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Analytical Solution for Longitudinal Anti-Push Stiffness of the Middle Tower of Cross-Cable Multi-Tower Cable-Stayed Bridge

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Abstract: The Queensferry Crossing in Scotland is the first multi-tower cable-stayed bridge with crossed cables in the world. This means that the conceptual design of a cross-cable multi-tower cable-stayed bridge has become a reality. In this paper, the cross-cable action mechanism is studied deeply. Suppose that the small displacement amount at the top of the side tower is ignored. The deformation coordination principle is used twice to analyze the relationship between the horizontal external force and the top displacement amount of the middle tower. Thus, derive the formula for estimating the anti-push stiffness of cross cables to the middle tower of the multi-tower cable-stayed bridge under the influence of the stiffness of the tower and beam. A numerical example is given to verify this. The research results show that the error between the analytical solution and the finite element solution is less than 8%, which meets the conceptual design requirements of cross-cable multi-tower cable-stayed bridges. The cross cables can reduce the deflection of the main beam and improve the stiffness of the bridge. After setting 10 pairs of cross cables, the displacement amount of the middle tower decreases by as high as 51%. The stiffness of the multi-tower cable-stayed bridge increases with an increase in the number of cross cables, but the increasing trend of stiffness gradually slows down. Increasing the stiffness of the tower or beam can improve the structural stiffness to a certain extent, but the effect is much less potent than that of the cross cable in the middle span.

Keywords: multi-tower cable-stayed bridge; cross cable; longitudinal anti-push stiffness; the principle of deformation coordination; analytical solution

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1. Introduction

In September 2017, the Queensferry Crossing (Original name Forth Replacement Crossing) was completed and put into operation on the Firth of Forth, Scotland, UK. The bridge is a three-tower cable-stayed bridge, and the main span is 650 m [1–4]. Setting the crossing cable in the middle of the span is adopted for the first time to improve the longitudinal anti-push stiffness of the middle tower. It marks the birth of the world's first cross-cable multi-tower cable-stayed bridge. Bianyuzhou Bridge is the first cross-cable multi-tower cable-stayed bridge in China. It is the second cable-stayed bridge with cross cables built in the world [5]. The construction of these two bridges shows that the notion of a cross-cable multi-tower cable-stayed bridge has shifted from a conceptual design to a reality.

As we all know, when one main span of the multi-tower cable-stayed bridge is full of live load and the other main span has no live load, the structural deformation reaches the maximum. Under the action of the unbalanced live load, the middle tower has a large deformation due to the lack of auxiliary piers or end anchor cables (Figure 1) [6–8].

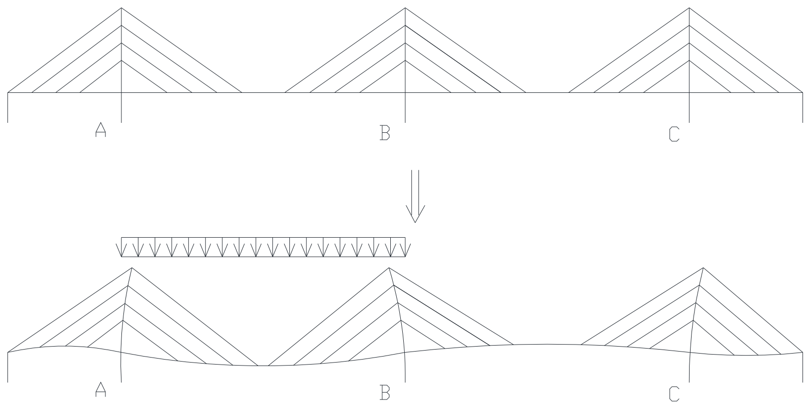

Figure 1. Deformation diagram of multi-tower cable-stayed bridge under unbalanced live load.

To solve this problem, many scholars have thoroughly explored methods to improve the stiffness of the middle tower [6–13]. The most obvious way to achieve this is to increase the stiffness of the main beam and tower, such as the Maracaibo Bridge, Millau Bridge, and Rio-Antirio Bridge in Greece, etc. One scholar analyzed a three-tower cable-stayed bridge with a main span of 580 m [13]. He increased the stiffness of the main beam by one to six times, and the displacement of the middle tower decreased accordingly. However, the stiffness of the main beam of the cable-stayed bridge with a dense cable system is very small relative to the span. Only by improving the stiffness of the main beam to a great extent can the structural stiffness be effectively increased. And the increase in the stiffness of the main beam is bound to increase the dead weight of the structure, which is not conducive to the bearing capacity of the structure. When the stiffness of the bridge tower is increased by 1 to 20 times, it is found that the influence on the displacement of the tower is not significant. And it is difficult to greatly improve the stiffness of the bridge tower. The use of an A-shaped bridge tower is suggested in the literature, but this method is bound to bring huge foundation construction problems. Some scholars have proposed using the method of modifying the cable system to improve the stiffness of the tower, including horizontal stiffening cable, inclined stiffening cable, cross cable, and so on [7,14]. The Ting Kau Bridge in China was designed with an inclined stiffening cable. For long-span multi-tower cable-stayed bridges, the effects of the first two stiffening methods are greatly reduced due to the sag effect of long cables. The cross cable has become more and more respected by scholars because its stiffening efficiency can be achieved by adjusting the number of settings. Through careful analysis, calculation, and theoretical research, they think that this method can effectively improve the stiffness of the structure [15–23].

At present, there are few studies on the action mechanism of crossed cables. Some scholars have carried out scale model tests on a three-tower railway cable-stayed bridge [24]. The stiffening efficiency of three kinds of cable systems have been compared in tests. One of these tests showed that the stiffening efficiencies of the horizontal cable and inclined cable are 18% and 11%, respectively. The stiffening efficiency of one pair of crossed cables is 6%, and the stiffening efficiency of two pairs of crossed cables is 10%. This test fully shows that the stiffening efficiency of crossed cables can be adjusted by changing the number of settings.

Some scholars have put forward the derivation formula of the constraint stiffness of the cross cable to the middle tower without considering the influence of the stiffness of the tower and the beam [23]. They believe that the action mechanism of the cross cable is that the weight of the mid-span beam is redistributed with crossed cables. This leads to a change in cable force, which restricts the deformation of the middle tower. In the process of derivation, it is assumed that the bottom of the tower is hinged and the stiffness of the main beam is very small. However, these assumptions are quite different from the actual structure. Inspired by this study, combined with the relevant methods of traditional cable-stayed bridge stiffness calculation [25,26], this paper derives the formula of longitudinal anti-push stiffness of the cross cable to the middle tower, considering the influence of the tower and beam stiffness itself, and the accuracy is verified by a finite element model.

2. Derivation of the Analytical Formula

In the process of derivation, the mid-span crossing cable is simplified to a pair of crossing cables (Figure 2). The influence of stay cables, except crossed cables, is not taken into account. Taking the three-tower cable-stayed bridge as an example, the common constraint mode, that is, the structural system in which the tower beam is consolidated at the middle tower and separated at the side tower, is adopted (Figure 3).

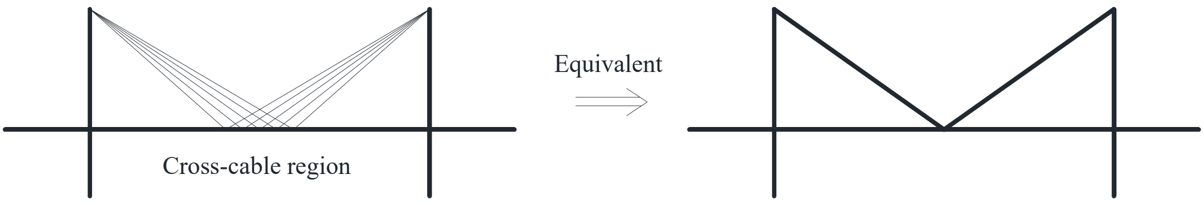

Figure 2. Simplified schematic diagram of a cross-cable bridge.

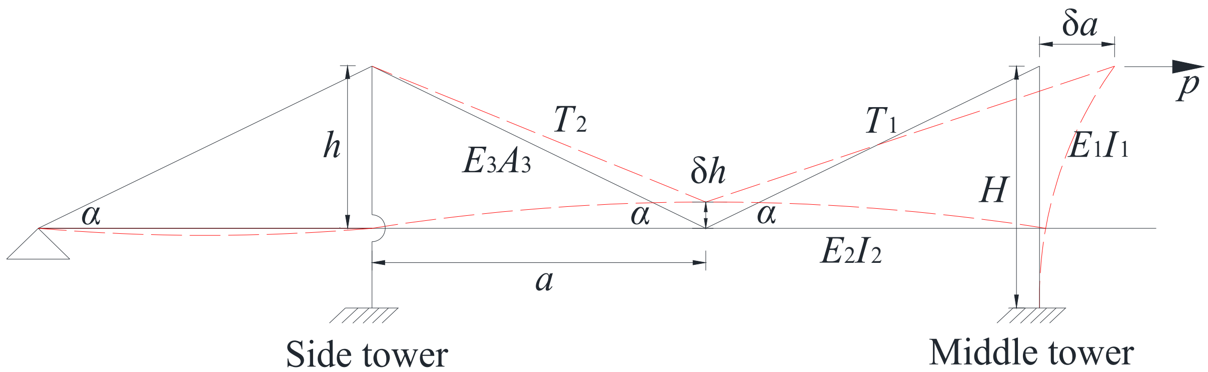

Figure 3. Diagram with structure deformation and parameter.

Due to the separation of the tower beam at the side tower and the constraint of the end anchor cable and auxiliary pier, the deformation of the side-span cable and side tower is small under the action of horizontal force. To simplify the derivation, the horizontal displacement amount at the top of the side tower is ignored. The formula derivation in this paper is based on the following assumptions:

(1) Under the action of live load, the deformation of the bridge tower and stay cable is small; thus, it is assumed that the inclination angle of the cable does not change.
(2) The cable material is linearly elastic. Consider that the deformation of the cable is elastic elongation.
(3) No longitudinal drift of the main beam.
(4) Ignore the axial deformation of the beam and the compression deformation of the tower.
(5) The horizontal displacement of the side tower is so small that it is ignored.

The change in displacement is expressed by δ, and the change in force is expressed by Δ. Based on the above assumptions, the structural deformation and parameters are shown in Figure 3.

The meaning of the parameters in Figure 3:

E_1, I_1—Elastic modulus and bending moment of inertia of the middle tower;
E_2, I_2—Elastic modulus and bending moment of inertia of the main beam;
E_3, A_3—Elastic modulus and cross-sectional area of crossed cables;
δa—Horizontal displacement of the middle tower under external force;
δh—Vertical displacement of the main beam in middle span under external force.

The cross cable force is T_1 and T_2. The length of the cable is l. The angle between the cable and the main beam is α. The height of the bridge tower is H. The height of the tower above the bridge deck is h. The main span is $2a$. The anti-push stiffness of the middle tower itself is K_T. The bending stiffness of the main beam itself is K_B. A horizontal external force p is applied at the top of the middle tower, and the longitudinal anti-push stiffness of the middle tower is obtained by studying the relationship between the horizontal force and the displacement of the top of the tower.

According to the knowledge of structural mechanics, the anti-push stiffness of the tower itself is

$$K_T = \frac{3E_1 I_1}{H^3} \tag{1}$$

The bending stiffness of the main beam in the middle of the span is

$$K_B = \frac{6E_2 I_2}{a^3} \tag{2}$$

The following is the rationale behind a solution that involves using the principle of deformation coordination twice.

(1) When the horizontal external force p acts on the middle tower, it can be divided into p_1 and p_2 parts. p_1 makes the deformation δa of the middle tower under the action of its own stiffness. p_2 makes the cable 1 elongate. The horizontal displacement amount of cable 1 near the end of the tower consists of two parts. One part is caused by the self-elongation of cable 1, and the other part is the horizontal displacement of cable 1 near the end of the middle tower due to the vertical deformation of the main beam. Taking the top of the middle tower as the research object, according to the principle of deformation coordination, the displacement of the middle tower under the action of p_1 should be equal to the displacement of cable 1 near the end of the tower under the action of p_2. The horizontal deformation values of both are δa.

(2) The influence of the change in cable 1 cable force can be divided into two parts. One part causes the vertical deformation of the beam, and the other part compresses the length of the cable 2. According to the previous assumption, the displacement of the top of the side tower is ignored. Taking the main beam in the middle of the span as the research object, according to the principle of deformation coordination, the vertical displacement of the beam is equal to the vertical deformation of cable 2 near the end of the beam. The vertical deformation values of both are δh.

By establishing the equilibrium equation according to the above idea, the value of δa can be solved. Furthermore, the longitudinal anti-push stiffness of the cross cable to the middle tower is obtained considering the influence of the stiffness of the tower and the beam.

We know

$$p = p_1 + p_2 \tag{3}$$

The displacement of middle tower under the action of p_1 is

$$\delta_a = \frac{p_1}{K_{\mathrm{T}}} \tag{4}$$

According to the balance principle of force, the variation of cable 1 cable force is

$$\Delta T_1 = \frac{p_2}{\cos\alpha} \tag{5}$$

The elongation of cable 1 is

$$\delta l_1 = \frac{\Delta T_1 l}{E_3 A_3} = \frac{p_2 l}{\cos\alpha E_3 A_3} \tag{6}$$

The horizontal displacement of the middle tower caused by the extension of cable 1 is δa_1, as shown in Figure 4.

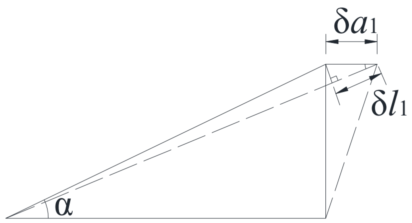

Figure 4. Diagram of solving δa_1.

As can be seen from Figure 4,

$$\delta a_1 = \frac{\delta l_1}{\cos\alpha} = \frac{p_2 l}{\cos^2\alpha E_3 A_3} \tag{7}$$

Taking the main beam in the mid-span as the research object, the mechanical analysis shown in Figure 5 clearly reflects the action mechanism of the cross cable. In the traditional cable-stayed bridge, the vertical component $\Delta T_{1\mathrm{y}}$ of cable 1 is completely used to make the main beam flex deformation, and the structure will produce greater deformation. After the cross cable is set, when there is a longitudinal displacement in the middle tower, the weight of the main beam is redistributed in the cross cable. Cable 1 cable force increases ΔT_1, and cable 2 cable force decreases ΔT_2. The vertical component of ΔT_1 is $\Delta T_{1\mathrm{y}}$, and part of it is balanced by the vertical component $\Delta T_{2\mathrm{y}}$ of cable 2. As a result, the upward bending force of the beam is reduced to $\Delta T_{1\mathrm{y}} - \Delta T_{2\mathrm{y}}$, so the deformation of the beam decreases and the structural stiffness increases.

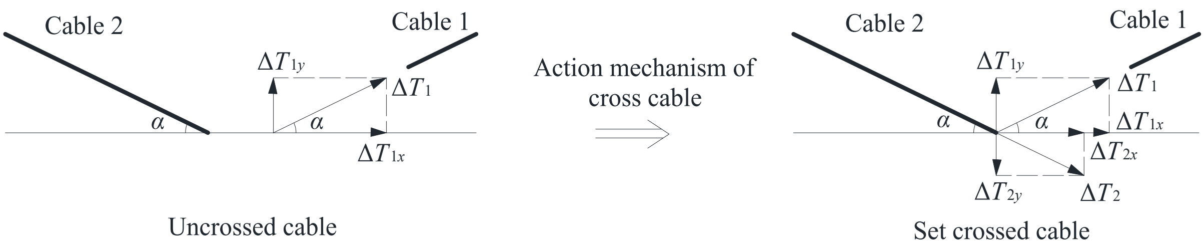

Figure 5. Decomposition of cable force.

Ignoring the axial deformation of the main beam, according to the balance principle of force, it can be known that the vertical force T_B on the main beam is

$$T_B = \Delta T_{1\mathrm{y}} - \Delta T_{2\mathrm{y}} \tag{8}$$

As can be seen from Figure 5,

$$\Delta T_{1\mathrm{y}} = \Delta T_1 \sin\alpha, \Delta T_{2\mathrm{y}} = \Delta T_2 \sin\alpha \tag{9}$$

Then,

$$\Delta T_1 \sin\alpha = \Delta T_2 \sin\alpha + T_B \tag{10}$$

The vertical deformation of the main beam in the middle span is

$$\delta h = \frac{T_B}{K_B} \tag{11}$$

The compression deformation of cable 2 is

$$\delta l_2 = \frac{\Delta T_2 l}{E_3 A_3} \tag{12}$$

As can be seen from Figure 6, the vertical component of the deformation of cable 2 is

$$\delta l_{2y} = \frac{\delta l_2}{\sin\alpha} = \frac{\Delta T_2 l}{E_3 A_3 \sin\alpha} \tag{13}$$

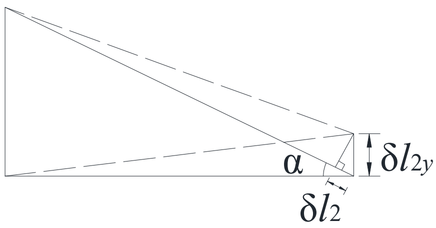

Figure 6. Diagram of solving $\delta \mathrm{l}_{2\mathrm{y}}$.

Due to the separation of the tower and the main beam at the side tower, the deformation of the top of the side tower is small under the action of p. Suppose that the small displacement amount at the top of the side tower is ignored. It can be obtained from the principle of deformation coordination that

$$\delta h = \delta l_{2y} \tag{14}$$

Then,

$$\frac{T_B}{K_B} = \frac{\Delta T_2 l}{E_3 A_3 \sin\alpha} \tag{15}$$

Due to the upward deflection of the main beam, cable 1 has a horizontal displacement near the end of the tower, as shown in Figure 7.

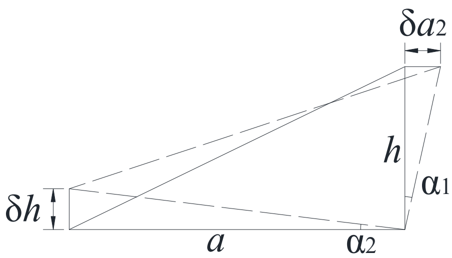

Figure 7. Diagram of solving δa_2.

Because the deformation is very small, we think

$$\alpha_1 \approx \alpha_2 \tag{16}$$

Then,

$$\tan\alpha_1 \approx \tan\alpha_2 \tag{17}$$

It can be seen from Figure 7 that

$$\frac{\delta a_2}{h} = \frac{\delta h}{a} \tag{18}$$

Then,

$$\delta a_2 = \delta h \frac{h}{a} \tag{19}$$

Taking the top of the middle tower as the research object, according to the principle of deformation coordination,

$$\delta a = \delta a_1 + \delta a_2 \tag{20}$$

By substituting Equations (4), (7), and (19) into (20), we can obtain the following:

$$\frac{p_1}{K_T} = \frac{p_2 l}{\cos^2 \alpha E_3 A_3} + \frac{T_B h}{K_B a} \tag{21}$$

The simultaneous Equations (3), (5), (10), (15) and (21) form a set of equations, which contain five unknowns—p_1, p_2, ΔT_1, ΔT_2, and T_B—and a total of five equations, which can be solved.

By substituting Equations (3) and (15) into (21), we can obtain the following:

$$\frac{p-p_2}{K_T} = \frac{p_2 l}{\cos^2 \alpha E_3 A_3} + \frac{\Delta T_2 \mathrm{lh}}{E_3 A_3 a \sin \alpha} \tag{22}$$

According to Equations (5), (10) and (15), it can be concluded that

$$\frac{p_2 \sin \alpha}{\cos \alpha} = \Delta T_2 \sin \alpha + \frac{\Delta T_2 l K_B}{E_3 A_3 \sin \alpha} \tag{23}$$

Simplify Equation (23). Then,

$$\Delta T_2 = \frac{p_2 E_3 A_3 \sin^2 \alpha}{\cos \alpha \left(E_3 A_3 \sin^2 \alpha + K_B l\right)} \tag{24}$$

According to the relationship of trigonometric function, we know

$$\sin \alpha = \frac{h}{l},\ \cos \alpha = \frac{a}{l} \tag{25}$$

By substituting Equations (1), (2), (24), and (25) into (22), we can obtain the following:

$$p - p_2 = p_2 \left[\frac{3E_1 I_1 l^3}{H^3 E_3 A_3 a^2} + \frac{3E_1 I_1 l^3 h^2 a}{H^3 (E_3 A_3 h^2 a^3 + 6E_2 I_2 l^3)} \right] \tag{26}$$

We specify

$$\gamma = \frac{3E_1 I_1 l^3}{H^3 E_3 A_3 a^2} + \frac{3E_1 I_1 l^3 h^2 a}{H^3 (E_3 A_3 h^2 a^3 + 6E_2 I_2 l^3)} \tag{27}$$

Then,

$$p_2 = p \frac{1}{1+\gamma} \tag{28}$$

When the structural parameters and materials are determined, γ is a constant that can be solved. Therefore, p_2 can be solved according to Formula (28), and p_1 can be solved according to Formula (3). That is,

$$p_1 = p \frac{\gamma}{1+\gamma} \tag{29}$$

When the tower top of the cross-cable three-tower cable-stayed bridge is subjected to a horizontal external force, the physical meaning of p_1 is the force distributed by the middle tower itself.

By substituting Equation (29) into (4), we can obtain the following:

$$\delta a = \frac{p_1}{K_T} = \frac{\gamma p}{(1+\gamma) K_T} \tag{30}$$

Then, the longitudinal anti-push stiffness of the middle tower of the cross-cable three-tower cable-stayed bridge is

$$K = \frac{p}{\delta a} = \frac{1+\gamma}{\gamma} K_T \tag{31}$$

The longitudinal anti-push stiffness of the middle tower solved here is the stiffness of the structural system in Figure 3. In other words, the longitudinal anti-push stiffness of the middle tower is solved by the interaction of the cross cable, the main beam, and the bridge tower. Then,

$$K = K_{T-T} + K_{T-B} + K_{T-C} \tag{32}$$

K_{T-T} is the contribution of the bridge tower itself to the longitudinal anti-push stiffness of the middle tower. K_{T-B} is the contribution of the main beam to the longitudinal anti-push stiffness of the middle tower. K_{T-C} is the contribution of the cross cable to the longitudinal anti-push stiffness of the middle tower. Because the influence of the ordinary stay cable is not taken into account in the theoretical derivation, it is necessary to solve K_{T-C} and then add the stiffness value of the middle tower of multi-tower cable-stayed bridge without the cross cable under the same conditions. In this way, the longitudinal anti-push stiffness of the middle tower can be obtained.

According to the above analysis,

$$K_{T-T} = \frac{p_1}{\delta a} = K_T \tag{33}$$

When the main beam is under the action of T_B, according to the balance principle of the moment, the force on the top of the middle tower F is

$$F = T_B \frac{a}{h} \tag{34}$$

According to Equations (11) and (19), it can be concluded that

$$K_{T-B} = \frac{F}{\delta a_2} = K_B \frac{a^2}{h^2} \tag{35}$$

Then,

$$K_{T-C} = K - K_{T-T} - K_{T-B} \tag{36}$$

Then,

$$K_{T-C} = \frac{\left(E_3 A_3 h^2 a^3\right)^2 - 6E_3 A_3 h^2 a^3 E_2 I_2 l^3 - 36\left(E_2 I_2 l^3\right)^2}{2l^3 h^2 a\left(E_3 A_3 h^2 a^3 + 3E_2 I_2 l^3\right)} \tag{37}$$

Considering the influence of the stiffness of the tower and the main beam itself, Formula (37) is the analytical solution of the longitudinal anti-push stiffness of the cross cable to the middle tower. It can be seen that the longitudinal anti-push stiffness of the middle tower contributed by the cross cable is related to the stiffness of the cross cable, the stiffness of the main beam, the length of the cross cable, the main span, and the height of the bridge tower. When the structural parameters are determined, the longitudinal anti-push stiffness of the cross cable to the middle tower can be solved according to Formula (37).

After the dimensions and parameters of the traditional multi-tower cable-stayed bridge are determined, the designer can solve the longitudinal anti-push stiffness K_0 of the middle tower through the model calculation or the reference [25,26] formula. If the designer needs to improve the structural stiffness by setting the cross cable, the longitudinal anti-push stiffness of the cross cable to the middle tower can be easily obtained by adjusting the material (E_3) or quantity (A_3) of the cross cable by using Formula (37). The longitudinal anti-push stiffness of the middle tower of the cross-cable multi-tower cable-stayed bridge is $K_0 + K_{T-C}$.

Formula (37) can arbitrarily adjust the material and quantity of the cross cable and quickly check whether the stiffness of the tower in the structure meets the force requirements. It can provide guidance and reference for the scheme design of improving the structural stiffness of a multi-tower cable-stayed bridge by setting cross cables.

3. Results and Discussion

3.1. Structural Parameters

In order to verify the accuracy of the formula derived in this paper, a finite element model must be established for comparative analysis. A model of a three-tower cable-stayed bridge with a span of (341 + 650 + 650 + 341) m is established. The constraint form of the structure is the consolidation of the tower and beam at the middle tower and the separation of the tower and beam at the side tower. The elevation layout is shown in Figure 8.

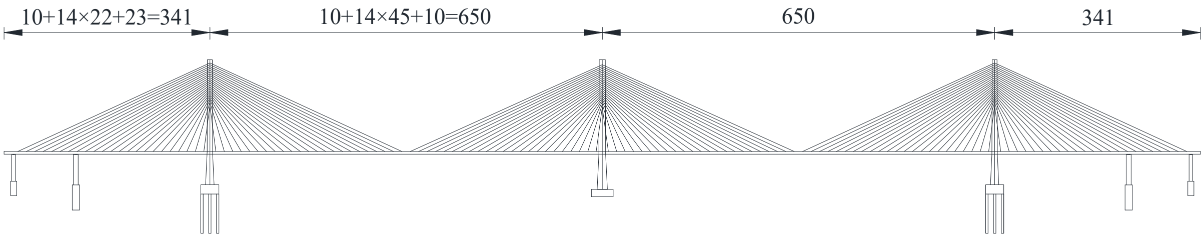

Figure 8. Elevation layout of three-tower cable-stayed bridge.

The main parameters of the bridge are as follows: The main beam adopts a steel box girder. The width of the cross-section of the beam is 37.6 m, and the height is 4.465 m. E_2 = 210 GPa, I_2 = 6.818 mm^4. The structure of the bridge tower is a single column, and the cross-section is a rectangular hollow box. The middle tower is 210.7 m high. The side tower is 203.3 m high. The bridge tower is made of C50 concrete. E_1 = 34.5 GPa, I_1 = 411.875 mm^4. The stay cables are made of steel strands. E_3 = 195 GPa. The area of a single cable is 0.011 m^2. The stay cable is a parallel double-cable plane.

The main parameters of the structural dimensions are as follows: H = 202.7 m, h = 141 m, 2a = 650 m, $l = \sqrt{h^2 + a^2}$.

3.2. Verification of Analytical Solution

Under the reasonable state of bridge completion, a horizontal external force p is applied to the top of the middle tower. The influence of structural nonlinearity is considered in the process of finite element calculation. The calculation results are shown in Table 1.

Table 1. Calculation results of the non-crossing cable model.

p (kN)	Horizontal Displacement of the Top of the Middle Tower (m)	Horizontal Displacement of the Top of the Side Tower (m)	Stiffness of the Middle Tower (kN·m^{-1})
20,000	0.486	−0.003	41,165.8

The accuracy of the analytical solution can be verified by model calculation. By adding stay cables to the middle of the span, the cross cables are formed in the middle of the model span. Overall, 2~10 pairs of crossed cables were added, respectively, and five new models were formed. Among them, the overall model diagram of the six pairs of crossed cables is shown in Figure 9. The detailed diagram in Figure 9 clearly reflects the spatial arrangement of the cross-anchored stay cables. The finite element model consists of 917 elements and 1074 nodes. Among them, the component properties of the bridge tower and the main beam are beam elements. The stay cables adopt tension-only cable elements.

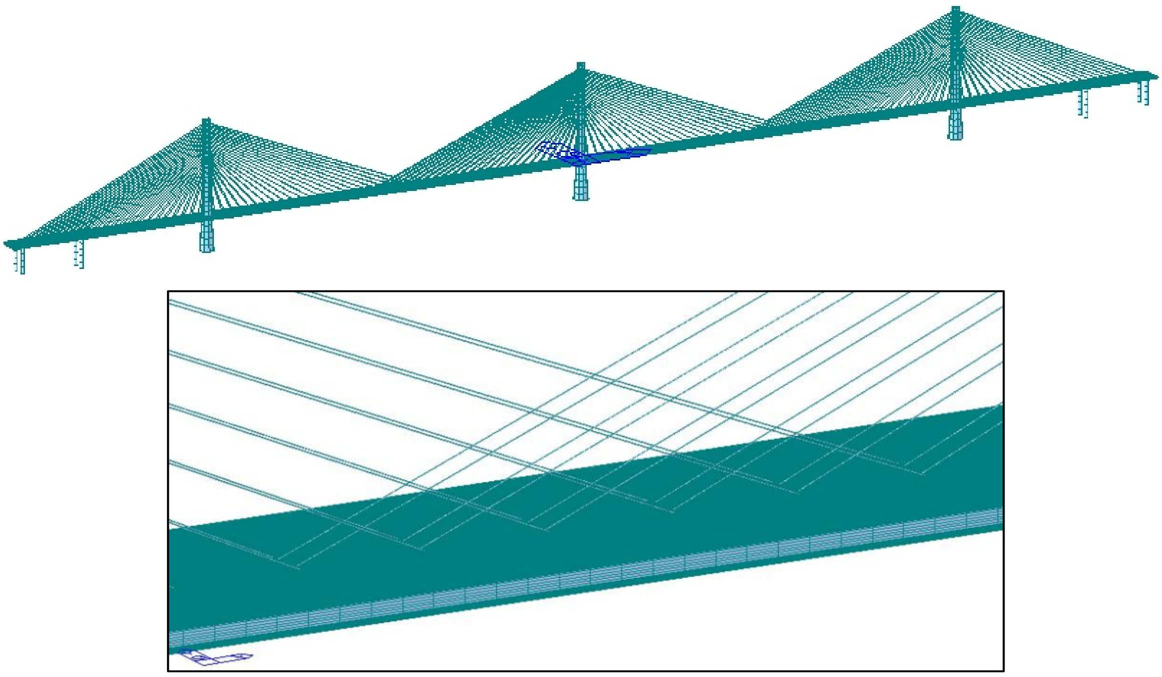

Figure 9. Crossed-cable model details.

The material and cross-section of the new cable are the same as the original cable parameters of the model. By adjusting the cable force of each model, the structure is in a reasonable state of bridge completion. The cable forces of each model are similar. Only in the vicinity of the cross cable, the cable force decreases and tends to achieve equilibrium.

Then, the same horizontal external force is applied to the top of the middle tower, and the calculation results of each finite element model are shown in Table 2.

Table 2. Calculation results of the crossed-cable model.

The Number of Pairs of Crossed Cables in Mid-Span	Horizontal Displacement of the Top of the Middle Tower (m)	Horizontal Displacement of the Top of the Side Tower (m)	Ratio of Horizontal Displacement at the Top of the Middle Tower to the Side Tower (%)	Stiffness of the Middle Tower (kN·m^{-1})
2	0.406	−0.011	2.71	49,261.1
4	0.345	−0.013	3.77	57,971.0
6	0.299	−0.015	5.02	66,889.6
8	0.264	−0.017	6.44	75,757.6
10	0.237	−0.022	9.28	84,388.2

From the calculation results of the finite element model in Table 2, it can be seen that, after the horizontal external force is applied to the middle tower, the horizontal displacement occurs in both the middle tower and the side tower, and the displacement amount of the side tower is much smaller than that of the middle tower. The calculation results show that the displacement amount of the side tower is only less than 10% of that of the middle tower. This shows that it is reasonable and feasible to ignore the displacement amount of the top of the side tower in the derivation of the formula.

The relevant parameters of the bridge are substituted into Formula (37). By changing the cross-sectional area of the cross cable in the formula, the longitudinal anti-push stiffness value of different numbers of the cross cable to the middle tower can be solved. Then, added with the longitudinal anti-push stiffness of the middle tower of the uncrossed cable model, the estimated value of the longitudinal anti-push stiffness of the middle tower of the cross-cable multi-tower cable-stayed bridge is obtained. The solution results and error analysis is shown in Table 3.

Table 3. Calculation accuracy of longitudinal anti-push stiffness of middle tower.

The Number of Pairs of Crossed Cables in Mid-Span	Stiffness of the Middle Tower (kN·m^{-1})		Error of Analytical Solution * (%)
	The Analytical Solution in This Paper	The Calculation Solution of Finite Element Model	
2	50,349.3	49,261.1	2.21
4	60,545.7	57,971.0	4.44
6	70,738.7	66,889.6	5.75
8	80,930.9	75,757.6	6.83
10	91,122.6	84,388.2	7.98

* Note: error of analytical solution = (analytical solution − finite element solution)/finite element solution × 100%.

Figure 10 shows a comparison between the analytical solution and the finite element solution of the horizontal displacement of the middle tower with different number of pairs of crossed cables in mid-span after 20,000 kN of horizontal force has been applied to the top of the middle tower.

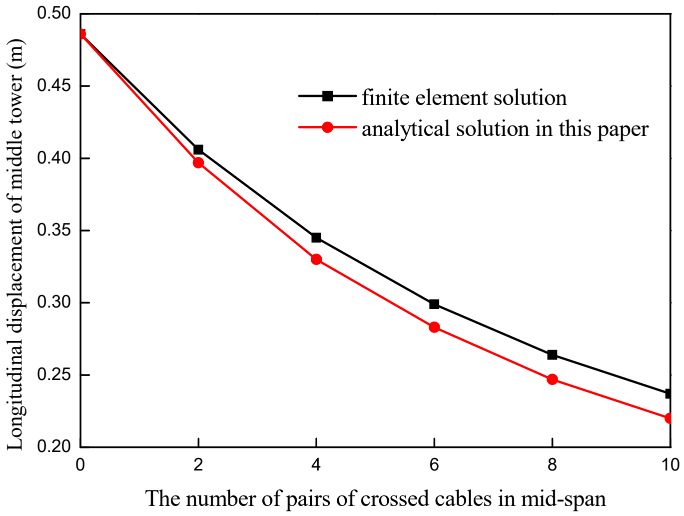

Figure 10. Comparison diagram of horizontal displacement of the middle tower.

According to the calculation results, we can see that the variation trend of the analytical solution is consistent with that of the finite element solution, the calculated results are close to each other, and the analytical solution of the stiffness of the middle tower is slightly larger than the finite element solution. The main reason for this is that, first of all, in the process of derivation, the longitudinal anti-push stiffness of the cross cable to the middle tower is slightly larger without considering the influence of the ordinary stay cables. Secondly, the small deformation of the top of the side tower and the influence of structural nonlinearity are ignored in the process of theoretical derivation.

For setting 2~10 pairs of crossed cables in mid-span, the error of the analytical solution is between 2% and 8%, and the calculation accuracy meets the requirements of the scheme design stage of a multi-tower cable-stayed bridge.

With the increase in the number of cross cables in the mid-span, the amount of cables used in the structure increases sharply. Considering the structural economy, the reference [22,27] proposes that too many cross cables should not be used, and the authors of [28] show that the reasonable range of the ratio of cable crossover is 15–35%. Therefore, it is more reasonable to select 2~10 pairs of crossed cables (the number used for this paper). It can be seen that the calculation accuracy of the analytical formula in this paper is enough to meet the requirements of engineering practice.

3.3. Analysis of the Influence of Tower and Main Beam on the Stiffness of Middle Tower

The analytical solution for the longitudinal anti-push stiffness of the middle tower of cross-cable multi-tower cable-stayed bridge is derived in this paper, which takes into account the influence of the stiffness of the tower and the main beam itself. As can be seen from Figure 10, the more cross cables are set, the smaller the horizontal displacement of the middle tower is. The stiffness of the structure increases continuously, but the efficiency decreases gradually. When 10 pairs of cross cables are set, the horizontal displacement amount of the middle tower is reduced by as much as 51%, which shows that the cross cables have a significant effect on improving the structural stiffness.

To further study the influence of the stiffness of the tower and the main beam itself on the structural stiffness of cross-cable multi-tower cable-stayed bridge, the main beam stiffness and tower stiffness are reduced by 10%, 20%, and 30%, respectively. Figures 11 and 12 show the calculation results of the horizontal displacement amount when 20,000 kN of horizontal force is applied at the top of the middle tower.

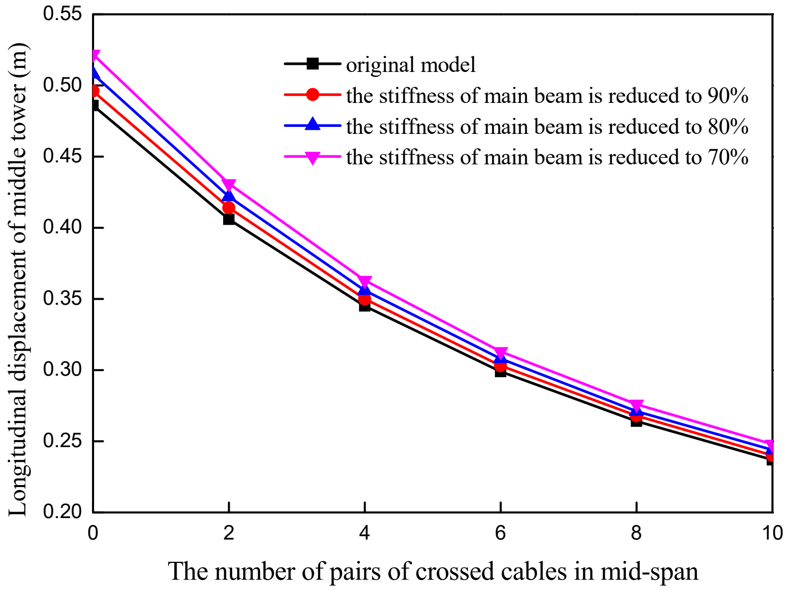

Figure 11. Influence of the stiffness of the main beam itself on the stiffness of the middle tower.

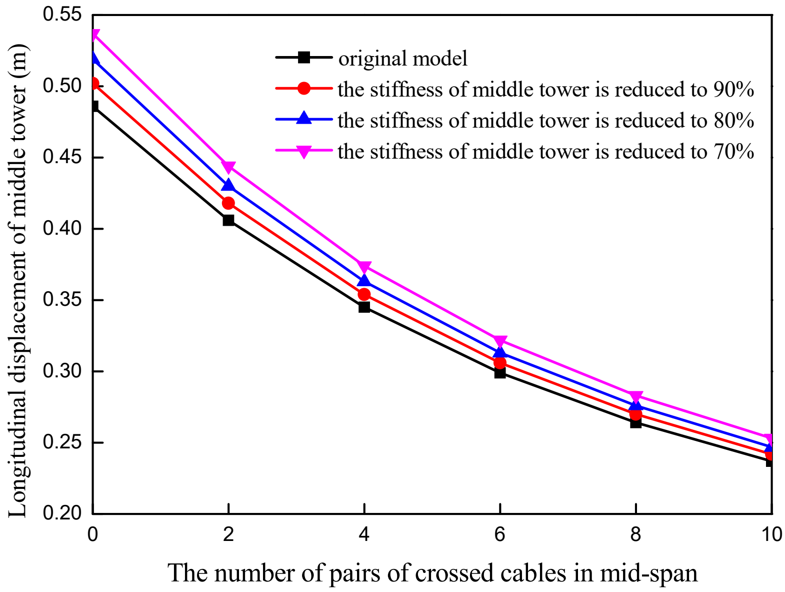

Figure 12. Influence of the stiffness of the tower itself on the stiffness of the middle tower.

It can be seen from the calculation results that the decrease in the stiffness of the main beam and the tower will lead to an increase in the horizontal displacement and a decrease in the longitudinal anti-push stiffness of the middle tower, and the influence of the tower on the structural stiffness is greater than that of the main beam. With the increase in the number of cross cables, the influence of the stiffness of the main beam and tower gradually decreases, and the structural stiffness increases significantly. This shows that the influence of the cross cable on the structural stiffness occupies a dominant position.

At the same time, Figures 10–12 show that, in order to improve the overall stiffness of bridges, the method of setting cross cables is more effective than improving the stiffness of the main beam and tower. For example, if two pairs of crossed cables are set, the displacement amount of the middle tower is reduced by 16.5%. When the stiffness of the main beam is increased by 30%, the displacement of the middle tower is only reduced by 7.4%, and when the stiffness of the bridge tower is increased by 30%, the displacement amount of the middle tower is only reduced by 10.5%.

4. Conclusions and Recommendations

Based on the principle of deformation coordination, the action mechanism of crossed cables has been revealed through theoretical analysis, and the analytical solution of longitudinal anti-push stiffness of the middle tower of cross-cable multi-tower cable-stayed bridge was derived. By building the finite element calculation model, the accuracy of the analytical solution was verified. The main conclusions are as follows:

(1) In this paper, the action mechanism of the cross cable is revealed as a result of mechanical analysis. That is, the cross cable in the middle of the span can weaken the vertical force that causes the deformation of the main beam and then reduce the overall deformation of the structure and improve the overall stiffness of the structure.
(2) Using the deformation coordination principle twice, we derive the analytical solution for calculating the longitudinal anti-push stiffness of the middle tower of cross-cable multi-tower cable-stayed bridges. The formula shows that the influence of the cross cable on the stiffness of the middle tower is related to its stiffness, the stiffness of the main beam, the main span, the height of the tower, and the length of the cross cable.
(3) The accuracy of the analytical solution is verified by an example of the finite element model. The analysis shows that the error of the analytical formula is less than 2–8%, which meets the accuracy requirements of the scheme design stage of the multi-tower cable-stayed bridge. The analytical formula can quickly determine the number of cross cables in the mid-span, with the stiffness of the middle tower being the target value. It can provide guidance and a reference for scheme designs which aim to improve the structural stiffness of multi-tower cable-stayed bridges by setting cross cables.
(4) The stiffness of the beam and tower itself has a certain influence on the overall stiffness of the structure, and the influence of the tower stiffness is greater than that of the beam. However, compared with setting the cross cables, the method of improving the structural stiffness by increasing the stiffness of the tower or the beam is inefficient and uneconomical.

In this paper, the middle tower stiffness of the cross-cable multi-tower cable-stayed bridge is analyzed from the aspect of static performance, and the setting amount of the cross cable which can meet the stiffness requirements can be quickly obtained through the analytical formula in the design stage. However, the bridge structure is not only subjected to static loads but also may bear dynamic loads such as seismic waves. The reasonable stiffness under the comprehensive consideration of the static and dynamic performance of the structure is worthy of further study in the future.

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