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Non-Similar Solutions of Dissipative Buoyancy Flow and Heat Transfer Induced by Water-Based Graphene Oxide Nanofluid through a Yawed Cylinder

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Abstract: The fluid flow through blunt bodies that are yawed and un-yawed frequently happens in many engineering applications. The practical significance of deep-water applications such as propagation control, splitting the boundary layer over submerged blocks, and preventing recirculation bubbles is explained by the fluid flow across a yawed cylinder. The current work examined the mixed convective flow and convective heat transfer by incorporating water-based graphene oxide nanofluid around a yawed cylinder with viscous dissipation and irregular heat source/sink. To investigate the heat diffusion across the system of buoyancy effects, the mathematical formulation of the problem was modeled in terms of coupled, nonlinear partial differential equations. The boundary value problem of the fourth-order (bvp4c) solver was operated to find the non-similarity solution. The outcomes indicated that the velocity in both directions enlarged owing to the higher impacts of yaw angle for the phenomenon of assisting flow but decreased for the instance of opposing flow, while the temperature of nanofluid increased because of heightened estimations of yaw angle for both assisting and opposing flows. In addition, with larger impacts of nanoparticle volume fraction, the shear stresses were enhanced by about 0.76% and 0.93% for the case of assisting flow, while for the case of opposing flow, they improved by almost 0.65% and 1.38%, respectively.

Keywords: nanofluid; yawed cylinder; mixed convection; convective heat condition



Citation: Khan, U.; Zaib, A.; Abu Bakar, S.; Soid, S.K.; Ishak, A.; Elattar, S.; Abed, A.M. Non-Similar Solutions of Dissipative Buoyancy Flow and Heat Transfer Induced by Water-Based Graphene Oxide Nanofluid through a Yawed Cylinder. *Lubricants* **2023**, *11*, 60. <https://doi.org/10.3390/lubricants11020060>

Received: 24 November 2022

Revised: 26 January 2023

Accepted: 30 January 2023

Published: 2 February 2023



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1. Introduction

Several significant research papers have been published over the past nearly six decades on the subject of mixed convection (MCN). Numerous scholars from around the world have produced amazing findings on mixed convection or buoyancy flow along a range of geometries. Surprisingly, very little research has investigated the mixed convective flow across a yawed cylinder (YC). The physical structure of heat exchangers must take into account the assessment of flow through yawed cylinders. In reality, this research aids in comprehending how the yaw angle parameter of a vertical cylinder affects the transport properties of the heat exchanger. The literature review reveals that few research articles on yawed cylinders have been published. The first author to explore boundary layer analysis along a yawed cylinder is Sears [1], who examined a variety of cylinder types

in this circumstance. The separation point was then demonstrated to be independent of the yaw angle by Chiu and Lienhard [2] in their study of the flow past a yawed cylinder. Later, Bucker and Lueptow [3] conducted an experimental analysis of the YC under turbulent flow conditions of the boundary layer. However, the research carried out by several researchers has made the many facets of the YC in the forced convection regime more accessible. For example, Roy [4] and Subhashini et al. [5] inspected the compressible flow through a cylinder with non-uniform enthalpy without and with suction, respectively, while Roy and Saikrishnan [6] used a non-uniform suction/injection slot past a yawed cylinder by incorporating water boundary-layer. Saikrishnan [7] and Revathi et al. [8] analyzed the steady and time-dependent problems through a mass transfer slot across a yawed cylinder. Recently, Patil et al. [9] scrutinized the free and forced convective flow through a yawed cylinder and presented a non-similar solution. They noticed that the yaw angle is a significant factor of raising the motion of the prescribed fluid in both the chordwise (CW) and spanwise (SW) directions for the instance of vertical heated sheet.

A nanofluid is a continuous phase of a solid–liquid mixture containing a nanometer-sized nanoparticle scattered in regular base fluids. The precise measurement of the thermal and physical parameters of the nanofluids, such as their specific heat, viscosity, and thermal conductivity is needed to understand their heat transfer behavior. Researchers frequently use well-known expressions or correlations to anticipate physical and thermal features of nanofluids to determine their convective heat transfer behaviors. In their studies, each researcher employed a distinct model of the thermophysical features. The study of convective heat transport comprising nanofluid is an area of interest in engineering and science. Several common fluids, such as ethylene glycol, water, mineral oils, toluene, etc., have relatively poor thermal conductivity in heat transfer operations. The nanofluid is a modern sort of fluid that contains nanometer-sized particles or fibres dispersed in the regular fluid. It was first introduced by Choi [10]. More about the significance of nanofluid can be observed in the book by Avramenko et al. [11].

Nanofluids unquestionably have benefits in their ability to be more stable, appropriate in terms of viscosity, and have better spreading, wetting, and dispersion capabilities on solid surfaces. Nanofluids are employed in a variety of technical fields, including microfluidics, microelectronics, transportation, solid-state lighting, and biomedicine. Additionally, the dispersion of metal nanoparticles is being produced for various additional uses, such as cancer treatment applications in medicine. The continuous impacts of buoyancy past a vertical/orthogonal flat plate encased in a porous media containing nanofluids were investigated by Ahmed and Pop [12]. Nazar et al. [13] investigated mixed convective flow that contained nanofluids through a circular cylinder immersed in a porous medium. Rashad et al. [14] investigated the mixed convection flow across a circular horizontal cylinder in a vertically upward stream encased in porous media containing a nanofluid. The significance of heat transfer using the single and the two-phase models of nanofluids was inspected by Turkyilmazoglu [15]. He presented an analytic solution and also showed that Ag/water is a good conductor of heat transfer. Sulochana and Naramgiri [16] considered the stagnation point flow of water-based copper nanoparticles along an exponential and horizontal movable cylinder with heat absorption/generation. Rekha et al. [17] examined the impacts of thermal radiation on heat transfer flow including nanoparticles through various geometries with thermophoretic particle deposition effects. The stagnation point flow induced by a uniform rotation of the disk and magnetohydrodynamic and radial stretching with hybrid nanoparticles over a spiraling disk via an asymptotic approach was discovered by Sarfaraz et al. [18]. Mabood et al. [19] utilized a water-based hybrid nanofluid to investigate the fully developed steady forced convection flow past a stretchable surface with melting heat transfer and irregular radiation effects. The 3D stagnation point flows with a buoyancy effect through a hybrid nanofluid past a vertical plate with suction and slips impact was examined by Wahid et al. [20]. Zangoee et al. [21] utilized the impact of slip surface and stagnation-point flow through a vertical plate with hybrid nanoparticles (Cu and Al_2O_3). Recently, Malekshah et al. [22] investigated experimentally as well as

numerically the impact of free convective flow and heat transfer within a cavity induced by a nanofluid. They observed that the thermal performance of the cavity is significantly impacted by the use of nanofluid and the thermal configuration of fins.

Convective heat transfer plays a significant role in processes involving high temperatures. Examples include nuclear power plants, gas turbines, thermal energy storage, etc. The convective boundary conditions (CBCs) are applicable to various technical and industrial processes, such as transpiration cooling, material drying, etc. Numerous scholars have investigated and published findings on this subject for viscous fluid because of the practical significance of CBCs. The Blasius flow and Sakiadis flow by utilizing CBCs in a viscous fluid were studied by Bataller [23]. Aziz [24] examined the steady flow and heat transfer by incorporating the CBCs and observed that a similarity solution is possible when the hot fluid is proportional to $x^{-1/2}$. Makinde [25] expanded the abovementioned work by including the buoyancy force with heat and mass transfer mechanisms. Wahid et al. [26] were able to obtain dual solutions for buoyancy effects on the magnetic time-dependent radiative flow of hybrid nanoparticles with the impact of CBCs. The features of fluid and heat transfer characteristics flow in the presence of Sisko fluid through a non-isothermal stretchable sheet with the convective condition were examined by Malik et al. [27]. Ramesh et al. [28] considered the 2D flow of dusty fluid through a convective boundary condition subject to the stretchable sheet. They observed that the temperature of the normal fluid as well as the dust fluid increased due to convective conditions. Rashid et al. [27] inspected the steady non-linear radiative tangent hyperbolic flow in hybrid nanoparticles past a stretchable sheet with CBCs and Lorentz forces. They discovered that as the magnetic parameter increases, the temperature and velocity accelerate and decelerate, respectively. Recently, Prasad et al. [28] utilized convective and surface conditions to investigate the magnetohydrodynamic flow past a Riga radially stretchable plate with a chemical reaction.

According to the aforementioned literature analysis, the investigation of buoyancy or mixed convection flow induced by water-based graphene oxide nanofluid involving irregular heat source/sink and convective boundary conditions through a yawed cylinder has not yet been conducted. Therefore, the goal of this study was to analyze the steady convection boundary-layer flow that is occurring across a yawed cylinder. The following are the novelties in the current analysis:

- Convective flow is driven by buoyancy force across a yawed cylinder.
- Impact of irregular heat source/sink, viscous dissipation and convective boundary condition is also one of the main objectives of the yawed problem.
- The influences of yawed angle on the velocity and temperature profiles.
- The contribution of water-based graphene oxide nanofluid enhances the requisite thermal characteristics or properties.
- The quantitative analysis of the shear stress and heat transfer for the influence of various distinct influential parameters.

Furthermore, the leading partial differential equations and boundary restriction are simplified using the non-similar variables and the local non-similarity solution technique. Then, these equations numerically solved using a boundary value problem solver. In addition, it is believed that the current study will be helpful in the practical significance of subsea applications such as boundary layer separation the above submerged blocks, transference control, and suppressing recirculating bubbles.

2. Materials and Methods

The physical model of the problem is shown schematically in Figure 1. Meanwhile, the influence of buoyancy or mixed convection flow conveying water-based graphene oxide nanofluid is assumed to flow past a vertically yawed cylinder (YC) with radius R_a , so that a yaw angle is taken between 0° and 60° . In addition, the irregular or non-uniform heat source/sink, convective boundary conditions and viscous dissipation impact is also incorporated in the given investigation. Since, the modeling is based on effects of mixed convection, the cylinder must be positioned inclined or vertically manner to experience the

effects of buoyancy force or mixed convection flow. The horizontal and vertical cylinders are specified by $\theta_a = 90^\circ$, and $\theta_a = 0^\circ$, respectively. In addition, the values of the yaw angle parameter above 60° would be closer to a stagnation-point flow, which is not the goal of the existing buoyancy or mixed convection flow analysis. The boundary layer formation is assumed along the y_a -direction and the respective z_a -axis direction is selected along the surface of the YC. The mainstream velocity is mathematically expressed as $u_E(x_a) = 2u_\infty \cos(R_a^{-1}x_a)$ with $w_\infty = u_\infty(\sin \theta_a)^{-1}$, in which u_∞ and w_∞ are constant velocities of the free-stream in x_a - and z_a -directions, respectively. Given that the cylinder's yawed shape is considered, the far field is expected to flow both the chord-wise (along the x_a -coordinates) and span-wise (along the z_a -coordinates) directions. The free-stream velocity $w_E = w_\infty \cos \theta_a$ is considered in the direction of x_a -axes. The velocities of fluid u_a , v_a , and w_a , taking into account the following x_a -, y_a -, and z_a -directions, respectively. All flows taken into consideration are fully developed and have an infinite spanwise extent. Thus, we are looking here for the outcomes with velocity and temperature fields independent of the coordinate z_a . In contrast to the fluid $T_f > T_\infty$ around it, the yawed cylinder surface should be hot. The following presumptions are used to further simplify the analysis of the problem.

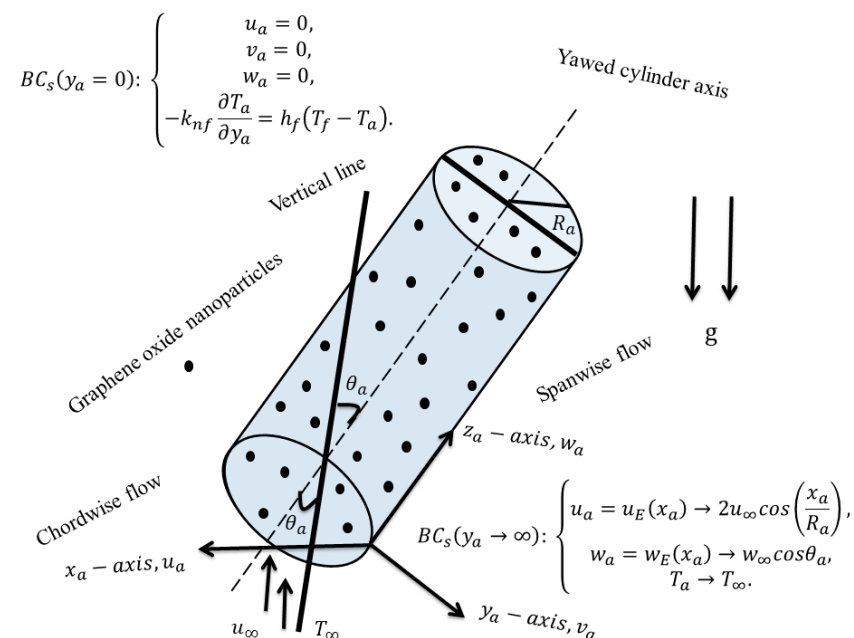


Figure 1. Physical model of the yawed cylinder embedded nanofluid.

- Steady flow
- Stagnation-point flow
- Convective boundary conditions
- Incompressible fluid
- Buoyancy or Mixed convection flow (Assisting and Opposing flows)
- Irregular heat sink/source
- Viscous dissipation
- Nanofluid
- Yawed Cylinder
- Boundary-layer approximations
- Boussinesq approximation

Using the above stated assumptions, the governing PDEs that differentiate the trend of the boundary-layer flow problem are given by [8,9]

$$\frac{\partial u_a}{\partial x_a} + \frac{\partial v_a}{\partial y_a} = 0, \quad (1)$$

$$u_a \frac{\partial u_a}{\partial x_a} + v_a \frac{\partial u_a}{\partial y_a} = v_{nf} \frac{\partial^2 u_a}{\partial y_a^2} + u_E \frac{du_E}{dx_a} + g\beta_f \frac{(\rho\beta)_{nf}/(\rho\beta)_f}{\rho_{nf}/\rho_f} (T_a - T_\infty) \sin \theta_a, \quad (2)$$

$$u_a \frac{\partial w_a}{\partial x_a} + v_a \frac{\partial w_a}{\partial y_a} = v_{nf} \frac{\partial^2 w_a}{\partial y_a^2} + g\beta_f \frac{(\rho\beta)_{nf}/(\rho\beta)_f}{\rho_{nf}/\rho_f} (T_a - T_\infty) \cos \theta_a, \quad (3)$$

$$u_a \frac{\partial T_a}{\partial x_a} + v_a \frac{\partial T_a}{\partial y_a} = \frac{k_{nf}}{(\rho c_p)_{nf}} \frac{\partial^2 T_a}{\partial y_a^2} + \frac{\mu_{nf}}{(\rho c_p)_{nf}} \left[\left(\frac{\partial u_a}{\partial y_a} \right)^2 + \left(\frac{\partial w_a}{\partial y_a} \right)^2 \right] + \frac{Q^{bbb}}{(\rho c_p)_{nf}}, \quad (4)$$

with the boundary conditions (BCs)

$$\begin{aligned} u_a = 0, v_a = 0, w_a = 0, -k_{nf} \frac{\partial T_a}{\partial y_a} &= h_f (T_f - T_a) \text{ at } y_a = 0, \\ u_a \rightarrow u_E(x_a), w_a \rightarrow w_E &= (\cos \theta_a) w_\infty, T_a \rightarrow T_\infty \text{ as } y_a \rightarrow \infty. \end{aligned} \quad (5)$$

The mathematical symbols used in the aforesaid equations, such as ν_{nf} , β_{nf} , k_{nf} , $(\rho c_p)_{nf}$, ρ_{nf} and h_f signify the kinematic viscosity, coefficient of thermal expansion, thermal conductivity, heat capacity, density, and convective heat transfer of nanofluids. The final terms in Equations (2) and (3) show the effect of buoyancy force, which has a positive sign, indicating the buoyancy-assisting flow (BAF) while the buoyancy-opposing flow (BOF) is shown by a negative sign. Moreover, the last two terms of the right hand of Equation (4) indicate the viscous dissipation and irregular heat source/sink, whilst the heat source/sink is defined as [29]

$$Q^{bbb} = \frac{k_{nf} u_E}{x_a v_{nf}} \left[A_a (T_f - T_\infty) e^{-\eta_a} + B_a (T_a - T_\infty) \right] \quad (6)$$

where the temperature-dependent heat sink/source and the exponentially decaying space coefficients are denoted by B_a and A_a , respectively. The phenomena of a heat source relate to the positive values of A_a and B_a , while the phenomenon of a heat sink correlates to the negative values of A_a and B_a . Additionally, we attempt a unique method to alter how researchers have recently approached the study of heat transfer in fluids. Generally, the nanofluids are used through the mixed composition of regular fluid and nanoparticles. It is believed that the present method will be an effective technique to modify the process of heat transfer in fluids. As a result, Table 1 lists the thermophysical properties of the regular fluid and graphene oxide nanoparticles. In the meantime, Table 2 offers the experimental characteristics of thermophysical properties.

Table 1. Thermophysical properties of nanofluids and regular fluid.

Properties	Nanofluid
Density	$\rho_{nf} = \{(1 - \varphi)\rho_f + \varphi\rho_{snp}\}$
Viscosity	$\mu_{nf} = \frac{\mu_f}{(1 - \varphi)^{2.5}}$
Thermal expansion	$(\rho\beta)_{nf} = [(1 - \varphi)(\rho\beta)_f + \varphi(\rho\beta)_{snp}]$
Thermal conductivity	$\frac{k_{nf}}{k_f} = \frac{k_{snp} + 2k_f - 2\varphi(k_f - k_{snp})}{k_{snp} + 2k_f + \varphi(k_f - k_{snp})}$
Heat capacity	$(\rho c_p)_{nf} = [(1 - \varphi)(\rho c_p)_f + \varphi(\rho c_p)_{snp}]$

Table 2. Thermophysical data of regular fluid and GO nanoparticle [30].

Characteristic Properties	H ₂ O	GO
ρ	997.1	1800
c_p	4179	717
k	0.613	5000
β	21	2.84×10^{-4}
Pr	6.2	-

Consequently, Equations (1) through (6) are transformed into a non-dimensional form using the following non-similarity variables [9]:

$$\zeta_a = \frac{1}{3} \int_0^{x_a} \left(\frac{u_E}{u_\infty} \right) d \left(\frac{x_a}{R_a} \right), \eta_a = y_a \left(\frac{u_E}{u_\infty} \right) \left(\frac{u_\infty}{2\nu_f \zeta_a R_a} \right)^{1/2}, w_a = w_E G(\zeta_a, \eta_a), \quad (7)$$

$$\psi(x_a, y_a) = u_\infty \left(\frac{2\nu_f \zeta_a R_a}{u_\infty} \right)^{1/2} F(\zeta_a, \eta_a), S = (T_a - T_\infty) / (T_f - T_\infty).$$

Equation (1) is satisfied while the other equations are transformed to

$$\frac{\mu_{nf}/\mu_f}{\rho_{nf}/\rho_f} F_{\eta_a \eta_a} + \frac{1}{3} (2\zeta_a F_{\zeta_a} + F) F_{\eta_a \eta_a} + \frac{\beta(\zeta_a)}{3} (1 - F_{\eta_a}^2) - \frac{2}{3} \zeta_a F_{\eta_a} F_{\zeta_a \eta_a} + \frac{(\rho\beta)_{nf}/(\rho\beta)_f}{\rho_{nf}/\rho_f} Ri_a p_a(\zeta_a) S \sin \theta_a = 0, \quad (8)$$

$$\frac{\mu_{nf}/\mu_f}{\rho_{nf}/\rho_f} G_{\eta_a \eta_a} + \frac{1}{3} (2\zeta_a F_{\zeta_a} + F) G_{\eta_a} - \frac{2}{3} \zeta_a F_{\eta_a} G_{\zeta_a} + \frac{(\rho\beta)_{nf}/(\rho\beta)_f}{\rho_{nf}/\rho_f} Ri_a q_a(\zeta_a) S \sin \theta_a = 0, \quad (9)$$

$$\frac{k_{nf}}{k_f} S_{\eta_a \eta_a} + Pr_a \frac{(\rho c_p)_{nf}/(\rho c_p)_f}{3} F S_{\eta_a} + 2\zeta_a Pr_a \frac{(\rho c_p)_{nf}/(\rho c_p)_f}{3} (F_{\zeta_a} S_{\eta_a} - F_{\eta_a} S_{\zeta_a}) + \frac{\mu_{nf}}{\mu_f} Pr_a Ec_a \left(4 \sin^2(\zeta_a) \sin^2(\theta_a) F_{\eta_a \eta_a}^2 + \cos^2(\theta_a) G_{\eta_a}^2 \right) + \frac{(k_{nf}/k_f)(\rho_{nf}/\rho_f)}{(\mu_{nf}/\mu_f)} J_a(\zeta_a) (A_a e^{-\eta_a} + B_a) = 0, \quad (10)$$

with the transformed BCs

$$\begin{cases} \text{At } \eta_a = 0 : F(\zeta_a, 0) + 2\zeta_a \frac{\partial F(\zeta_a, 0)}{\partial \zeta_a} = 0, F_{\eta_a}(\zeta_a, 0) = 0, \\ G(\zeta_a, 0) = 0, \frac{k_{nf}}{k_f} S_{\eta_a}(\zeta_a, 0) = -Bi_a L_a(\zeta_a) (1 - S(\zeta_a, 0)), \\ \text{As } \eta_a \rightarrow \infty : F_{\eta_a}(\zeta_a, \infty) \rightarrow 1, G(\zeta_a, \infty) \rightarrow 1, S(\zeta_a, \infty) \rightarrow 0, \end{cases} \quad (11)$$

where η_a and ζ_a are the transformation coordinates, $F(\zeta_a, \eta_a)$ and ψ indicate the stream functions, Ri_a the mixed convective, Pr_a the Prandtl number, Bi_a the Biot number, Ec_a the Eckert number, G and F specify the NDL (non-dimensional) velocities in the requisite directions of span-wise (SW) and chord-wise (CW), respectively, and S implies as the NDL temperature distribution. Moreover, the mathematical terminology of Ri_a , $\beta_a(\zeta_a)$, $p(\zeta_a)$, Pr_a , $J_a(\zeta_a)$, Ec_a , and $q_a(\zeta_a)$ are defined as

$$Ri_a = \frac{Gr}{Re^2} = \frac{Ra g \beta_f (T_f - T_\infty)}{u_\infty^2}, p_a(\zeta_a) = \frac{\zeta_a}{4 \cos^3 x_b}, \beta_a(\zeta_a) = \frac{2\zeta_a}{u_E} \frac{du_E}{d\zeta_a}, q_a(\zeta_a) = \frac{\zeta_a}{2 \cos^2 x_b},$$

$$J_a(\zeta_a) = \frac{\zeta_a}{\cos x_b}, Pr_a = \frac{\nu_f}{\alpha_f}, Ec_a = \frac{w_\infty^2}{(c_p)_f (T_f - T_\infty)}, Bi_a = \frac{h_f R_a}{\sqrt{2} Re^{1/2} k_f}, L_a(\zeta_a) = \frac{\sqrt{\zeta_a}}{\cos x_b},$$

$$Gr = \frac{g \beta_f (T_f - T_\infty) R_a^3}{\nu_f^2}, Re = \frac{u_\infty R_a}{\nu_f}, x_b = \frac{x_a}{R_a}.$$

Likewise, the components of velocity v_a , u_a , and w_a in simplified forms are as follows:

$$v_a = -\frac{u_\infty}{3} \left(\frac{2\nu_f \zeta_a}{u_\infty R_a} \right)^{1/2} \left\{ 2F_{\zeta_a} \cos x_b + \frac{F}{\zeta_a} (\cos x_b) - 3\eta_a F_{\eta_a} \tan x_b - \frac{\eta_a}{\zeta_a} \cos x_b F_{\eta_a} \right\}, u_a = u_E F_{\eta_a}(\zeta_a, \eta_a), \text{ and } w_a = w_E G(\zeta_a, \eta_a).$$

The aforesaid expressions of ζ_a , $\beta_a(\zeta_a)$, $q_a(\zeta_a)$, $J_a(\zeta_a)$, $p_a(\zeta_a)$ and $L_a(\zeta_a)$ in terms of x_b are given as:

$$\begin{aligned} \zeta_a &= \frac{2}{3} \sin x_b, \beta_a(x_b) = -2 \tan^2 x_b, q_a(x_b) = \frac{\sin x_b}{3 \cos^2 x_b}, J_a(x_b) = \frac{2}{3} \tan x_b, \\ L_a(x_b) &= \sqrt{\frac{2}{3}} \frac{\sqrt{\sin x_b}}{\cos x_b}, p_a(x_b) = \frac{\sin x_b}{6 \cos^3 x_b}. \end{aligned} \quad (12)$$

Furthermore, the relation between x_b and ζ_a is signifying as

$$\zeta_a \frac{\partial}{\partial \zeta_a} = C(x_b) \frac{\partial}{\partial x_b}, \text{ where } C(x_b) = \tan x_b. \quad (13)$$

Consequently, using Equations (12) and (13), Equations (8)–(10) through the boundary conditions are expressed as

$$\begin{aligned} \frac{\mu_{nf}/\mu_f}{\rho_{nf}/\rho_f} F_{\eta_a \eta_a} + \frac{1}{3} F F_{\eta_a \eta_a} + \frac{\beta_a(x_b)}{3} (1 - F_{\eta_a}^2) + \\ \frac{(\rho\beta)_{nf}/(\rho\beta)_f}{\rho_{nf}/\rho_f} Ri_a p_a(x_b) S \sin \theta_a = \frac{2C(x_b)}{3} (F_{\eta_a} F_{x_b \eta_a} - F_{\eta_a \eta_a} F_{x_b}), \end{aligned} \quad (14)$$

$$\frac{\mu_{nf}/\mu_f}{\rho_{nf}/\rho_f} G_{\eta_a \eta_a} + \frac{1}{3} F G_{\eta_a} + \frac{(\rho\beta)_{nf}/(\rho\beta)_f}{\rho_{nf}/\rho_f} Ri_a q_a(x_b) S \sin \theta_a = \frac{2C(x_b)}{3} (F_{\eta_a} G_{x_b} - F_{x_b} G_{\eta_a}), \quad (15)$$

$$\begin{aligned} \frac{k_{nf}}{k_f} S_{\eta_a \eta_a} + Pr_a \frac{(\rho c_p)_{nf}/(\rho c_p)_f}{3} F S_{\eta_a} + \frac{\mu_{nf}}{\mu_f} Pr_a Ec_a \left(4 \sin^2(x_b) \sin^2(\theta_a) F_{\eta_a \eta_a}^2 + \cos^2(\theta_a) G_{\eta_a}^2 \right) + \\ \frac{(k_{nf}/k_f)(\rho_{nf}/\rho_{nf})}{(\mu_{nf}/\mu_f)} J_a(x_b) (A_a e^{-\eta_a} + B_a) = 2 Pr_a \frac{(\rho c_p)_{nf}/(\rho c_p)_f}{3} C(x_b) (F_{\eta_a} S_{x_b} - F_{x_b} S_{\eta_a}), \end{aligned} \quad (16)$$

with BCs (11) and (13) become

$$\begin{cases} \text{At } \eta_a = 0 : F(x_b, 0) + 2C(x_b) \frac{\partial F(x_b, 0)}{\partial x_b} = 0, F_{\eta_a}(x_b, 0) = 0, \\ G(x_b, 0) = 0, \frac{k_{nf}}{k_f} S_{\eta_a}(x_b, 0) = -Bi_a L_a(x_b) (1 - S(x_b, 0)), \\ \text{As } \eta_a \rightarrow \infty : F_{\eta_a}(x_b, \infty) \rightarrow 1, G(x_b, \infty) \rightarrow 1, S(x_b, \infty) \rightarrow 0, \end{cases} \quad (17)$$

One can see that the leading Equations (14)–(16) with BCs (17) remain partial differential equations upon modification, with the $\frac{\partial}{\partial x_b}$ factor on the corresponding right-hand side (RHS) acting as the principal barrier to a solution. Additionally, the type of NS (non-similarity) velocity model, which in turn results in the NS thermal and momentum boundary layers.

3. Local Similarity Technique (Equation-1 Model)

It is helpful to examine Equations (14)–(17) from the perspective of local similarity before beginning the method to the local NS solution. According to this method, it is assumed that the RHS of Equations (14)–(17) are sufficiently small; therefore, they can be approximated by zero. It gives

$$\frac{\mu_{nf}/\mu_f}{\rho_{nf}/\rho_f} F_{\eta_a \eta_a} + \frac{1}{3} F F_{\eta_a \eta_a} + \frac{\beta_a(x_b)}{3} (1 - F_{\eta_a}^2) + \frac{(\rho\beta)_{nf}/(\rho\beta)_f}{\rho_{nf}/\rho_f} Ri_a p_a(x_b) S \sin \theta_a = 0, \quad (18)$$

$$\frac{\mu_{nf}/\mu_f}{\rho_{nf}/\rho_f} G_{\eta_a \eta_a} + \frac{1}{3} F G_{\eta_a} + \frac{(\rho\beta)_{nf}/(\rho\beta)_f}{\rho_{nf}/\rho_f} Ri_a q_a(x_b) S \sin \theta_a = 0, \quad (19)$$

$$\frac{k_{nf}}{k_f} S_{\eta_a \eta_a} + \text{Pr}_a \frac{(\rho c_p)_{nf}/(\rho c_p)_f}{3} F S_{\eta_a} + \frac{\mu_{nf}}{\mu_f} \text{Pr}_a \text{Ec}_a \left(4 \sin^2(x_b) \sin^2(\theta_a) F_{\eta_a \eta_a}^2 + \cos^2(\theta_a) G_{\eta_a}^2 \right) + \frac{(k_{nf}/k_f)(\rho_{nf}/\rho_{nf})}{(\mu_{nf}/\mu_f)} J_a(x_b) (A_a e^{-\eta_a} + B_a) = 0, \quad (20)$$

subject to BCs

$$\begin{cases} \text{At } \eta_a = 0 : F(x_b, 0) = 0, F_{\eta_a}(x_b, 0) = 0, G(x_b, 0) = 0, \\ \frac{k_{nf}}{k_f} S_{\eta_a}(x_b, 0) = -Bi_a L_a(x_b)(1 - S(x_b, 0)), \\ \text{As } \eta_a \rightarrow \infty : F_{\eta_a}(x_b, \infty) \rightarrow 1, G(x_b, \infty) \rightarrow 1, S(x_b, \infty) \rightarrow 0, \end{cases} \quad (21)$$

At any direction of flow at the place, the quantity x_b can be seen as an inevitable parameter. Because of this, even though the equations $F_{\eta_a \eta_a \eta_a}$, $G_{\eta_a \eta_a}$ and $S_{\eta_a \eta_a}$ are the requisite posited PDEs, they can be handled as ODEs and are solved using tried-and-true or any similarity techniques that work for boundary layers with similarity when the quantities involved, β_a , p_a , J_a , L_a and q_a , can be discovered for a determined constant factor at any specified or fixed position x_a (or x_b). The flow phenomenon in the SW and CW dependency of the ND temperature and ND velocity profiles can be expressed as a progression of x_b quantity. In order for Equations (14)–(17) and Equations (18)–(21) to be simplified to verify without requiring that x_a be small, it must be assumed that the amounts shown in the right-hand side brackets are insignificant. The Equation-1 model has a weakness because it is unclear whether or not this premise is true.

4. Local NS Solution Technique (Equation-2 Model)

First, it is helpful to define the x_b derivatives of F , G and S to remove their explicit occurrence in relation to the outcome of the LNS of Equations (14)–(17). Let

$$K_a(x_b, \eta_a) = \frac{\partial F}{\partial x_b}, h_a(x_b, \eta_a) = \frac{\partial G}{\partial x_b} \text{ and } X_a(x_b, \eta_a) = \frac{\partial S}{\partial x_b}. \quad (22)$$

Equations (14)–(17), which result from substituting these factors, become as

$$\frac{\mu_{nf}/\mu_f}{\rho_{nf}/\rho_f} F''' + \frac{1}{3} F F'' + \frac{\beta_a(x_b)}{3} (1 - F'^2) + \frac{(\rho\beta)_{nf}/(\rho\beta)_f}{\rho_{nf}/\rho_f} Ri_a p_a(x_b) S \sin \theta_a = \frac{2C(x_b)}{3} (F' K_a' - F'' K_a), \quad (23)$$

$$\frac{\mu_{nf}/\mu_f}{\rho_{nf}/\rho_f} G'' + \frac{1}{3} F G' + \frac{(\rho\beta)_{nf}/(\rho\beta)_f}{\rho_{nf}/\rho_f} Ri_a q_a(x_b) S \sin \theta_a = \frac{2C(x_b)}{3} (F' h_a - K_a G'), \quad (24)$$

$$\frac{k_{nf}}{k_f} S'' + \text{Pr}_a \frac{(\rho c_p)_{nf}/(\rho c_p)_f}{3} F S' + \frac{\mu_{nf}}{\mu_f} \text{Pr}_a \text{Ec}_a \left(4 \sin^2(x_b) \sin^2(\theta_a) F'^2 + \cos^2(\theta_a) G'^2 \right) + \frac{(k_{nf}/k_f)(\rho_{nf}/\rho_{nf})}{(\mu_{nf}/\mu_f)} J_a(x_b) (A_a e^{-\eta_a} + B_a) = 2 \text{Pr}_a \frac{(\rho c_p)_{nf}/(\rho c_p)_f}{3} C(x_b) (F' X_a - K_a S'). \quad (25)$$

Additionally, BCs (17) in reference to (22) are modified as:

$$\begin{cases} \text{At } \eta_a = 0 : F(x_b, 0) = -2C(x_b) K_a(x_b, 0), F'(x_b, 0) = 0, \\ G(x_b, 0) = 0, \frac{k_{nf}}{k_f} S'(x_b, 0) = -Bi_a L_a(x_b)(1 - S(x_b, 0)), \\ \text{As } \eta_a \rightarrow \infty : F'(x_b, \infty) \rightarrow 1, G(x_b, \infty) \rightarrow 1, S(x_b, \infty) \rightarrow 0 \end{cases} \quad (26)$$

Here, primes signify the partial change with respect to the variable η_a . Then, by differentiating Equations (23)–(26) in terms of a non-similarity variable x_b , we obtain

$$\frac{\mu_{nf}/\mu_f}{\rho_{nf}/\rho_f} K_a''' + \frac{1}{3} F K_a'' + \frac{F'' K_a}{3} \left(1 + 2 \frac{dC}{dx_b}\right) + \frac{1}{3} \frac{d\beta_a}{dx_b} (1 - F'^2) - \frac{2}{3} F' K_a' \left(\beta_a + \frac{dC}{dx_b}\right) - \frac{(\rho\beta)_{nf}/(\rho\beta)_f}{\rho_{nf}/\rho_f} Ri_a \sin \theta_a \left(\frac{dp_a}{dx_b} S + p_a(x_b) X_a\right) = \frac{2}{3} C(x_b) \frac{\partial}{\partial x_b} (F' K_a' - F'' K_a), \quad (27)$$

$$\begin{cases} \text{At } \eta_a = 0 & : K_a(x_b, 0) \left(1 + 2 \frac{dC}{dx_b}\right) = -2C(x_b) \frac{\partial K_a(x_b, 0)}{\partial x_b}, K_a'(x_b, 0) = 0, \\ \text{As } \eta_a \rightarrow \infty & : K_a'(x_b, \infty) \rightarrow 0, \end{cases} \quad (28)$$

$$\frac{\mu_{nf}/\mu_f}{\rho_{nf}/\rho_f} h_a'' + \frac{1}{3} F h_a' - \frac{2}{3} \frac{dC}{dx_b} F' h_a + \frac{1}{3} G' K_a \left(1 + 2 \frac{dC}{dx_b}\right) + \frac{(\rho\beta)_{nf}/(\rho\beta)_f}{\rho_{nf}/\rho_f} Ri_a \sin \theta_b \left(\frac{dq_a}{dx_b} S + q_a(x_b) X_a\right) = \frac{2}{3} C(x_b) \frac{\partial}{\partial x_b} (F' h_a - G' K_a), \quad (29)$$

$$\begin{cases} \text{At } \eta_a = 0 & : h_a(x_b, 0) = 0, \\ \text{As } \eta_a \rightarrow \infty & : h_a(x_b, \infty) \rightarrow 0, \end{cases} \quad (30)$$

$$\begin{aligned} & \frac{k_{nf}}{k_f} X_a'' + Pr_a \frac{(\rho c_p)_{nf}/(\rho c_p)_f}{3} F X_a' + Pr_a \frac{(\rho c_p)_{nf}/(\rho c_p)_f}{3} K_a S' \left(1 + 2 \frac{dC}{dx_b}\right) - \\ & 2Pr_a \frac{(\rho c_p)_{nf}/(\rho c_p)_f}{3} \frac{dC}{dx_b} F' X_a + \frac{(k_{nf}/k_f)(\rho_{nf}/\rho_{nf})}{(\mu_{nf}/\mu_f)} \frac{dJ_a}{dx_b} (A_a e^{-\eta_a} + B_a) + \\ & \frac{\mu_{nf}}{\mu_f} Pr_a Ec_a \left(4 \sin^2(\theta_a) \sin(2x_b) F'^2 + 8 \sin^2(\theta_a) \sin^2(x_b) F'' K_a'' + 2 \cos^2(\theta_a) G' h_a'\right) = \\ & 2Pr_a \frac{(\rho c_p)_{nf}/(\rho c_p)_f}{3} C(x_b) \frac{\partial}{\partial x_b} (F' X_a - K_a S'), \end{aligned} \quad (31)$$

$$\begin{cases} \text{At } \eta_a = 0 & : \frac{k_{nf}}{k_f} X_a'(x_b, 0) = -Bi_a \frac{dL_a}{dx_b} (1 - S(x_b, 0)) + Bi_a L_a(x_b) X_a(x_b, 0), \\ \text{As } \eta_a \rightarrow \infty & : X_a(x_b, \infty) \rightarrow 0, \end{cases} \quad (32)$$

when using explicit terms, the aforementioned equations RHS are overloaded with x_b derivatives. Equations (27)–(32) offer characteristics to the considered Equations (23)–(25) subject to BCs (26). For simultaneous treatment, the functions F , K_a , G and h_a are given in Equations (23), (24), (27) and (29). Similarly, functions S and X_a are presented in (25) and (31), requiring simultaneously the non-similar solution. To guarantee that all labels/classes in the leading equations and their BCs are retained, Equations (23)–(26) are not approximated. Hence, further proposed in the characteristic equations that the RHS of Equations (27), (29) and (30) are sufficiently small to allow for their neglect. As a result, the leading Equations (23), (24) and (26)–(30) for the problem of momentum fields in the CW and SW directions for the Equation-2 model (or LNSSM) can be combined as follows:

$$\frac{\mu_{nf}/\mu_f}{\rho_{nf}/\rho_f} F''' + \frac{1}{3} F F'' + \frac{\beta_a(x_b)}{3} (1 - F'^2) + \frac{(\rho\beta)_{nf}/(\rho\beta)_f}{\rho_{nf}/\rho_f} Ri_a p_a(x_b) S \sin \theta_a = \frac{2C(x_b)}{3} (F' K_a' - F'' K_a), \quad (33)$$

$$\frac{\mu_{nf}/\mu_f}{\rho_{nf}/\rho_f} K_a''' + \frac{1}{3} F K_a'' + \frac{F'' K_a}{3} \left(1 + 2 \frac{dC}{dx_b}\right) + \frac{1}{3} \frac{d\beta_a}{dx_b} (1 - F'^2) - \frac{2}{3} F' K_a' \left(\beta_a + \frac{dC}{dx_b}\right) - \frac{(\rho\beta)_{nf}/(\rho\beta)_f}{\rho_{nf}/\rho_f} Ri_a \sin \theta_a \left(\frac{dp_a}{dx_b} S + p_a(x_b) X_a\right) = 0, \quad (34)$$

$$\begin{cases} \text{At } \eta_a = 0.0 & : F(x_b, 0) = F'(x_b, 0) = 0, K_a(x_b, 0) = 0, K_a'(x_b, 0) = 0, \\ \text{As } \eta_a \rightarrow \infty & : F'(x_b, \infty) \rightarrow 1, K_a'(x_b, \infty) \rightarrow 0. \end{cases} \quad (35)$$

$$\frac{\mu_{nf}/\mu_f}{\rho_{nf}/\rho_f} G'' + \frac{1}{3} F G' + \frac{(\rho\beta)_{nf}/(\rho\beta)_f}{\rho_{nf}/\rho_f} Ri_a q_a(x_b) S \sin \theta_a = \frac{2C(x_b)}{3} (F' h_a - K_a G'), \quad (36)$$

$$\frac{\mu_{nf}/\mu_f}{\rho_{nf}/\rho_f} h_a'' + \frac{1}{3} F h_a' - \frac{2}{3} \frac{dC}{dx_b} F' h_a + \frac{1}{3} G' K_a \left(1 + 2 \frac{dC}{dx_b}\right) + \frac{(\rho\beta)_{nf}/(\rho\beta)_f}{\rho_{nf}/\rho_f} Ri_a \sin \theta_b \left(\frac{dq_a}{dx_b} S + q_a(x_b) X_a\right) = 0, \quad (37)$$

$$\begin{cases} \text{At } \eta_a = 0 & : G(x_b, 0) = 0, h_a(x_b, 0) = 0, \\ \text{As } \eta_a \rightarrow \infty & : G(x_b, \infty) \rightarrow 1, h_a(x_b, \infty) \rightarrow 0. \end{cases} \quad (38)$$

when x_b is taken into account as a uniform or fixed prescribable constraint at any fluid flow directions or locations, Equations (33)–(38), which are ODEs, can be handled as a requisite posted system by solving using the customary methods appropriate for similarity boundary layers. The leading Equations (25), (26), (31) and (32) for the thermal boundary layer similarly take place in the following form:

$$\frac{k_{nf}}{k_f} S'' + \text{Pr}_a \frac{(\rho c_p)_{nf}/(\rho c_p)_f}{3} F S' + \frac{\mu_{nf}}{\mu_f} \text{Pr}_a \text{Ec}_a (4 \sin^2(x_b) \sin^2(\theta_a) F'^2 + \cos^2(\theta_a) G'^2) + \frac{(k_{nf}/k_f)(\rho_{nf}/\rho_{nf})}{(\mu_{nf}/\mu_f)} J_a(x_b) (A_a e^{-\eta_a} + B_a) = 2 \text{Pr}_a \frac{(\rho c_p)_{nf}/(\rho c_p)_f}{3} C(x_b) (F' X_a - K_a S'), \quad (39)$$

$$\begin{aligned} & \frac{k_{nf}}{k_f} X_a'' + \text{Pr}_a \frac{(\rho c_p)_{nf}/(\rho c_p)_f}{3} F X_a' + \text{Pr}_a \frac{(\rho c_p)_{nf}/(\rho c_p)_f}{3} K_a S' \left(1 + 2 \frac{dC}{dx_b}\right) - \\ & 2 \text{Pr}_a \frac{(\rho c_p)_{nf}/(\rho c_p)_f}{3} \frac{dC}{dx_b} F' X_a + \frac{(k_{nf}/k_f)(\rho_{nf}/\rho_{nf})}{(\mu_{nf}/\mu_f)} \frac{dJ_a}{dx_b} (A_a e^{-\eta_a} + B_a) + \\ & \frac{\mu_{nf}}{\mu_f} \text{Pr}_a \text{Ec}_a (4 \sin^2(\theta_a) \sin(2x_b) F''^2 + 8 \sin^2(\theta_a) \sin^2(x_b) F'' K_a'' + 2 \cos^2(\theta_a) G' h_a') = 0, \end{aligned} \quad (40)$$

$$\begin{cases} \text{At } \eta_a = 0 & : \frac{k_{nf}}{k_f} S'(x_b, 0) = -\text{Bi}_a L_a(x_b) (1 - S(x_b, 0)), \\ & \frac{k_{nf}}{k_f} X_a'(x_b, 0) = -\text{Bi}_a \frac{dL_a}{dx_b} (1 - S(x_b, 0)) + \text{Bi}_a L_a(x_b) X_a(x_b, 0), \\ \text{As } \eta_a \rightarrow \infty & : S(x_b, \infty) \rightarrow 0, X_a(x_b, \infty) \rightarrow 0. \end{cases} \quad (41)$$

Moreover, Equations (39)–(41) can be thought of as a group of ODEs for a fixed variable x_b . The system can be easily solved using the same method as for similarity boundary layers, much like Equation-1 model.

4.1. Local NS Technique (Equation-3 Model)

Equations (27), (29) and (31) are now differentiated with respect to x_b in the three-equation model before the additional functions are introduced.

$$N_a = \frac{\partial K_a}{\partial x_b} = \frac{\partial^2 F}{\partial x_b^2}, R_a = \frac{\partial h_a}{\partial x_b} = \frac{\partial^2 G}{\partial x_b^2} \text{ and } H_a = \frac{\partial X_a}{\partial x_b} = \frac{\partial^2 S}{\partial x_b^2}. \quad (42)$$

The derivatives that directly involve x_b terms are clustered on the RHS and are seen to be

$$\frac{2}{3} C(x_b) \frac{\partial^2}{\partial x_b^2} (F' K_a' - F'' K_a), \frac{2}{3} C(x_b) \frac{\partial^2}{\partial x_b^2} (F' h_a - G' K_a)$$

and

$$\frac{2}{3} \text{Pr}_a \frac{(\rho c_p)_{nf}}{(\rho c_p)_f} C(x_b) \frac{\partial^2}{\partial x_b^2} (F' X_a - S' K_a).$$

Now it is expected that this amount is sufficiently small to be close to zero. However, no assumption is necessary to reach any of the terms in Equations (23)–(25), (27), (29) and (31). This result includes the following third truncation or the Equation-3 model for the momentum fields in the CW and SW directions, and the requisite energy equation with coupled BCs for convenience in subsequent computations are as follows:

$$\begin{aligned} & \frac{\mu_{nf}/\mu_f}{\rho_{nf}/\rho_f} F''' + \frac{1}{3} F F'' + \frac{\beta_a(x_b)}{3} (1 - F'^2) + \\ & \frac{(\rho \beta)_{nf}/(\rho \beta)_f}{\rho_{nf}/\rho_f} \text{Ri}_a p_a(x_b) S \sin \theta_a = \frac{2C(x_b)}{3} (F' K_a' - F'' K_a), \end{aligned} \quad (43)$$

$$\frac{\mu_{nf}/\mu_f}{\rho_{nf}/\rho_f} K_a''' + \frac{1}{3} F K_a'' + \frac{F'' K_a}{3} \left(1 + 2 \frac{dC}{dx_b}\right) + \frac{1}{3} \frac{d\beta_a}{dx_b} (1 - F'^2) - \frac{2}{3} F' K_a' \left(\beta_a + \frac{dC}{dx_b}\right) \\ \frac{(\rho\beta)_{nf}/(\rho\beta)_f}{\rho_{nf}/\rho_f} Ri_a \sin \theta_a \left(\frac{dp_a}{dx_b} S + p_a(x_b) X_a\right) = \frac{2}{3} C(x_b) (F' N_a' + K_a'^2 - F'' N_a - K_a K_a''), \quad (44)$$

$$\frac{\mu_{nf}/\mu_f}{\rho_{nf}/\rho_f} N_a''' + \frac{2}{3} F'' K_a \frac{d^2 C}{dx_b^2} + \frac{1}{3} \left(1 + 4 \frac{dC}{dx_b}\right) (F'' N_a + K_a K_a'') + \frac{1}{3} (F N_a'' + K_a N_a'') + \\ \frac{1}{3} \frac{d^2 \beta_a}{dx_b^2} (1 - F'^2) - \frac{4}{3} \frac{d\beta_a}{dx_b} F' K_a' - \frac{2}{3} \frac{d^2 C}{dx_b^2} F' K_a' - \frac{2}{3} (F' N_a' + K_a'^2) \left(\beta_a + 2 \frac{dC}{dx_b}\right) + \\ \frac{(\rho\beta)_{nf}/(\rho\beta)_f}{\rho_{nf}/\rho_f} Ri_a \sin \theta_a \left(\frac{d^2 p_a}{dx_b^2} S + 2 \frac{dp_a}{dx_b} X_a + p_a(x_b) H_a\right) = 0, \quad (45)$$

$$\begin{cases} \text{At } \eta_a = 0.0 : F'(x_b, 0) = F(x_b, 0) = 0, & K_a'(x_b, 0) = K_a(x_b, 0) = 0, \\ & N'(x_b, 0) = N_a(x_b, 0) = 0, \\ \text{As } \eta_a \rightarrow \infty : F'(x_b, \infty) \rightarrow 1, & K_a'(x_b, \infty) \rightarrow 0, & N_a'(x_b, \infty) \rightarrow 0 \end{cases} \quad (46)$$

$$\frac{\mu_{nf}/\mu_f}{\rho_{nf}/\rho_f} G'' + \frac{1}{3} F G' + \frac{(\rho\beta)_{nf}/(\rho\beta)_f}{\rho_{nf}/\rho_f} Ri_a q_a(x_b) S \sin \theta_a = \frac{2C(x_b)}{3} (F' h_a - K_a G'), \quad (47)$$

$$\frac{\mu_{nf}/\mu_f}{\rho_{nf}/\rho_f} h_a'' + \frac{1}{3} F h_a' - \frac{2}{3} \frac{dC}{dx_b} F' h_a + \frac{1}{3} G' K_a \left(1 + 2 \frac{dC}{dx_b}\right) + \\ \frac{(\rho\beta)_{nf}/(\rho\beta)_f}{\rho_{nf}/\rho_f} Ri_a \sin \theta_b \left(\frac{dq_a}{dx_b} S + q_a(x_b) X_a\right) = \frac{2}{3} C(x_b) (F' R_a + h_a K_a' - G' N_a - K_a h_a'), \quad (48)$$

$$\frac{\mu_{nf}/\mu_f}{\rho_{nf}/\rho_f} R_a'' + \frac{2}{3} \frac{d^2 C}{dx_b^2} G' K_a + \frac{1}{3} F R_a' - \frac{4}{3} \frac{dC}{dx_b} (F' R_a + h_a K_a') + \frac{1}{3} h_a' K_a + \\ \frac{1}{3} \left(1 + 4 \frac{dC}{dx_b}\right) (G' N_a + K_a h_a') - \frac{2}{3} \frac{d^2 C}{dx_b^2} F' h_a + \\ \frac{(\rho\beta)_{nf}/(\rho\beta)_f}{\rho_{nf}/\rho_f} Ri_a \sin \theta_a \left(\frac{d^2 q_a}{dx_b^2} S + 2 \frac{dq_a}{dx_b} X_a + q_a H_a\right) = 0, \quad (49)$$

$$\begin{cases} \text{At } \eta_a = 0 : G(x_b, 0) = 0, & h_a(x_b, 0) = 0, & R_a(x_b, 0) = 0, \\ \text{As } \eta_a \rightarrow \infty : G(x_b, \infty) \rightarrow 1, & h_a(x_b, \infty) \rightarrow 0, & R_a(x_b, \infty) \rightarrow 0 \end{cases} \quad (50)$$

$$\frac{k_{nf}}{k_f} S'' + Pr_a \frac{(\rho c_p)_{nf}/(\rho c_p)_f}{3} F S' + \frac{\mu_{nf}}{\mu_f} Pr_a Ec_a \left(4 \sin^2(x_b) \sin^2(\theta_a) F''^2 + \cos^2(\theta_a) G'^2\right) + \\ \frac{(k_{nf}/k_f)(\rho_{nf}/\rho_{nf})}{(\mu_{nf}/\mu_f)} J_a(x_b) (A_a e^{-\eta_a} + B_a) = 2 Pr_a \frac{(\rho c_p)_{nf}/(\rho c_p)_f}{3} C(x_b) (F' X_a - K_a S'), \quad (51)$$

$$\frac{k_{nf}}{k_f} X_a'' + Pr_a \frac{(\rho c_p)_{nf}/(\rho c_p)_f}{3} F X_a' + Pr_a \frac{(\rho c_p)_{nf}/(\rho c_p)_f}{3} K_a S' \left(1 + 2 \frac{dC}{dx_b}\right) - \\ 2 Pr_a \frac{(\rho c_p)_{nf}/(\rho c_p)_f}{3} \frac{dC}{dx_b} F' X_a + \frac{(k_{nf}/k_f)(\rho_{nf}/\rho_{nf})}{(\mu_{nf}/\mu_f)} \frac{dJ_a}{dx_b} (A_a e^{-\eta_a} + B_a) + \\ \frac{\mu_{nf}}{\mu_f} Pr_a Ec_a \left(4 \sin^2(\theta_a) \sin(2x_b) F''^2 + 8 \sin^2(\theta_a) \sin^2(x_b) F'' K_a'' + 2 \cos^2(\theta_a) G' h_a'\right) = \\ 2 Pr_a \frac{(\rho c_p)_{nf}/(\rho c_p)_f}{3} C(x_b) (F' H_a + X_a K_a' - S' N_a - K_a X_a'), \quad (52)$$

$$\frac{k_{nf}}{k_f} H_a'' + \frac{2}{3} Pr_a \frac{(\rho c_p)_{nf}}{(\rho c_p)_f} \frac{d^2 C}{dx_b^2} K_a S' - \frac{4}{3} Pr_a \frac{(\rho c_p)_{nf}}{(\rho c_p)_f} \frac{dC}{dx_b} (F' H_a + X_a K_a') + \\ \frac{1}{3} Pr_a \frac{(\rho c_p)_{nf}}{(\rho c_p)_f} \left(1 + 4 \frac{dC}{dx_b}\right) (K_a X_a' + S' N_a) + \frac{1}{3} Pr_a \frac{(\rho c_p)_{nf}}{(\rho c_p)_f} (F H_a' + K_a X_a') - \\ \frac{2}{3} Pr_a \frac{(\rho c_p)_{nf}}{(\rho c_p)_f} \frac{d^2 C}{dx_b^2} F' X_a + \frac{(k_{nf}/k_f)(\rho_{nf}/\rho_{nf})}{(\mu_{nf}/\mu_f)} \frac{d^2 J_a}{dx_b^2} (A_a e^{-\eta_a} + B_a) + \\ \frac{\mu_{nf}}{\mu_f} Pr_a Ec_a \left(\begin{array}{l} 8 \sin^2(\theta_a) \cos(2x_b) F''^2 + 16 \sin^2(\theta_a) \sin(2x_b) F'' K_a'' + \\ 8 \sin^2(\theta_a) \sin^2(x_b) F'' N_a'' + 8 \sin^2(\theta_a) \sin^2(x_b) K_a''^2 + \\ 2 \cos^2(\theta_a) G' R_a' + 2 \cos^2(\theta_a) h_a'^2 \end{array} \right) = 0, \quad (53)$$

$$\left\{ \begin{array}{l} \text{At } \eta_a = 0 : \frac{k_{nf}}{k_f} S'(x_b, 0) = -Bi_a L_a(x_b)(1 - S(x_b, 0)), \\ \frac{k_{nf}}{k_f} X_a'(x_b, 0) = -Bi_a \frac{dL_a}{dx_b}(1 - S(x_b, 0)) + Bi_a L_a(x_b) X_a(x_b, 0), \\ \frac{k_{nf}}{k_f} H_a'(x_b, 0) = -Bi_a \frac{d^2 L_a}{dx_b^2}(1 - S(x_b, 0)) + 2Bi_a \frac{dL_a}{dx_b} X_a(x_b, 0) + Bi_a L_a(x_b) H_a(x_b, 0), \\ \text{As } \eta_a \rightarrow \infty : S(x_b, \infty) \rightarrow 0, X_a(x_b, \infty) \rightarrow 0, H_a(x_b, \infty) \rightarrow 0. \end{array} \right. \quad (54)$$

In which

$$\begin{aligned} \beta_a(x_b) &= -2 \tan^2 x_b, \frac{d\beta_a}{dx_b} = -4 \tan x_b \sec^2 x_b, \frac{d^2\beta_a}{dx_b^2} = -4(2 \tan^2 x_b \sec^2 x_b + \sec^2 x_b), \\ q_a(x_b) &= \frac{\sin x_b}{3 \cos^2 x_b}, \frac{dq_a}{dx_b} = \frac{\cos^3 x_b + \sin 2x_b \sin x_b}{3 \cos^4 x_b}, \\ \frac{d^2 q}{dx_b^2} &= \frac{\sin 4x_b + 2 \cos^2 x_b \sin 2x_b + 2 \cos^3 x_b \sin x_b + 8 \sin^2 x_b \sin 2x_b}{6 \cos^5 x_b}, \frac{d^2 C}{dx_b^2} = 2 \sec^2 x_b \tan x_b, \\ p_a(x_b) &= \frac{\sin x_b}{6 \cos^3 x_b}, \frac{dp_a}{dx_b} = \frac{\cos^2 x_b + 3 \sin^2 x_b}{6 \cos^4 x_b}, \frac{d^2 p_a}{dx_b^2} = \frac{\cos x_b \sin 2x_b + 2(\sin x_b \cos^2 x_b + 3 \sin^3 x_b)}{3 \cos^5 x_b}, \\ J_a(x_b) &= \frac{2}{3} \tan x_b, \frac{dJ_a}{dx_b} = \frac{2}{3} \sec^2 x_b, \frac{d^2 J_a}{dx_b^2} = \frac{4}{3} \sec^2 x_b \tan x_b, C(x_b) = \tan x_b, \frac{dC}{dx_b} = \sec^2 x_b, \end{aligned}$$

and

$$\begin{aligned} L_a(x_b) &= \sqrt{\frac{2}{3}} \frac{\sqrt{\sin x_b}}{\cos x_b}, \frac{dL_a}{dx_b} = \frac{1 + \sin^2 x_b}{\sqrt{6} \cos^2(x_b) \sqrt{\sin x_b}}, \\ \frac{d^2 L_a}{dx_b^2} &= \frac{\cos^2 x_b \sin(2x_b) \sqrt{\sin x_b} - \left(\frac{\cos^3 x_b}{2\sqrt{\sin x_b}} - \sqrt{\sin x_b} \sin(2x_b) \right) (1 + \sin^2 x_b)}{\sqrt{6} \cos^4 x_b \sin x_b} \end{aligned} \quad (55)$$

The terms in Equation (55) will once more be treated as known constant parameters at any fixed value of x_b , allowing Equations (43)–(54) to be viewed as a collection of connected ODEs of the similarity type. These equations were then solved numerically via a bvp4c. However, the thermal energy itself and the initial subsidiary equations are both still intact. The subsidiary secondary terms to the field of thermal and velocity are eliminated throughout the development of this system since the thermal and velocity equations are truncated twice to make the reductions of the errors. Therefore, as noted from the Equation-3 model outcomes it should be more precise from the upshots found from the Equation-1 and Equation-2 models.

4.2. Gradients

The gradients are demonstrated in the following form

$$C_F = \frac{2\mu_{nf} \left(\frac{\partial u_a}{\partial y_a} \right) \Big|_{y_a=0}}{\rho_f u_\infty^2}, \quad C_G^* = \frac{2\mu_{nf} \left(\frac{\partial w_a}{\partial y_a} \right) \Big|_{y=0}}{\rho_f u_\infty^2}, \quad Nu = \frac{-Ra k_{nf} \left(\frac{\partial T_a}{\partial y_a} \right) \Big|_{y_a=0}}{k_f (T_f - T_\infty)}. \quad (56)$$

The aforementioned Equation (56) is further simplified to the following ND form by putting the requisite mentioned NST as follows:

$$\begin{aligned} \text{Re}^{1/2} C_F &= 4\sqrt{3} \frac{\mu_{nf}}{\mu_f} \frac{\cos^2 x_b}{\sqrt{\sin x_b}} F_{\eta_a \eta_a}(x_b, 0), \quad \text{Re}^{1/2} C_G^* = 2 \frac{\mu_{nf}}{\mu_f} \cot \theta_a \frac{\sqrt{3} \cos x_b}{\sqrt{\sin x_b}} G_{\eta_a}(x_b, 0), \\ \text{Re}^{-1/2} Nu &= -\frac{k_{nf}}{k_f} \frac{\sqrt{3} \cos x_b}{\sqrt{\sin x_b}} S_{\eta_a}(x_b, 0), \end{aligned} \quad (57)$$

where $\text{Re}^{1/2} C_F$ signifies the friction factor in the chord-wise direction and $\text{Re}^{1/2} C_G^*$ signifies in the span-wise direction, whilst the reduced heat transfer (RHT) represents $\text{Re}^{-1/2} Nu$ at the cylinder surface.

5. Analysis of the Results

The main goal of the current investigation was to produce precise numerical upshots for the mixed convective water-based graphene oxide nanoparticle flow via a YC experiencing the significant influence of the convective boundary conditions, viscous dissipation, and irregular heat source/sink. The impact of physical parameters such as solid nanoparticle volume fraction φ , yaw angle θ_a , mixed convection or buoyancy parameter Ri_a , Biot number Bi_a , heat source/sink parameter A_a , B_a , and the Eckert number Ec_a was profoundly measured when evaluating the given problem. Tables 1 and 2 represent the correlations and the experimental thermophysical data of the graphene oxide nanoparticles and the base (water) fluid, respectively. On the other hand, Table 3 shows the comparison of the given results obtained by the LNSSM (or the Equation-3 model) for the several values of the x_b and Ri_a with those of Kumari and Nath [31] in the absence of the yaw angle, viscous dissipation, convective boundary conditions, irregular heat source/sink parameter, and the nanoparticles volume fraction when $Pr_a = 0.7$. It was found that the results were extremely harmonious, which validates the current numerical scheme and the procedure's approach. Further, the utility of the new solution approach was strongly supported by a given exceptional comparison of the current and published results as shown in Table 3. Thus, it gives us confidence that the technique employed in the existing problem is accurate and also the new results found in the model are precisely useful and correct. However, the influence of these parameters on the velocity profiles in the chordwise and spanwise directions, temperature profiles, shear stresses in the chordwise and spanwise directions, and the heat transfer rate is graphically depicted in Figures 2–11, respectively. Meanwhile, the new quantitative data of the reduced shear stresses in both directions and heat transfer for the instance of buoyancy-assisting flow (BAF) and buoyancy-opposing flow (BOF) are given in the respective Tables 4–6. The value of the Prandtl number Pr_a is considered to be 6.2, which signifies water and is treated to be a fixed constant throughout the numerical scrutiny (see Refs. [32–35]). Additionally, the other influential parameters such as $\varphi = 0.025$, $Bi_a = 0.5$, $\theta_a = 15^\circ$, $x_b = 0.5$, $Ec_a = 0.5$, $A_a = 0.1$ and $B_a = 0.1$ are executed as a default in the whole or complete numerical simulations. The black solid lines in all the graphs signify the outcomes for the case of BAF while the red solid lines denote the case of solutions for the BOF.

Table 3. Assessment of $Re^{1/2}C_F$ and $Re^{-1/2}Nu$ with the results of [31] when $\varphi = 0$, $\theta_a = 0^\circ$, $Ec_a = 0$, $Bi_a = 0$, $A_a = 0$, $B_a = 0$ and $Pr_a = 0.7$.

x_b	Ri_a	Kumari and Nath [31]		Current Results	
		$Re^{1/2}C_F$	$Re^{-1/2}Nu$	$Re^{1/2}C_F$	$Re^{-1/2}Nu$
0.0	0.0	1.3281	0.5854	1.3281	0.5854
	1.0	4.9663	0.8219	4.9663	0.8219
	2.0	7.7119	0.9302	7.7119	0.9302
1.0	0.0	1.9167	0.8666	1.9167	0.8666
	1.0	5.2578	1.0617	5.2578	1.0617
	2.0	7.8863	1.1685	7.8863	1.1685
2.0	0.0	2.3975	1.0963	2.3975	1.0963
	1.0	5.6993	1.2712	5.6993	1.2712
	2.0	8.3555	1.3741	8.3555	1.3741

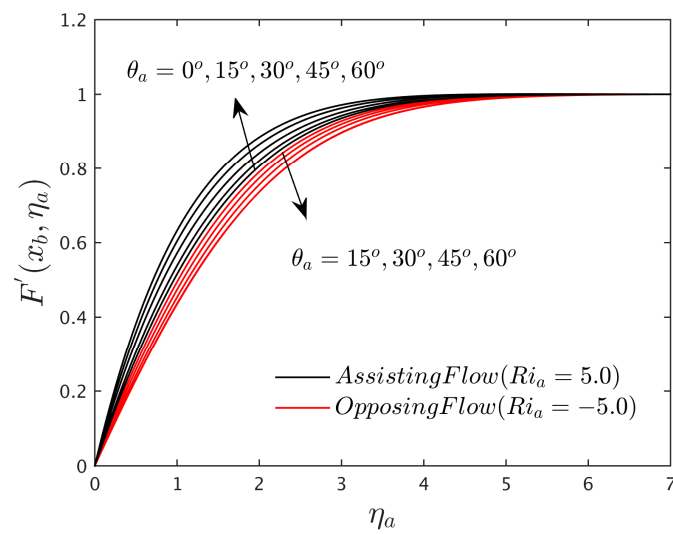


Figure 2. Influence of θ_a on $F'(x_b, \eta_a)$.

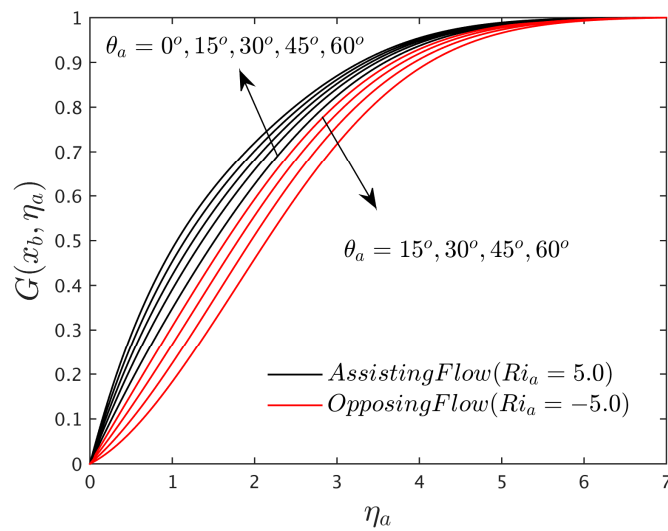


Figure 3. Influence of θ_a on $G(x_b, \eta_a)$.

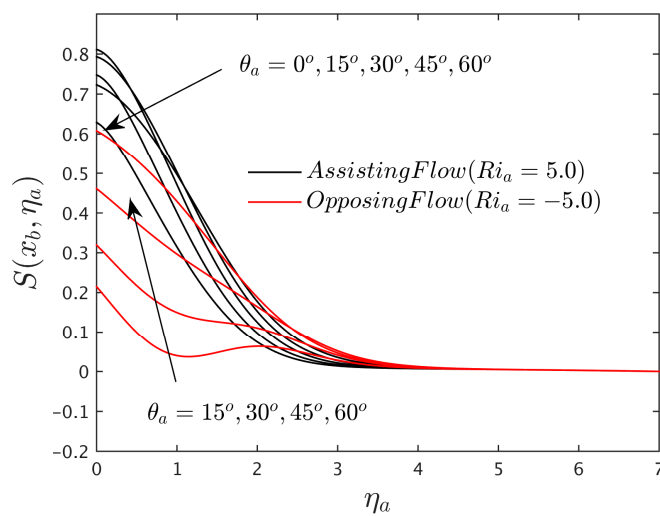


Figure 4. Influence of θ_a on $S(x_b, \eta_a)$.

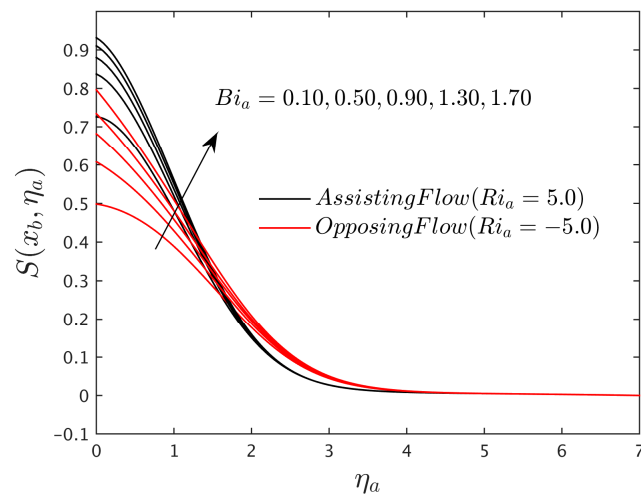


Figure 5. Influence of Biot number on $S(x_b, \eta_a)$.

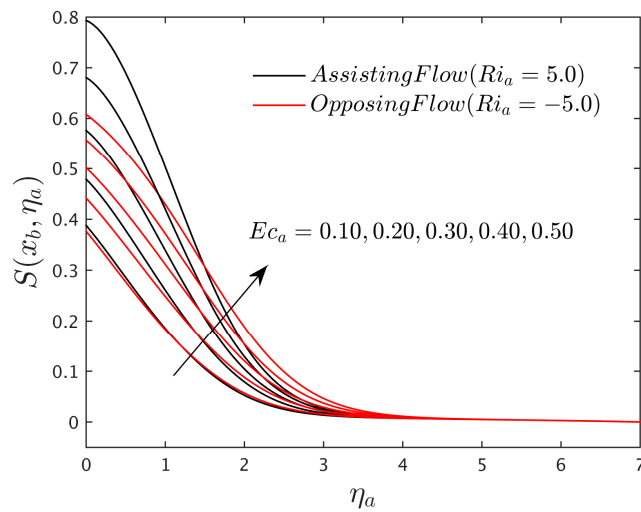


Figure 6. Influence of Eckert number on $S(x_b, \eta_a)$.

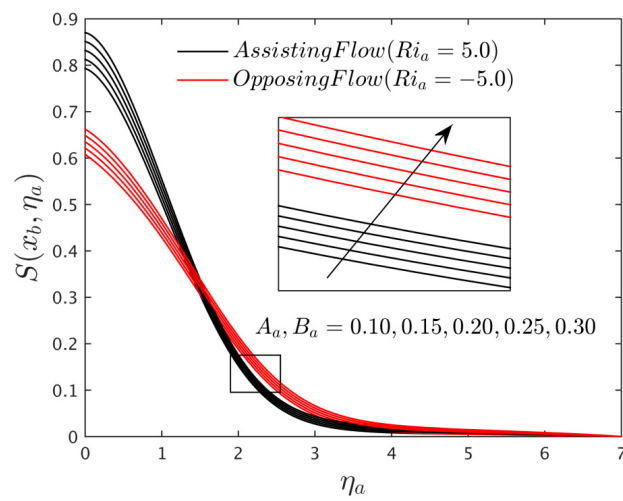


Figure 7. Influence of heat source factor on $S(x_b, \eta_a)$.

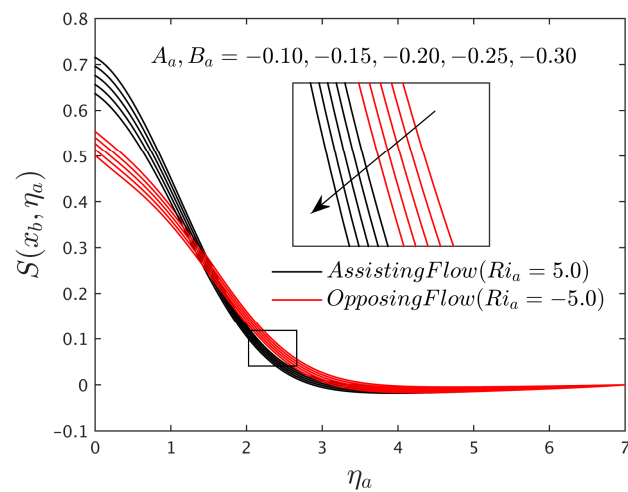


Figure 8. Influence of heat sink factor on $S(x_b, \eta_a)$.

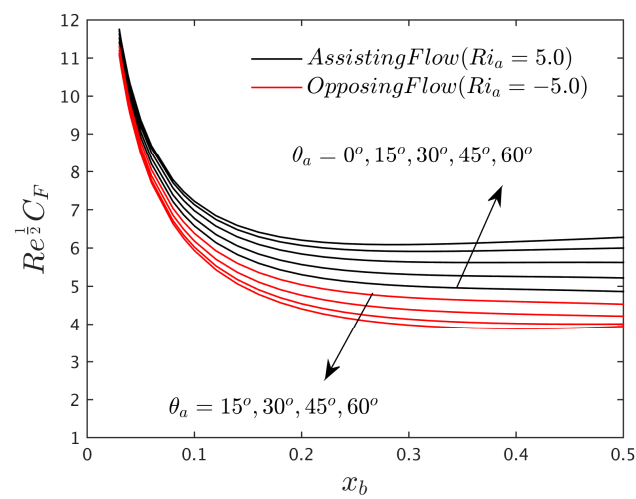


Figure 9. Influence of yaw angle on $Re^{1/2} C_F$.

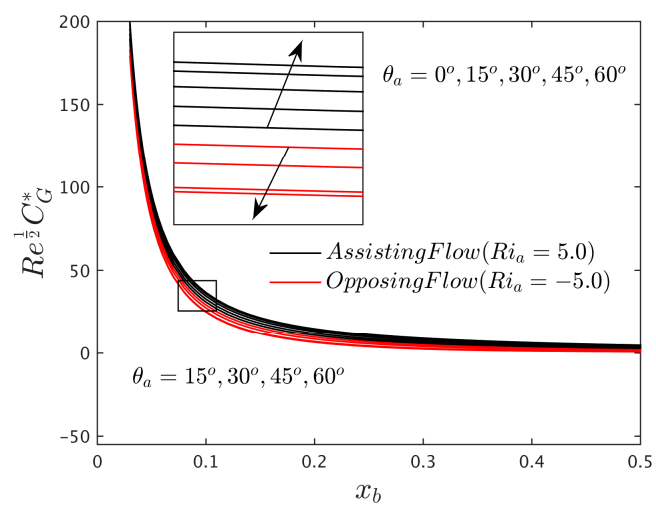


Figure 10. Influence of yaw angle on $Re^{1/2} C_G^*$.

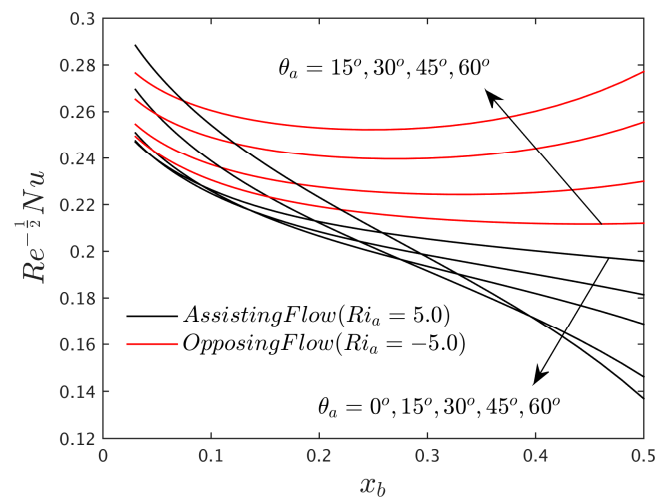


Figure 11. Influence of yaw angle on $Re^{-1/2}Nu$.

Table 4. Numerical values of $Re^{1/2}C_F$ and $Re^{1/2}C_G^*$ for the instance of BAF with variation in the values of φ and θ_a when $x_b = 0.5$, $A_a = 0.1$, $B_a = 0.1$, $Ec_a = 0.5$ and $Bi_a = 0.5$.

φ	θ_a	$Re^{1/2}C_F$	$Re^{1/2}C_G^*$
0.0	15°	5.6243618	3.5905573
0.030	-	5.6673802	3.6138515
0.035	-	5.7109067	3.6374371
0.025	15°	5.6243618	3.5905573
-	30°	6.2796404	4.6663076
-	45°	6.6225145	5.2286145

Table 5. Numerical values of $Re^{1/2}C_F$ and $Re^{1/2}C_G^*$ for the instance of BOF with variation in the values of φ and θ_a when $x_b = 0.5$, $A_a = 0.1$, $B_a = 0.1$, $Ec_a = 0.5$ and $Bi_a = 0.5$.

φ	θ_a	$Re^{1/2}C_F$	$Re^{1/2}C_G^*$
0.0	15°	4.2166338	1.2069270
0.030	-	4.2558146	1.2235525
0.035	-	4.2954308	1.2403568
0.025	15°	4.2166338	1.2069270
-	30°	4.0114303	0.8309678
-	45°	3.9431649	0.6997625

Table 6. The latest computational outputs of the RHT for the instances of BAF and BOF with variation in the values of several influential parameters when $x_b = 0.5$, $A_a = 0.1$, and $B_a = 0.1$.

φ	θ_a	Ec_a	Bi_a	Assisting Flow	Opposing Flow
0.025	15°	0.5	0.5	0.1448552	0.2758385
0.030	-	-	-	0.1455113	0.2765296
0.035	-	-	-	0.1461382	0.2771948
0.025	15°	0.5	0.5	0.1448552	0.2758385
-	30°	-	-	0.1368615	0.3102864
-	45°	-	-	0.1333446	0.3809967
0.025	15°	0.1	0.5	0.4327421	0.4409451
-	-	0.3	-	0.2996868	0.3525156
-	-	0.5	-	0.1461382	0.2758385
0.025	15°	0.5	0.1	0.0386987	0.0709445
-	-	-	0.3	0.1003166	0.1871035
-	-	-	0.5	0.1461382	0.2771948

Tables 4 and 5 illustrate the latest computational values of the reduced shear stresses (RSS) in both paths (chordwise and spanwise) for the case of BAF ($Ri_a = 5.0$) and OFs ($Ri_a = -5.0$) with diverse values of φ and θ_a when $x_b = 0.5$, $A_a = 0.1$, $B_a = 0.1$, $Pr_a = 6.2$, $Ec_a = 0.5$ and $Bi_a = 0.5$. Outcomes in the tabular form demonstrate that the RSS escalated for the case of BAF with varying values of φ and θ_a . On the other hand, for the case of BOF, the behavior of the reduced shear stresses looks similar to the case of BAF owing to the heightened values of φ but declined in both (CW and SW) directions for the higher values of yaw angle. Moreover, the shear stresses in both (chordwise and spanwise) directions upsurged by almost 0.76% and 0.65% for the case of BAF due to the deviations in the values of φ while it increased up to 11.65% and 29.96% with superior impacts of yaw angle. For the case of BOF, the RSS increased by almost 0.93% and 1.38% in the CW and SW directions due to the augmentation in the selected values of φ but decreased by almost 4.87% and 31.15% for the advanced values of θ_a .

Table 6 elucidates the numerical outputs of the RHT for the case of BAF and BOF with varying several parameters when $x_b = 0.5$, $A_a = 0.1$, and $B_a = 0.1$. Note that the results demonstrate that the heat transfer upsurging for both cases is due to the superior impact of the constraint φ and Bi_a while it is diminished for the case of BAF and BOF with upsurging values of Ec_a . Additionally, the larger values of the yaw angle decreased the importance of heat transfer for the phenomenon of BAF while it abruptly escalated for the case of BOF. percentage-wise, the heat transfer escalated almost 0.45% and 59.18% for the case of BAF due to augmentation in the impacts of φ and Bi_a while for the case of BOF, it WAS enhanced up to 0.25% and 63.79%, respectively. Additionally, the rate of heat transfer decreased up to 5.52% and 30.76% for the situation of BAF with deviation in the values of yaw angle and Eckert number, respectively, while it improved up to the level of 12.47% for the case of BOF but decreased by almost 20.05%.

Figures 2–4 portray the impression of the yaw angle θ_a on the velocity field in both (chordwise and spanwise) directions and the temperature profiles of the water-based graphene oxide nanoparticles against η_a , respectively. Results were generated for the instance of BAF and BOF cases. Growing values of θ_a augmented the velocity curves in the CW direction as well as in the SW direction for the case of BAF while the flow behavior in both graphs (see Figures 3 and 4) behaved oppositely for the situation of OFs. In a general scenario, the cylinder inclines as the yaw angle rises, allowing the fluid to move quickly alongside the cylinder simultaneously. The cylinder's angular orientation, which affects the fluid's inner pressure, is the reason for this propensity. Furthermore, the CW direction and surface friction are both made worse by the cylinder's slope or gradient. When the yaw angle is zero, the nanofluid curves converge to a single curve which shows the ideal vertical cylinder. As a result, as the yaw angle increases, and it is possible to see more fluctuations/oscillations away from the wall than at the surface. Alternatively, the temperature curves decline slightly for the case of BAF due to the higher impacts of the yaw angle. Meanwhile, the behavior significantly progresses for the case of BOF, as shown graphically in Figure 4. Additionally, the gap in both solution curves is clear by increasing the impression of the yaw angle.

The impact of the Biot number Bi_a and the Eckert number Ec_a on the temperature distribution profiles of the water-based graphene oxide nanoparticles for the case of BAF and BOF are graphically depicted in Figures 5 and 6, respectively. It is shown that the ND temperature profile curves developed for both cases due to the higher implementation of Bi_a and Ec_a parameters. Generally, it can be noticed that the convective heating upsurged with increasing the values of the Bi_a , i.e., $Bi_a \rightarrow \infty$ stimulates the isothermal surface, shown in Figure 5, where, $S(x_b, 0) = 1$, as $Bi_a \rightarrow \infty$. The ND temperature difference between the surface and the nanofluid became stronger when the Biot number was higher because strong surface convection generated more heat that was transferred to the cylinder's surface. As a result, the TTBL and the temperature was augmented with superior impacts of the Biot number. In contrast, growing the Eckert number increased the fluid temperature

because positive values of the Eckert number indicate an increment in kinetic energy, which contributes to the growing temperature distribution, as shown graphically in Figure 6.

Figures 7 and 8 indicate the temperature profiles of the water-based graphene oxide nanoparticles for the circumstance of BAF and BOF due to the variation in the internal heat source/sink factor, respectively. Outcomes reveal that the temperature solution curves escalated for both cases due to superior values of the internal heat source factor (IHSF) while decreased for both phenomena because of the higher internal heat sink factor. Moreover, the thickness of the temperature boundary layer was thicker in the case of BAF compared with the BOF. Physically, the rise of the IHSF generates more energy inside the boundary layer that causes the increment of the fluid temperature, see Figure 7. Meanwhile, the production of less energy inside the boundary layer is observed to determine the influence of the internal heat sink factor. As a result, the higher the internal heat sink factor influences the temperature deceleration, see Figure 8.

The influence of θ_a on the reduced shear stresses in both (chord-wise and span-wise) paths and the RHT versus x_b for the cases of BAF and BOF is illustrated in Figures 9–11, respectively. From Figures 9 and 10, it is practically seen that the shear stresses in the CW and SW directions increased for the situation of BAF while decreased for the case of BOF due to the higher employing values of the yaw angle. In addition, the shear stress is highly observed for the circumstance of BAF compared with the situation of the BOF. Additionally, for the case of BAF, the shear stresses in both directions increased by almost 11.65% and 29.96%, respectively, due to the higher values of yaw angle. However, the shear stresses decreased by up to 4.87% and 31.15% for the phenomenon of BOF. In contrast, the reduced heat transfer decelerated for the condition of BAF and increased for the BOF condition owing to advance values of the yaw angle θ_a (see Figure 11). Consequently, the reduced heat transfer upsurged by 12.47% for the case of BOF due to the higher impressions of the yaw angle but decreased by 5.52% for the case of BAF.

6. Conclusions

This study considered numerical solutions to the problem of a buoyancy or mixed convection nanofluid flow through a yawed cylinder with irregular heat sink/source effects. Additionally, the model comprised the impact of convective boundary conditions and viscous dissipation. The approach for finding LNSSs was exercised in the investigation. This technique's key characteristic is the retention of NS terms in the obtained equations without approximations. In that, the influence of varying strength of yaw angle, Biot number, the volume fraction of nanoparticle, Eckert number, internal heat sink/source factor, and the Richardson number on the velocity and shear stresses profile in both (CW and SW) directions, the non-dimensional temperature and heat transfer were analyzed employing distinct graphs and tables. After taking into consideration this thorough analysis, the results are described below:

- With variation in all the physical parameters, the solutions were obtained for the fixed constant values in the case of assisting and opposing flows.
- For enlarging magnitudes of the nanoparticles volume fraction, the shear stresses increased by about 0.76% and 0.93% for the instance of BAF, while for the case of BOF, they increased by almost 0.65% and 1.38%, respectively.
- Velocity and shear stress enhanced the case of assisting flow in both (chordwise and spanwise) directions due to the larger influences of the yaw angle, but both profile curves were diminished for the case of opposing flow.
- Shear stress decreased by almost 4.87% while it increased up to 11.65% for the case of opposing and assisting flows in the respective chordwise direction due to the higher impressions of yaw angle. However, it decreased and inclined spanwise by about 31.15% and 29.96%, respectively.
- The non-dimensional temperature upsurged for both assisting and opposing flow cases due to the higher values of Eckert and Biot numbers.

- For increasing values of Eckert number, the heat transfer rates decayed by about 30.76% and 20.05% for the phenomenon of AFs and OFs while increasing by around 0.45% and 0.25% with nanoparticles volume fraction.

Author Contributions: Conceptualization, S.E. and U.K.; methodology, S.E. and U.K.; software, S.E., A.M.A. and U.K.; validation, A.I., S.E., A.Z., S.A.B., S.K.S. and U.K.; formal analysis, A.Z., S.E. and A.M.A.; investigation, U.K. and A.M.A.; resources, A.I.; data curation, A.Z.; writing—original draft preparation, A.I., A.Z., U.K. and S.E.; writing—review and editing, A.I., A.Z., S.A.B., S.K.S. and A.M.A.; visualization, S.E.; supervision, A.I.; project administration, A.M.A. and A.I.; funding acquisition, A.I. and A.M.A. All authors have read and agreed to the published version of the manuscript.

Funding: This work was funded by Universiti Kebangsaan Malaysia (DIP-2020-001). Also, this work received support from Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2023R163), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia. In addition, this study is also funded by Prince Sattam bin Abdulaziz University project number (PSAU/2023/R/1444).

Data Availability Statement: Not applicable.

Acknowledgments: The authors are thankful for the support of Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2023R163), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia. Also, this work is supported via funding from Prince Sattam bin Abdulaziz University project number (PSAU/2023/R/1444).

Conflicts of Interest: The authors declare no conflict of interest.

Nomenclature

A_a	Exponentially decaying space coefficient
B_a	Temperature-dependent heat sink/source coefficient
Bi_a	Biot number
C_F	Friction factor in the chord-wise direction
C_G^*	Friction factor in the span-wise direction
Ec_a	Eckert number
G, F	Non-dimensionless velocities in span-wise and chord-wise directions, respectively
g	Gravity acceleration (m s^{-2})
h_f	Convective heat transfer
k_f	Thermal conductivity of base fluid ($\text{W m}^{-1} \text{K}^{-1}$)
k_{nf}	Thermal conductivity of nanofluid ($\text{W m}^{-1} \text{K}^{-1}$)
k_{snp}	Thermal conductivity of nanoparticles ($\text{W m}^{-1} \text{K}^{-1}$)
Nu	Nusselt number
Pr_a	Prandtl number
Q^{bbb}	Irregular heat source/sink
R_a	Radius of the cylinder (m)
Ri_a	Mixed convective parameter
Re	Reynolds number
S	Non-dimesional temperature
T_a	Temperature of the nanofluid (K)
T_∞	Free-stream temperature (K)
T_f	Temperature of the hot fluid (K)
u_a	Velocity projection for the chordwise direction (m s^{-1})
v_a	Velocity projection for the normal direction (m s^{-1})
w_a	Velocity projection for the spanwise direction (m s^{-1})
w_E, u_E	Variable Free-stream velocities (m s^{-1})
w_∞, u_∞	Free-stream velocities in z - and x - directions (m s^{-1})
(x_a, y_a, z_a)	Curvilinear coordinates (m)

Greek symbols

β_{nf}	Nanofluid thermal expansion (K^{-1})
β_f	Base fluid thermal expansion (K^{-1})
β_s	Nanoparticle thermal expansion (K^{-1})
η_a, ξ_a	Transformed variables
μ_{nf}	Nanofluid dynamic viscosity ($kg\ m\ s^{-1}$)
μ_f	Base fluid dynamic viscosity ($kg\ m\ s^{-1}$)
ν_{nf}	Nanofluid kinematic viscosity ($m^2\ s^{-1}$)
ν_f	Kinematic viscosity of base fluid ($m^2\ s^{-1}$)
ρ_{nf}	Nanofluid density ($kg\ m^{-3}$)
ρ_{snp}	Nanoparticles density ($kg\ m^{-3}$)
ρ_f	Density of base fluid ($kg\ m^{-3}$)
$(\rho c_p)_{nf}$	Nanofluid heat capacity ($Jkg^{-1}\ K^{-1}$)
φ	Volume fraction of nanoparticles
ψ, F	Dimensional and non-dimensional stream functions

Acronyms

BAF	Bouyancy assisting flow
BCs	Boundary conditions
BOF	Bouyancy opposing flow
CBCs	Convective boundar conditions
CW	Chordwise
GO	Graphene oxide
LNS	Local non-similarity
MCN	Mixed convection
NDL	Non-dimensional
ODEs	Ordinary differential equations
PDEs	Partial differential equations
RHT	Rate of heat transfer
SP	Spanwise
TTBL	Thickness of the thermal boundary layer
YC	Yawed Cylinder

Subscripts

f	Condition at free stream
nf	Nanofluid
snp	Solid nanoparticles
w	Wall boundary condition
∞	Free-stream condition

Superscripts

$'$	Derivative w.r.t. η
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