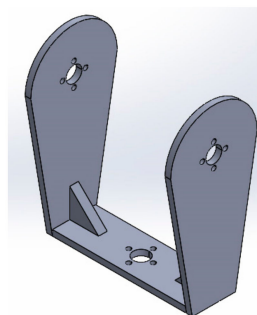


Figure A7. Axis link 4.

Finally, all flanges, links, and motors are assembled to form the robot with which they are going to work. This can be observed in Figure A8. For shaft 4, only the motor with its flange is placed on link 4. Finally, custom bearings are made to provide stability to link 3 with link 2, and in the same way, link 3 with link 4; this is seen in Figure A9.

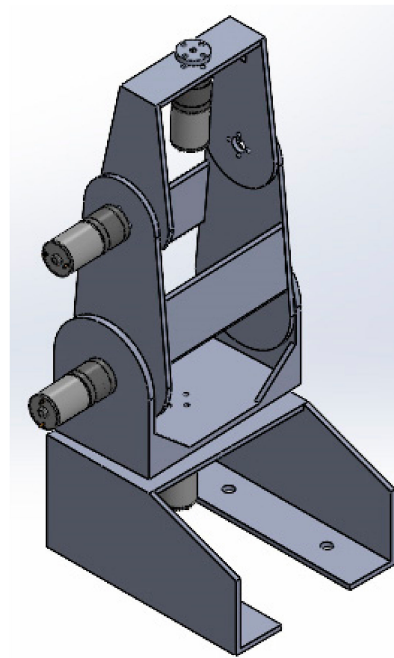


Figure A8. 4-DOF robot assembly.

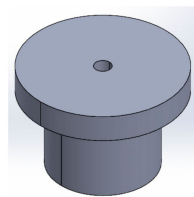


Figure A9. Bearing for links 2, 3, and 4.

Appendix A.2. Direct Kinematics of A Four-Degrees-of-Freedom Robot

The kinematics of a robot are the movements of the robot in a defined space with respect to a reference point. In this subsection, the direct kinematics for a four-degrees-of-freedom (4-DOF) robot is developed. This is with the aim of knowing the position of the robot when a certain angle is applied to each of its axes, resulting in the displacement on the “X” axis, the “Y” axis, and the “Z” axis in the same way indicates the rotational movement that the robot had.

To obtain the kinematics of the 4-DOF robot, the method of Denavit–Hartenberg is used, and this kinematics is checked by the library and instructions of Peter Corke developed for Matlab, the objective of which is to see that the kinematics are adequate since these kinematics will be used to develop its dynamics.

Denavit–Hartenberg’s algorithm

Based on Figure A8, a free-body drawing is made that indicates the number of links and joints and places the coordinate system for each axis in order to form a system that allows the Denavit–Hartenberg table to be made [33]. The steps are as follows.

1. Number the links starting with 1 (first moving link in the chain) and ending with “n” (last moving link). The fixed base of the robot is numbered link 0;
2. Number each joint starting with 1 (the one corresponding to the first degree of freedom) and ending with “n”;
3. Locate the axis of each joint. If it is rotating, the axis will be its own axis of rotation. If it is prismatic, it will be the axis along which the displacement occurs;
4. For i from 0 to $n - 1$, place the axis on z_i the axis of the joint $i + 1$;
5. Place the origin of the base system at $\{S_0\}$ any point on the axis z_0 . The axes x_0 and y_0 shall be positioned so as to form a right-handed system with z_0 ;

6. For i from 1 to $n - 1$, locate the origin of the system $\{S_i\}$ (attached to the link i) at the intersection of the axis z_i with the common normal line z_{i-1} and z_i . If both axes were cut, it would be located $\{S_i\}$ at the cutting point. If they were parallel $\{S_i\}$, it would be located at the joint $i + 1$;
7. Place x_i on the common normal line z_{i-1} and z_i ;
8. Place y_i so that they form a right-handed system with x_i and z_i ;
9. Place the system $\{S_n\}$ at the end of the robot so that z_n matches the address of z_{n-1} and x_n be normal to z_{n-1} and z_n ;
10. Obtain θ_i as the angle that must be rotated around z_{n-1} so that x_{i-1} and x_i remain parallel;
11. Obtain d_i as the distance, measured along z_{i-1} , that would have to be moved $\{S_{i-1}\}$ so that x_i and x_{i-1} were aligned;
12. Obtain a_i as the distance measured along x_i (which would now coincide with x_{i-1}) that would have to be shifted to the new $\{S_{i-1}\}$ one for its origin to match $\{S_i\}$;
13. Obtain α_i as the angle that would have to be rotated around x_i , so that the new $\{S_{i-1}\}$ totally coincided with $\{S_i\}$;
14. Obtain the transformation matrices A_i^{i-1} ;
15. Obtain the transformation matrix that relates the base system to that of the robot end $T = {}^0A_1 * {}^1A_2 * A_n^{n-1}$;
16. The matrix T defines the orientation (rotation submatrix) and position (translation submatrix) of the end and refers to the base as a function of the n articular coordinates.

Taking into account the above algorithm, a free-body diagram of the robot can be constructed, as shown in Figure A10.

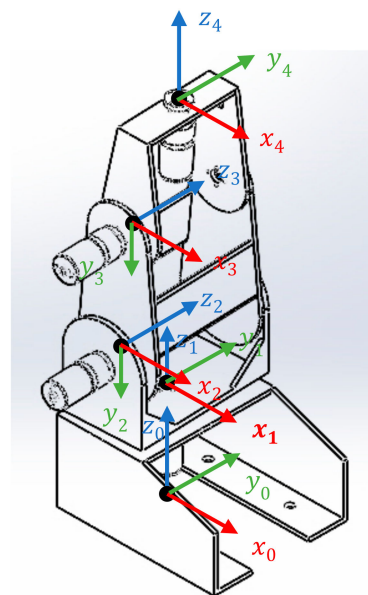


Figure A10. 4-DOF anthropomorphic robot.

Denavit–Hartenberg’s table (DH) is presented in Table A1.

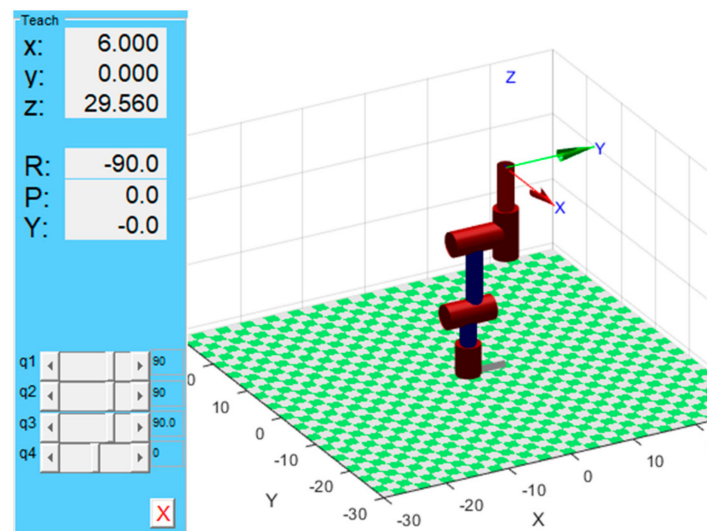
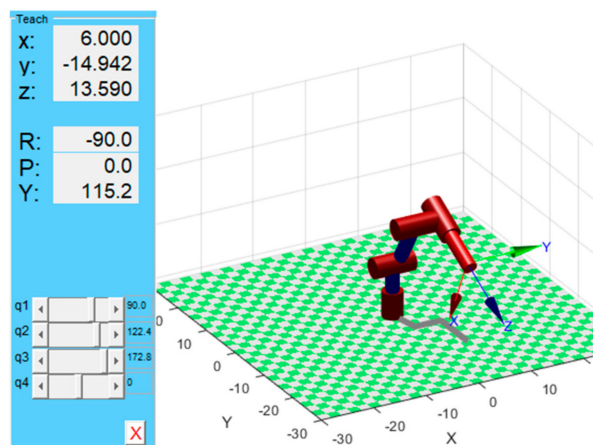
Table A1. Denavit–Hartenberg parameters corresponding to the robot of 4-DOF.

	θ_i	d_i	a_i	α_i
A_1^0	q_1	0.0856 m	0	90°
A_2^1	q_2	0.01 m	0.11 m	0°
A_3^2	q_3	0.05 m	0	0°
A_4^3	q_4	0.1 m	0	90°

Table A2. Meaning of DH parameters.

θ_i	It is the angle formed by the axes x_{i-1} and measured in a plane perpendicular to the axis z_{i-1} , using the right-hand ruler. This is a variable parameter in swivel joints.
d_i	It is the distance along the axis z_{i-1} from the origin of the coordinate system ($i-1$) to the intersection of the axis z_{i-1} with the axis x_i . It is a variable parameter in prismatic joints.
a_i	It is the distance along the axis x_i that goes from the intersection of the axis z_{i-1} with the axis x_i to the origin of the i -th system in the case of rotating joints. For prismatic joints, it is calculated as the shortest distance between the axes z_{i-1} and z_i .
α_i	It is the angle of separation of the axis z_{i-1} and z_i axis, measured in a plane perpendicular to the axis x_i , using the right-hand ruler.

An important detail is to check that Table A1 actually describes the kinematics of the robot you are looking for; in this case, that of an anthropomorphic robot with four DOF, to provide this veracity of the parameters of DH Peter Corke's libraries and Matlab software are used. The results obtained based on Table A1 are presented in Figures A11 and A12. If we compare Figures A1, A10 and A11, it can be observed that they are very similar, which provides veracity that the parameters of Table A1 are correct. In this way, you can start to develop the equations of direct kinematics, which are transformation matrices.

**Figure A11.** 4-DOF robot obtained by using Peter Corke's libraries, view 1.**Figure A12.** 4-DOF robot obtained by using Peter Corke's libraries, view 2.

Appendix A.3. Homogeneous Transformation Matrices

In this subsection, the homogeneous transformation matrices are obtained, which represent steps 14, 15, and 16 of DH. To obtain the homogeneous transformation matrices, Equation (A1) is considered.

$$A_1^0 = \begin{pmatrix} \cos(\theta_i) & -\cos\alpha_i \sin(\theta_i) & \sin(\alpha_i) \sin(\theta_i) & a_i \cos(\theta_i) \\ \sin(\theta_i) & \cos(\alpha_i) \cos(\theta_i) & -\sin(\alpha_i) \cos(\theta_i) & a_i \sin(\theta_i) \\ 0 & \sin\alpha_i & \cos(\alpha_i) & d_i \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (A1)$$

Taking into account Equation (A1) and Table A1, the homogeneous transformation matrix is performed for the first link, which is represented by Equation (A2).

$$A_1 = \begin{pmatrix} \cos q_1 & 0 & \sin q_1 & 0 \\ \sin q_1 & 0 & -\cos q_1 & 0 \\ 0 & 1 & 0 & 0.0856 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (A2)$$

In a very similar manner, it is performed for the other axes, resulting in Equations (A3)–(A5):

$$A_2 = \begin{pmatrix} \cos q_2 & -\sin q_2 & 0 & 0.11 \cos q_2 \\ \sin q_2 & \cos q_2 & 0 & 0.11 \sin q_2 \\ 0 & 0 & 1 & 0.1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (A3)$$

$$A_3 = \begin{pmatrix} \cos q_3 & -\sin q_3 & 0 & 0 \\ \sin q_3 & \cos q_3 & 0 & 0 \\ 0 & 0 & 1 & 0.05 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (A4)$$

$$A_4 = \begin{pmatrix} \cos q_4 & 0 & \sin q_4 & 0 \\ \sin q_4 & 0 & -\cos q_4 & 0 \\ 0 & 1 & 0 & 0.1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (A5)$$

To determine the general transformation matrix of the system, Equation (A6) is calculated.

$$T = A_1 * A_2 * A_3 * A_4 \quad (A6)$$

Equation (A6) can be written as follows:

$$a_{11} = (\cos q_1)((\cos q_2)(\cos q_3 \cos q_4 - \sin q_3 \sin q_4) - (\sin q_2)(\cos q_3 \sin q_4 + \cos q_4 \sin q_3))$$

$$a_{21} = (\sin q_1)((\cos q_2)(\cos q_3 \cos q_4 - \sin q_3 \sin q_4) - (\sin q_2)(\cos q_3 \sin q_4 + \cos q_4 \sin q_3))$$

$$a_{31} = (\cos q_2)(\cos q_3 \sin q_4 + \cos q_4 \sin q_3) + (\sin q_2)(\cos q_3 \cos q_4 - \sin q_3 \sin q_4)$$

$$a_{41} = 0$$

$$a_{12} = \sin q_1$$

$$a_{22} = -\cos q_1$$

$$a_{32} = 0$$

$$a_{42} = 0$$

$$a_{13} = (\cos q_1)((\cos q_2)(\cos q_3 \sin q_4 + \cos q_4 \sin q_3) + (\sin q_2)(\cos q_3 \cos q_4 - \sin q_3 \sin q_4))$$

$$a_{23} = (\sin q_1)((\cos q_2)(\cos q_3 \sin q_4 + \cos q_4 \sin q_3) + (\sin q_2)(\cos q_3 \cos q_4 - \sin q_3 \sin q_4))$$

$$a_{33} = (\sin q_2)(\cos q_3 \sin q_4 + \cos q_4 \sin q_3) - (\cos q_2)(\cos q_3 \cos q_4 - \sin q_3 \sin q_4)$$

$$\begin{aligned}
 a_{43} &= 0 \\
 a_{14} &= 0.25\sin q_1 + 0.11\cos q_1\cos q_2 \\
 a_{24} &= 0.11\cos q_2\sin q_1 - 0.25\cos q_1 \\
 a_{34} &= 0.11\sin q_2 + 0.0856 \\
 a_{44} &= 1
 \end{aligned}$$

From Equation (A6), the position of the robot in a coordinate system is described, where $x = a_{14}$, $y = a_{24}$, $z = a_{34}$.

Appendix A.4. Four-Degrees-of-Freedom Robot Dynamics

In this section, the dynamics of the robot are carried out using a Euler–Lagrange algorithm, which is based on an energy system.

It is worth mentioning that, for the development of this section, it is very important to properly develop direct kinematics.

These steps are called LN, where “L” refers to one step of the Lagrange equation and “N” is the number of steps.

For L1 and L2, it is the development of direct kinematics.

L3. The matrices where the equation is taken into consideration are obtained $U_{ij}(i = 1, j = 1)$

$$U_{ij} = \frac{\partial}{\partial q_j} A_i^0 \quad (A7)$$

where the result of Equation (A7) is shown in Equation (A8):

$$U_{11} = \frac{\partial}{\partial q_1} \begin{pmatrix} \cos q_1 & 0 & \sin q_1 & 0 \\ \sin q_1 & 0 & -\cos q_1 & 0 \\ 0 & 1 & 0 & 0.0856 \\ 0 & 0 & 0 & 1 \end{pmatrix} U_{11} = \begin{pmatrix} -\sin q_1 & 0 & \cos q_1 & 0 \\ \cos q_1 & 0 & \sin q_1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (A8)$$

For U_{12} , the result is shown in Equation (A9):

$$U_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (A9)$$

In a very similar way, it can be performed until it ends with U_{14} , resulting in the following equations:

$$\begin{aligned}
 U_{13} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
 U_{14} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
 \end{aligned}$$

For U_{21} and even U_{24} , it is taken into consideration that the transformation matrix is A_2^0 , giving $U_{21} = \frac{\partial}{\partial q_1} A_2^0$. Remember that it is not the same A_2^0 as A_2^1 .

Therefore,

$$A_2^0 = \begin{pmatrix} \cos q_1 \cos q_2 & -\cos q_1 \sin q_2 & \sin q_1 & 0.1 \sin q_1 + 0.11 \cos q_1 \cos q_2 \\ \cos q_2 \sin q_1 & -\sin q_1 \sin q_2 & -\cos q_1 & 0.11 \cos q_2 \sin q_1 - 0.1 \cos q_1 \\ \sin q_2 & \cos q_2 & 0 & 0.11 \sin q_2 + 0.0856 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$U_{21} = \begin{pmatrix} -\cos q_2 \sin q_1 & \sin q_1 \sin q_2 & \cos q_1 & 0.1 \cos q_1 - 0.11 \cos q_2 \sin q_1 \\ \cos q_1 \cos q_2 & -\cos q_1 \sin q_2 & \sin q_1 & 0.1 \sin q_1 + 0.11 \cos q_1 \cos q_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In a very similar way, it can be performed until it ends with U_{44} , resulting in the following equations:

$$U_{22} = \begin{pmatrix} -\cos q_1 \sin q_2 & -\cos q_1 \cos q_2 & 0 & -0.11 \cos q_1 \sin q_2 \\ -\sin q_1 \sin q_2 & -\cos q_2 \sin q_1 & 0 & -0.11 \sin q_1 \sin q_2 \\ \cos q_2 & -\sin q_2 & 0 & 0.11 \cos q_2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$U_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$U_{24} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$U_{31} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

where

$$\begin{aligned} a_{11} &= -(\sin q_1)(\cos q_2 \cos q_3 - \sin q_2 \sin q_3) \\ a_{12} &= (\sin q_1)(\cos q_2 \sin q_3 + \cos q_3 \sin q_2) \\ a_{13} &= \cos q_1 \\ a_{14} &= 0.15 \cos q_1 - 0.11 \cos q_2 \sin q_1 \\ a_{21} &= (\cos q_1)(\cos q_2 \cos q_3 - \sin q_2 \sin q_3) \\ a_{22} &= -(\cos q_1)(\cos q_2 \sin q_3 + \cos q_3 \sin q_2) \\ a_{23} &= \sin q_1 \\ a_{24} &= 0.15 \sin q_1 + 0.11 \cos q_1 \cos q_2 \end{aligned}$$

From a_{31} to a_{44} , the value it has is 0.

$$U_{32} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

$$\begin{aligned} a_{11} &= -(\cos q_1)(\cos q_2 \sin q_3 + \cos q_3 \sin q_2) \\ a_{12} &= -(\cos q_1)(\cos q_2 \cos q_3 - \sin q_2 \sin q_3) \\ a_{13} &= 0 \\ a_{14} &= -0.11 \cos q_1 \sin q_2 \end{aligned}$$

$$a_{21} = -(\sin q_1)(\cos q_2 \sin q_3 + \cos q_3 \sin q_2)$$

$$a_{22} = -(\sin q_1)(\cos q_2 \cos q_3 - \sin q_2 \sin q_3)$$

$$a_{23} = 0$$

$$a_{24} = -0.11 \sin q_1 \sin q_2$$

$$a_{31} = \cos q_2 \cos q_3 - \sin q_2 \sin q_3$$

$$a_{32} = -\cos q_2 \sin q_3 - \cos q_3 \sin q_2$$

$$a_{33} = 0$$

$$a_{34} = 0.11 \cos q_2$$

From a_{41} to a_{44} , the value it has is 0.

$$U_{33} = \begin{pmatrix} -(\cos q_1)(\cos q_2 \sin q_3 + \cos q_3 \sin q_2) & -(\cos q_1)(\cos q_2 \cos q_3 - \sin q_2 \sin q_3) & 0 & 0 \\ -(\sin q_1)(\cos q_2 \sin q_3 + \cos q_3 \sin q_2) & -(\sin q_1)(\cos q_2 \cos q_3 - \sin q_2 \sin q_3) & 0 & 0 \\ \cos q_2 \cos q_3 - \sin q_2 \sin q_3 & -\cos q_2 \sin q_3 - \cos q_3 \sin q_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$U_{34} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$U_{41} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$a_{11} = -(\sin q_1)((\cos q_2)(\cos q_3 \cos q_4 - \sin q_3 \sin q_4) - (\sin q_2)(\cos q_3 \sin q_4 + \cos q_4 \sin q_3))$$

$$a_{12} = \cos q_1$$

$$a_{13} = -(\sin q_1)((\cos q_2)(\cos q_3 \sin q_4 + \cos q_4 \sin q_3) + (\sin q_2)(\cos q_3 \cos q_4 - \sin q_3 \sin q_4))$$

$$a_{14} = 0.25 \cos q_1 - 0.11 \cos q_2 \sin q_1$$

$$a_{21} = (\cos q_1)((\cos q_2)(\cos q_3 \cos q_4 - \sin q_3 \sin q_4) - (\sin q_2)(\cos q_3 \sin q_4 + \cos q_4 \sin q_3))$$

$$a_{22} = \sin q_1$$

$$a_{23} = (\cos q_1)((\cos q_2)(\cos q_3 \sin q_4 + \cos q_4 \sin q_3) + (\sin q_2)(\cos q_3 \cos q_4 - \sin q_3 \sin q_4))$$

$$a_{24} = 0.25 \sin q_1 + 0.11 \cos q_1 \cos q_2$$

$$U_{42} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & 0 & a_{23} & a_{24} \\ a_{31} & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$a_{11} = -(\cos q_1)((\cos q_2)(\cos q_3 \sin q_4 + \cos q_4 \sin q_3) + (\sin q_2)(\cos q_3 \cos q_4 - \sin q_3 \sin q_4))$$

$$a_{12} = 0$$

$$a_{13} = (\cos q_1)((\cos q_2)(\cos q_3 \cos q_4 - \sin q_3 \sin q_4) - (\sin q_2)(\cos q_3 \sin q_4 + \cos q_4 \sin q_3))$$

$$a_{14} = -0.11 \cos q_1 \sin q_2$$

$$a_{21} = -(\sin q_1)((\cos q_2)(\cos q_3 \sin q_4 + \cos q_4 \sin q_3) + (\sin q_2)(\cos q_3 \cos q_4 - \sin q_3 \sin q_4))$$

$$a_{22} = 0$$

$$a_{23} = (\sin q_1)((\cos q_2)(\cos q_3 \cos q_4 - \sin q_3 \sin q_4) - (\sin q_2)(\cos q_3 \sin q_4 + \cos q_4 \sin q_3))$$

$$a_{24} = -0.11 \sin q_1 \sin q_2$$

$$a_{31} = (\cos q_2)(\cos q_3 \cos q_4 - \sin q_3 \sin q_4) - (\sin q_2)(\cos q_3 \sin q_4 + \cos q_4 \sin q_3)$$

$$a_{32} = 0$$

$$a_{33} = (\cos q_2)(\cos q_3 \sin q_4 + \cos q_4 \sin q_3) + (\sin q_2)(\cos q_3 \cos q_4 - \sin q_3 \sin q_4)$$

$$a_{34} = 0.11 \cos q_2$$

$$U_{43} = \begin{pmatrix} a_{11} & 0 & a_{13} & 0 \\ a_{21} & 0 & a_{23} & 0 \\ a_{31} & 0 & a_{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$a_{11} = -(\cos q_1)((\cos q_2)(\cos q_3 \sin q_4 + \cos q_4 \sin q_3) + (\sin q_2)(\cos q_3 \cos q_4 - \sin q_3 \sin q_4))$$

$$a_{12} = 0$$

$$a_{13} = (\cos q_1)((\cos q_2)(\cos q_3 \cos q_4 - \sin q_3 \sin q_4) - (\sin q_2)(\cos q_3 \sin q_4 + \cos q_4 \sin q_3))$$

$$a_{14} = 0$$

$$a_{21} = -(\sin q_1)((\cos q_2)(\cos q_3 \sin q_4 + \cos q_4 \sin q_3) + (\sin q_2)(\cos q_3 \cos q_4 - \sin q_3 \sin q_4))$$

$$a_{22} = 0$$

$$a_{23} = (\sin q_1)((\cos q_2)(\cos q_3 \cos q_4 - \sin q_3 \sin q_4) - (\sin q_2)(\cos q_3 \sin q_4 + \cos q_4 \sin q_3))$$

$$a_{24} = 0$$

$$a_{31} = (\cos q_2)(\cos q_3 \cos q_4 - \sin q_3 \sin q_4) - (\sin q_2)(\cos q_3 \sin q_4 + \cos q_4 \sin q_3)$$

$$a_{32} = 0$$

$$a_{33} = (\cos q_2)(\cos q_3 \sin q_4 + \cos q_4 \sin q_3) + (\sin q_2)(\cos q_3 \cos q_4 - \sin q_3 \sin q_4)$$

$$a_{34} = 0$$

$$U_{44} = \begin{pmatrix} a_{11} & 0 & a_{13} & 0 \\ a_{21} & 0 & a_{23} & 0 \\ a_{31} & 0 & a_{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$a_{11} = \frac{1}{2} \sin(q_1 - q_2 - q_3 - q_4) - \frac{1}{2} \sin(q_1 + q_2 + q_3 + q_4)$$

$$a_{12} = 0$$

$$a_{13} = \frac{1}{2} \cos(q_1 + q_2 + q_3 + q_4) + \frac{1}{2} \cos(q_1 - q_2 - q_3 - q_4)$$

$$a_{14} = 0$$

$$a_{21} = \frac{1}{2} \cos(q_1 + q_2 + q_3 + q_4) - \frac{1}{2} \cos(q_1 - q_2 - q_3 - q_4)$$

$$a_{22} = 0$$

$$a_{23} = \frac{1}{2} \sin(q_1 + q_2 + q_3 + q_4) + \frac{1}{2} \sin(q_1 - q_2 - q_3 - q_4)$$

$$a_{24} = 0$$

$$a_{31} = \cos(q_2 + q_3 + q_4)$$

$$a_{32} = 0$$

$$a_{33} = \sin(q_2 + q_3 + q_4)$$

$$a_{34} = 0$$

L4. Obtain the matrices U_{ijk} , defined by Equation (A10):

$$U_{ijk} = \frac{\partial U_{ij}}{\partial q_k} \quad (\text{A10})$$

where U_{111} is

$$U_{111} = \frac{\partial}{\partial q_1} U_{11}$$

$$U_{111} = \begin{pmatrix} -\cos q_1 & 0 & -\sin q_1 & 0 \\ -\sin q_1 & 0 & \cos q_1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In a very similar way, it can be performed until it ends with U_{444} , resulting in the following equations:

$$U_{112} = \frac{\partial}{\partial q_2} U_{11}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

For $U_{112} = U_{113} = U_{114} = U_{121} = U_{122} = U_{123} = U_{124}$.

In a similar way, it occurs to $U_{131} = U_{132} = U_{133} = U_{134} = U_{141} = U_{142} = U_{143} = U_{144}$.

For U_{211} ,

$$U_{211} = \frac{\partial}{\partial q_1} U_{21}$$

$$= \begin{pmatrix} -\cos q_1 \cos q_2 & \cos q_1 \sin q_2 & -\sin q_1 & -0.1 \sin q_1 - 0.11 \cos q_1 \cos q_2 \\ -\cos q_2 \sin q_1 & \sin q_1 \sin q_2 & \cos q_1 & 0.1 \cos q_1 - 0.11 \cos q_2 \sin q_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$U_{212} = \frac{\partial}{\partial q_2} U_{21}$$

$$= \begin{pmatrix} \sin q_1 \sin q_2 & \cos q_2 \sin q_1 & 0 & 0.11 \sin q_1 \sin q_2 \\ -\cos q_1 \sin q_2 & -\cos q_1 \cos q_2 & 0 & -0.11 \cos q_1 \sin q_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

For U_{213} and U_{214} , the result is a matrix of zeros.

Continuing with U_{221} ,

$$U_{221} = \frac{\partial}{\partial q_1} U_{22}$$

$$= \begin{pmatrix} \sin q_1 \sin q_2 & \cos q_2 \sin q_1 & 0 & 0.11 \sin q_1 \sin q_2 \\ -\cos q_1 \sin q_2 & -\cos q_1 \cos q_2 & 0 & -0.11 \cos q_1 \sin q_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$U_{222} = \frac{\partial}{\partial q_2} U_{22}$$

$$= \begin{pmatrix} -\cos q_1 \cos q_2 & \cos q_1 \sin q_2 & 0 & -0.11 \cos q_1 \cos q_2 \\ -\cos q_2 \sin q_1 & \sin q_1 \sin q_2 & 0 & -0.11 \cos q_2 \sin q_1 \\ -\sin q_2 & -\cos q_2 & 0 & -0.11 \sin q_2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

For U_{223} and U_{224} , the result is a matrix of zeros.

The same occurs with $U_{231} = U_{232} = U_{233} = U_{234} = U_{241} = U_{242} = U_{243} = U_{244} =$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

For U_{311} , the result is different.

$$U_{311} = \frac{\partial}{\partial q_1} U_{31} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$a_{11} = -(\cos q_1)(\cos q_2 \cos q_3 - \sin q_2 \sin q_3)$$

$$a_{12} = (\cos q_1)(\cos q_2 \sin q_3 + \cos q_3 \sin q_2)$$

$$a_{13} = -\sin q_1$$

$$a_{14} = -0.15 \sin q_1 - 0.11 \cos q_1 \cos q_2$$

$$a_{21} = -(\sin q_1)(\cos q_2 \cos q_3 - \sin q_2 \sin q_3)$$

$$a_{22} = (\sin q_1)(\cos q_2 \sin q_3 + \cos q_3 \sin q_2)$$

$$a_{23} = \cos q_1$$

$$a_{24} = 0.15 \cos q_1 - 0.11 \cos q_2 \sin q_1$$

$$U_{312} = \frac{\partial}{\partial q_2} U_{31} = \begin{pmatrix} a_{11} & a_{12} & 0 & a_{14} \\ a_{21} & a_{22} & 0 & a_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$a_{11} = (\sin q_1)(\cos q_2 \sin q_3 + \cos q_3 \sin q_2)$$

$$a_{12} = (\sin q_1)(\cos q_2 \cos q_3 - \sin q_2 \sin q_3)$$

$$a_{13} = 0$$

$$a_{14} = 0.11 \sin q_1 \sin q_2$$

$$a_{21} = -(\cos q_1)(\cos q_2 \sin q_3 + \cos q_3 \sin q_2)$$

$$a_{22} = -(\cos q_1)(\cos q_2 \cos q_3 - \sin q_2 \sin q_3)$$

$$a_{23} = 0$$

$$a_{24} = -0.11 \cos q_1 \sin q_2$$

$$U_{313} = \frac{\partial}{\partial q_3} U_{31}$$

$$= \begin{pmatrix} (\sin q_1)(\cos q_2 \sin q_3 + \cos q_3 \sin q_2) & (\sin q_1)(\cos q_2 \cos q_3 - \sin q_2 \sin q_3) & 0 & 0 \\ -(\cos q_1)(\cos q_2 \sin q_3 + \cos q_3 \sin q_2) & -(\cos q_1)(\cos q_2 \cos q_3 - \sin q_2 \sin q_3) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$U_{314} = \frac{\partial}{\partial q_4} U_{31} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$U_{321} = \frac{\partial}{\partial q_1} U_{32} = \begin{pmatrix} a_{11} & a_{12} & 0 & a_{14} \\ a_{21} & a_{22} & 0 & a_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$a_{11} = (\sin q_1)(\cos q_2 \sin q_3 + \cos q_3 \sin q_2)$$

$$a_{12} = (\sin q_1)(\cos q_2 \cos q_3 - \sin q_2 \sin q_3)$$

$$a_{13} = 0$$

$$a_{14} = 0.11 \sin q_1 \sin q_2$$

$$a_{21} = -(\cos q_1)(\cos q_2 \sin q_3 + \cos q_3 \sin q_2)$$

$$a_{22} = -(\cos q_1)(\cos q_2 \cos q_3 - \sin q_2 \sin q_3)$$

$$a_{23} = 0$$

$$a_{24} = -0.11 \cos q_1 \sin q_2$$

$$U_{322} = \frac{\partial}{\partial q_2} U_{32} = \begin{pmatrix} a_{11} & a_{12} & 0 & a_{14} \\ a_{21} & a_{22} & 0 & a_{24} \\ a_{31} & a_{32} & 0 & a_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$a_{11} = -(\cos q_1)(\cos q_2 \cos q_3 - \sin q_2 \sin q_3)$$

$$a_{12} = (\cos q_1)(\cos q_2 \sin q_3 + \cos q_3 \sin q_2)$$

$$a_{13} = 0$$

$$a_{14} = -0.11 \cos q_1 \cos q_2$$

$$a_{21} = -(\sin q_1)(\cos q_2 \cos q_3 - \sin q_2 \sin q_3)$$

$$a_{22} = (\sin q_1)(\cos q_2 \sin q_3 + \cos q_3 \sin q_2)$$

$$a_{23} = 0$$

$$a_{24} = -0.11 \cos q_2 \sin q_1$$

$$a_{31} = -\cos q_2 \sin q_3 - \cos q_3 \sin q_2$$

$$a_{32} = \sin q_2 \sin q_3 - \cos q_2 \cos q_3$$

$$a_{33} = 0$$

$$a_{34} = -0.11 \sin q_2$$

$$U_{323} = \frac{\partial}{\partial q_3} U_{32}$$

$$= \begin{pmatrix} -(\cos q_1)(\cos q_2 \cos q_3 - \sin q_2 \sin q_3) & (\cos q_1)(\cos q_2 \sin q_3 + \cos q_3 \sin q_2) & 0 & 0 \\ -(\sin q_1)(\cos q_2 \cos q_3 - \sin q_2 \sin q_3) & (\sin q_1)(\cos q_2 \sin q_3 + \cos q_3 \sin q_2) & 0 & 0 \\ -\cos q_2 \sin q_3 - \cos q_3 \sin q_2 & \sin q_2 \sin q_3 - \cos q_2 \cos q_3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$U_{324} = \frac{\partial}{\partial q_4} U_{32} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$U_{331} = \frac{\partial}{\partial q_1} U_{33}$$

$$= \begin{pmatrix} (\sin q_1)(\cos q_2 \sin q_3 + \cos q_3 \sin q_2) & (\sin q_1)(\cos q_2 \cos q_3 - \sin q_2 \sin q_3) & 0 & 0 \\ -(\cos q_1)(\cos q_2 \sin q_3 + \cos q_3 \sin q_2) & -(\cos q_1)(\cos q_2 \cos q_3 - \sin q_2 \sin q_3) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$U_{332} = \frac{\partial}{\partial q_2} U_{33}$$

$$= \begin{pmatrix} -(\cos q_1)(\cos q_2 \cos q_3 - \sin q_2 \sin q_3) & (\cos q_1)(\cos q_2 \sin q_3 + \cos q_3 \sin q_2) & 0 & 0 \\ -(\sin q_1)(\cos q_2 \cos q_3 - \sin q_2 \sin q_3) & (\sin q_1)(\cos q_2 \sin q_3 + \cos q_3 \sin q_2) & 0 & 0 \\ -\cos q_2 \sin q_3 - \cos q_3 \sin q_2 & \sin q_2 \sin q_3 - \cos q_2 \cos q_3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$U_{333} = \frac{\partial}{\partial q_3} U_{33}$$

$$= \begin{pmatrix} -(\cos q_1)(\cos q_2 \cos q_3 - \sin q_2 \sin q_3) & (\cos q_1)(\cos q_2 \sin q_3 + \cos q_3 \sin q_2) & 0 & 0 \\ -(\sin q_1)(\cos q_2 \cos q_3 - \sin q_2 \sin q_3) & (\sin q_1)(\cos q_2 \sin q_3 + \cos q_3 \sin q_2) & 0 & 0 \\ -\cos q_2 \sin q_3 - \cos q_3 \sin q_2 & \sin q_2 \sin q_3 - \cos q_2 \cos q_3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$U_{334} = \frac{\partial}{\partial q_4} U_{33} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$U_{341} = \frac{\partial}{\partial q_1} U_{34} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$U_{341} = U_{342} = U_{343} = U_{344}$$

$$U_{411} = \frac{\partial}{\partial q_1} U_{41} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$a_{11} = -\frac{1}{2} \cos(q_1 + q_2 + q_3 + q_4) - \frac{1}{2} \cos(q_1 - q_2 - q_3 - q_4)$$

$$a_{12} = -\sin q_1$$

$$a_{13} = \frac{1}{2} \sin(q_1 - q_2 - q_3 - q_4) - \frac{1}{2} \sin(q_1 + q_2 + q_3 + q_4)$$

$$a_{14} = -0.055 \cos(q_1 - 1.0q_2) - 0.055 \cos(q_1 + q_2) - 0.25 \sin q_1$$

$$a_{21} = -\frac{1}{2} \sin(q_1 + q_2 + q_3 + q_4) - \frac{1}{2} \sin(q_1 - q_2 - q_3 - q_4)$$

$$a_{22} = \cos q_1$$

$$a_{23} = \frac{1}{2} \cos(q_1 + q_2 + q_3 + q_4) - \frac{1}{2} \cos(q_1 - q_2 - q_3 - q_4)$$

$$a_{24} = 0.25 \cos q_1 - 0.055 \sin(q_1 + q_2) - 0.055 \sin(q_1 - 1.0q_2)$$

$$U_{412} = \frac{\partial}{\partial q_2} U_{41} = \begin{pmatrix} a_{11} & 0 & a_{13} & a_{14} \\ a_{21} & 0 & a_{23} & a_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$a_{11} = \frac{1}{2} \cos(q_1 - q_2 - q_3 - q_4) - \frac{1}{2} \cos(q_1 + q_2 + q_3 + q_4)$$

$$a_{12} = 0$$

$$a_{13} = -\frac{1}{2} \sin(q_1 + q_2 + q_3 + q_4) - \frac{1}{2} \sin(q_1 - q_2 - q_3 - q_4)$$

$$a_{14} = 0.055\cos(q_1 - 1.0q_2) - 0.055\cos(q_1 + q_2)$$

$$a_{21} = \frac{1}{2}\sin(q_1 - q_2 - q_3 - q_4) - \frac{1}{2}\sin(q_1 + q_2 + q_3 + q_4)$$

$$a_{22} = 0$$

$$a_{23} = \frac{1}{2}\cos(q_1 + q_2 + q_3 + q_4) + \frac{1}{2}\cos(q_1 - q_2 - q_3 - q_4)$$

$$a_{24} = -0.11\cos q_1 \sin q_2$$

$$U_{413} = \frac{\partial}{\partial q_3} U_{41} = \begin{pmatrix} a_{11} & 0 & a_{13} & 0 \\ a_{21} & 0 & a_{23} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$a_{11} = \frac{1}{2}\cos(q_1 - q_2 - q_3 - q_4) - \frac{1}{2}\cos(q_1 + q_2 + q_3 + q_4)$$

$$a_{12} = 0$$

$$a_{13} = -\frac{1}{2}\sin(q_1 + q_2 + q_3 + q_4) - \frac{1}{2}\sin(q_1 - q_2 - q_3 - q_4)$$

$$a_{14} = 0$$

$$a_{21} = \frac{1}{2}\sin(q_1 - q_2 - q_3 - q_4) - \frac{1}{2}\sin(q_1 + q_2 + q_3 + q_4)$$

$$a_{22} = 0$$

$$a_{23} = \frac{1}{2}\cos(q_1 + q_2 + q_3 + q_4) + \frac{1}{2}\cos(q_1 - q_2 - q_3 - q_4)$$

$$a_{24} = 0$$

$$U_{414} = \frac{\partial}{\partial q_4} U_{41} = \begin{pmatrix} a_{11} & 0 & a_{13} & 0 \\ a_{21} & 0 & a_{23} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$a_{11} = \frac{1}{2}\cos(q_1 - q_2 - q_3 - q_4) - \frac{1}{2}\cos(q_1 + q_2 + q_3 + q_4)$$

$$a_{12} = 0$$

$$a_{13} = -\frac{1}{2}\sin(q_1 + q_2 + q_3 + q_4) - \frac{1}{2}\sin(q_1 - q_2 - q_3 - q_4)$$

$$a_{14} = 0$$

$$a_{21} = \frac{1}{2}\sin(q_1 - q_2 - q_3 - q_4) - \frac{1}{2}\sin(q_1 + q_2 + q_3 + q_4)$$

$$a_{22} = 0$$

$$a_{23} = \frac{1}{2}\cos(q_1 + q_2 + q_3 + q_4) + \frac{1}{2}\cos(q_1 - q_2 - q_3 - q_4)$$

$$a_{24} = 0$$

$$U_{421} = \frac{\partial}{\partial q_1} U_{42} = \begin{pmatrix} a_{11} & 0 & a_{13} & a_{14} \\ a_{21} & 0 & a_{23} & a_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$a_{11} = \frac{1}{2}\cos(q_1 - q_2 - q_3 - q_4) - \frac{1}{2}\cos(q_1 + q_2 + q_3 + q_4)$$

$$a_{12} = 0$$

$$a_{13} = -\frac{1}{2}\sin(q_1 + q_2 + q_3 + q_4) - \frac{1}{2}\sin(q_1 - q_2 - q_3 - q_4)$$

$$a_{14} = 0.055\cos(q_1 - 1.0q_2) - 0.055\cos(q_1 + q_2)$$

$$a_{21} = \frac{1}{2}\sin(q_1 - q_2 - q_3 - q_4) - \frac{1}{2}\sin(q_1 + q_2 + q_3 + q_4)$$

$$a_{22} = 0$$

$$a_{23} = \frac{1}{2}\cos(q_1 + q_2 + q_3 + q_4) + \frac{1}{2}\cos(q_1 - q_2 - q_3 - q_4)$$

$$a_{24} = -0.11\cos q_1 \sin q_2$$

$$U_{422} = \frac{\partial}{\partial q_2} U_{42} = \begin{pmatrix} a_{11} & 0 & a_{13} & a_{14} \\ a_{21} & 0 & a_{23} & a_{24} \\ a_{31} & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$a_{11} = -\frac{1}{2}\cos(q_1 + q_2 + q_3 + q_4) - \frac{1}{2}\cos(q_1 - q_2 - q_3 - q_4)$$

$$a_{12} = 0$$

$$a_{13} = \frac{1}{2}\sin(q_1 - q_2 - q_3 - q_4) - \frac{1}{2}\sin(q_1 + q_2 + q_3 + q_4)$$

$$a_{14} = -0.11\cos q_1 \cos q_2$$

$$a_{21} = -\frac{1}{2}\sin(q_1 + q_2 + q_3 + q_4) - \frac{1}{2}\sin(q_1 - q_2 - q_3 - q_4)$$

$$a_{22} = 0$$

$$a_{23} = \frac{1}{2}\cos(q_1 + q_2 + q_3 + q_4) - \frac{1}{2}\cos(q_1 - q_2 - q_3 - q_4)$$

$$a_{24} = -0.11\cos q_2 \sin q_1$$

$$a_{31} = -\sin(q_2 + q_3 + q_4)$$

$$a_{32} = 0$$

$$a_{33} = \cos(q_2 + q_3 + q_4)$$

$$a_{34} = -0.11\sin q_2$$

$$U_{423} = \frac{\partial}{\partial q_3} U_{42} = \begin{pmatrix} a_{11} & 0 & a_{13} & 0 \\ a_{21} & 0 & a_{23} & 0 \\ a_{31} & 0 & a_{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$a_{11} = -\frac{1}{2}\cos(q_1 + q_2 + q_3 + q_4) - \frac{1}{2}\cos(q_1 - q_2 - q_3 - q_4)$$

$$a_{12} = 0$$

$$a_{13} = \frac{1}{2}\sin(q_1 - q_2 - q_3 - q_4) - \frac{1}{2}\sin(q_1 + q_2 + q_3 + q_4)$$

$$a_{14} = 0$$

$$a_{21} = -\frac{1}{2}\sin(q_1 + q_2 + q_3 + q_4) - \frac{1}{2}\sin(q_1 - q_2 - q_3 - q_4)$$

$$a_{22} = 0$$

$$a_{23} = \frac{1}{2}\cos(q_1 + q_2 + q_3 + q_4) - \frac{1}{2}\cos(q_1 - q_2 - q_3 - q_4)$$

$$a_{24} = 0$$

$$a_{31} = -\sin(q_2 + q_3 + q_4)$$

$$a_{32} = 0$$

$$a_{33} = \cos(q_2 + q_3 + q_4)$$

$$a_{34} = 0$$

$$U_{424} = \frac{\partial}{\partial q_4} U_{42} = \begin{pmatrix} a_{11} & 0 & a_{13} & 0 \\ a_{21} & 0 & a_{23} & 0 \\ a_{31} & 0 & a_{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$a_{11} = -\frac{1}{2}\cos(q_1 + q_2 + q_3 + q_4) - \frac{1}{2}\cos(q_1 - q_2 - q_3 - q_4)$$

$$a_{12} = 0$$

$$a_{13} = \frac{1}{2}\sin(q_1 - q_2 - q_3 - q_4) - \frac{1}{2}\sin(q_1 + q_2 + q_3 + q_4)$$

$$a_{14} = 0$$

$$a_{21} = -\frac{1}{2}\sin(q_1 + q_2 + q_3 + q_4) - \frac{1}{2}\sin(q_1 - q_2 - q_3 - q_4)$$

$$a_{22} = 0$$

$$a_{23} = \frac{1}{2}\cos(q_1 + q_2 + q_3 + q_4) - \frac{1}{2}\cos(q_1 - q_2 - q_3 - q_4)$$

$$a_{24} = 0$$

$$a_{31} = -\sin(q_2 + q_3 + q_4)$$

$$a_{32} = 0$$

$$a_{33} = \cos(q_2 + q_3 + q_4)$$

$$a_{34} = 0$$

$$U_{431} = \frac{\partial}{\partial q_1} U_{43} = \begin{pmatrix} a_{11} & 0 & a_{13} & 0 \\ a_{21} & 0 & a_{23} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$a_{11} = \frac{1}{2}\cos(q_1 - q_2 - q_3 - q_4) - \frac{1}{2}\cos(q_1 + q_2 + q_3 + q_4)$$

$$a_{12} = 0$$

$$a_{13} = -\frac{1}{2}\sin(q_1 + q_2 + q_3 + q_4) - \frac{1}{2}\sin(q_1 - q_2 - q_3 - q_4)$$

$$a_{14} = 0$$

$$a_{21} = \frac{1}{2}\sin(q_1 - q_2 - q_3 - q_4) - \frac{1}{2}\sin(q_1 + q_2 + q_3 + q_4)$$

$$a_{22} = 0$$

$$a_{23} = \frac{1}{2}\cos(q_1 + q_2 + q_3 + q_4) + \frac{1}{2}\cos(q_1 - q_2 - q_3 - q_4)$$

$$a_{24} = 0$$

$$U_{432} = \frac{\partial}{\partial q_2} U_{43} = \begin{pmatrix} a_{11} & 0 & a_{13} & 0 \\ a_{21} & 0 & a_{23} & 0 \\ a_{31} & 0 & a_{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$a_{11} = -\frac{1}{2}\cos(q_1 + q_2 + q_3 + q_4) - \frac{1}{2}\cos(q_1 - q_2 - q_3 - q_4)$$

$$a_{12} = 0$$

$$a_{13} = \frac{1}{2}\sin(q_1 - q_2 - q_3 - q_4) - \frac{1}{2}\sin(q_1 + q_2 + q_3 + q_4)$$

$$a_{14} = 0$$

$$a_{21} = -\frac{1}{2}\sin(q_1 + q_2 + q_3 + q_4) - \frac{1}{2}\sin(q_1 - q_2 - q_3 - q_4)$$

$$a_{22} = 0$$

$$a_{23} = \frac{1}{2}\cos(q_1 + q_2 + q_3 + q_4) - \frac{1}{2}\cos(q_1 - q_2 - q_3 - q_4)$$

$$a_{24} = 0$$

$$a_{31} = -\sin(q_2 + q_3 + q_4)$$

$$a_{32} = 0$$

$$a_{33} = \cos(q_2 + q_3 + q_4)$$

$$a_{34} = 0$$

$$U_{433} = \frac{\partial}{\partial q_3} U_{43} = \begin{pmatrix} a_{11} & 0 & a_{13} & 0 \\ a_{21} & 0 & a_{23} & 0 \\ a_{31} & 0 & a_{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$a_{11} = -\frac{1}{2}\cos(q_1 + q_2 + q_3 + q_4) - \frac{1}{2}\cos(q_1 - q_2 - q_3 - q_4)$$

$$a_{12} = 0$$

$$a_{13} = \frac{1}{2}\sin(q_1 - q_2 - q_3 - q_4) - \frac{1}{2}\sin(q_1 + q_2 + q_3 + q_4)$$

$$a_{14} = 0$$

$$a_{21} = -\frac{1}{2}\sin(q_1 + q_2 + q_3 + q_4) - \frac{1}{2}\sin(q_1 - q_2 - q_3 - q_4)$$

$$a_{22} = 0$$

$$a_{23} = \frac{1}{2}\cos(q_1 + q_2 + q_3 + q_4) - \frac{1}{2}\cos(q_1 - q_2 - q_3 - q_4)$$

$$a_{24} = 0$$

$$a_{31} = -\sin(q_2 + q_3 + q_4)$$

$$a_{32} = 0$$

$$a_{33} = \cos(q_2 + q_3 + q_4)$$

$$a_{34} = 0$$

$$U_{434} = \frac{\partial}{\partial q_4} U_{43} = \begin{pmatrix} a_{11} & 0 & a_{13} & 0 \\ a_{21} & 0 & a_{23} & 0 \\ a_{31} & 0 & a_{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$a_{11} = -\frac{1}{2}\cos(q_1 + q_2 + q_3 + q_4) - \frac{1}{2}\cos(q_1 - q_2 - q_3 - q_4)$$

$$a_{12} = 0$$

$$a_{13} = \frac{1}{2}\sin(q_1 - q_2 - q_3 - q_4) - \frac{1}{2}\sin(q_1 + q_2 + q_3 + q_4)$$

$$a_{14} = 0$$

$$a_{21} = -\frac{1}{2}\sin(q_1 + q_2 + q_3 + q_4) - \frac{1}{2}\sin(q_1 - q_2 - q_3 - q_4)$$

$$a_{22} = 0$$

$$a_{23} = \frac{1}{2}\cos(q_1 + q_2 + q_3 + q_4) - \frac{1}{2}\cos(q_1 - q_2 - q_3 - q_4)$$

$$a_{24} = 0$$

$$a_{31} = -\sin(q_2 + q_3 + q_4)$$

$$a_{32} = 0$$

$$a_{33} = \cos(q_2 + q_3 + q_4)$$

$$a_{34} = 0$$

$$U_{441} = \frac{\partial}{\partial q_1} U_{44} = \begin{pmatrix} a_{11} & 0 & a_{13} & 0 \\ a_{21} & 0 & a_{23} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$a_{11} = \frac{1}{2}\cos(q_1 - q_2 - q_3 - q_4) - \frac{1}{2}\cos(q_1 + q_2 + q_3 + q_4)$$

$$a_{12} = 0$$

$$a_{13} = -\frac{1}{2}\sin(q_1 + q_2 + q_3 + q_4) - \frac{1}{2}\sin(q_1 - q_2 - q_3 - q_4)$$

$$a_{14} = 0$$

$$a_{21} = \frac{1}{2}\sin(q_1 - q_2 - q_3 - q_4) - \frac{1}{2}\sin(q_1 + q_2 + q_3 + q_4)$$

$$a_{22} = 0$$

$$a_{23} = \frac{1}{2}\cos(q_1 + q_2 + q_3 + q_4) + \frac{1}{2}\cos(q_1 - q_2 - q_3 - q_4)$$

$$a_{24} = 0$$

$$U_{442} = \frac{\partial}{\partial q_2} U_{44} = \begin{pmatrix} a_{11} & 0 & a_{13} & 0 \\ a_{21} & 0 & a_{23} & 0 \\ a_{31} & 0 & a_{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$a_{11} = -\frac{1}{2}\cos(q_1 + q_2 + q_3 + q_4) - \frac{1}{2}\cos(q_1 - q_2 - q_3 - q_4)$$

$$a_{12} = 0$$

$$a_{13} = \frac{1}{2}\sin(q_1 - q_2 - q_3 - q_4) - \frac{1}{2}\sin(q_1 + q_2 + q_3 + q_4)$$

$$a_{14} = 0$$

$$a_{21} = -\frac{1}{2}\sin(q_1 + q_2 + q_3 + q_4) - \frac{1}{2}\sin(q_1 - q_2 - q_3 - q_4)$$

$$a_{22} = 0$$

$$a_{23} = \frac{1}{2}\cos(q_1 + q_2 + q_3 + q_4) - \frac{1}{2}\cos(q_1 - q_2 - q_3 - q_4)$$

$$a_{24} = 0$$

$$a_{31} = -\sin(q_2 + q_3 + q_4)$$

$$a_{32} = 0$$

$$a_{33} = \cos(q_2 + q_3 + q_4)$$

$$a_{34} = 0$$

$$U_{443} = \frac{\partial}{\partial q_3} U_{44} = \begin{pmatrix} a_{11} & 0 & a_{13} & 0 \\ a_{21} & 0 & a_{23} & 0 \\ a_{31} & 0 & a_{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$a_{11} = -\frac{1}{2}\cos(q_1 + q_2 + q_3 + q_4) - \frac{1}{2}\cos(q_1 - q_2 - q_3 - q_4)$$

$$a_{12} = 0$$

$$a_{13} = \frac{1}{2}\sin(q_1 - q_2 - q_3 - q_4) - \frac{1}{2}\sin(q_1 + q_2 + q_3 + q_4)$$

$$a_{14} = 0$$

$$a_{21} = -\frac{1}{2}\sin(q_1 + q_2 + q_3 + q_4) - \frac{1}{2}\sin(q_1 - q_2 - q_3 - q_4)$$

$$a_{22} = 0$$

$$a_{23} = \frac{1}{2}\cos(q_1 + q_2 + q_3 + q_4) - \frac{1}{2}\cos(q_1 - q_2 - q_3 - q_4)$$

$$a_{24} = 0$$

$$a_{31} = -\sin(q_2 + q_3 + q_4)$$

$$a_{32} = 0$$

$$a_{33} = \cos(q_2 + q_3 + q_4)$$

$$a_{34} = 0$$

$$U_{444} = \frac{\partial}{\partial q_4} U_{44} = \begin{pmatrix} a_{11} & 0 & a_{13} & 0 \\ a_{21} & 0 & a_{23} & 0 \\ a_{31} & 0 & a_{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$a_{11} = -\frac{1}{2}\cos(q_1 + q_2 + q_3 + q_4) - \frac{1}{2}\cos(q_1 - q_2 - q_3 - q_4)$$

$$a_{12} = 0$$

$$a_{13} = \frac{1}{2}\sin(q_1 - q_2 - q_3 - q_4) - \frac{1}{2}\sin(q_1 + q_2 + q_3 + q_4)$$

$$a_{14} = 0$$

$$a_{21} = -\frac{1}{2}\sin(q_1 + q_2 + q_3 + q_4) - \frac{1}{2}\sin(q_1 - q_2 - q_3 - q_4)$$

$$a_{22} = 0$$

$$a_{23} = \frac{1}{2}\cos(q_1 + q_2 + q_3 + q_4) - \frac{1}{2}\cos(q_1 - q_2 - q_3 - q_4)$$

$$a_{24} = 0$$

$$a_{31} = -\sin(q_2 + q_3 + q_4)$$

$$a_{32} = 0$$

$$a_{33} = \cos(q_2 + q_3 + q_4)$$

$$a_{34} = 0$$

L5. Obtain the pseudo-inertia matrices J_i for each element, which are defined by Equation (A11):

$$J_i = \begin{bmatrix} \int x_i^2 dm & \int x_i y_i dm & \int x_i z_i dm & \int x_i dm \\ \int y_i x_i dm & \int y_i^2 dm & \int y_i z_i dm & \int y_i dm \\ \int z_i x_i dm & \int z_i y_i dm & \int z_i^2 dm & \int z_i dm \\ \int x_i dm & \int y_i dm & \int z_i dm & \int dm \end{bmatrix} \quad (\text{A11})$$

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 7.3274 \times 10^{-3} m_1 & 0.0856 m_1 \\ 0 & 0 & 0.0856 m_1 & m_1 \end{pmatrix}$$

$$J_2 = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

$$a_{11} = m_2 (0.1 \sin q_1 + 0.11 \cos q_1 \cos q_2)^2$$

$$a_{12} = -m_2 (0.1 \cos q_1 - 0.11 \cos q_2 \sin q_1) (0.1 \sin q_1 + 0.11 \cos q_1 \cos q_2)$$

$$a_{13} = m_2 (0.11 \sin q_2 + 0.0856) (0.1 \sin q_1 + 0.11 \cos q_1 \cos q_2)$$

$$a_{14} = m_2 (0.1 \sin q_1 + 0.11 \cos q_1 \cos q_2)$$

$$a_{21} = -m_2 (0.1 \cos q_1 - 0.11 \cos q_2 \sin q_1) (0.1 \sin q_1 + 0.11 \cos q_1 \cos q_2)$$

$$a_{22} = m_2 (0.1 \cos q_1 - 0.11 \cos q_2 \sin q_1)^2$$

$$a_{23} = -m_2 (0.11 \sin q_2 + 0.0856) (0.1 \cos q_1 - 0.11 \cos q_2 \sin q_1)$$

$$a_{24} = -m_2 (0.1 \cos q_1 - 0.11 \cos q_2 \sin q_1)$$

$$a_{31} = m_2 (0.11 \sin q_2 + 0.0856) (0.1 \sin q_1 + 0.11 \cos q_1 \cos q_2)$$

$$a_{32} = -m_2 (0.11 \sin q_2 + 0.0856) (0.1 \cos q_1 - 0.11 \cos q_2 \sin q_1)$$

$$a_{33} = m_2 (0.11 \sin q_2 + 0.0856)^2$$

$$a_{34} = m_2 (0.11 \sin q_2 + 0.0856)$$

$$a_{41} = m_2 (0.1 \sin q_1 + 0.11 \cos q_1 \cos q_2)$$

$$a_{42} = -m_2 (0.1 \cos q_1 - 0.11 \cos q_2 \sin q_1)$$

$$a_{43} = m_2 (0.11 \sin q_2 + 0.0856)$$

$$a_{44} = m_2$$

$$J_3 = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

$$a_{11} = m_3 (0.15 \sin q_1 + 0.11 \cos q_1 \cos q_2)^2$$

$$a_{12} = -m_3 (0.15 \cos q_1 - 0.11 \cos q_2 \sin q_1) (0.15 \sin q_1 + 0.11 \cos q_1 \cos q_2)$$

$$a_{13} = m_3 (0.15 \sin q_1 + 0.11 \cos q_1 \cos q_2) (0.11 \sin q_2 + 0.0856)$$

$$a_{14} = m_3 (0.15 \sin q_1 + 0.11 \cos q_1 \cos q_2)$$

$$a_{21} = -m_3 (0.15 \cos q_1 - 0.11 \cos q_2 \sin q_1) (0.15 \sin q_1 + 0.11 \cos q_1 \cos q_2)$$

$$a_{22} = m_3 (0.15 \cos q_1 - 0.11 \cos q_2 \sin q_1)^2$$

$$a_{23} = -m_3 (0.15 \cos q_1 - 0.11 \cos q_2 \sin q_1) (0.11 \sin q_2 + 0.0856)$$

$$\begin{aligned}
 a_{24} &= -m_3(0.15\cos q_1 - 0.11\cos q_2 \sin q_1) \\
 a_{31} &= m_3(0.15\sin q_1 + 0.11\cos q_1 \cos q_2)(0.11\sin q_2 + 0.0856) \\
 a_{32} &= -m_3(0.15\cos q_1 - 0.11\cos q_2 \sin q_1)(0.11\sin q_2 + 0.0856) \\
 a_{33} &= m_3(0.11\sin q_2 + 0.0856)^2 \\
 a_{34} &= m_3(0.11\sin q_2 + 0.0856) \\
 a_{41} &= m_3(0.15\sin q_1 + 0.11\cos q_1 \cos q_2) \\
 a_{42} &= -m_3(0.15\cos q_1 - 0.11\cos q_2 \sin q_1) \\
 a_{43} &= m_3(0.11\sin q_2 + 0.0856)
 \end{aligned}$$

$$a_{44} = m_3$$

$$J_4 = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

$$\begin{aligned}
 a_{11} &= m_4(0.25\sin q_1 + 0.11\cos q_1 \cos q_2)^2 \\
 a_{12} &= -m_4(0.25\cos q_1 - 0.11\cos q_2 \sin q_1)(0.25\sin q_1 + 0.11\cos q_1 \cos q_2) \\
 a_{13} &= m_4(0.25\sin q_1 + 0.11\cos q_1 \cos q_2)(0.11\sin q_2 + 0.0856) \\
 a_{14} &= m_4(0.25\sin q_1 + 0.11\cos q_1 \cos q_2) \\
 a_{21} &= -m_4(0.25\cos q_1 - 0.11\cos q_2 \sin q_1)(0.25\sin q_1 + 0.11\cos q_1 \cos q_2) \\
 a_{22} &= m_4(0.25\cos q_1 - 0.11\cos q_2 \sin q_1)^2 \\
 a_{23} &= -m_4(0.25\cos q_1 - 0.11\cos q_2 \sin q_1)(0.11\sin q_2 + 0.0856) \\
 a_{24} &= -m_4(0.25\cos q_1 - 0.11\cos q_2 \sin q_1) \\
 a_{31} &= m_4(0.25\sin q_1 + 0.11\cos q_1 \cos q_2)(0.11\sin q_2 + 0.0856) \\
 a_{32} &= -m_4(0.25\cos q_1 - 0.11\cos q_2 \sin q_1)(0.11\sin q_2 + 0.0856) \\
 a_{33} &= m_4(0.11\sin q_2 + 0.0856)^2 \\
 a_{34} &= m_4(0.11\sin q_2 + 0.0856) \\
 a_{41} &= m_4(0.25\sin q_1 + 0.11\cos q_1 \cos q_2) \\
 a_{42} &= -m_4(0.25\cos q_1 - 0.11\cos q_2 \sin q_1) \\
 a_{43} &= m_4(0.11\sin q_2 + 0.0856) \\
 a_{44} &= m_4
 \end{aligned}$$

L6. Obtain the matrix of inertia $D = [d_{ij}]$, whose elements are defined by Equation (A12):

$$d_{ij} = \sum_{k=(i,j)}^n \text{Traza}(U_{kj}J_kU_{ki}^T) \quad (\text{A12})$$

For $i, j = 1, 2, \dots, n$

$$\begin{aligned}
 d_{11} &= \text{Traza}(U_{11} * J_1 * U_{11}^T) + \text{Traza}(U_{21} * J_2 * U_{21}^T) + \text{Traza}(U_{31} * J_3 * U_{31}^T) + \text{Traza}(U_{41} * J_4 * U_{41}^T) \\
 d_{12} &= \text{Traza}(U_{22} * J_2 * U_{21}^T) + \text{Traza}(U_{32} * J_3 * U_{31}^T) + \text{Traza}(U_{42} * J_4 * U_{41}^T)
 \end{aligned}$$

$$\begin{aligned}
 d_{13} &= \text{Traza}(U_{33} * J_3 * U_{31}^T) + \text{Traza}(U_{43} * J_4 * U_{41}^T) \\
 d_{14} &= \text{Traza}(U_{44} * J_4 * U_{41}^T) \\
 d_{21} &= \text{Traza}(U_{21} * J_2 * U_{22}^T) + \text{Traza}(U_{31} * J_3 * U_{32}^T) + \text{Traza}(U_{41} * J_4 * U_{42}^T) \\
 d_{22} &= \text{Traza}(U_{22} * J_2 * U_{22}^T) + \text{Traza}(U_{32} * J_3 * U_{32}^T) + \text{Traza}(U_{42} * J_4 * U_{42}^T) \\
 d_{23} &= \text{Traza}(U_{33} * J_3 * U_{32}^T) + \text{Traza}(U_{43} * J_4 * U_{42}^T) \\
 d_{24} &= \text{Traza}(U_{44} * J_4 * U_{42}^T) \\
 d_{31} &= \text{Traza}(U_{31} * J_3 * U_{33}^T) + \text{Traza}(U_{41} * J_4 * U_{43}^T) \\
 d_{32} &= \text{Traza}(U_{32} * J_3 * U_{33}^T) + \text{Traza}(U_{42} * J_4 * U_{43}^T) \\
 d_{33} &= \text{Traza}(U_{33} * J_3 * U_{33}^T) + \text{Traza}(U_{43} * J_4 * U_{43}^T) \\
 d_{34} &= \text{Traza}(U_{44} * J_4 * U_{43}^T) \\
 d_{41} &= \text{Traza}(U_{41} * J_4 * U_{44}^T) \\
 d_{42} &= \text{Traza}(U_{42} * J_4 * U_{44}^T) \\
 d_{43} &= \text{Traza}(U_{43} * J_4 * U_{44}^T) \\
 d_{44} &= \text{Traza}(U_{44} * J_4 * U_{44}^T)
 \end{aligned}$$

where

$$DI = \begin{pmatrix} d_{11} & d_{12} & d_{13} & d_{14} \\ d_{21} & d_{22} & d_{23} & d_{24} \\ d_{31} & d_{32} & d_{33} & d_{34} \\ d_{41} & d_{42} & d_{43} & d_{44} \end{pmatrix} \quad (A13)$$

L7. Obtain the terms h_{ikm} , defined by Equation (A14):

$$h_{ikm} = \sum_{j=(i,k,m)}^n \text{traza}(U_{jkm} J_j U_{ji}^T) \quad (A14)$$

$$\begin{aligned}
 h_{111} &= \text{traza}(U_{111} J_1 U_{11}^T) + \text{traza}(U_{211} J_2 U_{21}^T) + \text{traza}(U_{311} J_3 U_{31}^T) + \text{traza}(U_{411} J_4 U_{41}^T) \\
 h_{112} &= \text{traza}(U_{212} J_2 U_{21}^T) + \text{traza}(U_{312} J_3 U_{31}^T) + \text{traza}(U_{412} J_4 U_{41}^T) \\
 h_{113} &= \text{traza}(U_{313} J_3 U_{31}^T) + \text{traza}(U_{413} J_4 U_{41}^T) \\
 h_{114} &= \text{traza}(U_{414} J_4 U_{41}^T) \\
 h_{121} &= \text{traza}(U_{221} J_2 U_{21}^T) + \text{traza}(U_{321} J_3 U_{31}^T) + \text{traza}(U_{421} J_4 U_{41}^T) \\
 h_{122} &= \text{traza}(U_{222} J_2 U_{21}^T) + \text{traza}(U_{322} J_3 U_{31}^T) + \text{traza}(U_{422} J_4 U_{41}^T) \\
 h_{123} &= \text{traza}(U_{323} J_3 U_{31}^T) + \text{traza}(U_{423} J_4 U_{41}^T) \\
 h_{124} &= \text{traza}(U_{424} J_4 U_{41}^T) \\
 h_{131} &= \text{traza}(U_{331} J_3 U_{31}^T) + \text{traza}(U_{431} J_4 U_{41}^T) \\
 h_{132} &= \text{traza}(U_{332} J_3 U_{31}^T) + \text{traza}(U_{432} J_4 U_{41}^T) \\
 h_{133} &= \text{traza}(U_{333} J_3 U_{31}^T) + \text{traza}(U_{433} J_4 U_{41}^T) \\
 h_{134} &= \text{traza}(U_{434} J_4 U_{41}^T) \\
 h_{141} &= \text{traza}(U_{441} J_4 U_{41}^T) \\
 h_{142} &= \text{traza}(U_{442} J_4 U_{41}^T) \\
 h_{143} &= \text{traza}(U_{443} J_4 U_{41}^T) \\
 h_{144} &= \text{traza}(U_{444} J_4 U_{41}^T) \\
 h_{211} &= \text{traza}(U_{211} J_2 U_{22}^T) + \text{traza}(U_{311} J_3 U_{32}^T) + \text{traza}(U_{411} J_4 U_{42}^T) \\
 h_{212} &= \text{traza}(U_{212} J_2 U_{22}^T) + \text{traza}(U_{312} J_3 U_{32}^T) + \text{traza}(U_{412} J_4 U_{42}^T) \\
 h_{213} &= \text{traza}(U_{313} J_3 U_{32}^T) + \text{traza}(U_{413} J_4 U_{42}^T) \\
 h_{214} &= \text{traza}(U_{414} J_4 U_{42}^T)
 \end{aligned}$$

$$\begin{aligned}
h_{221} &= \text{traza}(U_{221}J_2U_{22}^T) + \text{traza}(U_{321}J_3U_{32}^T) + \text{traza}(U_{421}J_4U_{42}^T) \\
h_{222} &= \text{traza}(U_{222}J_2U_{22}^T) + \text{traza}(U_{322}J_3U_{32}^T) + \text{traza}(U_{422}J_4U_{42}^T) \\
h_{223} &= \text{traza}(U_{323}J_3U_{32}^T) + \text{traza}(U_{423}J_4U_{42}^T) \\
h_{224} &= \text{traza}(U_{424}J_4U_{42}^T) \\
h_{231} &= \text{traza}(U_{331}J_3U_{32}^T) + \text{traza}(U_{431}J_4U_{42}^T) \\
h_{232} &= \text{traza}(U_{332}J_3U_{32}^T) + \text{traza}(U_{432}J_4U_{42}^T) \\
h_{233} &= \text{traza}(U_{333}J_3U_{32}^T) + \text{traza}(U_{433}J_4U_{42}^T) \\
h_{234} &= \text{traza}(U_{434}J_4U_{42}^T) \\
h_{241} &= \text{traza}(U_{441}J_4U_{42}^T) \\
h_{242} &= \text{traza}(U_{442}J_4U_{42}^T) \\
h_{243} &= \text{traza}(U_{443}J_4U_{42}^T) \\
h_{244} &= \text{traza}(U_{444}J_4U_{42}^T) \\
h_{311} &= \text{traza}(U_{311}J_3U_{33}^T) + \text{traza}(U_{411}J_4U_{43}^T) \\
h_{312} &= \text{traza}(U_{312}J_3U_{33}^T) + \text{traza}(U_{412}J_4U_{43}^T) \\
h_{313} &= \text{traza}(U_{313}J_3U_{33}^T) + \text{traza}(U_{413}J_4U_{43}^T) \\
h_{314} &= \text{traza}(U_{414}J_4U_{43}^T) \\
h_{321} &= \text{traza}(U_{321}J_3U_{33}^T) + \text{traza}(U_{421}J_4U_{43}^T) \\
h_{322} &= \text{traza}(U_{322}J_3U_{33}^T) + \text{traza}(U_{422}J_4U_{43}^T) \\
h_{323} &= \text{traza}(U_{323}J_3U_{33}^T) + \text{traza}(U_{423}J_4U_{43}^T) \\
h_{324} &= \text{traza}(U_{424}J_4U_{43}^T) \\
h_{331} &= \text{traza}(U_{331}J_3U_{33}^T) + \text{traza}(U_{431}J_4U_{43}^T) \\
h_{332} &= \text{traza}(U_{332}J_3U_{33}^T) + \text{traza}(U_{432}J_4U_{43}^T) \\
h_{333} &= \text{traza}(U_{333}J_3U_{33}^T) + \text{traza}(U_{433}J_4U_{43}^T) \\
h_{334} &= \text{traza}(U_{434}J_4U_{43}^T) \\
h_{341} &= \text{traza}(U_{441}J_4U_{43}^T) \\
h_{342} &= \text{traza}(U_{442}J_4U_{43}^T) \\
h_{343} &= \text{traza}(U_{443}J_4U_{43}^T) \\
h_{344} &= \text{traza}(U_{444}J_4U_{43}^T) \\
h_{411} &= \text{traza}(U_{411}J_4U_{44}^T) \\
h_{421} &= \text{traza}(U_{421}J_4U_{44}^T) \\
h_{422} &= \text{traza}(U_{422}J_4U_{44}^T) \\
h_{444} &= \text{traza}(U_{444}J_4U_{44}^T)
\end{aligned}$$

L8. Obtain the Coriolis and centripetal force column matrix $H = [h_i]^T$, whose elements are defined by Equation (A15):

$$h_i = \sum_{k=1}^n \sum_{m=1}^n h_{ikm} \dot{q}_k \dot{q}_m \quad (\text{A15})$$

$$\begin{aligned}
h_1 &= (q'_1)^2 h_{1,1,1} + (q'_2)^2 h_{1,2,2} + (q'_3)^2 h_{1,3,3} + (q'_4)^2 h_{1,4,4} + q'_1 q'_2 h_{1,1,2} + q'_1 q'_2 h_{1,2,1} + q'_1 q'_3 h_{1,1,3} \\
&\quad + q'_1 q'_3 h_{1,3,1} + q'_1 q'_4 h_{1,1,4} + q'_1 q'_4 h_{1,4,1} + q'_2 q'_3 h_{1,2,3} + q'_2 q'_3 h_{1,3,2} + q'_2 q'_4 h_{1,2,4} \\
&\quad + q'_2 q'_4 h_{1,4,2} + q'_3 q'_4 h_{1,3,4} + q'_3 q'_4 h_{1,4,3} \\
h_2 &= (q'_1)^2 h_{2,1,1} + (q'_2)^2 h_{2,2,2} + (q'_3)^2 h_{2,3,3} + (q'_4)^2 h_{2,4,4} + q'_1 q'_2 h_{2,1,2} + q'_1 q'_2 h_{2,2,1} + q'_1 q'_3 h_{2,1,3} \\
&\quad + q'_1 q'_3 h_{2,3,1} + q'_1 q'_4 h_{2,1,4} + q'_1 q'_4 h_{2,4,1} + q'_2 q'_3 h_{2,2,3} + q'_2 q'_3 h_{2,3,2} + q'_2 q'_4 h_{2,2,4} \\
&\quad + q'_2 q'_4 h_{2,4,2} + q'_3 q'_4 h_{2,3,4} + q'_3 q'_4 h_{2,4,3} \\
h_3 &= (q'_1)^2 h_{3,1,1} + (q'_2)^2 h_{3,2,2} + (q'_3)^2 h_{3,3,3} + (q'_4)^2 h_{3,4,4} + q'_1 q'_2 h_{3,1,2} + q'_1 q'_2 h_{3,2,1} + q'_1 q'_3 h_{3,1,3} \\
&\quad + q'_1 q'_3 h_{3,3,1} + q'_1 q'_4 h_{3,1,4} + q'_1 q'_4 h_{3,4,1} + q'_2 q'_3 h_{3,2,3} + q'_2 q'_3 h_{3,3,2} + q'_2 q'_4 h_{3,2,4} \\
&\quad + q'_2 q'_4 h_{3,4,2} + q'_3 q'_4 h_{3,3,4} + q'_3 q'_4 h_{3,4,3} \\
h_4 &= (q'_1)^2 h_{4,1,1} + (q'_2)^2 h_{4,2,2} + (q'_3)^2 h_{4,3,3} + (q'_4)^2 h_{4,4,4} + q'_1 q'_2 h_{4,1,2} + q'_1 q'_2 h_{4,2,1} + q'_1 q'_3 h_{4,1,3} \\
&\quad + q'_1 q'_3 h_{4,3,1} + q'_1 q'_4 h_{4,1,4} + q'_1 q'_4 h_{4,4,1} + q'_2 q'_3 h_{4,2,3} + q'_2 q'_3 h_{4,3,2} + q'_2 q'_4 h_{4,2,4} \\
&\quad + q'_2 q'_4 h_{4,4,2} + q'_3 q'_4 h_{4,3,4} + q'_3 q'_4 h_{4,4,3}
\end{aligned}$$

The Coriolis matrix is shown in Equation (A16):

$$H = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{pmatrix} \quad (\text{A16})$$

L9. Obtain the column matrix of gravity forces $G = [g_i]^T$, whose elements are defined by Equation (A17):

$$g_i = \sum_{j=1}^n \left(-m_j * g * U_{ji} r_j^i \right) \quad (\text{A17})$$

where $i = 1, 2, \dots, n$, g is the gravity vector expressed in the base system $\{S_0\}$ and is expressed by $(g_{x0}, g_{y0}, g_{z0}, 0)$, and r_j^i is the homogeneous coordinate vector of the element's center of mass j expressed in the element's reference frame j .

$$\begin{aligned} g &= \begin{pmatrix} 0 & 0 & -g & 0 \end{pmatrix} \\ r_1^1 &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\ r_2^2 &= \begin{pmatrix} 0.11 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\ r_3^3 &= \begin{pmatrix} 0.1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\ r_4^4 &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\ g_1 &= \sum_{j=1}^4 \left[\left(-m_j g (U_{j,i} r_j^i) \right) \right]_{i=1} \\ g_2 &= \sum_{j=1}^4 \left[\left(-m_j g (U_{j,i} r_j^i) \right) \right]_{i=2} \\ g_3 &= \sum_{j=1}^4 \left[\left(-m_j g (U_{j,i} r_j^i) \right) \right]_{i=3} \\ g_4 &= \sum_{j=1}^4 \left[\left(-m_j g (U_{j,i} r_j^i) \right) \right]_{i=4} \end{aligned}$$

Resulting in Equation (A18):

$$G = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \end{pmatrix} \quad (\text{A18})$$

L10. The robot's dynamics are seen in Equation (A19):

$$\tau = DIq'' + Cq' + G \quad (\text{A19})$$

where τ is the torque or force exerted on the engine, DI is the inertia matrix of the system and this is defined in Equation (A13), C is the Coriolis matrix, which is defined in Equation

(A16), and G is the gravity matrix, which is represented by Equation (A18). It is worth mentioning that $\tau = [\tau_1, \tau_2, \tau_3, \tau_4]^T$, $q'' = [q_1'', q_2'', q_3'', q_4'']^T$, and $q' = [q_1', q_2', q_3', q_4']^T$.

In order to know the angular behavior of each of the motor axes, the following equation must be followed (Equation (A20)):

$$q'' = DI^{-1}(\tau - Cq' - G) \quad (\text{A20})$$

Appendix B

Appendix B.1. Programming

The programming is performed using the Arduino language, where the first thing you want is to be able to read the number of pulses that a motor revolution makes; then, through an analog input and an encoder, it is programmed that 0° is 0 volts and 360° is 5 volts of direct current. Subsequently, the actions of the buttons are conditioned, where the important instruction is “attachInterrupt” because it is the one that will interrupt when the PID controller button is activated.

For the development of the PID controller, it is necessary to perform an error, which is the subtraction between the desired value and the measured value. To perform the derivative, the value of the Arduino timer is divided by one million.

For the part of the derivative, the current value is measured minus a previous value, all this by “dt”.

For the integrative part, the previous error plus the current error is added and multiplied by “dt”. The codes are listed in Table A3.

Table A3. Code.

```
const int C1 = 3; //Sig A encoder
const int C2 = 2; //Sig B encoder

volatile int n = 0;
volatile byte ant = 0;
volatile byte act = 0;

const int Pin_config = 7;
const int Pin_m3_down = 8;
const int Pin_m3_up = 9;

int btn_config = 0;
int btn_m3Down = 0;
int btn_m3Up = 0;

const int Pin_m3A1 = 5;
const int Pin_m3A2 = 6;

int m3A1 = 0;
int m3A2 = 0;

unsigned long lasTime = 0;
unsigned long sampleTime = 100;

const int r = 1980; //9592 Number of steps in the route
const float recorrido = 360.0; //Distance of the route in m
float angulo = 0.0; //Distance calibration
float angulo_previo = 0.0;
bool status = false;

double setpoint = 0;
double respuesta = 0;
```

Table A3. *Cont.*

```

double Y1 = 0;
double Y2 = 0;
double Y3 = 0;

uint32_t timer = 0;
double dt = 0;
double U = 0;

double Gt = 35; //950
double Gd = 150; //0.3
double ki = 0.001;

void setup() {
  Serial.begin(9600);

  pinMode(C1, INPUT);
  pinMode(C2, INPUT);

  attachInterrupt(digitalPinToInterrupt(C1), encoder, CHANGE);
  attachInterrupt(digitalPinToInterrupt(C2), encoder, CHANGE);

  pinMode(Pin_config, INPUT);
  pinMode(Pin_m3_down, INPUT);
  pinMode(Pin_m3_up, INPUT);

  pinMode(Pin_m3A1, OUTPUT);
  pinMode(Pin_m3A2, OUTPUT);

  timer = micros();
}

void loop() {

  btn_config = digitalRead(Pin_config);
  btn_m3Down = digitalRead(Pin_m3_down);
  btn_m3Up = digitalRead(Pin_m3_up);
  //delay(200);
  if(btn_config == HIGH){
    status = true;
    delay(1000);
  }

  if (btn_m3Down == HIGH and btn_m3Up == LOW)
  {
    analogWrite(Pin_m3A1,0);
    analogWrite(Pin_m3A2,250);
  } else if(btn_m3Down == LOW and btn_m3Up == HIGH)
  {
    analogWrite(Pin_m3A1,250);
    analogWrite(Pin_m3A2,0);
  }else
  {
    analogWrite(Pin_m3A1,0);
    analogWrite(Pin_m3A2,0);
  }
  setpoint = (analogRead(A0)*0.023)*(15,652)/1023;
  angulo = n*recorrido/r;

  Serial.print("Modo manual: ");
  Serial.print(angulo,4);
  Serial.println(" , setpoint = " + String(setpoint));

```

Table A3. *Cont.*

```

U = 0;
timer = micros();
while(status){
int atd = 150 ;
setpoint = (analogRead(A0)*0.023)*(15,652)/1023;

angulo_previo = angulo;
angulo = n*recorrido/r;
Y1 = angulo—setpoint;

dt = (double)(micros()-timer)/100000; // original 1000000
timer = micros();
Y2 = (angulo — angulo_previo)/dt;
// Y3 = Y3 + (10*ki*Y1);

U = —(10*Gt*Y1) —(0 —((1*Gd*Y2);

If(U < -atd){
U = —(atd;
}
else if(U > atd){
U = atd;
}

if(U <= —(15.0001){
analogWrite(Pin_m3A1,map(abs(U),5.001,165,2,250));
analogWrite(Pin_m3A2,0);
}
else if(U >= 15.0001){
analogWrite(Pin_m3A1,0);
analogWrite(Pin_m3A2,map(U,5.001,165,2,250));
}
else{
analogWrite(Pin_m3A1,0);
analogWrite(Pin_m3A2,0);
}

Serial.println("Ref = " + String(setpoint) + ", error = " + String(Y1) + ", posicion = " + String(angulo) + ", U = "
+ String(U) + ", velocidad = " + String(Y2));

}

}

void encoder(void)
{
Ant = act;

if(digitalRead(C1)) bitSet(act,1); else bitClear(act,1);
if(digitalRead(C2)) bitSet(act,0); else bitClear(act,0);

if(ant == 2 && act ==0) n++;
if(ant == 0 && act ==1) n++;
if(ant == 3 && act ==2) n++;
if(ant == 1 && act ==3) n++;

if(ant == 1 && act ==0) n--;
if(ant == 3 && act ==1) n--;
if(ant == 0 && act ==2) n--;
if(ant == 2 && act ==3) n--;
}

```

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