buildings

MDPI

Article

Practical Calculation Method for Overall Stability of Bent Frames and Frame Structures Based on Critical Force Redistribution

Shuwei Lan [1], Xiangyu Xu [2,3,*], Difei Zhao [4,*], Wei Zhang [5], Jiansheng Zhang [6] and Hongyu Chen [1]

1 School of Civil Engineering and Mechanics, Kunming University, Kunming 650214, China; lanshuwei@kmu.edu.cn (S.L.); 385603495@kmu.edu.cn (H.C.)
2 Dali Wansheng Real Estate Co., Ltd., Dali 671000, China
3 School of Public Safety and Emergency Management, Kunming University of Science and Technology, No. 68 Wenchang Road, Wuhua Zone, Kunming 650093, China
4 School of Computer Science and Technology/School of Artificial Intelligence, China University of Mining & Technology, Xuzhou 221116, China
5 Department of Spatial Planning, TU Dortmund University, 44227 Dortmund, Germany; wei8.zhang@tu-dortmund.de
6 School of Construction Engineering, Yunnan Vocational University of Land and Resources, Kunming 650504, China; 201310164@yngtxy.edu.cn
* Correspondence: 20232139008@stu.kust.edu.cn (X.X.); difei.zhao@cumt.edu.cn (D.Z.)

Abstract

The effective length factor method cannot account for variations in axial forces between columns and the mutual assistance among columns, resulting in overestimation of the critical load of columns that receive assistance and underestimation of that of columns that provide assistance. Moreover, this method is not readily applicable to bent frames and frame structures with multi-story tie beams. The phenomenon of critical load redistribution among columns under non-uniform loading conditions is revealed in this paper, and its underlying mechanical mechanism is elucidated. Based on this, a two-stage loading procedure for critical load redistribution analysis is proposed, in which the instability process of irregular structures is divided into two stages: independent loading of individual columns and combined loading of the assembly. By superimposing the critical load of columns in regular structures with the remaining load capacity, a formula for evaluating the overall stability critical capacity of bent and frame structures is established. The proposed method effectively accounts for the restraint provided by tie beams between columns, eliminating the need for iterative solutions of transcendental equations or the construction of complex total potential energy equations, thereby significantly simplifying the computational process. Comparative analyses of numerical examples demonstrate that the proposed method achieves high accuracy and enables quantitative assessment of mutual assistance between columns. It provides a conceptually clear, computationally efficient, and reliable practical tool for the stability design of bent and frame structures under complex loading conditions.

Keywords: bent frames; frame structures; critical load redistribution; overall stability; critical load capacity; the effective length factor

Academic Editor: Emanuele Brunesi

Received: 11 May 2026
Revised: 4 June 2026
Accepted: 8 June 2026
Published: 9 June 2026

1. Introduction

Bent frames and frame structures are widely used in bridges and buildings due to their explicit load transfer mechanisms and convenience of construction. In such structures, columns serve as the primary vertical load-bearing members, and their stability is directly related to the overall safety of the structure. Determining structural instability requires establishing the critical load capacity, and the applied loads must remain below this capacity. Especially when the structure is subjected to non-uniform vertical loads, the uneven distribution of axial forces among columns significantly affects the overall critical load capacity. Thus, accurately and efficiently determining the critical capacity of such structures remain a key concern. In engineering design, the primary method for determining the overall stability of bent frames and frame structures is the effective length factor method. When calculating the overall critical load capacity of a bent frame, the bent column is typically treated as a cantilever column with an effective length factor taken as 2. This approach cannot account for differences in axial forces among bent columns. When analyzing the overall stability of frame structures, the current *Steel Structure Design Standard* [1–4] provides calculation tables for the effective length factors of frame columns. Similarly, the *Standard for Design of Highway Reinforced Concrete and Prestressed Concrete Bridges and Culverts* [5,6] presents stability calculation formulas for compression members with one end fixed and the other end subjected to rotational and horizontal elastic restraints. However, the effective length factor method has significant limitations in practical applications for determining the critical load capacity of overall structures. Firstly, it is mainly applicable to regular structures with uniform loads and consistent column parameters; its validity for irregular structures with varying axial forces on the same floor remains unproven. Secondly, for bent frames and frame structures with multi-story tie beams, the mutual bracing between columns cannot be reasonably captured, leading to deviations from actual behavior. To address these issues, numerous studies have been conducted. Tong [7] investigated the mutual assistance between stories in a double-column single-span frame. By establishing rotational restraint equations at column ends, the exact solution for the effective length factor of columns was obtained and an analytical approach for multi-story frames was proposed. Furthermore, he also pointed out that the active and passive bifurcation of member instability are crucial to the overall stability mechanism: active bifurcation leads to an immediate degradation of the capacity, whereas passive bifurcation depends on prior deformation or external constraints, thereby fundamentally altering the post-buckling path and the interaction between columns. Hao [8] derived the overall stability calculation formula for multi-story sway frames using the concept of equivalent negative stiffness; however, the formula is lengthy and involves numerous parameters. Geng [9] employed the equilibrium method to derive formulas for the effective length factor of frame columns under various end restraint conditions and provided corresponding nomographs. Li [10] derived the stability matrix for single-story structural units and extended it to the overall structure, obtaining the governing equations for the elastic buckling load and proposing a simplified stability analysis method. Slimani [11,12] established the governing equilibrium equations of the structure based on the effective length factor method, derived the global stiffness matrix of the structure, and determined the overall critical load capacity of the frame structure with the aid of computer software. In addition, the influence of the boundary conditions at the column ends above and below the column under consideration on the effective length factor of the frame column was evaluated. Webber [13] considered the effect of variations in axial forces in adjacent columns on the effective length factor of frame columns and, by modifying the effective length factor, enabled the buckling loads of adjacent columns to be taken into account. Lan [14] transformed the solution of the critical load of frame structures into the calculation of story drifts of the frame and presented a formula for the critical load capacity of multi-story frame structures; however, the solution procedure is relatively cumbersome. Farajian [15] determined the elastic buckling critical load of modular frame columns by deriving the governing equations and solving them numerically based on the effective length factor method, and proposed a modified formula that accounts for the influence of joint stiffness. Xiao [16] derived formulas for the internal and external stiffness of separated columns based on the spring-rocking column model, transforming the determination of the critical load capacity of frame structures into the calculation of the overall lateral stiffness of the frame structure. This approach requires the assembly and computation of the global structural stiffness, making the calculation process relatively complex. Shafan [17] used finite element software to analyze the axial deformation effects in sway frame structures to evaluate their overall stability. Du [18] developed a finite element model of an aqueduct bent structure based on the OpenSees V2.0 platform, derived its seismic response characteristics through nonlinear dynamic time-history analysis, and determined the overall stability of the structure under seismic excitation using incremental dynamic analysis. Slack [19] evaluated the overall stability of steel frame structures by considering a combination of sway imperfections and member bow imperfections, and by introducing imperfections using either the direct definition method or the eigenmode scaling method. Although the aforementioned studies have conducted extensive and valuable explorations into methods for calculating the overall critical load capacity of bent frames and frame structures, certain limitations remain. Most studies focus on modifying and improving the effective length factor method by employing equilibrium methods or energy methods, which involve solving complex transcendental equations or total potential energy equations, making the procedures cumbersome and lengthy. The effective length factor method specified in current codes is primarily applicable to regular structures with uniformly distributed loads and consistent column parameters; its applicability to bent frames and frame structures with multi-story tie beams and complex column-to-column restraint relationships is limited, and it cannot account for the effects of variations in axial forces among columns. Furthermore, when finite element methods are used to evaluate the overall stability of bent frames and frame structures, the underlying computational theories and methods are embedded within the software, and engineers often have limited insight into the software kernel, resulting in a lack of design guidance that balances accuracy and convenience. To address the limitations of existing methods for calculating the overall critical load capacity of bent frames and frame structures, The influence of uneven axial force distribution among columns on the overall critical load capacity is investigated, the fundamental principles and mechanical mechanism underlying the phenomenon of critical load redistribution are elucidated, and a two-stage calculation method for the overall stability analysis of irregular bent frames and frame structures is established. The method is based on the critical load capacity of regular structures, from which the actual critical loads of individual columns are derived, eliminating the need to formulate and solve transcendental equations or construct total potential energy equations. This greatly simplifies the calculation process and provides a conceptually clear, computationally convenient, and accurate manual calculation tool for the stability design of bent frames and frame structures.

2. Phenomenon of Critical Load Redistribution

2.1. Concept of the Critical Load Redistribution Method

A reinforced concrete single-story single-span bent frame is illustrated in Figure 1. The concrete columns have a cross-section of 60×80 cm, a moment of inertia I = 2,560,000 cm^4, an elastic modulus E = 2000 kN/cm^2, and a Poisson's ratio of 0.2. The roof beam is an HN600 $\times$ 200 $\times$ 12 $\times$ 16 (Yuxi Yukun Iron and Steel Group Co., Ltd., Yuxi, Yunnan, China), with an elastic modulus E = 20,600 kN/cm^2 and a Poisson's ratio of 0.3. The column height is h = 800 cm, and the span length is l = 1200 cm. Loads are applied as concentrated nodal forces, with a vertical load P = 250 kN. Elastic buckling analysis is performed using the finite element software Ansys 17.0 to investigate the variation in the critical load capacity of the bent frame under different load distributions.

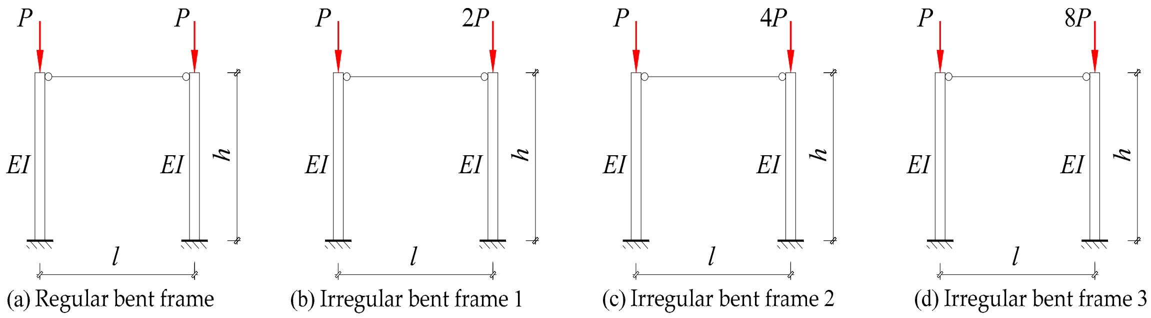

Figure 1. Single-story single-span bent frames.

During the finite element analysis using Ansys 17.0, the bent columns are simulated using BEAM188 elements, while the roof beams are modeled using LINK1 elements. The beam-to-column connections were assumed to be pinned, and the column bases are considered as fixed nodes, with axial deformations neglected. For mesh generation, each beam, column, and brace is divided into a minimum of 10 elements, with mesh sizes automatically optimized based on member length and cross-sectional dimensions. The Block Lanczos eigenvalue solver is employed to extract the first ten buckling modes of the structure and their corresponding critical load factors [20]. The overall critical load capacity of the structure is obtained as the product of the critical load factor and the applied load. For comparative analysis, the results obtained from the effective length factor method and the finite element analysis using Ansys 17.0 are summarized in Table 1.

Table 1. Comparison of critical loads obtained by the effective length factor method and Ansys 17.0.

Type		Type (a)	Type (b)	Type (c)	Type (d)
Structural type		Regular	Irregular	Irregular	Irregular
Ansys 17.0 ①/kN		$P^*_{cr,l} = 19,739$ $P^*_{cr,r} = 19,739$ $\Sigma P^*_{cr} = 39,478$	$P^*_{cr,l} = 13,127$ $P^*_{cr,r} = 26,254$ $\Sigma P^*_{cr} = 39,381$	$P^*_{cr,l} = 7834$ $P^*_{cr,r} = 31,336$ $\Sigma P^*_{cr} = 39,170$	$P^*_{cr,l} = 4329$ $P^*_{cr,r} = 34,662$ $\Sigma P^*_{cr} = 38,961$
The effective length factor method ②/kN		$P_{cr,l} = 19,739$ $P_{cr,r} = 19,739$ $\Sigma P_{cr} = 39,478$	$P_{cr,l} = 19,739$ $P_{cr,r} = 19,739$ $\Sigma P_{cr} = 39,478$	$P_{cr,l} = 19,739$ $P_{cr,r} = 19,739$ $\Sigma P_{cr} = 39,478$	$P_{cr,l} = 19,739$ $P_{cr,r} = 19,739$ $\Sigma P_{cr} = 39,478$
②/①	$P_{cr,l}/P^*_{cr,l}$	1.000	1.504	2.520	4.560
	$P_{cr,r}/P^*_{cr,r}$	1.000	0.752	0.630	0.569
	$\Sigma P_{cr}/\Sigma P^*_{cr}$	1.000	1.002	1.008	1.013
Mutual assistance		No assistance occurs	The left column provides assistance for the right column	The left column provides assistance for the right column	The left column provides assistance for the right column
Critical load redistribution		No redistribution occurs	Redistribution occurs	Redistribution occurs	Redistribution occurs

Note: $P_{cr,l}$ and $P_{cr,r}$ are the critical loads of the left and right columns, respectively, and ΣP_{cr} c is the total critical load of the bent frame structure.

In type (a), the axial forces in the left and right columns are identical, representing a regular bent frame with uniformly distributed loads. In this case, no critical load redistribution occurs during buckling analysis. The Ansys 17.0 results in Table 1 show that the critical loads of the left and right columns are equal, indicating no mutual assistance between the two columns. For types (b), (c), and (d), the axial forces in the left and right columns differ, representing irregular bent frames with non-uniformly distributed loads. When the axial force in the right column reaches the critical load of a column in a regular bent frame, the axial force in the left column remains below its critical load. If the right column buckles upon reaching its critical load, lateral deformation would occur; however, the left column still possesses load capacity reserve and remains stable, maintaining its vertical alignment. Since the left and right columns are connected by a roof beam, they provide mutual assistance to each other, preventing the right column from deforming independently. Consequently, the right column remains vertical under the assistance provided by the left column. At this stage, both columns can continue to be loaded until the remaining reserve of the left column is fully exhausted, at which point the entire structure experiences simultaneous instability. It should be noted that the mechanical essence of mutual bracing between columns lies in the following: when the axial force of a certain column first reaches its critical value for independent buckling, its flexural stiffness begins to degrade significantly, generating a tendency for lateral deformation. However, because the adjacent columns are still in the elastic stage and possess load-carrying reserves, they provide lateral restraint to that column through axial force transfer via beams or tie beams, forcing the columns to deform cooperatively. During this process, the buckling of the column subjected to a higher compressive load is delayed, while the reserve capacity of the column with a lower compressive load is gradually dissipated, ultimately leading to global instability. This mechanism is analogous to the internal force redistribution phenomenon observed in statically determinate structures; therefore, this phenomenon is referred to as the critical load redistribution phenomenon in this paper.

It can also be observed from Table 1 that for type (a), the critical loads of the left and right columns obtained by the effective length factor method are equal to those obtained by Ansys 17.0, indicating that the effective length factor method yields results in good agreement with the critical load of the bent frame under uniformly distributed loads. For types (b), (c), and (d), which are irregular bent frames with non-uniformly distributed loads, the critical load of the left column obtained by the effective length factor method is 50.4% larger than that obtained by Ansys 17.0, while that of the right column is 24.8% smaller for type (b); for type (c), the left column critical load is 152% larger and the right column is 37% smaller; for type (d), the left column critical load is 356% larger and the right column is 43.1% smaller. The primary reason for these discrepancies is that the effective length factor method determines the critical load capacity of the structure based on the assumption of uniformly distributed loads and the absence of mutual bracing between columns. For types (b), (c), and (d), however, mutual bracing occurs between the left and right columns. The effective length factor method cannot account for this mutual assistance, leading to significant errors: the critical load of the right column that receives assistance is underestimated, while that of the left column that provides assistance is overestimated, resulting in a conservative estimate for the column that receives assistance and an unsafe estimate for the column that provides assistance. For types (a) through (d), the total critical loads of the bent frames vary only slightly, within a range of 1.5%, and the results obtained by the effective length factor method are in good agreement with those from Ansys 17.0.

Based on the above analysis and extensive finite element calculations, the characteristics of the systems to which critical load redistribution is applicable can be summarized as follows:

(1) The total critical load of the structure remains unchanged.
(2) When the axial force in the more heavily loaded column reaches the critical load of a column in a regular structure, buckling does not occur.
(3) The less heavily loaded column provides assistance to the more heavily loaded column through the beam, allowing the irregular structure to be further loaded until the reserve capacity of the column that provides assistance is completely exhausted.

2.2. Calculation Procedure of the Critical Load Redistribution Method

The single-story single-span bent frame shown in Figure 1a has equal loads and stiffnesses for the left and right columns, with no mutual assistance occurring between them. This represents a regular bent frame, where each bent column acts as an independent cantilever column and the beam at the column tops does not contribute to assistance. The critical loads of the left and right columns are identical, and the critical load of the regular bent frame columns can be obtained using the effective length factor method as follows:

$$P_{cr0} = \frac{\pi^2 EI}{(\mu h)^2} = \frac{3.1416^2 \times 2000 \times 2,560,000}{(2 \times 800)^2} = 19,739 \text{ kN} \tag{1}$$

where μ is the effective length factor of the column, and EI is the flexural stiffness of the bent frame column.

The single-story single-span bent frame shown in Figure 1a is a regular structure, in which simultaneous buckling of the left and right columns occurs without mutual assistance when the overall structure loses stability. For the three types of irregular bent frames shown in Figure 1b–d, the critical load redistribution can be divided into two stages. Figure 1b is taken as an example.

First loading stage (individual columns subjected to independent loading): The left and right columns are loaded simultaneously. The axial force in the right column reaches the critical load first, i.e., $2P_0 = P_{cr0}$, while the axial force in the left column is only half of its critical load, i.e., $P_0 = P_{cr0}/2 = 9870$ kN. If the right column buckles upon reaching its critical load, lateral deformation would occur. However, at this stage, the left column still possesses half of its load capacity reserve ($P_{cr0}/2$) and remains stable, maintaining its vertical alignment. Since the left and right columns are connected by a beam, the right column cannot deform independently; therefore, it remains vertical under the assistance provided by the left column.

Second loading stage (columns subjected to combined loading): The left and right columns can be further loaded until the remaining load capacity reserve of the left column is fully exhausted, i.e., $\Delta P + 2\Delta P = P_{cr0}/2$. Therefore, the redistributed applied loads are $\Delta P = P_{cr0}/6 = 3290$ kN for the left column and $2\Delta P = 6580$ kN for the right column. Total loads on the left and right columns during the two stages:

$$P_{cr,l} = P_0 + \Delta P = 9870 + 3290 = 13,160 \text{ kN}; \; P_{cr,r} = 2P_0 + 2\Delta P = 26,320 \text{ kN}$$

Following the aforementioned calculation procedure for critical load redistribution, the two types of irregular bent frames shown in Figure 1c,d are also calculated, and the results are summarized in Table 2.

Table 2. Calculation process of structural critical capacity by the critical load redistribution method.

Type	Type (a)	Type (b)	Type (c)	Type (d)
The critical load redistribution method/kN	$P_0 = 19,739$ $\Delta P = 0$ $P_{cr,l} = 19,739$ $P_{cr,r} = 19,739$ $\Sigma P_{cr} = 39,478$	$P_0 = 9870$ $\Delta P = 3290$ $P_{cr,l} = 13,160$ $P_{cr,r} = 26,320$ $\Sigma P_{cr} = 39,480$	$P_0 = 4935$ $\Delta P = 2961$ $P_{cr,l} = 7896$ $P_{cr,r} = 31,584$ $\Sigma P_{cr} = 39,480$	$P_0 = 2467$ $\Delta P = 1919$ $P_{cr,l} = 4386$ $P_{cr,r} = 35,088$ $\Sigma P_{cr} = 39,474$

Note: P_0 and ΔP are the loads that can be sustained during the first and second loading stages of the critical load redistribution method, respectively.

As can be seen from Table 2: Type (a) is a regular bent frame. In the first loading stage, the left and right columns reach the critical load simultaneously, with no load-carrying reserve, so $\Delta P = 0$ in the second loading stage. Types (b), (c), and (d) are irregular bent frames. The right column first reaches the critical load P_{cr0} of the regular bent frame column, while the left column still has load capacity reserve, which constitutes the first loading stage. In the second loading stage, the left and right columns continue to be loaded proportionally by ΔP until the remaining load capacity reserve of the left column is exhausted and buckling occurs. Therefore, by using the critical load redistribution method to determine the critical load capacity of bent frames, once the critical load capacity of the regular bent frame is determined, the critical loads of the columns in the irregular bent frame can be easily obtained. Furthermore, it can be observed that the maximum error between the critical loads of the four types of bent frames in Figure 1 calculated by the critical load redistribution method and those obtained from Ansys 17.0 is 1.3%, indicating good agreement. This also verifies the reliability and accuracy of the critical load redistribution method.

3. Calculation of Critical Load Capacity of Bent Frames

The focus is on bent structures, where tie beams are hinged to columns: flexural stiffness is neglected, axial stiffness is assumed infinite, and only horizontal constraints are provided. As shown in Figure 2, for a multi-story multi-span bent frame, loads are applied at the tops of the bent columns, and the loads at each column top may vary, i.e., $P_1 = \xi_1 P$, $P_2 = \xi_2 P$, $P_j = \xi_j P$, $P_n = \xi_n P$.

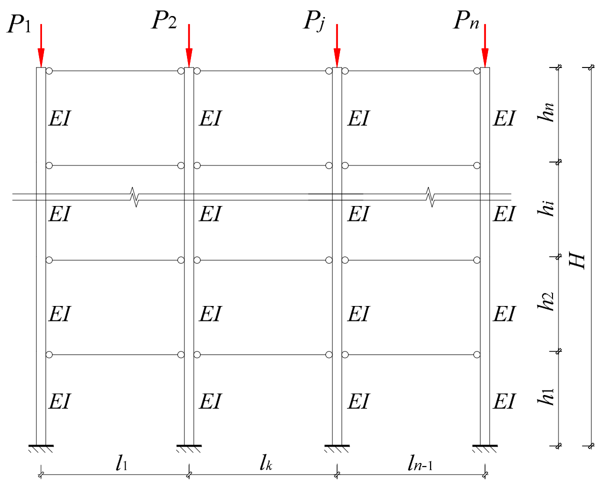

Figure 2. Multi-story multi-span bent frame.

For a multi-story bent frame, if the loads applied at the tops of all bent columns are equal, i.e., $\xi_1 = \xi_2 = \xi_j = \xi_n$, then no mutual assistance occurs among the bent columns, and it is considered a regular bent frame. By extending the method used for determining the overall stability of a single-story single-span bent frame via the critical load redistribution method to multi-story multi-span bent frames, for a regular multi-story multi-span bent frame, all columns reach their critical loads simultaneously during the first loading stage with no load capacity reserve, and thus, during the second loading stage, $\Delta P = 0$ and further loading is not possible. When the loads applied to the bent columns are unequal, during the first loading stage, the column subjected to the largest load reaches the critical load P_{cr0} of a regular bent frame column first, while the remaining bent columns still have load capacity reserve, with the column subjected to the smallest load having the greatest reserve. During the second loading stage, all columns continue to be loaded proportionally by ΔP until the remaining load capacity reserve of the column with the smallest load is exhausted, leading to instability. To determine the overall stability of a bent frame using the critical load redistribution method, the total critical load of the regular bent frame can first be obtained using the effective length factor method. The irregular bent frame then undergoes instability simultaneously after the two loading stages, and its critical load capacity can be determined based on the proportional relationships of the loads applied to each column.

$$\sum_{j=1}^{m} P_{cr0j} = \frac{m \cdot \pi^2 EI_j}{(\mu_j H)^2} \tag{2}$$

$$N_{crj} = \frac{\xi_j}{\sum_{j=1}^{m} \xi_j} \sum_{j=1}^{m} P_{cr0j} \tag{3}$$

where P_{cr0j} is the critical load of the j-th column in the regular bent frame; EI_j is the flexural stiffness of the j-th column; μ_j is the effective length factor of the j-th column; N_{crj} is the critical load of the j-th column in the bent frame; ξ_j is the load proportion coefficient of the j-th column; m is the total number of columns in the bent frame; and H is the total height of the bent column.

From Equation (3), the effective length factor of each bent column can also be obtained using the following formula:

$$\mu_j = \sqrt{\frac{\pi^2 EI_j}{N_{crj} H^2}} \tag{4}$$

4. Calculation of Critical Load Capacity of Single-Span Frame Structures

The analysis focuses on frame structures featuring rigid beam-column connections, in which beam stiffness affects the rotational restraint at column ends. The single-story single-span frame shown in Figure 3 has loads applied at the tops of the frame columns. The loads at each column top may vary, with $P_2 = \xi P_1$ taken as an example.

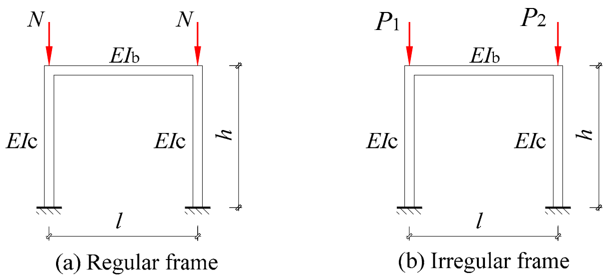

Figure 3. Single−story single−span frame.

The regular frame shown in Figure 3a has equal axial forces in the left and right columns, with no mutual assistance occurring between them. In the first loading stage, both columns reach the critical load simultaneously, and there is no load capacity reserve; therefore, in the second loading stage $\Delta P = 0$, further loading is not possible. The total critical load of the regular frame columns can be obtained using the effective length factor method as follows:

$$P_{cr0} = \frac{\pi^2 EI_c}{(\mu h)^2} \tag{5}$$

where μ is the effective length factor of the column, which can be obtained from Equation (6); EI_c is the flexural stiffness of the frame column; and h is the height of the frame column.

When the flexural stiffness EI_c of the regular frame column and the flexural stiffness EI_b of the frame beam are determined, the effective length factor of the regular frame column can be calculated using the following formula [21]:

$$\mu = \sqrt{\frac{1.52 + 4(K_1 + K_2) + 7.5K_1K_2}{K_1 + K_2 + 7.5K_1K_2}} \tag{6}$$

where K_1 and K_2 are the ratios of the sum of the flexural stiffness of the beams at the column top and bottom to the sum of the flexural stiffness of the columns, respectively.

Using the critical load redistribution method to analyze the irregular frame shown in Figure 3b, during the first loading stage, each column is subjected to independent loading. Assuming a load proportion coefficient $\xi > 1$, the left and right columns are loaded simultaneously, with the right column, which carries a larger load, first reaching the critical load of a regular frame column, i.e., $\xi P_0 = P_{cr0}$. At this point, the axial force in the left column is $1/\xi$ times the critical load, and it still retains some load capacity reserve. As the right column is connected to the left column through the beam, it remains vertical due to the assistance provided by the left column. In the second loading stage, the left and right columns continue to be loaded proportionally until the remaining load capacity reserve of the left column is exhausted, i.e., $\Delta P + \xi\Delta P = (1 - 1/\xi)P_{cr0}$. Consequently, the redistributed critical load of the left column is $\Delta P = \frac{(\xi-1)}{\xi(\xi+1)}P_{cr0}$. Based on the analysis of the two loading stages, the loads in the two stages are superimposed to obtain the following:

$$\text{Left column}: \ P_{cr,l} = P_0 + \frac{(\xi - 1)}{\xi(\xi + 1)}P_{cr0} = \frac{2}{\xi + 1}P_{cr0} \tag{7}$$

$$\text{Right column}: \ P_{cr,r} = \xi P_0 + \xi\frac{(\xi - 1)}{\xi(\xi + 1)}P_{cr0} = \frac{2\xi}{\xi + 1}P_{cr0} \tag{8}$$

where P_{cr0} is the critical load of the regular frame column, which can be obtained from Equation (5); ξ is the load proportion coefficient.

The above analysis method is applied to a double-story single-span frame, with its regular and irregular frames illustrated in Figure 4.

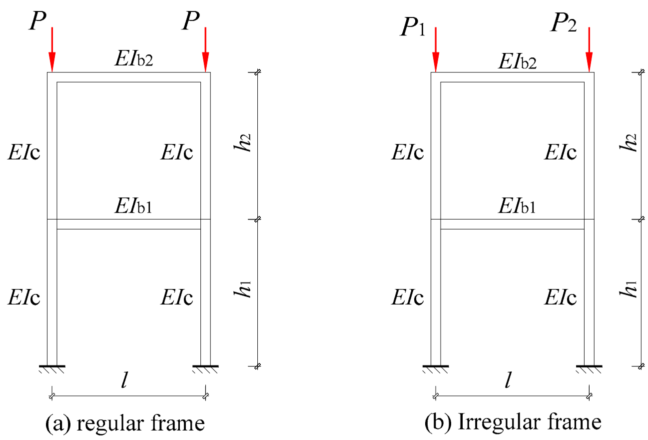

Figure 4. Double−story single−span frame.

When the critical load redistribution method is applied to double-story frames, it is suitable for analyzing the non-uniform distribution of axial forces between columns. When the beam-to-column stiffness ratios at different stories of the frame structure do not differ significantly, the method yields good accuracy and reliability in calculating the structural critical load capacity. For a regular double-story single-span frame, the critical load capacity can be obtained by calculating the critical loads of the columns in the upper and lower stories using the effective length factor method and then taking the average. Accordingly, the critical load of the columns in a regular double-story single-span frame can be determined using the following formula.

$$P_{cr0} = [\frac{1}{(\mu_1 h_1)^2} + \frac{1}{(\mu_2 h_2)^2}]\frac{\pi^2 EI_c}{2} \tag{9}$$

where μ_1 and μ_2 are the effective length factors of the frame columns in the first and second stories, respectively, which can be obtained from Equation (6), and h_1 and h_2 are the story heights of the first and second stories, respectively.

After the critical loads of the columns in the regular double-story single-span frame are determined, they are substituted into Equations (7) and (8) to obtain the critical loads of the left and right columns in the irregular frame using the critical load redistribution method.

After the critical load of the frame columns is determined by Equations (7) and (8), the effective length factor of each frame column can also be obtained using the following formula:

$$\mu = \sqrt{\frac{\pi^2 EI_c}{P_{cr}h^2}} \tag{10}$$

5. Application Examples

5.1. Example 1

A two-span double-story reinforced concrete bent frame structure is as illustrated in Figure 5. The concrete columns have a cross-section of 60 × 80 cm, an elastic modulus of E = 2000 kN/cm^2, and a Poisson's ratio of 0.2. The roof beams are HN400 × 200 × 8 × 13 (Yuxi Yukun Iron and Steel Group Co., Ltd., Yuxi, Yunnan, China), with an elastic modulus of E = 20,600 kN/cm^2 and a Poisson's ratio of 0.3. The story height is h = 500 cm, and the span length is l = 1000 cm. Loads are applied as concentrated nodal forces, with a vertical load P = 300 kN. The critical load redistribution method is used to determine the overall stability of the structure.

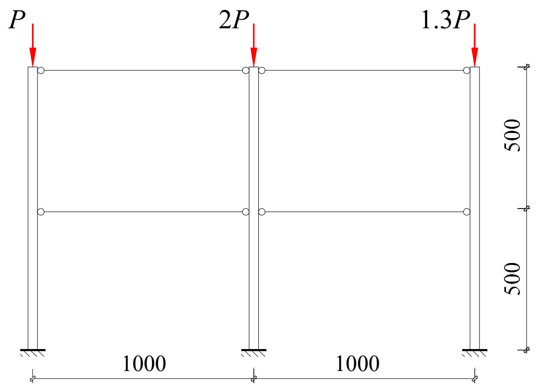

Figure 5. Two−span double−story reinforced concrete bent frame.

Calculation steps of the proposed method:

(1) Calculation of the critical load of each bent column

Based on the known parameters, the flexural stiffness of each bent frame column can be determined: $EI = 2000 \times \frac{1}{12} \times 60 \times 80^3 = 5,120,000,000$ kN·cm^2. Substituting the obtained flexural stiffness into Equation (2) yields

$$\sum_{j=1}^{m} P_{cr0j} = \frac{m \cdot \pi^2 EI_j}{(\mu_j H)^2} = \frac{3 \times 3.1416^2 \times 5,120,000,000}{(2 \times 1000)^2} = 37,899 \text{ kN}$$

Substituting the above calculation results into Equation (3) yields the critical load for each bent column.

$$N_{crL} = \frac{1}{1 + 2 + 1.3} \times 37,899 = 8814 \text{ kN (Left column)}$$

$$N_{crM} = \frac{2}{1+2+1.3} \times 37,899 = 17,627 \text{ kN (Middle column)}$$

$$N_{crR} = \frac{1.3}{1+2+1.3} \times 37,899 = 11,458 \text{ kN (Right column)}$$

(2) Calculation of the effective length factor of each bent column

The resulting critical loads are then substituted into Equation (4) to obtain the effective length factor of each bent column.

$$\mu_L = \sqrt{\frac{\pi^2 EI_L}{N_{crL}H^2}} = \sqrt{\frac{3.1416^2 \times 5,120,000,000}{8814 \times 1000^2}} = 2.394 \text{ (left column)}$$

$$\mu_M = \sqrt{\frac{\pi^2 EI_M}{N_{crM}H^2}} = \sqrt{\frac{3.1416^2 \times 5,120,000,000}{17,627 \times 1000^2}} = 1.693 \text{ (middle column)}$$

$$\mu_R = \sqrt{\frac{\pi^2 EI_R}{N_{crR}H^2}} = \sqrt{\frac{3.1416^2 \times 5,120,000,000}{11,458 \times 1000^2}} = 2.100 \text{ (right column)}$$

The calculated results of the proposed method in this paper are presented in Table 3.

Table 3. Compare results of bucking loads and effective length factors for bent frame columns.

Column	Column Axial Force N_j/kN	N_{crj}/kN					μ_j				
		The Effective Length Factor Method ①	The Proposed Method ②	Ansys 17.0 ③	①/③③	②/③	The Effective Length Factor Method ④	The Proposed Method ⑤	Ansys 17.0 ⑥	④/⑥	⑤/⑥
Right	390	12,633	11,458	11,456	1.103	1.000	2.000	2.100	2.100	0.952	1.000
Middle	600	12,633	17,627	17,624	0.717	1.000	2.000	1.693	1.693	1.181	1.000
Left	300	12,633	8814	8812	1.433	1.000	2.000	2.394	2.395	0.835	1.000

For comparative analysis, two additional methods are employed. The first is the effective length factor method, in which the effective length factor of each bent column (right column, middle column and left column) is taken as $\mu = 2$ according to the relevant standard, and the critical load of each bent column is directly calculated using Euler's formula: $N_{crl} = N_{crM} = N_{crR} = \frac{\pi^2 EI_c}{(\mu H)^2} = \frac{3.1416^2 \times 5,120,000,000}{(2\times1000)^2} = 12,633$ kN. The second is elastic buckling analysis using the finite element software Ansys 17.0: columns are modeled with BEAM188 elements, while roof beams and tie beams are modeled with LINK1 elements; column bases are fixed; and the elastic buckling loads are extracted using the Block Lanczos eigenvalue solver. The results obtained from both methods are also listed in Table 3.

As can be seen from Table 3, the results calculated by the proposed method are in good agreement with those obtained from Ansys 17.0, with minimal errors, indicating that the assistance among the bent columns and the influence of tie beams between them are adequately considered. In contrast, the results obtained by the effective length factor method show significant deviations from the Ansys 17.0 results. For the left bent column, the critical load is 1.433 times that obtained from Ansys 17.0, and the effective length factor is 16.5% smaller, indicating an overestimation of the critical load for this column, which is unsafe. For the right bent column, the critical load is 1.103 times that from Ansys 17.0, with an effective length factor 4.8% smaller, also overestimating its critical load. For the middle bent column, the critical load is only 0.717 times that from Ansys 17.0, and the effective length factor is 18.1% larger, underestimating its critical load. This is primarily because the left and right columns provide assistance, which reduces their own critical loads, whereas the middle column receives assistance, thereby increasing its critical load.

5.2. Example 2

An urban pedestrian bridge adopting a pier-girder rigidly connected system is as illustrated in Figure 6. The piers of the pedestrian bridge have a box-section of 250 × 250 × 10 mm, the capping beam is HN600 × 200 × 11 × 17, and the tie beam between piers is HN300 × 150 × 6 × 9 (Yuxi Yukun Iron and Steel Group Co., Ltd., Yuxi, Yunnan, China). The steel has an elastic modulus of E = 20,600 kN/cm^2, a yield strength f_y = 235 Mpa, and a Poisson's ratio of 0.3. The applied load P is 120 kN, the story height h = 300 cm, and the span length l =400 cm. Loads are applied as concentrated nodal forces. The critical load redistribution method is used to determine the overall structural stability capacity of the pedestrian bridge.

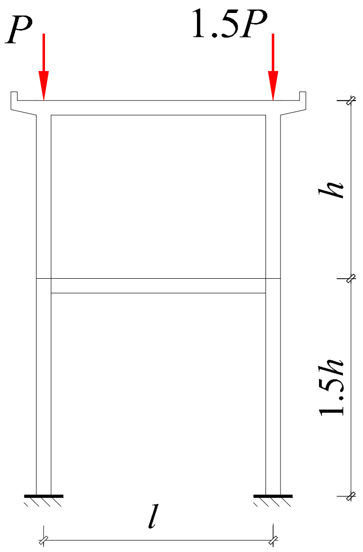

Figure 6. Calculation diagram of the urban pedestrian bridge.

The critical load of the columns in the regular frame structure is calculated using Equation (9), and then substituted into Equations (7) and (8) to obtain the critical loads of the left and right piers of the pedestrian bridge. The effective length factors of the piers are determined using Equation (10), and the calculation results are presented in Table 4. For comparison, calculations are also performed using the effective length factor method and the finite element software Ansys 17.0, with the corresponding results also listed in Table 4. In the Ansys 17.0 analysis, both columns and beams are modeled using BEAM188 elements, all beam–column connections are rigid, and the column bases are fixed.

Table 4. Compare results of critical loads and effective length factors for bridge piers.

Bridge Pier	Pier Column Axial Force N_{ij}/kN	$(N_{ij})_{cr}$/kN			①/③	②/③	μ_{ij}			④/⑥	⑤/⑥
		The Effective Length Factor Method ①	The Proposed Method ②	Ansys 17.0 ③			The Effective Length Factor Method ④	The Proposed Method ⑤	Ansys 17.0 ⑥		
Right-2	180	10,442	9196	10,047	1.039	0.915	1.373	1.463	1.399	0.981	1.046
Right-1	180	4867	9196	10,047	0.484	0.915	1.340	0.975	0.933	1.436	1.045
Left-2	120	10,442	6131	6698	1.559	0.915	1.373	1.791	1.714	0.801	1.045
Left-1	120	4867	6131	6698	0.727	0.915	1.340	1.194	1.143	1.172	1.045

It can be seen from Table 4 that the results calculated by the proposed method are in good agreement with those obtained from Ansys 17.0, with an error of 8.5% in the critical load of the piers and 5.5% in the effective length factor, indicating that the proposed method effectively accounts for the assistance between the left and right piers and the influence of the tie beams between them. In contrast, the results obtained by the effective length factor method show significant deviations from the Ansys 17.0 results. For the lower left pier (Left-1), the critical load is 27.3% smaller than that from Ansys 17.0, and the effective length factor is 19.1% larger, indicating that this pier receives assistance and its critical load is underestimated. For the upper left pier (Left-2), the critical load is 55.9% larger than that from Ansys 17.0, and the effective length factor is 19.9% smaller, indicating that this pier provides assistance and its critical load is overestimated. For the lower right pier (Right-1), the critical load is 51.6% smaller than that from Ansys 17.0, and the effective length factor is 43.6% larger, indicating that this pier receives assistance and its critical load is underestimated. For the upper right pier (Right-2), the critical load is 3.9% larger than that from Ansys 17.0, and the effective length factor is 1.9% smaller, indicating that this pier provides assistance and its critical load is overestimated.

6. Conclusions

(1) The phenomenon of critical load redistribution among columns under non-uniform loading conditions is revealed. When the distribution of axial forces among columns is non-uniform, the more heavily loaded column does not immediately undergo instability upon reaching the critical load of a regular structural column. Instead, it receives assistance through connections such as beams or tie beams with the less heavily loaded column, and overall instability of the structure occurs only after the load capacity reserve of the latter is completely exhausted. The core mechanism underlying this phenomenon lies in the internal force redistribution and mutual bracing effect between columns.

(2) A method for critical load redistribution analysis based on a two-stage loading procedure is proposed. The method divides the buckling process of irregular structures into two stages: independent loading of individual columns and combined loading of the assembly. By superimposing the critical load of columns in regular structures with the remaining load capacity reserve, the actual critical load of each column at the point of overall instability can be accurately calculated. Comparison with finite element results from Ansys 17.0 demonstrates that the proposed method yields smaller calculation errors and significantly outperforms the effective length factor method, while offering explicit physical concepts and computational simplicity.

(3) The critical load redistribution method is applicable to bent frames and frame structures characterized by non-uniform load distribution and reliable lateral connections between columns, and its core premise is that the total critical load of the structure remains constant. Examples confirm that this method effectively quantifies the mutual assistance between columns, avoiding the overestimation of the critical load of columns that provide assistance and the underestimation of the critical load of columns that receive assistance inherent in the effective length factor method. It thus provides a more accurate analytical tool for the stability design of tall piers, bent frames, and frame structures under complex loading conditions.

(4) The proposed method is based on linear elastic assumptions and is applicable only to plane unbraced bent and frame structures with sufficiently large slenderness ratios such that buckling occurs within the elastic range. Inelastic buckling, spatial effects, and bracing systems are not considered. Future work will include experimental

validation and extend the method to geometric and material nonlinear stability as well as seismic overall stability analysis to enhance its engineering applicability.

Author Contributions: Conceptualization, S.L. and X.X.; Methodology, S.L.; Software, D.Z. and H.C.; Validation, D.Z. and H.C.; Formal analysis, S.L. and D.Z.; Investigation, S.L., D.Z., X.X., D.Z., W.Z. and H.C.; Resources, D.Z. and H.C.; Data curation, S.L., J.Z. and H.C.; Writing—original draft, S.L.; Writing—review and editing, D.Z.; Visualization, W.Z.; Supervision, D.Z.; Project administration, S.L.; Funding acquisition, S.L. All authors have read and agreed to the published version of the manuscript.

Funding: This study was financially supported by Basic Research Project in Yunnan province of China (202401AT070031), the Special Basic Cooperative Research Programs of Yunnan Provincial Undergraduate Universities' Association (202501BA070001-075, 202501BA070001-083) and the National Natural Science Foundation of China (51868034).

Data Availability Statement: The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding author.

Conflicts of Interest: Author Xiangyu Xu was employed by the company Dali Wansheng Real Estate Co., Ltd. The remaining authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

References

1. *GB 50017-2017*; Standard for Design of Steel Structures. China Architecture & Building Press: Beijing, China, 2018.
2. *ANSI/AISC 360-16*; Specification for Structural Steel Buildings. AISC: Chicago, IL, USA, 2016.
3. *Eurocode 3*; Design of Steel Buildings. BSI: London, UK, 2005.
4. *AS 4100*; Building Code of Australia—Steel Buildings. Standards Australia: Sydney, Australia, 1998.
5. *JTG 3362-2018*; Specifications for Design of Highway Reinforced Concrete and Prestressed Concrete Bridges and Culverts. China Architecture & Building Press: Beijing, China, 2018.
6. *AASHTO LRFDBD-9*; AASHTO LRFD Bridge Design Specifications. American Association of State Highway and Transportation Officials: Washington, DC, USA, 2020.
7. Tong, G. *Plane Stability of Steel Structures*; China Architecture & Building Press: Beijing, China, 2005; pp. 181–185.
8. Hao, J.; Tian, W.; Wang, X. Simple calculation method for overall stability of multi-story sway frame. *J. Build. Struct.* **2011**, *32*, 183–188. [CrossRef]
9. Geng, X.; Zhou, D. Universal charts for determining the effective length of compression columns. *Eng. Mech.* **2014**, *31*, 154–160. [CrossRef]
10. Li, Q.; Zhou, A. A simplified method for stability analysis of multi-story frames considering vertical interactions between stories. *Adv. Struct. Eng.* **2016**, *19*, 599–610. [CrossRef]
11. Slimani, A.; Ammari, F.; Adman, R. The effective length of columns in in unsymmetrical frames asymmetrically loaded. *Asian J. Civ. Eng.* **2018**, *19*, 487–499. [CrossRef]
12. Slimani, A.; Belaid, T.; Saidani, M.; Ammari, F.; Adman, R. The effective length factor of columns in steel framed buildings with end restraint effects. *Structures* **2023**, *58*, 105521. [CrossRef]
13. Webber, A.; Orr, J.J.; Shepherd, P.; Crothers, K. The effective length of columns in multi-storey frames. *Eng. Struct.* **2015**, *102*, 132–143. [CrossRef]
14. Lan, S.; Zhou, D.; Shuang, C. Calculation method of critical force of sway frame based on deflection method. *J. Huazhong Univ. Sci. Technol. (Nat. Sci. Ed.)* **2019**, *47*, 122–127. [CrossRef]
15. Farajian, M.; Sharafi, P.; Kildashti, K. The influence of inter-module connections on the effective length of columns in multi-story modular steel frames. *J. Constr. Steel Res.* **2021**, *177*, 106450. [CrossRef]
16. Xiao, J.; Shuang, C.; Pan, W.; Su, H.; Sun, G.; Han, C. Solving the overall critical load capacity of sway frame based on the spring-pendulum model. *Structures* **2024**, *61*, 106013. [CrossRef]
17. Shafan, T. Addressing Differential Axial Deformation Effect and Sway in 3D Building Frame Analysis. *East Afr. J. Eng.* **2024**, *7*, 74–92. [CrossRef]
18. Du, M.; Zhang, S.; Wang, C.; She, L.; Li, J.; Lu, T. Seismic fragility assessment of aqueduct bent structures subjected to mainshock-aftershock sequences. *Eng. Struct.* **2023**, *292*, 116505. [CrossRef]
19. Slack, H.; Walport, F.; Chan, H.U.; Wadee, M.A.; Gardner, L. A consistent approach to the definition of initial geometric imperfections for in-plane stability design of steel moment frames. *Structures* **2024**, *65*, 23–39. [CrossRef]

20. Azizi, H.; Ahmadi, J.; Kazemi, F. Enhancing seismic resilience in steel frames with synchronized double-stage yield buckling-restrained braces. *Soil Dyn. Earthq. Eng.* **2025**, *198*, 18–36. [CrossRef]
21. Lan, S.; Zhang, J.; Xu, B. An analytical method for determining the global buckling load of shear-braced steel frame-concrete core tube structures. *Structures* **2026**, *86*, 111434. [CrossRef]