Supplementary Information

1. Significance of parameter symbols

Table S1 Symbols and Meanings

Symbols	Meanings
A , B , G , R	Stochastic variables.
μ_a , μ_b , μ_g , μ_r	Sample average.
σ_a , σ_b , σ_g , σ_r	Standard difference.
S_a , S_b , S_g , S_r	Sample standard difference.
n_g , n_r	Sample sequence of G, R.
μ_{S_a} , μ_{S_b} , μ_{S_g} , μ_{S_r}	Sample average of S_a , S_b , S_g , S_r.
$\delta_g = S_g / \mu_g$, $\delta_r = S_r / \mu_r$	Vary coefficient of G, R.
$C = \mu_r / \mu_g$	Sample averages' ratio of G, R.
k_a , k_b	Upper α_a and α_b fractile of standard normal distribution for stochastic variables G and A , R and B , respectively.
Z_g and Z_a , Z_r and Z_b	Stochastic variables of characteristic value with $1-\alpha_a$ and $1-\alpha_b$ assurance factor for stochastic variables G and A , R and B , respectively.
Z_c	Difference between Z_b and Z_a

2. Possibility density function of sample standard difference

Suppose $G \in N(\mu_g, \sigma_g^2)$ and $R \in N(\mu_r, \sigma_r^2)$. And $G \in \left\{G_1, G_2, ..., G_{n_g}\right\}$, $E(g_i) = \mu_g$, $D(g_i) = \sigma_g^2$; $R \in \left\{R_1, R_2, ..., R_{n_r}\right\}$, $E(r_i) = \mu_r$, $D(r_i) = \sigma_r^2$, respectively. The sample variances are as follows:

$$\begin{cases} S_g^2 = \dfrac{1}{n_g - 1}\left[\displaystyle\sum_{i=1}^{n_g} g_i^2 - n_g \mu_g^2\right] \\ S_r^2 = \dfrac{1}{n_r - 1}\left[\displaystyle\sum_{i=1}^{n_r} r_i^2 - n_r \mu_r^2\right] \end{cases} \tag{1}$$

In which, $\mu_g = \dfrac{1}{n_g}\sum_{i=1}^{n_g} g_i$, $\mu_r = \dfrac{1}{n_r}\sum_{i=1}^{n_r} r_i$.

From Bure and Parilina [1], $(n_g - 1)S_g^2 / \sigma_g^2 \sim \chi^2(n_g - 1)$, $\sigma_g > 0$, $S_g > 0$ and $n_g - 1 > 0$, there are

$$\begin{cases} F(S) = \displaystyle\int_0^{\frac{(n_g-1)S^2}{\sigma_g^2}} \frac{1}{2^{\frac{n_g-1}{2}}\Gamma(\frac{n_g-1}{2})} y^{\frac{n_g-1}{2}-1} e^{-\frac{y}{2}} dy \\ \Gamma(g) = \displaystyle\int_0^{+\infty} t^{g-1} e^{-t} dt, \quad g \in (0, +\infty) \end{cases} \tag{2}$$

According to Eq. (2), there is the following relationship:

$$f_i(S)\big|_{(i=g,r)} = \begin{cases} 2(\frac{n_i-1}{2})^{\frac{n_i-1}{2}} \dfrac{S^{n_i-2}}{\sigma_i^{n_i-1}\Gamma(\frac{n_i-1}{2})} e^{-\frac{(n_i-1)S^2}{2\sigma_i^2}}, S>0 \\ 0, \qquad S\le 0 \end{cases} \tag{3}$$

From Eq. (3), let $\lambda^2 = (n_i-1)S^2/(2\sigma_i^2), (i=g,r)$, then we can get,

$$\begin{cases} \mu_{S_i} = E(S_i) = M_i\sigma_i = 2\sqrt{\dfrac{2}{n_i-1}} \dfrac{\sigma_i}{\Gamma(\frac{n_i-1}{2})} \int_0^\infty \lambda^{n_i-1} e^{-\lambda^2} d\lambda, (i=g,r) \\ M_i = \sqrt{2}\Gamma(\frac{n_i}{2}) \Big/ [\sqrt{n_i-1}\Gamma(\frac{n_i-1}{2})], \qquad (i=g,r) \end{cases} \tag{4}$$

From Eq. (4), there are $\sigma_i = E(S_i/M_i) = E(S_i)/M_i, (i=g,r)$.

3. Possibility density function of difference between two characteristic values

3.1 Dimensionless stochastic vars. A and B

Assume that μ_g and μ_r are known, new stochastic variables are constructed, as follows:

$A: a_i = g_i/[0.5(\mu_g+\mu_r)], (i=1,2,\cdots\cdots,n_g)$

$B: b_i = r_i/[0.5(\mu_g+\mu_r)], (i=1,2,\cdots\cdots,n_r)$

Then by Eq. (1), there are

$$\begin{cases} \mu_a = \dfrac{1}{n_g}\sum_{i=1}^{n_g} a_i = \dfrac{2}{1+\mu_r/\mu_g} = \dfrac{2}{1+C} \\ S_a^2 = \dfrac{1}{n_g-1}\sum_{i=1}^{n_g}(a_i - \mu_a)^2 = \dfrac{4\delta_g^2}{(1+C)^2} \end{cases} \tag{5}$$

$$\begin{cases} \mu_b = \dfrac{1}{n_r}\sum_{i=1}^{n_r} b_i = \dfrac{2}{1+\mu_g/\mu_r} = \dfrac{2C}{1+C} \\ S_b^2 = \dfrac{1}{n_r-1}\sum_{i=1}^{n_r}(b_i - \mu_b)^2 = \dfrac{4C^2\delta_r^2}{(1+C)^2} \end{cases} \tag{6}$$

Due to $A \sim N(\mu_a, \sigma_a^2)$ and $B \sim N(\mu_b, \sigma_b^2)$ in Bure and Parilina [1], by the same derivation as in Section 2, the possibility density function and average value of sample standard difference for stochastic variables A and B are respectively as

$$f_j(S)\big|_{(j=a,i=g;j=b,i=r)} = \begin{cases} 2(\frac{n_i-1}{2})^{\frac{n_i-1}{2}} \dfrac{S^{n_i-2}}{\sigma_j^{n_i-1}\Gamma(\frac{n_i-1}{2})} e^{-\frac{(n_i-1)S^2}{2\sigma_j^2}}, S>0 \\ 0, \qquad S\le 0 \end{cases} \tag{7}$$

$$\mu_{S_j} = E(S_j) = \int_0^\infty S f_j(S) dS = M_j\sigma_j, (j=a,b) \tag{8}$$

In which $M_a = M_g$ and $M_b = M_r$.

From Eq. (8), the unbiased estimations of σ_a and σ_b are S_a/M_a and S_b/M_b, respectively.

3.2 Characteristic value of stochastic variables G, R, A and B

Stochastic variables $G \in N(\mu_g, \sigma_g^2)$ and $A \sim N(\mu_a, \sigma_a^2)$, get $(a-\mu_a)/\sigma_a \sim N(0,1)$. There are

$$\alpha_a = \begin{cases} P\{g \le \mu_g - k_a\sigma_g\} \\ P\{(g-\mu_g)/\sigma_g \le -k_a\} = P\{a \le \mu_a - k_a\sigma_a\} \end{cases} \tag{9}$$

Therefore, the upper α_a fractile k_a of stochastic variables G and A has the same probability α_a, and the two characteristic values with $1-\alpha_a$ assurance factor are respectively as

$$\mu_g - k_a\sigma_g \qquad (10\text{-}1) \qquad\qquad \mu_a - k_a\sigma_a \qquad (10\text{-}2)$$

Similarly, for the stochastic variables R and B, there are respectively as

$$\mu_r - k_b\sigma_r \qquad (11\text{-}1) \qquad\qquad \mu_b - k_b\sigma_b \qquad (11\text{-}2)$$

3.3 Derivation of possibility density function of characteristic value of stochastic variables A and B

Define stochastic variable

$$\overline{a} = \sum_{i=1}^{n_g} a_i \Big/ n_g \tag{12}$$

There are

$$\begin{cases} E(\overline{a}) = \dfrac{1}{n_g}\sum_{i=1}^{n_g} E(a_i) = \dfrac{1}{n_g} n_g\mu_a = \mu_a \\ D(\overline{a}) = \dfrac{1}{n_g^2}\sum_{i=1}^{n_g} D(a_i) = \dfrac{1}{n_g^2} n_g\sigma_a^2 = \dfrac{\sigma_a^2}{n_g} \end{cases} \tag{13}$$

For Eq. (10-2), the $\overline{a}$ is used instead of μ_a, and S_a/M_a is used instead of σ_a, getting stochastic variable $Z_a = \overline{a} - k_a S_a/M_a$. And there is

$$E(Z_a) = \mu_a - k_a M_a \sigma_a / M_a = \mu_a - k_a\sigma_a$$

Therefore, average value $E(Z_a)$ of stochastic variable Z_a is an unbiased estimation of Eq. (10-2).

Due to $\overline{a} \sim N(\mu_a, \sigma_a^2/n_g)$ in Bure and Parilina [1], the possibility density function of $\overline{a}$ is

$$f_{\overline{a}}(a) = \frac{1}{\sqrt{2\pi}}\frac{1}{\sigma_a/\sqrt{n_g}} e^{-\frac{(a-\mu_a)^2}{2\sigma_a^2/n_g}}, (-\infty < a < +\infty) \tag{14}$$

Assume that the possibility density function of stochastic variable Z_a is $g_{Z_a}(a,S)$. And since $\overline{a}$ and S_a are independent of each other, then $g_{Z_a}(a,S) = f_{\overline{a}}(a) f_a(S)$. Let $u = a - k_a S/M_a$, then there is

$$F_{Z_a}(z) = \int_{-\infty}^{\infty}[\int_{-\infty}^{z+k_aS/M_a} g_{Z_a}(a,S)da]dS = \int_{-\infty}^{z}[\int_{-\infty}^{\infty} f_{\overline{a}}(u + k_a S/M_a) f_a(S)dS]du \tag{15}$$

From the above formula, the possibility density function of stochastic variable Z_a is:

$$\begin{cases} f_{Z_a}(z) = H_a \int_0^{\infty} S^{n_g-2} e^{-\frac{1}{2\sigma_a^2}[n_g(k_aS/M_a+z-\mu_a)^2+(n_g-1)S^2]} dS \\ H_a = \sqrt{\dfrac{2n_g}{\pi}} \dfrac{(\frac{n_g-1}{2})^{\frac{n_g-1}{2}}}{\sigma_a^{n_g}\Gamma(\frac{n_g-1}{2})} \end{cases} \tag{16}$$

Define stochastic variable

$$\overline{b} = \sum_{i=1}^{n_r} b_i \Big/ n_r \tag{17}$$

Same as above, getting stochastic variable $Z_b = \overline{b} - k_b S_b/M_b$, its possibility density function is

$$\begin{cases} f_{Z_b}(z) = H_b \int_0^{\infty} S^{n_r-2} e^{-\frac{1}{2\sigma_b^2}[n_r(k_b S/M_b + z - \mu_b)^2 + (n_r-1)S^2]} dS \\ H_b = \sqrt{\frac{2n_r}{\pi}} \frac{(\frac{n_r-1}{2})^{\frac{n_r-1}{2}}}{\sigma_b^{n_r}\Gamma(\frac{n_r-1}{2})} \end{cases} \tag{18}$$

3.4 Derivation of possibility density distribution of difference between characteristic values of stochastic variables A and B

Let $Z_c = Z_b - Z_a$. According to the independence of stochastic variables Z_a and Z_b, the possibility density function of stochastic variable Z_c is $g_{Z_c}(Z_a, Z_b)$, then

$$F_{Z_c}(z) = \int_{-\infty}^{z} [\int_{-\infty}^{\infty} g_{Z_c}(Z_b - u, Z_b) dZ_b] du \tag{19}$$

Therefore, the possibility density function of Z_c is

$$f_{Z_c}(u) = \int_{-\infty}^{\infty} f_{Z_a}(Z_b - u) f_{Z_b}(Z_b) dZ_b \tag{20}$$

Then, the probability of $Z_c > Z_\alpha$ (Z_α is any real number data) is

$$P\{Z_c > Z_\alpha\} = 1 - P\{Z_c \le Z_\alpha\} = 1 - F_{Z_c}(Z_\alpha) \tag{21}$$

3.5 Probability calculation of difference between characteristic values of stochastic vars. G and R

Define stochastic variable

$$\overline{g} = \sum_{i=1}^{n_g} g_i \Big/ n_g \tag{22}$$

There are

$$\begin{cases} E(\overline{g}) = \mu_g \\ D(\overline{g}) = \frac{\sigma_g^2}{n_g} \end{cases} \tag{23}$$

For Eq. (10-1), $\overline{g}$ and S_g/M_g are used instead of μ_g and σ_g respectively, thus stochastic variable $Z_g = \overline{g} - k_a S_g / M_g$ is obtained, and there is

$$E(Z_g) = E(\overline{g}) - \frac{k_a}{M_g} E(S_g) = \mu_g - k_a \sigma_g$$

Therefore, average value $E(Z_g)$ of stochastic variable Z_g is unbiased estimation of Eq. (10-1).

Define stochastic variable

$$\overline{r} = \sum_{i=1}^{n_r} g_i \Big/ n_r \tag{24}$$

Then, there are

$$\begin{cases} E(\overline{r}) = \frac{1}{n_r} \sum_{i=1}^{n_r} E(r_i) = \frac{1}{n_r} n_r \mu_r = \mu_r \\ D(\overline{r}) = \frac{1}{n_r^2} \sum_{i=1}^{n_y} D(r_i) = \frac{1}{n_r^2} n_r \sigma_r^2 = \frac{\sigma_r^2}{n_r} \end{cases} \tag{25}$$

For Eq. (11-1), $\overline{r}$ and S_r/M_r are used instead of μ_r and σ_r respectively, thus stochastic variable $Z_r = \overline{r} - k_b S_r / M_r$ is obtained, then there is

$$E(Z_r) = E(\overline{r}) - \frac{k_b}{M_r} E(S_r) = \mu_r - k_b \sigma_r$$

Therefore, average value $E(Z_r)$ of stochastic variable Z_r is unbiased estimation of Eq. (11-1).

According to the definition of stochastic variables Z_g and Z_r, there is

$$(Z_r - Z_g)/[0.5(\mu_g + \mu_r)] = \overline{b} - k_b S_b / M_b - \overline{a} + k_a S_a / M_a = Z_b - Z_a$$

Considering Eq. (21), the probability that Z_r is greater than $Z_g + 0.5Z_\alpha(\mu_g + \mu_r)$ is

$$P\{Z_r - Z_g > 0.5Z_\alpha(\mu_g + \mu_r)\} = 1 - F_{Z_c}(Z_\alpha) \tag{26}$$

When Z_α is a positive number and close to 0, it can be considered as

$$P\{Z_r > Z_g\} = P\{Z_b > Z_a\} = 1 - P\{Z_c \le Z_\alpha\} = 1 - F_{Z_c}(Z_\alpha) \tag{27}$$

4. Verification of validity of formulas by calculation examples

Without loss of generality, in the following example, it is assumed that the sample standard differences of stochastic variables G and R are the same. Moreover, the number of samples for stochastic variables G and R is $n_g = n_r = 6$, respectively. Generally, the characteristic value is with 95% assurance factor for a stochastic variable, so take $\alpha_a = \alpha_b = 1 - 95\% = 5\%$. Then, the upper α_a fractile for stochastic vars. G and A, and the upper α_b fractile for stochastic vars. R and B are $k_a = k_b = 1.645$, respectively.

4.1 When $Z_g > Z_r$

When $Z_g > Z_r$, the selected params and calculated results are shown in Table S2, and the possibility density function $f_{Z_c}(u)$ of Z_c is pictured in Figure S1. From the calculation results, the probability, stochastic variable Z_r of characteristic value of stochastic variable R is greater than stochastic variable Z_g of characteristic value of stochastic variable G, is 0.1079, that is, the characteristic value of stochastic var. R is smaller than that of stochastic var. G.

Table S2 $Z_g > Z_r$

Stochastic variable G	Sample average μ_g	Sample standard difference S_g	Variation coefficient δ_g
	87.3	9.9	0.1134
Stochastic variable R	Sample average μ_r	Sample standard difference S_r	Variation coefficient δ_r
	69.5	9.9	0.1424
C	0.7961		
Stochastic variable A	Sample average μ_a	Sample standard difference S_a	
	1.1135	0.1263	
Stochastic variable B	Sample average μ_b	Sample standard difference S_b	
	0.8865	0.1263	
Z_α	0.001	$P\{Z_r > Z_g\}$	0.1079

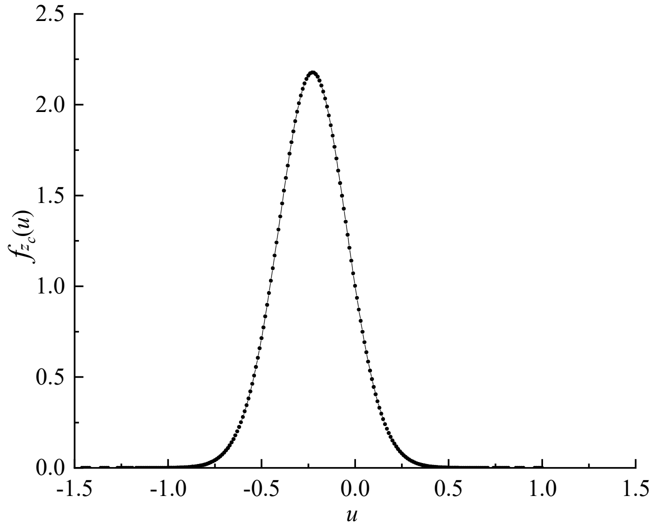

Figure S1 When $Z_g > Z_r$, the possibility density function $f_{Z_c}(u)$ of Z_c

4.2 When $Z_g = Z_r$

When $Z_g = Z_r$, the selected params and calculated results are shown in Table S3, and the possibility density function $f_{Z_c}(u)$ of Z_c is pictured in Figure S2. From the calculated results, the probability, stochastic variable Z_r of characteristic value of stochastic variable R is greater than stochastic variable Z_g of characteristic value of stochastic variable G, is 0.5000, that is, the characteristic value of stochastic variable R is not greater than that of stochastic variable G. In addition, from the symmetry of $f_{Z_c}(u)$ about $u = 0$ in Figure S2, it can be concluded that if the characteristic value of stochastic var. R is greater than that of stochastic variable G, the probability $P\{Z_r > Z_g\}$ must be greater than 0.5.

Table S3 $Z_g = Z_r$

Stochastic variable G	Sample average μ_g	Sample standard difference S_g	Variation coefficient δ_g
	87.3	9.9	0.1134
Stochastic variable R	Sample average μ_r	Sample standard difference S_r	Variation coefficient δ_r
	87.3	9.9	0.1134
C	1.0		
Stochastic variable A	Sample average μ_a	Sample standard difference S_a	
	1.0	0.1134	
Stochastic variable B	Sample average μ_b	Sample standard difference S_b	
	1.0	0.1134	
Z_α	0.0000001	$P\{Z_r > Z_g\}$	0.5000

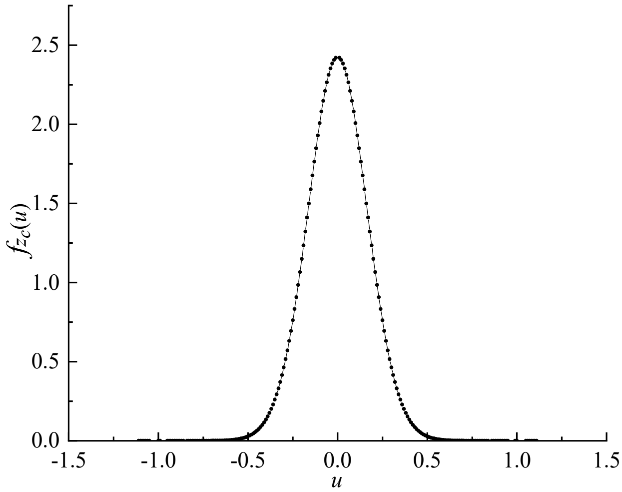

Figure S2 When $Z_g = Z_r$, the possibility density function $f_{Z_c}(u)$ of Z_c

4.3 When $Z_g < Z_r$

Table S4 $Z_g < Z_r$

Stochastic variable G	Sample average μ_g	Sample standard difference S_g		Variation coefficient δ_g
	87.3	9.9		0.1134
Stochastic variable R	Sample average μ_r	Sample standard difference S_r		Variation coefficient δ_r
	110.4	9.9		0.0897
C	1.2646			
Stochastic variable A	Sample average μ_a	Sample standard difference S_a		
	0.8832	0.1002		
Stochastic variable B	Sample average μ_b	Sample standard difference S_b		
	1.1168	0.1002		
Z_α	0.001	$P\{Z_r > Z_g\}$		0.9438

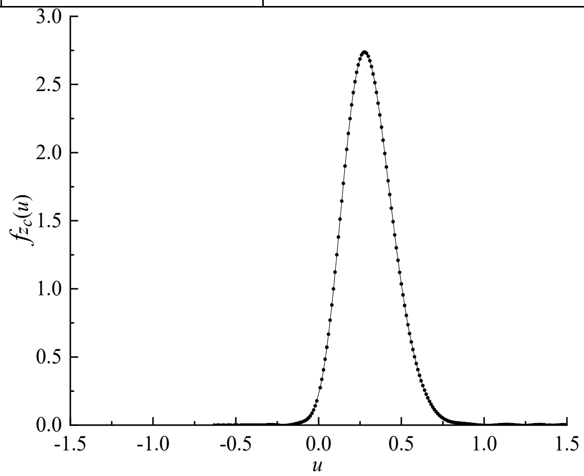

Figure S3 When $Z_g < Z_r$, the possibility density function $f_{Z_c}(u)$ of Z_c

When $Z_g < Z_r$, the selected params and calculated results are shown in Table S4, and the possibility density function $f_{Z_c}(u)$ of Z_c is pictured in Figure S3. From the calculated results, the probability, stochastic variable Z_r of characteristic value of stochastic variable R is greater than stochastic variable Z_g of characteristic value of stochastic variable R, is 0.9438, that is, the characteristic value of stochastic var. R is greater than that of stochastic var. G.

5. Results

From above-mentioned calculation examples, it can be concluded as follows: 1) when the probability that stochastic variable Z_r of characteristic value of stochastic variable R is greater than stochastic variable Z_g of characteristic value of stochastic variable G, is smaller than 0.5, the characteristic value of stochastic variable R is smaller than that of stochastic variable G; when the probability is equal to 0.5, the characteristic value of stochastic variable R is equal to that of stochastic variable G; when the probability is greater than 0.5, the characteristic value of stochastic variable R is greater than that of stochastic variable G.

The theoretical formula of statistics developed above is complicated, the calculation is mainly based on integral, and the amount of calculation is large. Since formulas (16) and (18) have no analytical solutions, Mathematica 12.0 scientific calculation software was adopted, and in this software the numerical calculation of theoretical formulas was completed through programming. In the program, $f_{Z_a}(z)$ in formula (16) and $f_{Z_b}(z)$ in formula (18) were obtained by 20 polynomials (i.e., 1, z, $z^2 \cdots z^{19}$) fitting, respectively, but which made the subsequent integral calculation amount large. On an ordinary Win10 system computer, about 40 minutes were taken for the calculation of a complete theoretical formula at one time. In order to facilitate the application of theoretical formula in this study, μ_g, μ_r, σ_g, σ_r, n_g, n_r, k_a and k_b can be taken as the unknown params to regress and fit the calculation formula of $f_{Z_c}(u)$ in formula (20) in the future.

References

[1] Bure V, Parilina E. Probability Theory and Mathematical Statistics. World Scientific, 2013.