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# Gear Root Bending Strength: A New Multiaxial Approach to Translate the Results of Single Tooth Bending Fatigue Tests to Meshing Gears

Franco Concli \*<sup>®</sup>, Lorenzo Fraccaroli and Lorenzo Maccioni \*<sup>®</sup>

Faculty of Science and Technology, Free University of Bozen-Bolzano, 39100 Bolzano, Italy; lorenzo.fraccaroli@unibz.it

\* Correspondence: franco.concli@unibz.it (F.C.); lorenzo.maccioni@unibz.it (L.M.); Tel.: +39-0471-017748 (F.C.)

**Abstract:** Developing accurate design data to enable the effective use of new materials is undoubtedly an essential goal in the gear industry. To speed up this process, Single Tooth Bending Fatigue (STBF) tests can be conducted. However, STBF tests tend to overestimate the material properties with respect to tests conducted on Running Gears (RG). Therefore, it is common practice to use a constant correction factor  $f_{korr}$ , of value 0.9 to exploit STBF results to design actual gears, e.g., through ISO 6336. In this paper, the assumption that this coefficient can be considered independent from the gear material, geometry, and loading condition was questioned, and through the combination of numerical simulations with a multiaxial fatigue criterion, a method for the calculation of  $f_{korr}$  was proposed. The implementation of this method using different gear geometries and material properties shows that  $f_{korr}$  varies with the gears geometrical characteristics, the material fatigue strength, and the load ratio (R) set in STBF tests. In particular, by applying the Findley criterion, it was found that, for the same gear geometry,  $f_{korr}$  depends on the material as well. Specifically,  $f_{korr}$  increases with the ratio between the bending and torsional fatigue limits. Moreover, through this method it was shown that the characteristics related to the material and the geometry have a relevant effect in determining the critical point (at the tooth root) where the fracture nucleates.

Keywords: STBF; FEM; Findley; gears; multiaxial fatigue; material characterization

# 1. Introduction

Gears are widespread components commonly used for transferring mechanical power between noncoaxial rotating shafts [1]. The working principle is based on the meshing of teeth with a conjugate profile that, on the one hand, allows the transmission of torque and motion and, on the other hand, undergoes the teeth to fatigue and to different failure modes [2,3]. According to [4], the repeated contacts between gear flanks lead to fatigue failure modes such as scuffing [5], wear [6], pitting [7], and micropitting [8]. Whereas the repeated pulsating bending loading of the teeth root leads to a failure mode called Tooth (Root) Bending Fatigue (TBF) [9].

The TBF phenomenon emerges due to the variation of the load entity (transmitted force) and position along the active profile of the tooth [9]. In other words, the rolling/sliding contact between the tooth flanks leads the tooth root fillet stresses to vary continuously, pulsating from zero to a maximum. Moreover, the stress value is amplified by the tooth root notch effect [10]. The crack propagation leads to the tooth root breakage, and therefore, to the endangerment of the entire system [11]. For this reason, the TBF is considered one of the most dangerous failure modes, and it could potentially lead to catastrophic consequences [12]. Therefore, TBF life, through the calculation of Tooth Bending Strength (TBS) to determine the load capacity ISO 6336-1 [13], is a fundamental aspect in gear design [14].



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**Copyright:** © 2021 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https:// creativecommons.org/licenses/by/ 4.0/). To rate the TBS, standards exploit an uniaxial strength criterion, i.e., the maximum tensile (positive) stress  $\sigma_F$  at the tooth root due to pure bending has to not exceed the permissible bending stress  $\sigma_{FP}$  ISO 6336-3 [15]. According to the Method B of ISO 6336-3 [15],  $\sigma_{FP}$  is a function of the material strength  $\sigma_{Flim}$ . The maximum tensile stress due to bending  $\sigma_F$  can be determined through several calculation methods, e.g., (ISO 6336-3 [15]; ANSI/AGMA [16]), starting from the gear geometry and the applied loads. Reference values for  $\sigma_{Flim}$  for different materials are available in the same standards, e.g., (ISO 6336-5 [17]; ANSI/AGMA [16]). However, the values reported in [17] are representative of common materials and finishing treatments only. For new materials and/or more reliable values, the standard suggests performing dedicated tests. Therefore, it is common practice for newly developed materials (or specific chemical compositions) to characterize the material strength  $\sigma_{Flim}$  through experimental testing.

Three kind of tests can be found in the literature to characterize the TBS. (1) tests on Running Gears (RG), e.g., [14,18]; (2) tests on Single Tooth Bending Fatigue (STBF), e.g., [19–21], and (3) tests on notched specimens, e.g., [22–24].

Tests on RG can reproduce the exact stress state of the actual gears, so through this method, it is possible to characterize the fatigue behavior of the material with excellent reliability [25]. The actual TBS obtained from tests on RG ( $\sigma_{FlimRG}$ ) is exactly the value  $\sigma_{Flim}$  as intended in the standard. However, to perform these kinds of tests, specific test rigs are needed, e.g., [14,26]. Moreover, the breakage of a tooth makes the gear unusable for further tests, and since each tooth is loaded once every rotation of the tested gear, the experimental campaign results are particularly long and expensive.

On the other hand, tests on notched specimens can be performed quickly on any universal testing machines in a relatively economical manner. However, due to the different geometries and loading modalities, the stress state history in these specimens is very different to the one on actual gears. Therefore, to be used in combination with the standard, the results obtained from these tests require a series of corrective coefficients. As a direct consequence, this approach leads to the greatest uncertainty [27].

A good tradeoff for the effectiveness of the experimental campaign in terms of time/costs, and reliability of results is to conduct STBF tests. Using gears as samples allows having a specimen that is representative of the same manufacturing process used on actual gears, i.e., it encompasses surface finishing effects, etc.; like the tests on RG. The concept behind STBF tests is to apply a variable force to two teeth of the same gear. In particular, the forces are applied through two anvils with parallel faces. These forces are tangent to the base circumference, and at the same time, normal to the tested teeth flanks (Wildhaber property). This kind of test can be carried out by means of a universal testing machine (e.g., a pulsator). In addition, the STBF configuration allows for the performance of multiple tests on a single gear specimen (depending on the number of teeth) with evident cost savings. Moreover, STBF tests do not require lubrication, ensuring a much more effective management with respect to RG tests. On the other hand, experimental evidence showed that the results of STBF tests in terms of material properties ( $\sigma_{FlimSTBF}$ ) cannot be directly compared to the ones obtained on RG  $\sigma_{FlimRG}$ , even if the two values should represent the same material property  $\sigma_{Flim}$  [28,29]. Indeed, as better explained in the following sections, RG and STBF have nonidentical stress states. In particular, the results of an inverse application of the standard [15] in terms of  $\sigma_{FlimSTBF}$  and  $\sigma_{FlimRG}$  by setting a consistent force (in STBF) and torque (in RG) highlight that  $\sigma_{FlimSTBF} > \sigma_{FlimRG}$ . To compensate this effect, Rettig first [28] and then Stahl [29] proposed to use the constant correction coefficient  $(f_{korr} = 0.9)$  defined as  $f_{korr} = \sigma_{FlimRG} / \sigma_{FlimSTBF}$ . However, this coefficient was estimated without investigating the effect of modulus, geometry, and material. Indeed, conducting an experimental campaign to investigate these effects would be extremely time-consuming and expensive if not impossible due to the mostly infinite possible combinations.

The present paper has a twofold objective. Firstly, to describe an innovative method for the estimation of  $f_{korr}$ . In particular, through the combination of (1) material data obtained with standard tests (i.e., bending fatigue, torsional fatigue, STBF); (2) results of

FEM simulations of STBF tests and RG conditions, and (3) the exploitation of a multiaxial fatigue criterion able to take into account nonproportional load histories, it is possible to obtain a more accurate estimation of  $f_{korr}$ . Secondly, to prove that the correction coefficient  $f_{korr}$  is a function of the geometry of the gears as well as the material properties. Indeed, by implementing the proposed method by varying the abovementioned parameters, it emerged that  $f_{korr}$  is not constant.

#### 2. Background

# 2.1. Causes of Different Stress States in STBF and RG Tests

STBF tests are widely exploited in technical and scientific literature; there are many papers presenting the design and the results of STBF tests carried out on gears with different modules and geometries and made of different materials. In Table 1, there are more than 20 references classified based on the normal module tested. Considering that the results of those tests should be used as the results performed on RG, in combination with standards (e.g., [15]), it is important to have a reliable and effective method to translate these data in the form adopted by the standards, namely  $\sigma_{Flim}$ .

Table 1. Relevant literature presenting Single Tooth Bending Fatigue (STBF) tests.

Normal Module <i>m<sub>n</sub></i>	<b>Relevant Papers Presenting STBF Tests</b>
$m_n < 1$	[21,30]
$1 \le m_n < 2$	[31,32]
$2 \le m_n < 3$	[33]
$3 \le m_n < 4$	[27,34–38]
$4 \le m_n < 5$	[34,39–43]
$m_n = 5$	[20,26,32,44–46]
$m_n > 5$	[32,47,48]

The results of the STBF and RG tests differ for two main reasons. Firstly, in STBF tests, to avoid undesired displacement of the gear, a minimum compressive load is usually present; the typical ratio between the minimum and maximum force applied to the teeth is (R = 0.1), e.g., the STBF tests conducted in [26,37,45,47,49]. Naturally, this differs from RG, where R = 0, and therefore, it alters the average stress present in the tooth [18,19,50,51].

Secondly, the loading condition in STBF tests are similar but not identical to the RG situation (as illustrated in Figure 1). On the one hand, in STBF tests, the loads are applied with a fixed direction and position, and they vary in a sinusoidal way with a constant amplitude. In addition, the relative angle between the force and the loaded tooth axis ( $\alpha_{Fen}$ ), which depends on the number of teeth included in the Wildhaber distance, is constant for the entire tests and can be significantly different from the one in the Outer Point of Single pair tooth Contact (OPSC) used for the calculation of  $\sigma_F$  on RG (according to the standard). This leads to the occurrence of a different share between pure bending and pure compressive stresses (the latter are neglected by the ISO 6336 approach). On the other hand, in RG tests, not only the force magnitude, but also the force direction is variable. This is related to the position of the contact, which moves along the tooth flank (while the standard is applied to the most critical engagement position only, namely the OPSC). Moreover, the variable number of mating teeth pairs leads to an uneven force sharing. Consequently, the stress time history at the tooth root is not sinusoidal as in the STBF tests [18,19]. Therefore, in RG the stress state is multiaxial and nonproportional [52] with the stress components varying as shown in Figure 1a. In Figure 1, it is possible to notice the theoretical trends of each component of the stress tensor calculated, along the time, in a point within the tooth root region in RG and STBF conditions. In these graphs, differences in terms of trends and nonproportionality are evident.



Figure 1. Difference between Running Gears (RG) (a) and STBF (b) in terms of stress state.

#### 2.2. Multiaxial Fatigue Criteria in Nonproportional Loading Conditions

From a general perspective, predicting the fatigue life of structural components under multiaxial loads is one of the most challenging tasks in the engineering field. Different scholars proposed numerous fatigue criteria with the aim of considering the influence of multiaxial conditions, e.g., [53–72]. Among the others, the most advanced models are based on the critical plane approach, i.e., the fatigue failure is assumed to take place on a specific plane on which both normal and shear components contribute to the fatigue failure. However, only a subset of methods is capable of considering nonproportional loadings [56–65,67,68,70–72] while others were just limited to cases where the loading is proportional. In addition, only few papers can be found in the literature investigating the phenomenon of tooth root bending through multiaxial fatigue criteria. Among them, Benedetti et al. [20] exploited the Sines fatigue criterion [55] where the stress field due to external loading was determined by means of finite element (FE) modeling of the gear tooth. However, the applicability of the Sines criterion is limited to proportional loading, and therefore, its application to rotating gears requires the assumption to consider the stresses amplitude proportional among them. As shown in Figure 1, this assumption is true only for STBF tests with R = 0. A further step in the state of the art was presented by [73] and [74] that introduced a multiaxial fatigue criterion capable of considering the nonproportionality of stresses. Specifically, in [73], the load carrying capacity of hypoid gears was determined through a method combining the Liu & Mahadevan fatigue criterion [67] and an FE approach. In [74,75], the Crossland fatigue criterion [54] was adopted to characterize the state of stress in spur gears. It is notable that while the Liu & Mahadevan fatigue criterion is based on the critical plane approach, the Crossland fatigue criterion is not.

# 3. Materials and Methods

#### 3.1. Description of the General Framework

According to the standard ISO 6336-3 [15], the maximum tensile stress  $\sigma_F$  at the tooth root is the cause of the failure for TBF, and therefore, can be considered the critical damage parameter. Due to its intrinsic formulations, the standard is effective only as far as mating gears are concerned. Consequently, when used to convert results from STBF tests, this assumption's results are too simplistic. This is due to the following reasons. Firstly, the standard considers only the tensile stress due to bending, neglecting the radial compression force. This is acceptable in RG because it does not affect the calculation (the shear between the tangential and the radial force is almost constant for typical gears geometries). In STBF tests, the ratio between the radial-force to total-force changes, and therefore, neglecting the compression is no longer acceptable. Secondly, the standard does not consider the whole stress tensor; namely, it approximates the gears as planar (2D), neglecting the out of plane stresses. Thirdly, it is acknowledged that for a reliable calculation, the complete stress cycle should be considered. As mentioned in the introduction, RG and STBF tests show different loading cycles, and therefore, the related stress histories (more in general nonproportional).

To convert the experimental results obtained through STBF tests into usable data in the standard, the  $f_{korr}$  was introduced [29]. Although this coefficient is representative of the ratio of the two stresses responsible for the TBF failure in RG and in STBF, its estimation was only carried out empirically. The present paper aims to propose a numerical approach to calculate the  $f_{korr}$  for any possible configuration. The method proposed involves two steps.

In the first step, by means of FE analysis, it is possible to obtain the stress histories in the two loading conditions (i.e., RG and STBF), and therefore, to consider the overall stresses exerted on the tooth root along the loading cycles. Naturally, the modelled RG test and an STBF test have the same gear geometry and material properties. It is fundamental to impose a loading condition (torque in the RG simulation and force in the STBF simulation) that, according to the standard [15], leads to the same maximum stress  $\sigma_F$  at the tooth root. Moreover, to simulate a critical condition,  $\sigma_F$  should achieve values comparable to the permissible stress  $\sigma_{FP}$ .

In this way, the two configurations are identical from the perspective of the standard, but as mentioned in the previous sections, the two stress histories are not equivalent. Therefore, by means of FE simulations, it is possible to quantify these differences. To do this, the stress tensor history  $\overline{\overline{\sigma}}(t)$  (a symmetric matrix showed in Equation (1)) must be extracted for each point where fracture could nucleate, i.e., each point within the root fillet region for both the FE analyses. Thus, through the numerical analysis, it is possible to obtain the stress tensor  $\overline{\overline{\sigma}}(t)$  of the most critical points that can be analyzed through a fatigue criterion.

$$\overline{\overline{\sigma}}(t) = \begin{bmatrix} \sigma_{xx}(t) & \tau_{xy}(t) & \tau_{xz}(t) \\ \tau_{yx}(t) & \sigma_{yy}(t) & \tau_{yz}(t) \\ \tau_{zx}(t) & \tau_{zy}(t) & \sigma_{zz}(t) \end{bmatrix}$$
(1)

Indeed, the second step of the method lies in the analysis of the extracted  $\overline{\sigma}(t)$  through a multiaxial fatigue criterion. As discussed in the previous sections, the most advanced fatigue criteria are based on the critical plane. The selected one must be able to consider the nonproportional loading condition. One of the criteria that fulfills these characteristics is the Findley criterion, which will be introduced in more detail in Section 3.3. However, any multiaxial fatigue criterion capable of accounting for nonproportional loading conditions can be implemented in the presented method.

Through the study of  $\overline{\sigma}(t)$  using an appropriate criterion, it is possible to identify the most critical point and quantify the damage parameter responsible of the TBF failure in the two simulated conditions. The ratio between the most critical damage parameters emerged in the STBF condition, and those that emerged in the RG condition are equivalent to the *f<sub>korr</sub>*.

By repeating the presented procedure for different gear geometries and for different materials, it is possible to study if and how the  $f_{korr}$  is related to these parameters. To calculate  $f_{korr}$ , several typical gear materials (according to [76]) were exploited. The simulations and the application of the fatigue criteria were conducted for two kinds of STBF tests, i.e., R = 0 and R = 0.1.

Details on the FE analysis performed in this work can be found in Section 3.2, while the  $\overline{\overline{\sigma}}(t)$  elaboration through the Findley criterion to calculate the  $f_{korr}$  is presented in Section 3.3.

#### 3.2. Finite Element Analysis

In the present paper, nine different gear geometries were simulated in both the RG and STBF conditions. The main geometrical parameters were picked up from the scientific literature presenting STBF tests. In Table 2, the main geometrical characteristics of each gear tested (A to I) are listed as well as the reference paper presenting STBF tests on such geometries. Based on these geometrical parameters, the three-dimensional models were

created through the commercial software KISSsoft<sup>®</sup> and imported into the open source FE software (Salome—Meca/Code\_Aster).

Construction 1 Provident	Sum	Gears								
Geometrical Parameters	Sym ·	Α	В	С	D	Ε	F	G	Н	Ι
Normal module [mm]	$m_n$	0.45	1	2	3	3	4	5	5	8
Normal pressure angle [°]	$\alpha_n$	20	20	20	20	20	20	20	20	20
Number of teeth	z	29	19	26	24	23	28	24	24	32
Face width [mm]	b	6.75	10.3	10	15	30	30	10	30	20
Profile shift coefficient	x	0.45	0	0.3	0	0.442	0	-0.2	0	0.223
Dedendum coefficient	$h_{fP}^*$	1.25	1.25	1.25	1.25	1.25	1.25	1.25	1.25	1.25
Root radius factor	$\rho_{fP}^{*}$	0.38	0.38	0.38	0.38	0.38	0.38	0.38	0.38	0.38
Addendum coefficient	$h_{aP}^{*}$	1	1	1	1	1	1	1	1	1
Wildhaber	w	4	4	5	5	4	4	3	3	4
Angle STBF [°]	$\alpha_{Fen}$	19	28	27.7	22	23	19	15	15	17
Reference		[21]	[31]	[77]	[35]	[27]	[42]	[44]	[20]	[48]

 Table 2. Geometrical characteristics of simulated gears.

In the model, symmetries were exploited to reduce the computational effort, i.e., in STBF simulations a quarter of each gear was meshed while in the RG ones the whole gear profile was modelled for half of the width. For each gear geometry, an extruded mesh was created. The mesh quality was improved in teeth subjected to loads in terms of mesh density and exploiting hexahedral elements in that region (as illustrated in Figure 2). Nonlinear simulations were carried out setting 40 time-steps for each loading cycle.



Figure 2. Finite Element (FE) modeling of RG (a) and STBF (b) tests.

In Figure 2a, it is possible to see the configuration of an RG simulation, while in Figure 2b, the configuration of an STBF simulation is shown. With respect to the RG simulations, the mating gears were positioned with the appropriate center distance and the axes of rotation were fixed. The motion was assigned to the driving gear and a resistant torque to the driven once accordingly. In the STBF simulation, the radial symmetry was exploited as a fixed constraint, while a pulsating force was applied to the anvil. The force (in STBF) and the torque (in RG) were set to lead to the same  $\sigma_F$ , according to the standard [15]. For each simulation, the  $\overline{\sigma}(t)$  in the nodes within the tooth root fillet subjected to bending were extracted. These nodes are not involved in the meshing area and are highlighted in red in Figure 2.

# 3.3. Implementation of the Findley Criterion for the Calculation of $f_{korr}$

As mentioned in the previous sections, the method proposed requires elaborating the  $\overline{\overline{\sigma}}(t)$  through a fatigue criterion capable of considering nonproportional loading condition.

In this section, the concepts behind the fatigue criteria based on the critical plane are illustrated as well as the steps to apply the Findley criterion.

The stress exerted on a plane defined by a normal vector **n**, of spherical coordinates  $\phi_n$ ,  $\theta_n$ , is constituted by a vector  $P_n$  having modulus and direction varying in time (as illustrated in Figure 3a).  $P_n$  can be determined through the relation showed in Equation (2).

$$\boldsymbol{P}_{\boldsymbol{n}}(\phi_{\boldsymbol{n}},\theta_{\boldsymbol{n}},t) = \overline{\boldsymbol{\sigma}}(t) \, \boldsymbol{n}(\phi_{\boldsymbol{n}},\theta_{\boldsymbol{n}}) \tag{2}$$

In Figure 3a,  $P_n$  can be decomposed into a normal component  $\sigma_n$ , with time-varying modulus and fixed direction, and a tangential component  $\tau_n$  with time-varying modulus and direction. In the case of periodic stresses, the vertex of the vector  $P_n$  describes a three-dimensional closed curve in the space, whose minimum and maximum distance from the plane correspond to the values of  $\sigma_{n,min}$  and  $\sigma_{n,max}$ , respectively (as illustrated in Figure 3b). In Figure 3b, it is possible to notice that the projection of this three-dimensional curve into the plane, which corresponds to the positions of  $\tau_n$  assumed in a load cycle, is defined as  $\Gamma_n$ .



**Figure 3.** (a): components of vector  $P_n(\phi_n, \theta_n, t)$  on plane defined by  $n(\phi_n, \theta_n)$ . (b): definition of curve  $\Gamma_n$  that is projection of positions of  $P_n(\phi_n, \theta_n, t)$  assumed in a load cycle in plane defined by  $n(\phi_n, \theta_n)$  or positions of  $\tau_n(\phi_n, \theta_n, t)$  assumed in a load cycle. Individuation values of maximum and minimum normal stress.

Through the geometrical properties of the curve  $\Gamma_n$ , it is possible to determine that in the plane with normal n, the alternative (average) component of the tangential stress  $\tau_{n,a}$  ( $\tau_{n,m}$ ) could be representative of the entire shear cycle. Several methods can be found in the literature to determine the  $\tau_{n,a}$  and  $\tau_{n,m}$  values for a given curve  $\Gamma_n$ . However, the most diffused method is the Minimum Circumscribed Circle (MCC) [78], i.e.,  $\tau_{n,a}$  is calculated as the radius of the smallest circle that can entirely contain the curve  $\Gamma_n$ , while  $\tau_{n,m}$  correspond to the distance between the center of the MCC and the origin of  $\tau_n$  (as illustrated in Figure 4).



**Figure 4.** Minimum Circumscribed Circle method applied at the curve  $\Gamma_n$  to determine the values of  $\tau_{n,a}$  and  $\tau_{n,m}$ .

At this point, for each plane having normal n that can be defined by varying the parameters ( $\phi_n$ ,  $\theta_n$ ), it is possible to evaluate the relevant stress values, e.g.,  $\tau_{n,a}$  and  $\sigma_{n,max}$ . More specifically, for the critical plane, the corresponding spherical coordinates and the related stresses will be labelled with the subscript c (Equations (3) and (4)).

$$\sigma_{c,max} = \sigma_{n,max}(\phi_C, \theta_C) \tag{3}$$

$$\tau_{c,a} = \tau_{n,a}(\phi_C, \theta_C) \tag{4}$$

The determination of the critical plane ( $\phi_c$ ,  $\theta_c$ ) differs for the different fatigue criteria. For example, the Findley criterion, shown in Equation (5), defines the critical plane as the ones in which the damage parameter (Equation (6)) is maximum. It can be individuated iteratively by varying  $\phi$  and  $\theta$  in the range [0,  $\pi/2$ ].

$$\pi_{c,a} + k\sigma_{c,max} \le f \tag{5}$$

$$(\phi_C, \theta_C) \to \max_{\phi \theta} \{ \tau_{n,a}(\phi, \theta) + k\sigma_{n,max}(\phi, \theta) \}$$
(6)

where *k* is a constant related to the different response of the material to bending and torsion (in terms of fracture propagation) (Equation (7)) and *f* is a constant related to the fatigue limits of the material (Equation (8)). Materials with marked ductility, for which the propagation of the damage is little affected by the normal stress, typically show low *k* values. Both these constants can be calculated by knowing the material fatigue limit at symmetrical alternating bending loading ( $\sigma_f$ ), and the material fatigue limit at symmetrical alternating torsional loading ( $\tau_f$ ) (Equation (9)).

$$k = \frac{2r_{\tau/\sigma} - 1}{2\sqrt{r_{\tau/\sigma} - r_{\tau/\sigma}^2}} \tag{7}$$

$$f = \frac{1}{2\sqrt{r_{\tau/\sigma} - r_{\tau/\sigma}^2}} \tau_f \tag{8}$$

$$r_{\tau/\sigma} = \frac{\tau_f}{\sigma_f} \tag{9}$$

Applying the Findley criterion, it is possible to evaluate the damage parameter on each point of the tooth root (where data on  $\overline{\overline{\sigma}}(t)$  were extracted from the FE analysis). While  $\sigma_F$  (ISO 6336) is representative of the most critical stress state within both RG and STBF tests (and lies in the same position for both configurations), the representative damage parameter is calculated as the maximum among all the calculated ones within the fillet regions. Therefore, the most critical geometrical point, where the cracks are expected

$$f_{korr} = \frac{(\tau_{c,a} + k\sigma_{c,max})|_{STBF}}{(\tau_{c,a} + k\sigma_{c,max})|_{RG}}$$
(10)

The framework presented in this section was implemented in a Matlab routine. In this way, starting from the  $\overline{\overline{\sigma}}(t)$  of each grid point within the root fillet region (extracted by both the FE analysis), it was possible to evaluate the maximum values of the damage parameters (according to the Findley criterion) in RG and STBF conditions, and therefore, to calculate the  $f_{korr}$ . To have a wider overview, the algorithm was applied to different materials typically used for gear applications (as illustrated in Table 3). The properties of these materials in terms of  $\sigma_f$  and  $\tau_f$  were collected from [76], while the *k* values were calculated through Equation (7). It is possible to notice that these materials cover the range from k = 0.2 to  $k \approx 0.3$ .

Table 3. Fatigue limits and resulting Findley's k parameter according to [76].

Material	$\sigma_{f}$	$ au_{f}$	k
42CrMo4	525.7	336.3	0.29
20MnCr5	410.0	258.0	0.27
34Cr4	410.0	256.0	0.26
30NCD16	690.0	428.0	0.25
C35N	250.0	150.0	0.20

# 4. Results and Discussion

In Table 4, the maximum Huber-Mises stresses within the tooth root fillet region extracted from the FE simulations are reported. The  $\sigma_{Eq}|_{RG}$  is the maximum Huber–Mises stress recorded in the OPSC condition in RG simulations while  $\sigma_{Eq}|_{STBF}$  is the maximum Huber-Mises stress observed when the applied force has achieved its maximum value in STBF simulations. It is interesting that for some gear geometries (i.e., A, C, E, and I) the  $\sigma_{Eq}|_{STBF} > \sigma_{Eq}|_{RG}$ , while for others (i.e., B, D, F, and G), the  $\sigma_{Eq}|_{STBF} < \sigma_{Eq}|_{RG}$ . With respect to the gear geometry H, the two values of maximum Huber-Mises stresses are the same. More specifically, taking as reference  $\sigma_{Eq}|_{STBF}$ , the  $\Delta \sigma_{Eq} \%$  is the percentage difference among the measured stresses in the STBF and RG simulations. This value ranges from a minimum of -9.5% (Gear B) to a maximum of +8.4% (Gear E). The Huber-Mises stress does not consider the stress history but includes the compression stress that is neglected by the standard. While the RG and STBF are equivalent from the perspective of the standard, by considering not only the tensile stress induced by bending but also the effect of the compression force, it emerges that the loading conditions could significantly differ. Moreover, the differences do not show a typical trend but oscillate depending on the considered geometry. The reason behind a positive or negative value of  $\Delta \sigma_{Eq} \%$  can be individuated in the share between pure bending and pure compressive stresses (the latter are neglected by the ISO 6336 approach), namely, different  $\alpha_{Fen}$  values in the most critical loading position. While this evidence already shows a limitation of the present standard, the Huber-Mises approach still does not consider the stress cycle which could be considered by applying the Findley's criterion.

Gear	$\sigma_{Eq}$   $_{RG}$	$\sigma_{Eq}$   $_{STBF}$	$\Delta \sigma_{Eq}$ %
A	394	402	2.0%
В	475	430	-9.5%
С	380	402	5.8%
D	471	436	-7.4%
Е	332	360	8.4%
F	428	399	-6.8%
G	522	483	-7.5%
Н	408	408	0.0%
Ι	409	420	2.7%

**Table 4.** Maximum Huber-Mises stresses within tooth root fillet region in critical position for STBF and RG simulations.

In Figure 5, the results of  $f_{korr}$  for different gear modulus, materials, and load ratio are reported. The two graphs are related to two different load ratios, i.e., R = 0 (left) and R = 0.1 (right). The results are shown for five different materials (according to Table 3) and for seven different modules (according to Table 2). It is noticeable that the application of the presented method leads to the estimation of  $f_{korr}$ , which assumes values even different from 0.9. For the materials and the geometries studied, all results for R = 0.1 do not exceed values of 1. This value is slightly exceeded in some cases of R = 0 (effect of the different shares between pure bending and pure compressive stresses).



**Figure 5.**  $f_{korr}$  values for different modules, materials, and load ratios. In (**a**) R = 0 and (**b**) R = 0.1.

The results vary according to load ratio and the material properties. Gears with the same modulus (i.e., Gear D, E having  $m_n = 3$  and G, H having  $m_n = 5$ ) present different values of  $f_{korr}$ . With respect to  $m_n = 3$ , the lowest values of  $f_{korr}$  are related to the Gear E, while for  $m_n = 5$ , the lowest values of  $f_{korr}$  are related to the Gear G. This evidence points out the importance of having a general-purpose method to estimate the appropriate value of  $f_{korr}$  for the specific configuration used in the tests.

With respect to the effect of the load ratio, the relation  $f_{korrR=0.1} < f_{korrR=0}$  is always valid for the same combination of material and gear geometry. Indeed, a minimum load of zero leads to zero tangential stresses on the critical plane, and this results in a possible extension of the curve  $\Gamma_c$ . This effect is translated into an increase in the radius of the MMC, and therefore, to a greater damage parameter in the STBF condition due to the increase of  $\tau_{c,a}$ . The difference between the two loading conditions in terms of  $f_{korr}$  is more pronounced for materials having low *k* values. For instance, the C35N shows differences of 7% between the two loading ratios, while for 42CrMo4, the maximum difference achieved for the two R is around 5%. This is because as *k* decreases, the  $\tau_{c,a}$  becomes predominant in the value of the damage parameter.

Considering results on the same gear geometry, it is noticeable that materials having higher k values results in higher value of  $f_{korr}$ . Indeed, the damage parameter is proportional to k.

Through the method proposed in this paper, it is possible to individuate (for each simulation performed) the node within the tooth root fillet in which the damage parameter is maximum. We observed that for most of the studied gears' geometries, the critical point is near the center of the fillet arc, and the RG and STBF conditions lead to similar results in terms of position of the critical point. However, in some cases, for the same gear geometry, the position of the critical point in STBF and RG simulations differs. For instance, the gear geometries G, E, and H present the higher difference.

In general, most of the combinations of geometry-material show a  $f_{korr}$  value that for the R = 0 STBF tests lies above 0.9. It's safe to use a constant value of 0.9; however, this could lead to significant underestimations of the real material performances. Most importantly, there are some specific combinations for which the values of  $f_{korr}$  significantly decreased below 0.9, leading to an overestimation of the material strength, and consequently, safety issues.

# 5. Conclusions

To date, a constant correction coefficient ( $f_{korr} = 0.9$ ) is usually applied to use  $\sigma_{Flim}$ values obtained through STBF tests in the ISO 6336. This correction coefficient is representative of the different loading conditions and stress states that lead to failures in STBF and RG tests. In the present paper, a method for estimating  $f_{korr}$  by combining numerical results through a fatigue criterion was presented. The concept behind the method proposed is to simulate, for the same geometry, the STBF and the RG conditions with applied loads that lead, according to ISO 6336, to the same  $\sigma_F$ . In this way, through the results of the FE models, it is possible to obtain the stress histories (in terms of stress tensors) for all the nodes that discretize the fillet of the tooth root. At this point, by analyzing the stress histories through a fatigue criterion, in this case the Findley one, it is possible to individuate the critical plane for each point, to evaluate the damage parameter in each critical plane, and therefore, to identify the critical point at which the damage parameter assumes the maximum value. This process can be followed for the RG and the STBF simulation. The ratio between the maximum damage parameter observed in the STBF condition and the one recorded for the RG condition corresponds to the  $f_{korr}$ , as it represents the ratio between the different effects that cause failure for TBF.

This method was applied to nine different gear geometries combined with five different gear materials. The geometries were selected considering what was already used in the past for STBF tests. The materials are representative of the most diffused gear steels. The dynamic FE simulations were carried out through Salome–Meca/Code\_Aster, and the Findley criterion was implemented in a Matlab routine developed ad hoc to elaborate the FE results. Two typical load ratios exploited in the STBF tests were considered, i.e., R = 0and R = 0.1. Preliminary results showed that  $f_{korr}$  is not a constant but varies according to the gear geometry, the load ratio, and the ratio between the torsional  $\tau_f$  and bending  $\sigma_f$ fatigue limits of the material. The results achieved can lead to some considerations about the effect of material properties and the load ratio. For instance, STBF tests carried out with R = 0 lead to higher  $f_{korr}$  values (the results are close to the one performed on RG) with respect to the STBF tests conducted with R = 0.1 on the same geometries and materials. This is because R = 0 can lead to alternative tangential stresses (on the critical plane) of higher value than those arising with R = 0.1. Another interesting result is that for the same gear geometry, materials with a higher k present higher  $f_{korr}$ . This is because as k increases, the damage parameter increases accordingly. However, no specific relation between the analyzed geometrical parameters and  $f_{korr}$  emerged. One of the main reasons for the differences between the two configurations can be individuated in the different angles  $\alpha_{Fen}$  and the related share between pure bending and pure compressive stresses that arise between RG and STBF. Indeed, it affects the compressive stresses at the tooth root that, currently, are not accounted for by the ISO 6336. Moreover, in the presented method, the Findley criterion was exploited since it can take into account multiaxial stress states and nonproportional loading conditions. However, other criteria with the same

capability can be simply implemented, such as the criterion of Matake [57], McDiarmid [59], or Papadopoulos [62]. Future studies will focus on comparing the results achieved with that of the application of different fatigue criteria.

In general, most of the studied configurations (geometry + material) show a  $f_{korr}$  value (for the R = 0 STBF tests) above 0.9. A fixed valued of 0.9 is on the side of safety, but it could lead to significant underestimations of the real material performances. Moreover, for some specific combinations,  $f_{korr}$  was found to be below 0.9. This leads to an overestimation of the material strength, and consequently, safety issues. For this reason, having a general-purpose method capable of predicting the value of  $f_{korr}$  for a specific configuration is fundamental to reliably assessing the material properties throughout STBF tests.

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#### Nomenclature

TBF	Tooth Bending Fatigue
TBS	Tooth Bending Strength
STBF	Single Tooth Bending Fatigue
RG	Running Gears
FE	Finite Element
m <sub>n</sub>	Normal module
α <sub>n</sub>	Normal pressure angle
z	Number of teeth
b	Face width
x	Profile shift coefficient
$h_{fP}^*$	Dedendum coefficient
$\rho_{fP}^*$	Root radius factor
$h_{aP}^{*}$	Addendum coefficient
w	Wildhaber
α <sub>Fen</sub>	Relative angle between the force and the loaded tooth axis
R	Load ratio
OPSC	Outer Point of Single pair tooth Contact
МСС	Minimum Circumscribed Circle
$\sigma_F$	Maximum tensile stress
$\sigma_{FP}$	Permissible bending stress
$\sigma_{Flim}$	Material strength
$\sigma_{FlimSTBF}$	Material strength calculated through Single Tooth Bending Fatigue tests
$\sigma_{FlimRG}$	Material strength calculated through Running Gear tests
f <sub>korr</sub>	Correction coefficient
$\overline{\overline{\sigma}}(t)$	Stress tensor history
$P_n$	Stress exerting on a plane defined by a normal vector n
$\phi_n, \theta_n$	Spherical coordinates of the plane defined by a normal vector n
$\sigma_n$	Stress component normal to the plane defined by a normal vector n

Stress component tangential to the plane defined by a normal vector n
Minimum value assumed by $\sigma_n$
Maximum value assumed by $\sigma_n$
Curve determined by $\tau_n$ along the time
Alternating tangential stress on the plane defined by a normal vector n
Average tangential stress on the plane defined by a normal vector n
Maximum stress component normal to the critical plane
Alternating tangential stress on the critical plane
Material constant related to the different response to bending and torsion
Constant related to the fatigue limits of the material
Material fatigue limit at symmetrical alternating bending loading
Material fatigue limit at symmetrical alternating torsional loading
Ratio between $\tau_f$ and $\sigma_f$
Equivalent stress
Percentage variation of the equivalent stress

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