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Astrophysical Neutrinos in Testing Lorentz Symmetry

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Abstract: An overview of searches related to neutrinos of astronomical and astrophysical origin performed within the framework of the Standard-Model Extension is provided. For this effective field theory, key definitions, intriguing physical consequences, and the mathematical formalism are summarized within the neutrino sector to search for effects from a background that could lead to small deviations from Lorentz symmetry. After an introduction to the fundamental theory, examples of various experiments within the astronomical and astrophysical context are provided. Order-of-magnitude bounds of SME coefficients are shown illustratively for the tight constraints that this sector allows us to place on such violations.

Keywords: Lorentz violation; models beyond the standard model; neutrino interactions; cosmic-ray interactions; elementary particle processes; relativity and gravitation

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1. Introduction

The neutrino astronomy journey began in 1965 with the solar neutrino detector in the Homestake mine in South Dakota [1], contributing to the earliest questions related to neutrinos. The properties that make neutrinos specifically interesting for astrophysical searches is that they have low mass, have no charge, react very weakly with other particles, and are not affected by magnetic fields in their propagation. They can escape from dense astrophysical environments such as the interior of the sun.

Neutrinos have been observed to arrive on Earth from the sun (solar) and from astrophysical objects besides the sun such as supernovae or active galactic nuclei (AGN), or they are generated from cosmic rays interacting with the atmosphere. The neutrinos produced by interactions of ultrahigh-energy cosmic rays with surrounding photons or matter are referred to as astrophysical neutrinos and constitute the main focus of this work. They could be messengers delivering information about extragalactic accelerators.

Neutrino physics is popular and has seen much change over the last century. The elusive particle is still producing surprises and anomalies, making it a worthwhile area of research; see for example [2]. Neutrinos are found in all sections of particle physics research, and in the case of astroparticle physics, the field is just beginning to expand. Neutrinos belong to the few types of particles that can convey information from less-known distant places and can access much higher energy ranges.

They are also an excellent tool for studying space-time subtleties such as Planck-scale effects related to different quantum gravity or unification scenarios beyond the Standard Model (SM) and General Relativity (GR). Such effects can arise in string theory [3–11], other quantum-gravity scenarios [12–26], noncommutative field theories [27–33], random dynamics [34], or multiverses [35]. One of the ideas in the string theory context was to explore possible space-time symmetry breaking motivated by the presence of tensor-valued backgrounds and to introduce minute relativity violations [3–11].

At low energies compared to the Planck scale, the SM is an excellent fit to nearly all observable processes; therefore, in the framework developed, this Lorentz-symmetry breaking is assumed to be highly suppressed. True to the idea of small violations at low energy, the theory is developed in the form of an effective quantum field theory called the Standard-Model Extension (SME), based on the SM [7–11,36,37] and GR [38–47]. The fundamentals of the extension have been theoretically developed for spontaneous and explicit symmetry breaking for various space-time metrics [25,48–58].

The SME only breaks particle Lorentz symmetry but keeps the gauge-symmetry structure of the SM in place. The Dirac fermion sector of the nonminimal SME presented here contains partial derivatives that translate to a lack of gauge symmetry. In studies of neutrino propagation where interactions can be neglected, that approximation is acceptable. The fully gauge-invariant minimal SME is presented in [37], and the nonminimal SME is published in [59,60].

In its minimal version, the SME is power-counting renormalizable (mSME). Lorentz violation (LV) is a feature that occurs only as an effect due to particle transformations of local fields compared to a symmetry-breaking background. Observer Lorentz symmetry is maintained. Here, all couplings to the background are assumed to be independent of space-time position, leading to energy–momentum conservation.

To date, the SME and frameworks alike examining Lorentz- and CPT-violation have grown into their own field, convincing experimentalists to conduct precision searches in every sector of physics. Due to the best bounds being obtained with neutrinos, this sector is at the forefront of this research. Most of the findings are cataloged within the SME framework and the substantial amount of data collected compiles into appropriate tables [61].

On the theoretical front, there is a steady expansion of ideas that now provide deeper understanding of spontaneous LV and CPTV as well as new geometries that yield a different space-time symmetry structure. These considerations merge into insightful results in general relativity and quantum field theory, broadening the understanding of symmetry breaking in general. It leads to discovering theoretical and experimental boundaries of Lorentz invariance and places its possible violation into connection with other new physics extensions of the SM and GR.

The neutrino sector is an excellent illustration of the SME itself, as we will see below. Its relation to neutrino astroparticle physics is centered around flavor and kinematic studies, the latter mostly in a flavor-blind and oscillation-free model. However, the interferometric oscillations of neutrinos influenced by Lorentz violation are also included in a brief discussion for their theoretical basics and usefulness in providing comparative data.

Neutral particle interferometry was one of the first in searches for CPT and Lorentz violation, with studies beginning in neutral-meson oscillations [9–11,62–83] providing early stringent constraints on the size of such symmetry-breaking effects. Neutrino physics brought substantial improvements to that. The wide field of neutrino oscillations over far larger propagation distances and over a much bigger energy spectrum, and the neutrino's role in interactions contributed to even better constraints for their sector, having been improved by many orders of magnitude.

This article cannot hope to explain all of these scenarios even at the most basic level. The discussions are taken only far enough to summarize a phenomenological core connecting to astrophysical and astronomical neutrino experiments. Earth-based experiments and theory are introduced only insofaras their results support the astrophysical areas. The goal is to provide a concise summary of the basic setup of this sector.

The assumption is made that observable Lorentz-violating effects at attainable energies by means of the well-tested SM and GR should be trackable with an all-general extension regardless of the new physics proposed as the source of the effect. Here, the appropriate operators and coefficients of the SME are described.

This paper almost exclusively presents studies and methods related to theoretically understanding and experimentally constraining coefficients of the SME framework. Other approaches to LV are not discussed. However, the SME is constructed in a way that it can match any possible situation within LV with appropriately adjusted SME coefficients. The references are correspondingly also tied to this approach; however, it is recognized that there are many other worthwhile endeavors in all aspects of these investigations.

The structure of the paper is as follows: after the Introduction, Section 2 gives the full neutrino-specific SME formalism, but in a greatly reduced way compared to the basic literature. Only the key reasonings, assumptions, and equations that serve to provide a crutch to lean on when reading the sections relevant to the astrophysical context are given. A subsection on transitioning from arbitrary operator dimensions to mass dimensions 3 and 4 is included.

Section 3 is a short overview of neutrino phenomenology. Section 4 contains all important ingredients in the area of oscillations, relevant specifics from the formalism introduced in Section 2, a description of short- and long-baseline perturbative methods, direction dependence considerations, and then some interesting details from the solar-neutrino experiments.

Section 5 is where all of the above culminate in a discussion of the astrophysical context. Its three main subsections cover flavor studies, kinematics-related methods, and atmospheric spectral analysis. The kinematics is further split into time-of-flight experiments and threshold effect studies, including the introduction of maximum attainable velocities. Section 6 provides a summary.

2. SME Framework for Neutrino Searches

This work addresses the all-general framework for the neutrino sector based on eleven important articles that laid out the theory and presented several applications and calculations of many constraints derived from then-available experimental results [84–93]. Some of the methods have been developed for the photon or general fermionic sectors but have been successfully adapted, where appropriate, to the neutrino formalism.

The full SME Lagrange density is constructed to be a coordinate-independent scalar under observer Lorentz transformations. Each term of it is a Lorentz-violating operator of a given mass dimension d contracted with a controlling coefficient forming an observer scalars [36,37,84]. Under local particle transformations, these terms break Lorentz symmetry according to the couplings described by the Lorentz-violating coefficients, reflecting a scenario where SM fields couple to a nontrivial constant space-time background causing local or global violation of Lorentz symmetry.

The goal of this section is to present an effective hamiltonian at first order in LV and mass, allowing for searches for such violations in neutrino physics. The most general Lorentz-violating theory contains any number of neutrino species with all possible Majorana- and Dirac-type couplings of left- and right-handed neutrinos, some of which also violate CPT symmetry, the combination of parity, time reversal, and charge conjugation [84,91].

These two publications cited differ when presenting the minimal and nonminimal SME. The distinction is connected to power-counting renormalizability of the effective field theory introduced. The LV operators in the minimal theory keep mass dimensions to renormalizable domains. Here, the most general effective hamiltonian is introduced, and Section 2.2 takes the formalism over into in the minimal setting.

2.1. General SME Framework for Neutrinos

The starting point is a quadratic Lagrange density for free fermion fields in a noninteracting scenario that arrives at an effective hamiltonian describing the propagation, oscillation, and mixing of three generations of left-handed neutrinos. The most general form of the theory considers Majorana-type, Dirac-type, and sterile neutrinos with Lorentz-violating operators of arbitrary dimension and neutrino generations.

This broad framework can be fitted to experimental searches in phenomenological studies. The LV effective hamiltonian is added to the conventional hamiltonian, including massive neutrinos via a standard seesaw mechanism [94–97]. Following Reference [91], the general starting Lagrange density has the following form:

$$\begin{aligned} S &= \int \mathcal{L}\, \mathrm{d}^4x, \\ \mathcal{L} &= \tfrac{1}{2}\overline{\Psi}_A(\gamma^\nu p_\nu \delta_{AB} - M_{AB} + \widehat{\mathcal{Q}}_{AB})\Psi_B + \text{h.c.} \end{aligned} \tag{1}$$

This has the structure of a kinetic term, an arbitrary mass matrix M_{AB}, and a Lorentz-violating operator $\widehat{\mathcal{Q}}_{AB}$. The latter has a component structure of a general 4×4 matrix in spinor space and a $2N \times 2N$ matrix in flavor space acting on a $2N$-dimensional multiplet of spinors Ψ_A that combines N spinors ψ_a together with their N charge-conjugates $\psi_a^C = C\overline{\psi}_a^T$.

$$\Psi_A = \begin{pmatrix} \psi_a \\ \psi_a^C \end{pmatrix}. \tag{2}$$

Hence, A ranges over $2N$ values while a ranges over N. This construct allows us to accommodate possible Majorana-type masses.

Conditions to ensure hermiticity of the Lagrangian are detailed in the relevant paper [91]. The literature includes general discussions of the space-time dependence of $\widehat{\mathcal{Q}}_{AB}$ and its origin in explicit and spontaneous symmetry breaking. Here, it is assumed that there is no significant space-time dependence, so energy and momentum are conserved. In the heart of the searches for LV is a decomposition of $\widehat{\mathcal{Q}}_{AB}$ in the basis of the 16 Dirac matrices γ_I.

$$\begin{aligned} \widehat{\mathcal{Q}}_{AB} &= \sum_I \widehat{\mathcal{Q}}^I_{AB}\gamma_I \\ &= \widehat{\mathcal{S}}_{AB} + \mathrm{i}\widehat{\mathcal{P}}_{AB}\gamma_5 + \widehat{\mathcal{V}}^\mu_{AB}\gamma_\mu + \widehat{\mathcal{A}}^\mu_{AB}\gamma_5\gamma_\mu + \tfrac{1}{2}\widehat{\mathcal{T}}^{\mu\nu}_{AB}\sigma_{\mu\nu}, \end{aligned} \tag{3}$$

The $\widehat{\mathcal{Q}}^I_{AB}$ are $2N \times 2N$ matrix operators that depend on derivatives $p_\mu = \mathrm{i}\partial_\mu$. These hermitian operators in flavor space carry the derivative-dependence through a sum of operators of definite mass dimension d,

$$\widehat{\mathcal{Q}}^I_{AB} = \sum_{d=3}^{\infty} \mathcal{Q}^{(d)I\alpha_1\alpha_2\ldots\alpha_{d-3}}_{AB} p_{\alpha_1}p_{\alpha_2}\ldots p_{\alpha_{d-3}}, \tag{4}$$

where the coefficients $\mathcal{Q}^{(d)I\alpha_1\alpha_2\ldots\alpha_{d-3}}_{AB}$ have mass dimension $4-d$ and are space-time constants corresponding to the space-time independence of $\widehat{\mathcal{Q}}_{AB}$.

There is a more practical way of presenting this decomposition that is typical throughout the treatment of fermion fields of the SME [36,37], which separates $\widehat{\mathcal{Q}}_{AB}$ in the following way:

$$\gamma^\nu p_\nu \delta_{AB} - M_{AB} + \widehat{\mathcal{Q}}_{AB} = \widehat{\Gamma}^\nu_{AB} p_\nu - \widehat{M}_{AB}. \tag{5}$$

This splits $\widehat{\mathcal{Q}}_{AB}$ into even and odd mass dimensions, allowing for a grouping of terms with similar physical content to be treated systematically in the phenomenological context. Some terms of the SME Lagrangian change sign under CPT transformations and hence are CPT-odd, while others preserve CPT and are CPT-even. The C, P, T, and CPT properties of Dirac bilinears is given for instance in [98]. In the nonminimal extension, one must also pay attention to the number of partial derivatives contained in an operator. The above separation also facilitates a clearer theoretical treatment based on CPT-even and CPT-odd operators and dealing with neutrinos and antineutrinos in certain applications.

$$\begin{aligned}\widehat{\Gamma}^{\nu}_{AB}p_{\nu} &= \gamma^{\nu}p_{\nu}\delta_{AB} + \widehat{c}^{\mu}_{AB}\gamma_{\mu} + \widehat{d}^{\mu}_{AB}\gamma_5\gamma_{\mu} \\ &\quad + \widehat{e}_{AB} + \mathrm{i}\widehat{f}_{AB}\gamma_5 + \tfrac{1}{2}\widehat{g}^{\kappa\lambda}_{AB}\sigma_{\kappa\lambda}, \\ \widehat{M}_{AB} &= m_{AB} + \mathrm{i}m_{5AB}\gamma_5 + \widehat{m}_{AB} + \mathrm{i}\widehat{m}_{5AB}\gamma_5 \\ &\quad + \widehat{a}^{\mu}_{AB}\gamma_{\mu} + \widehat{b}^{\mu}_{AB}\gamma_5\gamma_{\mu} + \tfrac{1}{2}\widehat{H}^{\mu\nu}_{AB}\sigma_{\mu\nu},\end{aligned} \tag{6}$$

where the contraction with p_{ν} in $\widehat{\Gamma}^{\nu}_{AB}p_{\nu}$ was absorbed. The CPT-contracted operators of the general case [91] correspond to those in the minimal model [84]. $\widehat{m}, \widehat{m}_5, \widehat{c}, \widehat{d}$, and $\widehat{H}$ are CPT-even, while $\widehat{a}, \widehat{b}, \widehat{e}, \widehat{f}$, and $\widehat{g}$ are CPT-odd. The appropriate operators relate back to the Dirac matrix decomposition of Equation (3) as

$$\begin{aligned}\widehat{\mathcal{S}}_{AB} &= \widehat{e}_{AB} - \widehat{m}_{AB}, \quad \widehat{\mathcal{P}}_{AB} = \widehat{f}_{AB} - \widehat{m}_{5AB}, \\ \widehat{\mathcal{V}}^{\mu}_{AB} &= \widehat{c}^{\mu}_{AB} - \widehat{a}^{\mu}_{AB}, \quad \widehat{\mathcal{A}}^{\mu}_{AB} = \widehat{d}^{\mu}_{AB} - \widehat{b}^{\mu}_{AB}, \\ \widehat{\mathcal{T}}^{\mu\nu}_{AB} &= \widehat{g}^{\mu\nu}_{AB} - \widehat{H}^{\mu\nu}_{AB}.\end{aligned} \tag{7}$$

The hamiltonian formalism can be found as the sum of a Lorentz-invariant and Lorentz-violating part starting from a modified Dirac equation and taking into account the LV effects on the time derivatives. Without the full reasoning, which is well described in the references, here, only the final form of the hamiltonian formalism is given to be adapted to the neutrino sector.

The effective $2N \times 2N$ hamiltonian H_{AB} at leading order in Lorentz violation is

$$H_{AB} = (H_0)_{AB} - \gamma_0\big(\widehat{\mathcal{S}}_{AB} + \mathrm{i}\widehat{\mathcal{P}}_{AB}\gamma_5 + \widehat{\mathcal{V}}^{\mu}_{AB}\gamma_{\mu} + \widehat{\mathcal{A}}^{\mu}_{AB}\gamma_5\gamma_{\mu} + \tfrac{1}{2}\widehat{\mathcal{T}}^{\mu\nu}_{AB}\sigma_{\mu\nu}\big)\big|_{E\to E_0}, \tag{8}$$

where $(H_0)_{AB}$ is the usual hamiltonian with conventional energy E_0 modified by a perturbative LV piece evaluated at leading order also at E_0.

The solution found in the hamiltonian formalism needs modifications to match the chirality of the SM neutrinos, so a projection is carried out onto left-handed fields. In the nonminimal approach that retains all leading-order LV terms to arbitrary mass dimensions, terms linear in neutrino mass are also included.

To incorporate Dirac- and Majorana-type masses, we make use of the fact that Ψ^C_A of Equation (2) obeys the following relation:

$$\Psi^C_A = \mathcal{C}\Psi_A, \quad \mathcal{C} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{9}$$

where $\mathcal{C}$ is a $2N \times 2N$ matrix $\mathcal{C}$ with $N \times N$ blocks in flavor space.

The left- and right-handed mass matrices relate to the full mass matrix M via the usual chiral projection operators $P_L = (1-\gamma_5)/2$ and $P_R = (1+\gamma_5)/2$ as

$$M = m_L P_L + m_R P_R, \tag{10}$$

where m_L and m_R satisfy $m_R = (m_L)^{\dagger} = m + \mathrm{i}m_5$ and $m_R = m_L^{\dagger}$.

Dirac- or Majorana-type masses can be identified by separating m_R into four $N \times N$ submatrices according to

$$m_R\mathcal{C} = \begin{pmatrix} L & D \\ D^T & R \end{pmatrix}, \tag{11}$$

where R and L refer to symmetric right- and left-handed Majorana-mass matrices, while D is the Dirac-mass matrix. All three matrices R, L, and D are complex, and R and L are symmetric.

It is assumed that either $D = 0$ signifying no mixing between right- and left-handed neutrinos or the standard seesaw mechanism accounts for suppressing such a mixing. This allows introducing an effective left-handed symmetric matrix m_l as follows:

$$m_l = L - DR^{-1}D^T. \tag{12}$$

The latter construction aligns with the experimental observation that propagating physical neutrinos are left-handed. Similarly, four $N \times N$ block-matrices following the structure of Equation (11) are found for the component operators in the expansion of $\widehat{\mathcal{Q}}_{AB}$. The Lorentz-invariant hamiltonian

$$h_0 = |\boldsymbol{p}|\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2|\boldsymbol{p}|}\begin{pmatrix} m_l m_l^\dagger & 0 \\ 0 & m_l^\dagger m_l \end{pmatrix} \tag{13}$$

is modified by a general effective LV hamiltonian as a small perturbation

$$h_{\text{eff}} = h_0 + \delta h. \tag{14}$$

Without the details found in the appropriate references, the final form of the LV effective Hamiltonian modifying the Lorentz-invariant Hamiltonian to order $O(m_l)$ is

$$\delta h = \frac{1}{|\boldsymbol{p}|}\begin{pmatrix} \widehat{a}_{\text{eff}} - \widehat{c}_{\text{eff}} & -\widehat{g}_{\text{eff}} + \widehat{H}_{\text{eff}} \\ -\widehat{g}^\dagger_{\text{eff}} + \widehat{H}^\dagger_{\text{eff}} & -\widehat{a}^T_{\text{eff}} - \widehat{c}^T_{\text{eff}} \end{pmatrix}, \tag{15}$$

where conjugation and transposition are flavor-space operations.

One additional consideration is the polarization of the neutrino. Using arbitrary unit vectors $\hat{\boldsymbol{e}}_1$ and $\hat{\boldsymbol{e}}_2$, the polarization vector ϵ^μ is defined as

$$\epsilon^\mu = \tfrac{1}{\sqrt{2}}(0; \hat{\boldsymbol{e}}_1 + \mathrm{i}\hat{\boldsymbol{e}}_2), \quad \epsilon_\mu(-\boldsymbol{p}) = \epsilon^*_\mu(\boldsymbol{p}), \tag{16}$$

where $\{\hat{\boldsymbol{p}}, \hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2\}$ form a right-handed orthonormal triad. The $N \times N$ Hamiltonian blocks are listed below according to whether they are CPT-even or -odd. The CPT-odd and even property of these terms can be inferred by examining the combinations in which they occur. The CPT-even and -odd behavior of the operators involved are described in the context of Equation (6) above. The CPT-odd parts take the following form:

$$\begin{aligned} \widehat{a}_{\text{eff}} &= p_\mu \widehat{a}^\mu_L - \widehat{e}_l + 2\mathrm{i}\epsilon_\mu\epsilon^*_\nu \widehat{g}^{\mu\nu}_l, \\ \widehat{g}_{\text{eff}} &= \mathrm{i}\sqrt{2}\, p_\mu \epsilon_\nu \widehat{g}^{\mu\nu}_{M+} + \sqrt{2}\,\epsilon_\mu \widehat{a}^\mu_l, \end{aligned} \tag{17}$$

while the CPT-even terms are

$$\begin{aligned} \widehat{c}_{\text{eff}} &= p_\mu \widehat{c}^\mu_L - \widehat{m}_l + 2\mathrm{i}\epsilon_\mu\epsilon^*_\nu \widehat{H}^{\mu\nu}_l, \\ \widehat{H}_{\text{eff}} &= \mathrm{i}\sqrt{2}\, p_\mu \epsilon_\nu \widehat{H}^{\mu\nu}_{M+} + \sqrt{2}\,\epsilon_\mu \widehat{c}^\mu_l. \end{aligned} \tag{18}$$

The following operators are independent of the mass matrix m_l:

$$\begin{aligned} \widehat{a}^\mu_L &= \widehat{a}^\mu_D + \widehat{b}^\mu_D, \quad \widehat{g}^{\mu\nu}_{M+} = \tfrac{1}{2}(\widehat{g}^{\mu\nu}_M + \mathrm{i}\widetilde{\widehat{g}}^{\mu\nu}_M), \\ \widehat{c}^\mu_L &= \widehat{c}^\mu_D + \widehat{d}^\mu_D, \quad \widehat{H}^{\mu\nu}_{M+} = \tfrac{1}{2}(\widehat{H}^{\mu\nu}_M + \mathrm{i}\widetilde{\widehat{H}}^{\mu\nu}_M). \end{aligned} \tag{19}$$

The operators linear in m_l are summarized as

$$
\begin{aligned}
\widehat{m}_l &= \tfrac{1}{2}\big(\widehat{m}_M + \mathrm{i}\widehat{m}_{5M}\big) m_l^\dagger + \tfrac{1}{2} m_l \big(\widehat{m}_M + \mathrm{i}\widehat{m}_{5M}\big)^\dagger, \\
\widehat{a}_l^\mu &= \tfrac{1}{2}\widehat{a}_L^\mu m_l + \tfrac{1}{2} m_l \big(\widehat{a}_L^\mu\big)^T, \\
\widehat{c}_l^\mu &= \tfrac{1}{2}\widehat{c}_L^\mu m_l - \tfrac{1}{2} m_l \big(\widehat{c}_L^\mu\big)^T, \\
\widehat{e}_l &= \tfrac{1}{2}\big(\widehat{e}_M + \mathrm{i}\widehat{f}_M\big) m_l^\dagger + \tfrac{1}{2} m_l \big(\widehat{e}_M + \mathrm{i}\widehat{f}_M\big)^\dagger, \\
\widehat{g}_l^{\mu\nu} &= \tfrac{1}{2}\widehat{g}_{M+}^{\mu\nu} m_l^\dagger + \tfrac{1}{2} m_l \big(\widehat{g}_{M+}^{\mu\nu}\big)^\dagger, \\
\widehat{H}_l^{\mu\nu} &= \tfrac{1}{2}\widehat{H}_{M+}^{\mu\nu} m_l^\dagger + \tfrac{1}{2} m_l \big(\widehat{H}_{M+}^{\mu\nu}\big)^\dagger.
\end{aligned} \tag{20}
$$

The notation employed is explained as follows: The caret on top signifies a sum of operators with definite mass dimension involving derivatives $\mathrm{i}\partial_\mu$ in the same expansion as shown for $\widehat{\mathcal{Q}}_{AB}^I$. The variables D and M stand for show Majorana- and Dirac-like blocks in the respective matrices, while l indicates combinations containing m_l.

These operators and coefficients are somewhat involved but take on practical meaning when examined in the context of experimentation. In what follows, the applications show appropriately simplified forms of the full model adapted to the relevant physics.

One practical presentation of the coefficients is the spherical-harmonics decomposition focusing on the significance of rotations and searches for isotropy violations. These lead to tight bounds when applied to certain particle processes at cosmic or galactic scales.

For neutrinos, the p_μ-dependent combinations of δh of Equation (15) are expanded in spherical harmonics. The terms of the diagonal blocks of δh are rotational scalars expanded via standard spherical harmonics $Y_{jm} \equiv {}_0Y_{jm}$. These involve six types of coefficients: $\big(a_L^{(d)}\big)_{jm}^{ab}$,$\big(c_L^{(d)}\big)_{jm}^{ab}$ with mass dimension $4-d$, and coefficients $\big(m_l^{(d)}\big)_{jm}^{ab}$, $\big(e_l^{(d)}\big)_{jm}^{ab}$, $\big(g_l^{(d)}\big)_{jm}^{ab}$, and $\big(H_l^{(d)}\big)_{jm}^{ab}$ of mass dimension $5-d$.

The off-diagonal blocks of δh that mix neutrinos and antineutrinos with opposite helicities must be expanded in spin-weighted spherical harmonics ${}_sY_{jm}$. Details of this formalism can be found in [91] and in Appendix A of Reference [88]. These expansions cover another four types of coefficients: $\big(H_{M+}^{(d)}\big)_{jm}^{ab}$, $\big(c_l^{(d)}\big)_{jm}^{ab}$, $\big(g_{M+}^{(d)}\big)_{jm}^{ab}$, and $\big(a_l^{(d)}\big)_{jm}^{ab}$. Here, $\big(g_{M+}^{(d)}\big)_{jm}^{ab}$, and $\big(H_{M+}^{(d)}\big)_{jm}^{ab}$ have mass dimension $4-d$, while the mass-induced coefficients $\big(a_l^{(d)}\big)_{jm}^{ab}$ and $\big(c_l^{(d)}\big)_{jm}^{ab}$ have dimension $5-d$.

However, since experiments are sensitive to the combinations $\widehat{a}_{\mathrm{eff}}$, $\widehat{c}_{\mathrm{eff}}$, $\widehat{g}_{\mathrm{eff}}$, and $\widehat{H}_{\mathrm{eff}}$, only the detailed expanded forms of those are shown here as follows:

$$
\begin{aligned}
\widehat{a}_{\mathrm{eff}}^{ab} &= \sum_{djm} |\boldsymbol{p}|^{d-2}\, Y_{jm}(\hat{\boldsymbol{p}})\big(a_{\mathrm{eff}}^{(d)}\big)_{jm}^{ab}, \\
\widehat{c}_{\mathrm{eff}}^{ab} &= \sum_{djm} |\boldsymbol{p}|^{d-2}\, Y_{jm}(\hat{\boldsymbol{p}})\big(c_{\mathrm{eff}}^{(d)}\big)_{jm}^{ab}, \\
\widehat{g}_{\mathrm{eff}}^{ab} &= \sum_{djm} |\boldsymbol{p}|^{d-2}\, {}_{+1}Y_{jm}(\hat{\boldsymbol{p}})\big(g_{\mathrm{eff}}^{(d)}\big)_{jm}^{ab}, \\
\widehat{H}_{\mathrm{eff}}^{ab} &= \sum_{djm} |\boldsymbol{p}|^{d-2}\, {}_{+1}Y_{jm}(\hat{\boldsymbol{p}})\big(H_{\mathrm{eff}}^{(d)}\big)_{jm}^{ab}.
\end{aligned} \tag{21}
$$

These effective spherical coefficients are related to the above ten by

$$\begin{aligned}
(a_{\text{eff}}^{(d)})_{jm}^{ab} &= (a_L^{(d)})_{jm}^{ab} - (e_l^{(d+1)})_{jm}^{ab} + (g_l^{(d+1)})_{jm}^{ab}, \\
(c_{\text{eff}}^{(d)})_{jm}^{ab} &= (c_L^{(d)})_{jm}^{ab} - (m_l^{(d+1)})_{jm}^{ab} + (H_l^{(d+1)})_{jm}^{ab}, \\
(g_{\text{eff}}^{(d)})_{jm}^{ab} &= (g_{M+}^{(d)})_{jm}^{ab} + (a_l^{(d+1)})_{jm}^{ab}, \\
(H_{\text{eff}}^{(d)})_{jm}^{ab} &= (H_{M+}^{(d)})_{jm}^{ab} + (c_l^{(d+1)})_{jm}^{ab}.
\end{aligned} \tag{22}$$

The indices a and b range over neutrino flavors, and the d superscript on the effective coefficients for LV might be different from the dimension of the underlying operator.

Interesting aspects of this theory include relating coefficients initially assumed to be independent via symmetries or radiative corrections. These are not discussed here, but due to the high sensitivities yielded in the neutrino sector, the influence of such connections cannot be underestimated.

Here, the general neutrino LV extension is considered with the ultimate goal of placing astrophysical searches into the framework. As we shall see, these searches return some of the most stringent bounds on Lorentz violation.

2.2. Restricting Higher-Dimensional Analysis to the Minimal SME

The first considerable simplification is from arbitrary dimensions to the more conservative renormalizable subset represented by the mSME. Specifically, what is meant below is a formalism with mass-independent operators. This subsection gives a comparison between the theory of arbitrary mass dimensions developed later [91] and the earlier one [84].

Many experimental investigations prefer to focus on the latter. There are parallels between the approaches and the form of these two treatments, but a direct correspondence is more subtle to decipher. The full understanding is best made based on the source papers [84,91]. To reduce the complexity from the SME to the mSME, let us also reduce the number of neutrinos to 3. The perturbation δh of the nonminimal LV effective hamiltonian reads:

$$\delta h = \frac{1}{|\boldsymbol{p}|}\begin{pmatrix} \widehat{a}_{\text{eff}} - \widehat{c}_{\text{eff}} & -\widehat{g}_{\text{eff}} + \widehat{H}_{\text{eff}} \\ -\widehat{g}_{\text{eff}}^{\dagger} + \widehat{H}_{\text{eff}}^{\dagger} & -\widehat{a}_{\text{eff}}^{T} - \widehat{c}_{\text{eff}}^{T} \end{pmatrix}. \tag{23}$$

The CPT-odd parts take the following form:

$$\begin{aligned}
\widehat{a}_{\text{eff}} &= p_\mu \widehat{a}_L^\mu - \widehat{e}_l + 2\mathrm{i}\epsilon_\mu \epsilon_\nu^* \widehat{g}_l^{\mu\nu}, \\
\widehat{g}_{\text{eff}} &= \mathrm{i}\sqrt{2}\, p_\mu \epsilon_\nu \widehat{g}_{M+}^{\mu\nu} + \sqrt{2}\,\epsilon_\mu \widehat{a}_l^\mu,
\end{aligned} \tag{24}$$

while the CPT-even terms are

$$\begin{aligned}
\widehat{c}_{\text{eff}} &= p_\mu \widehat{c}_L^\mu - \widehat{m}_l + 2\mathrm{i}\epsilon_\mu \epsilon_\nu^* \widehat{H}_l^{\mu\nu}, \\
\widehat{H}_{\text{eff}} &= \mathrm{i}\sqrt{2}\, p_\mu \epsilon_\nu \widehat{H}_{M+}^{\mu\nu} + \sqrt{2}\,\epsilon_\mu \widehat{c}_l^\mu.
\end{aligned} \tag{25}$$

For comparison with Equation (15), δh of the minimal LV effective hamiltonian cast in a similar form would be

$$\delta h = \frac{1}{|\boldsymbol{p}|}\begin{pmatrix} \hat{a}_{\text{min}} - \hat{c}_{\text{min}} & -\hat{g}_{\text{min}} + \hat{H}_{\text{min}} \\ -\hat{g}_{\text{min}}^{*} + \hat{H}_{\text{min}}^{*} & -\hat{a}_{\text{min}}^{*} - \hat{c}_{\text{min}}^{*} \end{pmatrix}. \tag{26}$$

Now, corresponding expressions to Equations (24) and (25) can be given in the minimal form:

$$
\begin{aligned}
\hat{a}_{\min} &= (a_L)^{\mu}_{ab} p_{\mu}, \\
\hat{g}_{\min} &= \mathrm{i}\sqrt{2} p_{\mu} (\epsilon_+)_{\nu} (g^{\mu\nu\sigma}\mathcal{C})_{ab} p_{\sigma}, \qquad (27)
\end{aligned}
$$

$$
\begin{aligned}
\hat{c}_{\min} &= (c_L)^{\mu\nu}_{ab} p_{\mu} p_{\nu}, \\
\hat{H}_{\min} &= \mathrm{i}\sqrt{2} p_{\mu} (\epsilon_+)_{\nu} H^{\mu\nu}_{ab} \mathcal{C}. \qquad (28)
\end{aligned}
$$

Here, neutrino–neutrino mixing is described by the coefficient combinations $(a+b)^{\mu}_{ab}$ and $(c+d)^{\mu\nu}_{ab}$. These are SU(3) $\times$ SU(2) $\times$ U(1) gauge-invariant and lepton-number preserving. The $(c_L)^{\mu\nu}_{ab}$ and $(a_L)^{\mu}_{ab}$ coefficients are defined accordingly as

$$
\begin{aligned}
(c_L)^{\mu\nu}_{ab} &\equiv (c+d)^{\mu\nu}_{ab}, \\
(a_L)^{\mu}_{ab} &\equiv (a+b)^{\mu}_{ab}. \qquad (29)
\end{aligned}
$$

The coefficients $(g^{\mu\nu\sigma}\mathcal{C})_{ab}$ and $(H^{\mu\nu}\mathcal{C})_{ab}$ relate to Majorana-like couplings and describe neutrino–antineutrino mixing. They violate the gauge symmetry as well as lepton-number conservation. According to Reference [84], using the definition

$$
\begin{aligned}
\widetilde{g}^{\nu\sigma} &\equiv g^{0\nu\sigma} + \tfrac{\mathrm{i}}{2}\epsilon^{0\nu}{}_{\gamma\rho} g^{\gamma\rho\sigma}, \\
\widetilde{H}^{\nu} &\equiv H^{0\nu} + \tfrac{\mathrm{i}}{2}\epsilon^{0\nu}{}_{\gamma\rho} H^{\gamma\rho}, \qquad (30)
\end{aligned}
$$

the physically observable combinations of $g^{\mu\nu\sigma}$ and $H^{\mu\nu}$ in h_{eff} are

$$
\begin{aligned}
p_{\mu}(\epsilon_+)_{\nu} g^{\mu\nu\sigma} &= |\varphi| (\epsilon_+)_{\nu} \widetilde{g}^{\nu\sigma}, \\
p_{\mu}(\epsilon_+)_{\nu} H^{\mu\nu} &= |\varphi| (\epsilon_+)_{\nu} \widetilde{H}^{\nu}. \qquad (31)
\end{aligned}
$$

Equations (17) and (18) are simply repeated for clarity as the key equations in demonstrating a connection between renormalizable mass-independent coefficients and the arbitrary-dimension formalism. Note also that, in Equations (24) and (25), for the operators of arbitrary mass dimensions $\widehat{a}^{\mu}, \widehat{c}^{\mu}, \widehat{g}^{\mu\nu}$, and $\widehat{H}^{\mu\nu}$, upper- and lower-case l, L or M^{+}, l indices appear. These signify mass-dependent and mass-independent operator expressions, as can be seen in detail in Equations (19) and (20).

Following the deduction of Reference [91], the counterparts of the renormalizable coefficients in the extended formalism are the following Cartesian coefficient matches: $a_L^{(3)\mu} \equiv (a_L)^{\mu}$, $c_L^{(4)\mu\nu} \equiv (c_L)^{\mu\nu}$, $g_{M+}^{(4)\mu\nu\rho} \equiv g^{\mu\nu\rho}$, and $H_{M+}^{(3)\mu\nu} \equiv H^{\mu\nu}$. This hints at further types of physical effects to be observed.

3. Areas to Search for LV in the Neutrino Sector

The main interest in observational SME physics in the neutrino sector is to translate theory into practical use for the vast areas of research out there. These areas include LV signals from neutrino oscillations, unconventional energy dependence, sidereal and annual variations observed in connection to neutrino processes, compass asymmetries and dependence on the propagation direction, time-of-flight experiments, neutrino–antineutrino mixing, phase space studies of neutrino reactions, and neutrino resonances.

These are performed in reactor [99–102], accelerator short- [103–113] and long-baseline [107,114–117] experiments via energy and directional spectra of atmospheric neutrino oscillations [115,116,118–122], solar neutrino investigations [123–125], beta decays [126–136], and kinematics [21,24,91,137–151].

The starting point of these searches are comprehensive papers on the full theory, which already include discussions on adapting it to phenomenological situations [84–93]. At the writing of these papers, an even larger set of related works was published.

These foundations and their further works contain the core explanations of how the theory is built as well as dedicated subsections for experimental research. An additional excellent overview of the coefficients, their properties, and the results from experiments is provided in the SME data tables [61], which is a supporting summary and catalog of up-to-date LV and CPTV constraints.

As a general navigation chart, a good first step is to reduce the information to an appropriate simplified hamiltonian that aligns with the physical scenario and to identify the relevant operators or coefficients for studying and determining the mass dimensions. The minimal and nonminimal approaches are often separated in these treatments or when discussing results.

In some situations, looking at symmetries, both physical and mathematical, can help in the simplification. These can also be a tool for easier manipulation of expressions, accounting for and managing coefficients and components, reducing complexity, connecting to the physics itself, or uncovering connections that can lead to multiple information from one measurement. These methods are all demonstrated in the source papers.

The operators must then be linked to the parameters of the phenomenology that return measured quantities or functions from the experiments, placing corresponding constraints. Thus far, no LV signal has been found, so this information appears as bounds on the coefficients.

To work in this inherently frame-, orientation-, and boost-dependent context, it is important to identify the frames playing a role, to set up coordinates appropriate to describe the physics based on these frames, to declare any special frames where physics is assumed to be rotationally invariant, and to specify transformations between the frames.

4. Neutrino Oscillations

While neutrino oscillations are not a typical tool in the realm of astroparticle physics, they are important as a basis for comparisons and in atmospheric and solar neutrino studies. Here, we consider three groupings. Strictly Earth-based experiments such as accelerator- or reactor-based experiments as well as decay processes such as double-beta decay searches receive little attention. Solar and atmospheric neutrinos bear more relevance and are summarized accordingly. The main focus is on high-energy (HE) and ultrahigh-energy (UHE) astrophysical neutrinos.

It should be noted here that the SME formalism encompasses CPT violation (CPTV) as well, and in this framework, CPTV implies LV [152]. The theory contains CPT-odd and CPT-even coefficients, and they are always indicated when significant. Some searches solely focus on CPTV.

It should also be added that double beta decay is a current hot research topic with several new results of both experimental and theoretical nature. It also represents a search for an effect known as a countershaded effect which is hard to test. Due to this, the double beta decay is discussed as an example in the context of oscillation-free models below.

This section is a condensed summary of oscillation-type searches. The interested reader should conduct further reading where more details are provided. The references here are reduced only to those directly contributing to a discussion. The number of useful resources published are far larger, and each topic requires further branching into needed articles.

In the vast land of the SME with a rich neutrino landscape, here, the key features of specific search paths are outlined and illustrated. Many articles that contain relevant data and experimental descriptions are not presented in this review if they do not address specific issues of Lorentz or CPT violation. Nevertheless, upon proper follow up, they can provide invaluable information and the necessary up-to-date data, since this dynamic field steadily brings new findings.

Different collaborations have been publishing various works for a long time, since SME searches as well as neutrino experiments now span decades. Those included here appeared either due to the methodologically field-defining presentation of their work or because they present the most recent results. These sets do not always overlap, and reviewing the full extent of their research is left to those applying it.

The oscillations in this subsection start with looking at neutrino–neutrino or antineutrino–antineutrino oscillations. Where there are no substantial differences in the formalism of the two, only the neutrino–neutrino equations are displayed for brevity. These experiments are rooted in the way different neutrino flavors propagate and take advantage of interferometric sensitivity as well as directional analysis to detect changes caused by LV.

There is a certain common-sense simplicity for studying the oscillation phenomena. The main importance of LV research is tied to energy dependence E, baseline length L, the subset of neutrinos relevant to the physical environment, as well as frame and directional dependence.

For baseline properties, there are indications throughout, but the core issue is short versus long baselines or a distance too long for oscillations to be the main issue. The energy ranges vary greatly for the experimental situations, and the SME operators also cause notable energy dependence of the oscillations that differ from the conventional effects. This too must be clarified in the situation in question. Finally, the main issue of direction dependence must be presented.

4.1. Lorentz-Violation Specific Signals in Neutrino Oscillations

Lorentz-violation-specific signal analyses in oscillations include spectral profiles deviating from conventional expectations. The LV energy dependence is influenced by the mass dimension of the LV operators. Different mass dimensions of the coefficients, masses, and matter potentials produce a more subtle scheme for the interference phenomena. In an LV background, it could also be that the constant mixing angles and phases determining the oscillation in the conventional context become energy-dependent.

In the standard massive-neutrino approach, the oscillation is determined by the ratio $\Delta m^2 L/E$. Here, Δm^2 is the mass-squared difference between two oscillating neutrino flavors; there is an inverse proportionality to the first power of the energy and direct proportionality to the length of the baseline. Table 1 of Reference [153] is an example of a comparison of baseline and energy ranges for different settings. Neutrinos of nuclear reactions and solar neutrinos carry similar energies, while atmospheric and accelerator neutrinos have energies orders of magnitude higher.

As we will see, very high energy ranges are reached in astrophysical neutrinos. Neutrinos generated by ultrahigh-energy cosmic rays (UHECRs) of energies above 10^{19} eV interacting with the cosmic photon background are called cosmogenic or GZK neutrinos. They fall into an energy region of $\sim 10^{16}$ eV, and neutrinos above that are designated as having extremely high energy (EHE).

The baselines are sorted as long and short, and sometimes, the same collaboration targets both. That of solar neutrinos is quite different, about 10^8 km compared to a typical short baseline Earth-based experiment of around 1 km. Due to their charge neutrality and weak interactions, astrophysical neutrinos can travel galactic distances between their production and the detectors. In the atmospheric setting, the zenith baseline is about 10 km and is substantially higher for neutrinos coming through the Earth.

Which neutrinos are produced in a relevant starting process, which neutrino flavors are mainly involved, which typical neutrino ratios are expected, and what effects caused by passing through matter are other initial conditions that need to be specified. Once these are clarified, the SME framework offers a fitting adaptation.

The LV hamiltonian affects neutrino oscillations with the following baseline and energy dependencies: $a^\mu L$, $b^\mu L$, $H^{\mu\nu} L$, $c^{\mu\nu} LE$, $d^{\mu\nu} LE$, $g^{\mu\nu\sigma} LE$, and $(k_d)^{\lambda\cdots} LE^d$ for nonrenormalizable dimensions of $n = d + 3$ for the $(k_d)^{\lambda\cdots}$ coefficient [84,91]. The first three indicated are constants in the neutrino energy, while the second three show direct proportionality with it. Based on this, several types of spectral analyses are implemented to search for LV effects.

Reference [25] uses a new approach based on a modified space-time geometry and metric. The corresponding modified dispersion relations include LV changes. In Sections 5 and 6, a detailed calculation of the phase difference of two neutrino mass eigenstates in the presence of LV effects as well as the energy dependence are found. Similar to the SME, this is an unusual energy dependence that differs from the conventional case. What is also demonstrated is that LV must be different for these mass eigenstates if it is to change the oscillation. This important point is iterated again when discussing modifications of the dispersion relation.

An example is the Super-Kamiokande atmospheric neutrino search discussed below [122]. An excellent illustrative chart comparing energy dependence sensitivities can be found in Figure 1 and the following discussion of [84] based on a wide range of experiments for solar [1,154–164], atmospheric [122], reactor-based [125,165–168], and accelerator-based [111–113,169–176] neutrino experiments.

Tailoring the SME to neutrino oscillations was performed in all papers published about it but it is introduced here in general based on Reference [90]. It reduces the formalism to minimal LV coefficients associated with quadratic operators.

4.2. Hamiltonian and Perturbation Methods for Short- and Long-Baseline Neutrino Oscillations

As outlined above, the hamiltonian appropriate for oscillations of neutrinos to neutrinos in three flavors is sought. A full analogy is valid here for the simplification method for antineutrinos and is often not said explicitly. The relevant coefficients are identified as $(a_L)^{\mu}_{ab}$ and $(c_L)^{\mu\nu}_{ab}$ of neutrino–neutrino mixings. The hamiltonian can be separated into Lorentz-invariant h_0 and -violating parts δh, with each as a 6×6 matrix for three neutrinos and three antineutrinos with 3×3 blocks.

The conventional hamiltonian piece h_0 is assumed to be block-diagonal with no neutrino–antineutrino mixing. Neutrino–neutrino mixing is governed by mass squared differences and has the conventional inverse proportionality with energy E. Antineutrino–antineutrino mixing is similar:

$$h_0 = \tfrac{1}{2E}\begin{pmatrix} \Delta m^2_{ab} & 0 \\ 0 & \Delta m^2_{\bar{a}\bar{b}} \end{pmatrix}. \tag{32}$$

Following Reference [90], indices $a, b, \ldots = e, \mu, \tau$ refer to the flavors of neutrinos while barred indices $\bar{a}, \bar{b}, \ldots = \bar{e}, \bar{\mu}, \bar{\tau}$ refer to the flavors of antineutrinos. CPT symmetry holds in the conventional part, giving the following relation:

$$\Delta m^2_{\bar{a}\bar{b}} = \Delta m^{2\,*}_{ab}. \tag{33}$$

The Lorentz-violating term δh can be written in the following form:

$$\delta h = \begin{pmatrix} \delta h_{ab} & \delta h_{a\bar{b}} \\ \delta h_{\bar{a}b} & \delta h_{\bar{a}\bar{b}} \end{pmatrix}. \tag{34}$$

For Lorentz-violating operators of renormalizable dimension, the upper-left diagonal block that pertains to ν-ν oscillations takes the following form:

$$\delta h_{ab} = \frac{1}{|\vec{p}|}[(a_L)^{\mu} p_{\mu} - (c_L)^{\mu\nu} p_{\mu} p_{\nu}]_{ab}. \tag{35}$$

Here, $(a_L)^{\mu}_{ab} \equiv (a+b)^{\mu}_{ab}$ and $(c_L)^{\mu\nu}_{ab} \equiv (c+d)^{\mu\nu}_{ab}$ as above, and the neutrino energy-momentum 4-vector is denoted $p_{\alpha} = (E, -\vec{p}) \approx E(1, -\hat{p})$ [84]. This is substantially simplified from the general full form of Equation (15) and was investigated in numerous experiments [103–107,107–110,113–121].

For antineutrino oscillations,

$$\delta h_{\bar{a}\bar{b}} = \frac{1}{|\vec{p}|}[-(a_L)^\mu p_\mu - (c_L)^{\mu\nu} p_\mu p_\nu]^*_{ab}, \tag{36}$$

using hermiticity to relate δh components for the antineutrino block.

Further focus pertains to the oscillation baseline, the distances from the neutrino source to the detector. In the SME for short-baseline experiments [103–110,113], a method was developed that is different from the long-baseline approach.

If baseline is small compared to the oscillation length, the oscillations can be treated as perturbations of an oscillation-free scenario [86]. Whether this approximation can be applied actually depends on a combination of the baseline and the energy. Baselines are regarded as long or short based on their L/E ratio. $L/E \sim 0.01 = 10\,\mathrm{km/GeV}$ and $L/E \sim 100 = 10{,}000\,\mathrm{km/GeV}$ are taken as short and long baselines, respectively [153].

The difference from unity in the transition amplitude is proportional to the LV hamiltonian δh according to

$$S(L) \simeq 1 - \mathrm{i}\delta h L/(\hbar c) - \tfrac{1}{2}\delta h^2 L^2/(\hbar c)^2 + \cdots. \tag{37}$$

For the probabilities, this yields the general form of leading-order short-baseline approximation seen in detector and accelerator applications.

$$P_{\nu_b \to \nu_a} \simeq \begin{cases} 1 - \sum_{c,c\neq a} P_{\nu_a \to \nu_c}, & a = b\,, \\ |(h_{\mathrm{eff}})_{ab}|^2 L^2/(\hbar c)^2, & a \neq b\,. \end{cases} \tag{38}$$

Solar-neutrino experiments [123,124,154–164] and atmospheric neutrino oscillations [115,116,118–122] are inherently long-baseline, as are some of the Earth-based detectors [107,114–117]. For this, a different time-dependent perturbative method is used for determining the transition amplitude given in the expanded form [90]

$$S(t) = S^{(0)}(t) + S^{(1)}(t) + S^{(2)}(t) + \cdots. \tag{39}$$

Here, the zeroth-order transition amplitude $S^{(0)}(t)$ is Lorentz invariant and $S^{(n)}$ is the nth-order perturbation in δh. For the first order, there is no neutrino–antineutrino mixing, and that is what is considered here.

With the details of the full perturbative calculations omitted, the corresponding first-order probabilities for ν to ν and $\bar{\nu}$ to $\bar{\nu}$ have the following form:

$$\begin{aligned} P^{(1)}_{\nu_b\to\nu_a} &= 2t\,\mathrm{Im}\left((\mathcal{S}^0)_{ab})^*\mathcal{H}^1_{ab}\right), \\ P^{(1)}_{\bar{\nu}_b\to\bar{\nu}_a} &= 2t\,\mathrm{Im}\left((\mathcal{S}^0)_{\bar{a}\bar{b}})^*\mathcal{H}^1_{\bar{a}\bar{b}}\right), \\ P^{(1)}_{\nu_b\to\bar{\nu}_a} &= P^{(1)}_{\bar{\nu}_b\to\nu_a} = 0, \end{aligned} \tag{40}$$

with $\mathcal{H}^1_{ab}$ being expressed in terms of experimentally determined factors and the modifying hamiltonian as

$$\mathcal{H}^1_{ab} = \sum_{cd} \left(\mathcal{M}^{(1)}_{ab}\right)_{cd} \delta h_{cd}. \tag{41}$$

Those for the antineutrinos are omitted since this is an outline of the procedure without the completeness of the source article. The time-dependent matrix $\mathcal{M}^{(1)}_{ab}$ is a factor carrying the dependence on energy, baseline, the conventional mixing angles, and masses in the first-order calculation. It contains nine complex constants for $\nu - \nu$ mixing and nine for $\bar{\nu} - \bar{\nu}$ with $\left(M(1)_{ab}\right)_{cd}$, providing the relevant coefficient combinations for the mixings. The time t is taken equal to the baseline distance L.

This means that coefficients of Equation (35) must be rewritten in an experimentally relevant form as well, with a corresponding combination of the a_L and c_L coefficient components as

$$(\widetilde{a}_L)^{\alpha}_{ab} = \sum_{cd} \left(\mathcal{M}^{(1)}_{ab}\right)_{cd} (a_L)^{\alpha}_{cd},$$
$$(\widetilde{c}_L)^{\alpha\beta}_{ab} = \sum_{cd} \left(\mathcal{M}^{(1)}_{ab}\right)_{cd} (c_L)^{\alpha\beta}_{cd}. \tag{42}$$

The hamiltonian is then better written as

$$\mathcal{H}^1_{ab} = \frac{1}{E}\left[(\widetilde{a}_L)^{\alpha} p_\alpha - (\widetilde{c}_L)^{\alpha\beta} p_\alpha p_\beta\right]_{ab}. \tag{43}$$

4.3. Directional Dependence

The mainstream of LV research is tied to the rotation of the Earth as well as its revolution around the sun. These motions allow for observations of sidereal or annual variations. Binning techniques can further look at general directional dependence. Differences in neutrino fluxes from around compass directions or annual variations of solar neutrino oscillations would signal anisotropies and hence the violation of symmetries under spatial rotations.

This necessitates the identification of the appropriate frames. In Earth-based, searches the usual standard inertial frame defined for the theory is a sun-centered celestial-equatorial frame (SCCEF) with a convention of coordinates (X, Y, Z, T), which is well described in all relevant work [61,177].

It has a Z axis defined by the rotational axis of the Earth and pointing north, with the X axis directed from the sun towards the vernal equinox and the Y axis completing a right-handed system. There is a time origin appointed as the vernal equinox 2000. Here, the local sidereal time and sidereal frequency from Earth's rotation are denoted by $T_\oplus$ and $\omega_\oplus$, respectively.

Momentum dependence is inherent to the formalism of neutrino oscillation studies and is tracked in reference to this frame. When laboratory coordinates are used in an experiment, appropriate transformations are necessary. In specific presentations of results from the different collaborations, the reader will find sidereal effects described with five amplitudes defined as

$$\begin{aligned}\mathcal{H}^1_{ab} = {} & (\mathcal{C}^{(1)})_{ab} + (\mathcal{A}^{(1)}_s)_{ab} \sin\omega_\oplus T_\oplus + (\mathcal{A}^{(1)}_c)_{ab} \cos\omega_\oplus T_\oplus \\ & + (\mathcal{B}^{(1)}_s)_{ab} \sin 2\omega_\oplus T_\oplus + (\mathcal{B}^{(1)}_c)_{ab} \cos 2\omega_\oplus T_\oplus.\end{aligned} \tag{44}$$

The probability for ν-ν of Equation (40) is linear in $\mathcal{H}^1_{ab}$, and Equation (43) shows that the dependence on the momentum is up to quadratic, so the sidereal frequency dependence appears up to the second harmonic $2\omega_\oplus$. This type of expression is the typical starting point of numerous experiments looking for sidereal variations resulting in detailed bounds for the a and c coefficients [99–107,107–110,113–119].

When the momentum direction of particular neutrinos is known for instance as a line of sight from an astronomical or cosmic origin, the sun, or direction given by an accelerator beam, further analysis includes expressing components of the SCCEF unit vector parallel to the momentum in appropriate laboratory frame coordinates tied to the experiment or polar directions toward an object. For Earth-based experiments, this is described as a function of the colatitude and some local north-east coordinate system in conjunction with suitable detector coordinates. It is then straightforward to express the known lab momentum in the SCCEF.

4.4. Solar Neutrinos

Some of the methods can be well illustrated with selected searches in the context of solar neutrinos. The sun is the most significant astronomical source of neutrinos. Its neutrinos are of interest in understanding the physics of the sun itself, conventional neutrino physics, and oscillations. However, the annual variations due to the Earth's revolution around the sun, the long baseline, and energy ranges permit constraining LV coefficients of neutrino–neutrino oscillations and allow us to place bounds on LV-generated neutrino to antineutrino oscillations.

The sun produces electron neutrinos that oscillate into the other two flavors, leaving about a third of them in the electron flavor by the time they reach the Earth. One excellent example of the above discussion is an impressive work conducted at the Sudbury Neutrino Observatory (SNO) [124], which made precise measurements of this electron neutrino fraction and undertook the observation of annual directional changes as neutrinos propagate from sun to Earth over one year.

Lorentz violation predicts directional changes in neutrino propagation as one of its possible consequences due to a (flavor-specific) coupling to some fixed tensor background. This could influence neutrino flavor oscillations. As the Earth moves in the sun's frame, the propagation direction of neutrinos toward Earth from the sun sweeps a full circle.

Adiabatic propagation inside the sun and vacuum propagation both have influences on the overall probabilities determining the oscillations. There is matter perturbation to the mixing matrix at the creation of the neutrino depending on the radial distance from the center. A Lorentz-violating modification is assumed small compared to both the mass and matter terms and can be treated as a perturbation to the conventional description of the propagation. This assumption of perturbation is necessary in light of the unusual energy dependence of the oscillation phase which is proportional to LE for SME coefficients $c^{\mu\nu}LE$, $d^{\mu\nu}LE$, $g^{\mu\nu\sigma}LE$ [90] and for the LV coefficient in the model of Reference [25].

From the effective hamiltonian of Equation (15), the upper left block is identified as relevant since only active neutrinos are expected to play a role. These are used in the spherical-harmonics expansion of Equation (21) for the $\widehat{a}^{ab}_{\rm eff}$ and $\widehat{c}^{ab}_{\rm eff}$ operators:

$$\begin{aligned} \widehat{a}^{ab}_{\rm eff} &= \sum_{djm} |\boldsymbol{p}|^{d-2}\, Y_{jm}(\hat{\boldsymbol{p}})\big(a^{(d)}_{\rm eff}\big)^{ab}_{jm}, \\ \widehat{c}^{ab}_{\rm eff} &= \sum_{djm} |\boldsymbol{p}|^{d-2}\, Y_{jm}(\hat{\boldsymbol{p}})\big(c^{(d)}_{\rm eff}\big)^{ab}_{jm}. \end{aligned} \tag{45}$$

They are further simplified to the renormalizable coefficients in these investigations on the assumption that, in this setting, they have the highest influence. The full form of the result of the modification to the oscillation probabilities is detailed in Reference [124] and presented in Equation (12). A similar separation is made as in the perturbative treatment of Section 4.2 of a factor related to mixing angles, masses, and the matter potential of the sun but independent of the size of LV.

This factor is energy dependent, and with the dominant linear energy dependence factored out, it gives a practical end result for the modification of the oscillation probability separated into energy dependence, LV coefficients, and the factor $w^{\beta\alpha}_{\gamma\delta}$.

$$\delta P = \mathrm{Re} \sum_{jm\gamma\delta} Y_{jm} w^{\beta\alpha}_{\gamma\delta}(\hat{\boldsymbol{p}})\big(E\big(a^{(3)}_{\rm eff}\big)^{\gamma\delta}_{jm} - E^2\big(c^{(4)}_{\rm eff}\big)^{\gamma\delta}_{jm}\big). \tag{46}$$

Detailed calculations and data analysis were performed for the solar neutrino energy range of 1–20 MeV. The results yielded comparable constraints on a-type coefficients of mass dimension one to the order of magnitude of the best results in the neutral-meson system, $\sim 10^{-18} - 10^{-21}$ GeV, and for the dimensionless c-type coefficients, $\sim 10^{-18} - 10^{-19}$.

As solar neutrinos are produced strictly as neutrinos, tight constraints can be placed on LV from the lack of neutrino–antineutrino oscillations. This was performed in [123].

Electron antineutrino appearance could be interpreted as a space-time background effect that is Lorentz-violating. If such effect is not seen, appropriate, LV constraints can be placed on coefficients tied to this LV mechanism.

The relevant SME hamiltomian for this effect will contain operators from the off-diagonal block of Equation (15):

$$\delta H_{\bar{\alpha}\beta} = \mathrm{i}\sqrt{2}(\epsilon_+)^*_\lambda \left[\hat{p}_\sigma E\, \widetilde{g}^{\lambda\sigma}_{\bar{\alpha}\beta} - \widetilde{H}^{\lambda}_{\bar{\alpha}\beta} \right], \tag{47}$$

where $\bar{\alpha}$ refers to an antineutrino flavor and β refers to that of the neutrino. The coefficients for Lorentz violation to describe neutrino–antineutrino oscillations are $\widetilde{g}^{\lambda\sigma}_{\bar{\alpha}\beta}$ and $\widetilde{H}^{\lambda}_{\bar{\alpha}\beta}$, E is the neutrino energy, $(\epsilon_+)_\lambda$ is a polarization 4-vector, and $\hat{p}_\sigma = (1, -\hat{\mathbf{p}})$ is a unit vector along the momentum [84,90].

Here, again, the Mikheyev–Smirnov–Wolfenstein (MSW) matter effects of the sun must be considered and appear in the oscillation probability. The matter potential is $V(r) = \sqrt{2}\, G_F N_e(r)$, where $N_e(r)$ is the electron density. Taking this into account, the mixing angles become a function of three factors: the energy of the neutrino, the radial position in the sun's interior, and the vacuum mixing angles.

As a way of treating this phenomenon, the transition probability of $\nu_e \to \bar{\nu}_e$ is factorized for the two parts of the path inside the sun and from the sun to Earth.

$$P_{\nu_e \to \bar{\nu}_e} = \sum_j P^{\odot}_{\nu_e \to \nu_j}\, P^{\mathrm{LV}}_{\nu_j \to \bar{\nu}_e}. \tag{48}$$

The directional geometry is different from Earth-based laboratories. The relevant frame is still the SCCEF frame. For the baseline, the eccentricity of Earth's orbit must be taken into account, and for the propagation direction, the annual changes in the unit vector of the connecting line from sun to the Earth are needed:

$$\hat{\mathbf{p}} = \big(-\cos\Omega_\odot T, -\cos\eta\sin\Omega_\odot T, -\sin\eta\sin\Omega_\odot T \big). \tag{49}$$

Instead of Earth's sidereal frequency $\omega_\oplus \simeq 2\pi/(23\,\mathrm{h}\,56\,\mathrm{m})$ above, here, Earth's annual frequency $\Omega_\odot \simeq 2\pi/(365.25\,\mathrm{d})$ describes the periodic change. The parameter $\eta \simeq 23.5°$ is included for the inclination of the orbital plane with respect to the plane of the celestial equator.

The details of setting up the coordinates and calculating the baseline as well as the final form of the probability are presented in [123]. Taking a limit for the $\nu_e \to \bar{\nu}_e$ oscillation probability set by KamLAND as an example, bounds of order 10^{-27} were set on the components of $\widetilde{g}^{\lambda\sigma}_{\bar{a}b}$ from an experimental limit on the transition probability of [125]:

$$\langle P_{\nu_e \to \bar{\nu}_e} \rangle_{\mathrm{exp}} < 5.3 \times 10^{-5} \qquad (90\%\ \mathrm{C.L.}). \tag{50}$$

This is a much tighter bound than those obtained for instance with the spectral-distortion analyses of reactor experiments, partly due to the significant differences in baseline. Other experiments that placed bounds on LV neutrino–antineutrino oscillations are presented in [102,117].

5. Astrophysical Neutrinos

Neutrinos play a central role in astrophysical searches, serving as probes of cosmological significance. The IceCube observatory found neutrinos up to 1–2 PeV energies but none above 10 PeV [178,179], which in itself can be used to verify or constrain novel models.

A generic measure relating to structure formation in the development of the universe is the star formation rate. Observations carried by neutrinos delivering information about cosmological evolution of UHECR sources compared to the star formation rate have key significance. These observations also allow for the study of LV using the high-energy neutrinos as investigations of anisotropy in spectral analysis, threshold effects in neutrino decays and interactions, and time-of-flight experiments.

Ultrahigh-energy particles are the most energetic particles in the universe, with energy $\sim 10^{20}$ eV. They are characterized by their energy, particle composition, and anisotropy. These particles are influenced by their interaction with astrophysical backgrounds, specifically the cosmic microwave background (CMB), and are influenced by the cosmological evolution of the universe. With such energies, their path can span baselines in the range of Gpc.

Active galactic nuclei jets, one of the possible cosmic accelerators that could send PeV-level neutrinos toward Earth, could produce a number of protons or nuclei. Interactions at the source of these particles with matter or radiation produce pions and, by further decay, UHE cosmic neutrinos. These neutrinos are classified as astrophysical, and they can pass undeflected to Earth.

Considering the proton as the cosmic-ray particle that is the primary source of neutrinos, the processes that the proton undergoes are pair production and scattering with CMB photons. Other neutrinos are generated when UHE cosmic rays propagating through the universe interact with the CMB or extragalactic background light (EBL).

In particular, cosmogenic neutrinos come from protons interacting with the CMB through a GZK process. The UHECRs, cosmic-ray protons above 5×10^{19} eV, are subject to the GZK cutoff and do not propagate beyond about 100 Mpc of the source of the cosmic ray. The neutrinos are produced within this GZK horizon. They align well with the source. See for instance [180] for further explanation and connection to LV.

A typical proton–photon reaction producing neutrinos is

$$p + \gamma \to \Delta \to \pi + n \to \pi + p + \nu_e + e. \tag{51}$$

Below, there are illustrative examples on how to use the SME formalism in the context of astrophysical neutrinos as possible signals for LV. After a brief discussion of neutrino flavor analysis in the astrophysical domain, kinematics-type searches are discussed such as threshold effects and time-of-flight. The discussions end with a short summary on atmospheric-neutrino-based spectral analysis.

5.1. Flavor Studies of Astrophysical Neutrinos

In the context of astrophysical neutrinos, oscillation searches have been tied to those of solar origin or to those produced by astroparticles in the atmosphere, referred to as atmospheric. However, a flavor study for astrophysical neutrinos was also performed and is presented in this section.

These neutrinos propagate over distances in the order of hundreds Mpc, over which they are expected to become incoherent. A more thorough study of coherence loss and coherent broadening of the spectrum of cosmic neutrinos is given in [181].

Lorentz violation tests of flavor composition of astrophysical neutrinos arriving at Earth's detectors could reveal LV background interactions. One tool for probing LV is to create plots based on characteristic observables and the energy ranges to show allowable energy levels for a specific coefficient. At IceCube, the astrophysical flavor studies use a triangle chart to analyze flavor content with or without LV. The triangle representation, shown in References [182,183], is a convenient tool to represent flavor ratios for the three neutrino flavors.

These investigations carry excellent sensitivity for higher-dimensional LV operators, reaching the highest level for dimensions five, six, and seven of any sector. They can probe for LV up to Planck-scale physics [182]. Their investigation is focused on an LV-modified mixing formalism as opposed to standard oscillations. It takes into account the typical processes with the characteristic neutrino flavor ratios stemming from the models that describe astrophysical objects generating neutrinos that can reach Earth.

A conventional oscillation would be expected to specify a small region in a flavor triangle representation signifying a close to equal ratio of each flavor (ϕ_e:ϕ_μ:ϕ_τ) = (1:1:1) [183–190].

This is expected over these distances regardless of the understanding of the source of neutrinos.

As mentioned earlier, this standard oscillation in vacuum is governed by the squared mass differences and the inverse of the energy. Neutrino–antineutrino oscillations and lepton number violation were excluded. The starting point is the mixing matrix $V(E)$ between mass $|\nu_i\rangle$ and flavor $|\nu_\alpha\rangle$ eigenstates:

$$|\nu_\alpha\rangle = \sum_i V_{\alpha i}(E)|\nu_i\rangle \, . \tag{52}$$

The standard formalism of neutrino oscillations can be adapted for astrophysical neutrinos due to the long baseline. This allows for averaging one flavor state to another, resulting in the following factored form:

$$\bar{P}_{\nu_\alpha \to \nu_\beta}(E) = \sum_i |V_{\alpha i}(E)|^2 |V_{\beta i}(E)|^2 \, , \tag{53}$$

from one flavor state $|\nu_\alpha\rangle$ to another flavor state $|\nu_\beta\rangle$ and with the probability depending on $|V_{\alpha i}(E)|$, which in turn depends on the energy. The equation above allows for calculating fluxes arriving at Earth from fluxes at the production.

The oscillation hamiltonian for standard Lorentz-invariant oscillations is known. Given a model about neutrino production with the processes involved and the information gained from current experiments, a flavor content at detection can be calculated for astrophysical neutrinos.

However, in the IceCube analysis, a modified hamiltonian is taken with general operators allowing for LV. A generic new-physics hamiltonian incorporates effective operators O_n and the new-physics related mixing matrix $\tilde{U}_n$

$$\delta H = \left(\frac{E}{\Lambda_n}\right)^n \tilde{U}_n O_n \tilde{U}_n^\dagger, \tag{54}$$

where O_n and Λ_n set the scale of the new physics. The operators can connect to various scenarios, but here, the attention is on LV and CPTV within the SME framework. The SME correspondence $n = 0$ is to CPT-odd Lorentz-violating operators [91], while $n = 1$ connects to the CPT-even ones. Orders indicated with different n correspond to different mass dimensions in the SME.

To set the scales of the operators, the best bounds from established neutrino oscillation experiments are obtained such as those of Super-Kamiokande and IceCube atmospheric neutrino searches. As an example, let us take $O_0 = 10^{-23}\,\text{GeV}$ for the $n = 0$ operator and $O_1 = 10^{-23}\,\text{GeV}$ for the $n = 1$ operator, which, taking the energy scale $\Lambda_1 = 1\,\text{TeV}$, yields $\dfrac{O_1}{\Lambda_1} = 10^{-27}$.

The idea is to investigate new physics effects depending on flavor content, creating various visual images via the flavor-triangle method. This can be performed based on new-physics mixing effects, constraints on relevant SME operators from previous experiments, and the neutrino energy and by using typical processes for flavor content at production, which determine the expected content at detection.

The result is to identify allowable regions of flavor content with or without LV narrowed within these triangles. For elegant details, the reader should consult the references given, where an analysis is performed for four different production processes.

5.2. Kinematics-Based Searches in the Astrophysical Domain

A more typical class of investigation in the astrophysical domain relates to kinematic investigations such as time-of-flight studies and threshold effects. The flux of astrophysical and cosmogenic neutrinos is small compared to other neutrinos, and detecting them is a formidable experimental challenge.

They are scrutinized for the power law of their fluxes, which has certain predicted robust characteristic energy dependencies based on kinematic considerations and thresholds identified in the reactions involved. These characteristics are not the focus of this summary; instead, consequences of LV are discussed that can change these characteristics.

Time-of-flight experiments

Neutrino oscillations examine the propagation of a neutrino of one flavor into other neutrino flavors. Time-of-flight studies compare the group velocities to those of other particles. This is often performed with a flavor-blind oscillation-free set of coefficients.

In these models, oscillation is not the prevalent tool. The effective hamiltonian in this case can be reduced to a single flavor. All coefficients antisymmetric in flavor space are omitted, as are the lepton-number-violating ones. Further details are given in Reference [91].

An even more restricted case of oscillation-free models and models in general is provided by the search for oscillation-free countershaded models. Some SME coefficients represent only an energy shift and can be removed via a phase shift of the fermion field. The a_μ coefficient is one such coefficient. It can only be observed as differences of particle-specific energy shifts in experiments such as the neutral mesons where quark fields are coupled via the weak interaction [64,66]. It can also be observed in some GR scenarios well described in [47].

Recent studies with new results target the oscillation-free countershaded "$\mathring{a}^{(3)}_{\text{of}}$" coefficients. These special searches are performed in the double beta decay summed energy spectra of electrons. Notable novel active efforts are being made in studies of the double beta decay, which are geared also toward the search for neutrinoless decays. These investigations are beyond the general aim of this work on extraterrestrial neutrinos; however, they represent some of the newest work rich both in theoretical aspects and on experimental undertakings [131–135].

As indicated earlier, a characteristic consequence of LV is direction dependence on the propagation direction. With restricted directional information as a simplifying measure, an isotropic limit can be taken for a given reference frame. An isotropic model is confined to coefficients with $j = 0$ in the spherical-harmonics expansion and includes no neutrino-antineutrino mixing terms.

Since most considerations below focus on either dispersion relation changes or propagation velocities, the two most important LV-modified quantities listed here are the oscillation-free energy, denoted as E^{of}_ν for the neutrino and $E^{\text{of}}_{\bar{\nu}}$ for the antineutrino and the oscillation-free group velocity below. They are expanded in spherical harmonics.

$$E^{\text{of}}_\nu = |\boldsymbol{p}| + \frac{|m_l|^2}{2|\boldsymbol{p}|} + \sum_{djm} |\boldsymbol{p}|^{d-3} Y_{jm}(\hat{\boldsymbol{p}}) \Big[\big(a^{(d)}_{\text{of}}\big)_{jm} - \big(c^{(d)}_{\text{of}}\big)_{jm} \Big]. \tag{55}$$

In going from neutrino to antineutrino, there is a change in the sign of the CPT-odd $\big(a^{(d)}_{\text{of}}\big)_{jm}$ coefficients, a notable feature that can be taken advantage of when setting certain bounds. Another important feature of the oscillation-free case is that indices are limited to $d \geq 3$ and 4 and $d - 2 \geq j \geq 0$.

In the astrophysics-related domain, the supernova SN1987A studies are taken here to guide us through the general method. The treatment uses only the neutrino-related LV. The coefficients are identified in a spherical-harmonics decomposition with d and j values indicated in Table 1.

Table 1. Spherical coefficients for flavor-blind models. Adapted with permission from [91]. Copyright 2011 by The American Physical Society.

Coefficient	d	j	Number
$\left(a_{\text{of}}^{(d)}\right)_{jm}$	odd, ≥ 3	$d-2 \geq j \geq 0$	$(d-1)^2$
$\left(c_{\text{of}}^{(d)}\right)_{jm}$	even, ≥ 4	$d-2 \geq j \geq 0$	$(d-1)^2$

The group velocity is defined as $v^{\text{of}} = \partial E_\nu^{\text{of}}/\partial|\boldsymbol{p}|$. In terms of the coefficients listed in Table 1, we obtain the following:

$$v^{\text{of}} = 1 - \frac{|m_l|^2}{2\boldsymbol{p}^2} + \sum_{djm}(d-3)|\boldsymbol{p}|^{d-4}\, Y_{jm}(\hat{\boldsymbol{p}})\left[\left(a_{\text{of}}^{(d)}\right)_{jm} - \left(c_{\text{of}}^{(d)}\right)_{jm}\right]. \tag{56}$$

The antineutrino group velocity can be found analogously when needed. Here, we focus on astrophysical neutrinos, so this expression for the group velocity can be directly applied without worrying about sidereal variations. The typical frame is still the SCCEF, in which the source-specific direction can be analyzed, but details of the analysis are omitted here.

Supernova SN1987A produced an antineutrino burst arriving on Earth propagating over a large distance that took about 5×10^{12} s to cross, arriving within about 10 s with energies ranging between 7.5 and 40 MeV. The difference between light speed and antineutrino speed was experimentally constrained to $|\delta v| < 2 \times 10^{-9}$ in [191–193]. Taking an energy value of 10 MeV, this can be translated to specific coefficient bounds using

$$\left|\sum_{djm}(d-3)|\boldsymbol{p}|^{d-4}\, Y_{jm}\left[\left(a_{\text{of}}^{(d)}\right)_{jm} + \left(c_{\text{of}}^{(d)}\right)_{jm}\right]\right| < 2 \times 10^{-9}. \tag{57}$$

Since these involve isotropic and anisotropic coefficients, bounds for different m values could be extracted using the known propagation direction in the SCCEF frame expressed with polar angles $\theta = 20.7°$, $\phi = 263.9°$.

Similarly, dispersion bounds could also be placed on the velocities based on the limited spread of antineutrino energies. The arrival times and the travel time place a restriction on a maximum spread in the speeds across the range of energies of $\delta v < 2 \times 10^{-12}$.

This constraint on the spread of the antineutrino speed allows for the calculation of further coefficient bounds. The range between two energies denoted by $|\boldsymbol{p}_1|$ to $|\boldsymbol{p}_2|$ can be defined as $\Delta(|\boldsymbol{p}|^{d-4}) = |\boldsymbol{p}_2|^{d-4} - |\boldsymbol{p}_1|^{d-4}$. The bound found was

$$\left|\sum_{djm}(d-3)\Delta(|\boldsymbol{p}|^{d-4})Y_{jm}\left[\left(a_{\text{of}}^{(d)}\right)_{jm} + \left(c_{\text{of}}^{(d)}\right)_{jm}\right]\right| < 2 \times 10^{-12}. \tag{58}$$

For detailed results, see Table XII of Reference [91] and the data tables [61].

An analysis restricted to the isotropic oscillation-free case gives supernova bounds derived from [191–193]:

$$\left|\sum_{d}(d-3)|\boldsymbol{p}|^{d-4}\,(\mathring{a}^{(d)} + \mathring{c}^{(d)})\right| < 2 \times 10^{-9}, \tag{59}$$

and the dispersion bound

$$\left|\sum_{d}(d-3)\Delta(|\boldsymbol{p}|^{d-4})(\mathring{a}^{(d)} + \mathring{c}^{(d)})\right| < 2 \times 10^{-12}, \tag{60}$$

where the diacritic notation indicates isotropic coefficients.

For detailed results, see Table XIII of Reference [91] and the data tables [61]. While extracting those limits, only one term at a time was taken to be nonzero, another way to eliminate the large complexity.

Threshold effects

A key feature of threshold processes is to introduce some small change to the neutrino dispersion relation based on an LV scenario [91]. These changes can produce effects [137–146] that become more significant at higher energies. The dispersion relation modifications in the SME are tied to the appropriate coefficients as in, for instance [148,149].

The argument is that tight bounds can also be set from the energy studies of decay processes of particles involved, which might be forbidden without LV. Such a reasoning provides complementary constraints to spectral studies achieving high accuracy from high energy, not from high precision [147]. In these investigations, the flavor-blind energy modification of Equation (55) in the spherical-harmonics expansion is used and it is assumed that only the neutrino feels the effect of LV.

This assumption is purely a simplification aimed at producing a neutrino-sector bound. The kinematics processes in question require that the dispersion relation modifications are different for different particles.

The authors of References [194,195] examined some issues on how this setting appears in the context of gauge invariance.

These deeper theoretical studies go beyond kinematics studies and address gauge symmetry considerations broadening the discussion to introduce a pseudo-metric in analogy to general relativity but without curvature. Modified gamma matrices dependent on the magnitude of the velocity but not on direction enter the Dirac equations and are analyzed in the electroweak theory of the SM.

Based on this, the conditions for relevant kinematics processes are analyzed in relation to the electroweak gauge structure. The analysis also gives a discussion of the energy ranges of relevance. A direct connection to the SME framework is shown with Equation (5) of Section 2 and specifically to the expression for the CPT-even $c_{\mu\nu}$ coefficients, which forms the bases of the typical SME searches. This term dominates at high energies over the CPT-odd one.

Most research for threshold effects relating to LV in astrophysical neutrinos comes from the study of high-energy particles. The IceCube collaboration found significant numbers of extraterrestrial neutrinos with energy above $\sim$60 TeV, corresponding to a cosmic-neutrino signal above the atmospheric background at a high confidence level [178,179]. Two neutrinos were identified at the PeV level and several in the $\sim$0.1 PeV range with $\sim 4\sigma$ above the atmospheric background.

The determination of cosmic origin involves some conditions such as the expected isotropy seen in the distribution with no enhancement in the galactic plane and the implied peak in the energy spectrum, hinting at photopion production with subsequent pion decay typical for active galactic nuclei or gamma ray burst models.

Of the two PeV neutrinos, at least one is strongly believed to be of extragalactic origin, since its direction of arrival is not from the galactic plane. Its specific origin is still an open question. Reports on this vary, and the source of the highest energy neutrinos is subject to further research [196–207]. The idea of the threshold is illustrated here with a positive pion decay into a muon and a muon antineutrino

$$\pi^+ \to \mu^+ + \bar{\nu}_\mu. \tag{61}$$

Following Reference [91], one arrives at the insight under reasonable assumptions that, above some threshold energy, the decay would become forbidden should there be an LV energy contribution to the neutrino. If such a cutoff is not seen by experiments regarding high-energy muon neutrinos, that puts a constraint on the size of the LV.

As usual, the latter still needs to be placed into the coefficient-based catalog by directional markers and mass dimension. The kinematic calculations were performed in the reference, establishing a threshold condition.

Splitting the energy of the neutrino into a conventional part E_0 defined by $E_0^2(\boldsymbol{p}) = p^2 + m_\nu^2$, where the momentum of the neutrino is denoted by $\boldsymbol{p}$ and where $\delta E(\boldsymbol{p})$ is the LV modification, we can write

$$E(\boldsymbol{p}) = E_0(\boldsymbol{p}) + \delta E(\boldsymbol{p}). \tag{62}$$

The following conditions can be given for the LV energy contribution when this process becomes forbidden:

$$\delta E(\boldsymbol{p}) \leq \frac{\Delta M^2 - m_\nu^2}{2E_0} \leq \frac{\Delta M^2}{2|\boldsymbol{p}|}. \tag{63}$$

In a kinematical analysis, this depends on the pion and muon mass difference $\Delta M = M_\pi - M_\mu$ and the neutrino momentum. Taking the expression for the oscillation-free case for the neutrino energy E_ν^{of} and the CPT-conjugate for $E_{\bar{\nu}}^{\text{of}}$ yields

$$\delta E^{\text{of}} = \sum_{djm} |\boldsymbol{p}|^{d-3}\, Y_{jm}(\hat{\boldsymbol{p}}) \Big[\pm \big(a_{\text{of}}^{(d)}\big)_{jm} - \big(c_{\text{of}}^{(d)}\big)_{jm} \Big], \tag{64}$$

for the neutrino energy E_ν^{of} and for the antineutrino energy $E_{\bar{\nu}}^{\text{of}}$, respectively. Note the sign flip on the CPT-odd a coefficient. This finding can be taken advantage of in setting two one-sided bounds using

$$\sum_{djm} |\boldsymbol{p}|^{d-2}\, Y_{jm}(\hat{\boldsymbol{p}}) \Big[\pm \big(a_{\text{of}}^{(d)}\big)_{jm} - \big(c_{\text{of}}^{(d)}\big)_{jm} \Big] < \tfrac{1}{2}\Delta M^2. \tag{65}$$

Similar to the time-of-flight case, this inequality has a directional dependence. In directional searches, one would search for a reduced number of muons arriving as a function of the polar angles (θ, ϕ), for instance.

A simplification can also be performed, however, in an isotropic scenario in a special frame where rotational invariance can be assumed. For the isotropic oscillation-free case, the neutrino energy is denoted by a diacritic $\mathring{E}_\nu$ and the two conditions present as follows:

$$\delta\mathring{E} = \sum_d |\boldsymbol{p}|^{d-3} (\pm \mathring{a}^{(d)} - \mathring{c}^{(d)}), \tag{66}$$

which leads to the two one-sided bounds

$$\sum_d |\boldsymbol{p}|^{d-2} (\pm \mathring{a}^{(d)} - \mathring{c}^{(d)}) < \tfrac{1}{2}\Delta M^2. \tag{67}$$

For pion decays, the numerical value of the right-hand side is $\frac{1}{2}\Delta M_\pi^2 = 5.7 \times 10^{-4}\,\text{GeV}^2$. Taking 400 TeV for the HE neutrino from [91], the bound yielded is

$$\sum_d (400\ \text{TeV})^{d-2} (\pm \mathring{a}^{(d)} - \mathring{c}^{(d)}) < 7.5 \times 10^{-2}\ \text{GeV}^2. \tag{68}$$

The results are shown in Table 2 to illustrate the strength of these two-sided bounds for individual isotropic CPT-odd a coefficients of various dimensions. Note that only the CPT-odd coefficient can be bound in a two-sided way. For the CPT-even one, only a one-sided lower negative bound can be extracted this way.

Table 2. Estimated bounds on isotropic oscillation-free coefficients from a threshold analysis of IceCube data. Units are GeV^{4-d}. Reprinted with permission from [91]. Copyright 2011 by The American Physical Society.

Coefficient	Bound	Coefficient	Bound
$\lvert \mathring{a}^{(3)} \rvert$	$<1.9 \times 10^{-7}$	$\mathring{c}^{(4)}$	$>-4.7 \times 10^{-13}$
$\lvert \mathring{a}^{(5)} \rvert$	$<1.2 \times 10^{-18}$	$\mathring{c}^{(6)}$	$>-2.9 \times 10^{-24}$
$\lvert \mathring{a}^{(7)} \rvert$	$<7.3 \times 10^{-30}$	$\mathring{c}^{(8)}$	$>-1.8 \times 10^{-35}$
$\lvert \mathring{a}^{(9)} \rvert$	$<4.6 \times 10^{-41}$	$\mathring{c}^{(10)}$	$>-1.1 \times 10^{-46}$

The arrival itself of neutrinos at $\sim$PeV permits the placement of new, even stricter bounds on LV. This method received particular attention when the OPERA results showed faster-than-light results, and threshold arguments were one of the ways to contradict its validity [21] with techniques already known in the threshold approach [147–149].

Neutrinos with maximum attainable velocity (MAV) higher than the velocity of light would rapidly lose energy by vacuum decays. This presents the core idea of this class of investigations. Three main processes relevant for threshold effects in astrophysical neutrinos are vacuum Čerenkov radiation ($\nu \to \nu\gamma$), vacuum electron–positron emission ($\nu \to \nu e^+ e^-$), and neutrino splitting ($\nu \to \nu\nu\bar{\nu}$) [21].

References [194,195] scrutinized these particular decays with specific conditions under which they fit into the SM gauge symmetries. The neutrino splitting process becomes kinematically allowed if LV causes a different MAV for the muon and electron neutrinos. References [194,195] deduced, under certain assumptions, the restricted gauge transformation under which this is realized. The situation is similar for the case of vacuum pair production where these differential LVs need to occur between leptons and neutrinos of the same generation.

This novel research marks the newest tendencies in theoretical investigations in the development of the SME. Symmetries including gauge symmetries, conservation laws, and discrete symmetries are also used to connect different SME sectors and to place bounds across the sectors. In Reference [208], for instance, sensitivities on flavor off-diagonal Lorentz violation in charged leptons are obtained from known sensitivities in the neutrino sector and sensitivities on flavor-diagonal Lorentz violation in neutrinos are computed from corresponding sensitivities in the charged-lepton sector.

Different papers present slightly different approaches to this important topic. In Equation (55), we already saw the full flavor-blind oscillation-free spherical-harmonic formalism for energy modification appropriate for threshold studies, including a-type and c-type coefficients. From this, various physical scenarios can even reduce to only one component of one coefficient under suitable assumptions. In between, there is a variety of ways that the SME description can be adapted.

There are reasons to include only the CPT-even c-type, mass dimension-four term in some cases while neglecting the CPT-odd a-type coefficients due to either high-energy consideration or because neutrino–antineutrino asymmetries are disregarded. A generic c-only form for the SME-modified dispersion relation given for HE neutrinos with momentum $\boldsymbol{p}$ and energy E in a relativistic, oscillation-free, and CPT-even limit is a straightforward reduction of the above presentation:

$$E(\boldsymbol{p}) = |\boldsymbol{p}| - \sum_{djm} |\boldsymbol{p}|^{d-3} Y_{jm}(\hat{\boldsymbol{p}}) \big(c_{\text{of}}^{(d)}\big)_{jm}, \tag{69}$$

with mass dimensions $d = 4, 6, 8, \ldots$ in the spherical-harmonics expansion.

Note the missing mass term. These scenarios involve EHE neutrinos. MAV studies are inherently velocity studies, and LV energy modifications are a function of the momentum with a small mass correction. The ultrarelativistic energy–momentum relation is modified to $E = (1+\delta)p$, giving a MAV $\frac{\mathrm{d}E}{\mathrm{d}p} = 1 + \delta$.

The modified isotropic dispersion relation is

$$E(\boldsymbol{p}) = |\boldsymbol{p}| - \sum_{d} |\boldsymbol{p}|^{d-3} \mathring{c}^{(d)}, \tag{70}$$

where $\mathring{c}^{(d)} \equiv \left(c_{\text{of}}^{(d)}\right)_{00} / \sqrt{4\pi}$ defines the coefficients $\mathring{c}^{(d)}$. Often, this is further reduced to just the dimension-four c coefficient.

References [148,149] also give a spin-dependent full expression for spin $\frac{1}{2}$ fermions for the parameter δ_w in terms of time and spatial components of the dimension-four c and d coefficients. The index w highlights the fact that the δ values are for a specific particle w. The modified dispersion relation is cast in the following form:

$$E_w(\boldsymbol{p}) = \sqrt{m_w^2 + [1 + 2\delta_w(\hat{\boldsymbol{p}})]\boldsymbol{p}^2}, \tag{71}$$

with δ_w expressed as

$$\delta_w(\hat{\boldsymbol{p}}) = -c_{00}^{w} - c_{(0j)}^{w} \hat{\boldsymbol{p}}_j - c_{jk}^{w} \hat{\boldsymbol{p}}_j \hat{\boldsymbol{p}}_k + s d_{00}^{w} + s d_{(0j)}^{w} \hat{\boldsymbol{p}}_j + s d_{jk}^{w} \hat{\boldsymbol{p}}_j \hat{\boldsymbol{p}}_k.$$

The main characterization of this kind of Lorentz violation at high energies is the slope of the function $E(p)$. Depending on the dimension of the operator, this is not necessarily linear, but when dimension-four coefficients are analyzed in MAV discussions in an isotropic limit, the energy–momentum relation is proportional to p. As mentioned above, as energy increases, the mass term can be neglected. A further reduction is made by approaching the isotropic frame and the isotropic limit, which is suitable for simply evaluating whether the order-of-magnitude values involved are capable of producing observable physics.

The MAV is defined in a frame that is assumed to have rotational invariance, in which the isotropic assumption is allowed. In all other frames, they remain anisotropic and proper transformation must be carried out between different frames. The frame taken as preferred depends on the physical situation. In the case of the astrophysical bounds, this frame is taken to be the CMB frame. An Earth-bound laboratory frame moves relative to it at about $u = 10^{-3}$ in units of the MAV of a certain particle.

It must be noted that some models exist that are aimed at preserving the overall isotropy of space-time such as homogeneously modified special relativity [56]. The bases of this is a new space-time structure that, in this review, falls under explicit-type symmetry breaking mentioned in the Introduction. Discussing the possible origin of LV is beyond the scope of this work but it is worth mentioning this alternative that would not tie isotropy to a special frame.

To clarify, the MAV of the neutrino as well as those of other particles are frame-dependent and direction-dependent effective values that can be superluminal. In an LV context, due to species- and direction-dependent couplings to the background, it is possible for each particle to have its own "velocity of light" or MAV [25,147].

Here, we conclude the threshold discussion with another notable decay that received wide discussion in the literature. The vacuum pair production ($\nu \to \nu e^+ e^-$) is the dominating process of the above three and can lead to constraints on LV via the threshold method. In the presence of LV, otherwise forbidden proton decay could happen in a UHE scenario.

Suppose the MAV is defined by $v_\nu = 1 + \delta_\nu$, δ_ν for the neutrino signifying the difference above light speed $c = 1$. Following the reasoning of Reference [151], it is clear that the condition [91,93] on the vacuum electron–positron decay to be kinematically allowed is

$$E_\nu \geq m_e \sqrt{\frac{2}{\delta_{\nu e}}}, \tag{72}$$

where m_e is the electron mass and $\delta_{\nu e}$ is the difference of the neutrino and electron MAV correction $\delta_{\nu e} = \delta_\nu - \delta_e$. Note that in these scenarios, one of the key components is that the

MAV and hence its modifying factor are different for different particles participating in a decay, here the electron and the neutrino. Note also the SME correspondence of $\delta_\nu = -\mathring{c}^{(4)}$.

Using this simple condition allows for expressing the predicament in a visual way. Taking the experimentally determined value $\delta_{\nu_e} = 5.6 \times 10^{-19}$ for $E_\nu = 1\,\mathrm{PeV}$, it is argued that superluminal neutrinos with multi-PeV or greater energies that survive to a terminal energy $E_T \sim 1\,\mathrm{PeV}$ at Earth and losing energy to electron–positron emission will not be able to traverse a distance greater than 32 Mpc. This is about the typical distance to the local supercluster. However, the contemplated extragalactic sources for the 1 PeV neutrino probably are a distance beyond the local supercluster. In their later work, the authors constrained δ_ν with a limit on δ_{ν_e} and δ_e to 10^{-20} [24,150], entertaining the idea that an $\sim$2 PeV cutoff in the E_ν^{-2} power-law spectrum of the neutrino would be due to LV.

5.3. Atmospheric Spectral Analysis

The discussion here considers spectral studies in the atmospheric setting by IceCube and Super-Kamiokande [118,120–122]. These were investigating zenith angle and energy distribution and could achieve excellent sensitivity due to the large baseline variation from 10 to 13,000 km and energy from 100 MeV to 10 TeV. Unlike the case with one or two PeV neutrinos, here, there are more neutrinos arriving from all directions.

The study here is still an energy-based study, since spectral analysis focuses only on the isotropic terms. In case of LV, the unusual energy dependence is to be scrutinized as described at the beginning of the paper.

Super-Kamiokande investigated the real and imaginary parts of the a and c coefficients. IceCube focused on the c coefficients up to mass dimension eight. Both specifically studied isotropic coefficients for the $\mu\tau$ oscillation, as opposed to previous sidereal studies.

The investigation performs energy sampling of neutrinos in different flavors, with the highest energy samples coming from up-going muon events. Binning is used in energy and in zenith angle. One IceCube spectral analysis, for instance, uses 10 linearly spaced bins of the cosine of the zenith angle between -1.0 and 0.0, and 17 logarithmically spaced bins in reconstructed muon energy from 400 GeV to 18 TeV [118]. Super-Kamiokande uses a combination of momentum and energy binning based on the number of Čerenkov rings seen, with 480 bins in all [120–122]. Binned data are compared to simulated results based on atmospheric models and assumed LV values. The simulated data consider different values for the different relevant coefficients and allows bounds to be placed on the basis of the comparison.

The exact hamiltonian in this case cannot be handled with perturbative methods due to the wide range of path lengths and energies. The full hamiltonian of three-flavor oscillations, including the conventional oscillation hamiltonian, the matter part, and the a and c modifications, must be diagonalized. The conventional hamiltonian reads as follows:

$$h_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \dfrac{\Delta m_{21}^2}{2E} & 0 \\ 0 & 0 & \dfrac{\Delta m_{21}^2}{2E} \end{pmatrix}. \tag{73}$$

The matter hamiltonian

$$h_{\mathrm{matter}} = \sqrt{2} G_F \begin{pmatrix} N_e & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{74}$$

where G_F is the Fermi constant and N_e is the average electron density along the neutrino's path. The LV hamiltonian for a coefficients is

$$\delta h = \begin{pmatrix} 0 & a^T_{e\mu} & a^T_{e\tau} \\ (a^T_{e\mu})^* & 0 & a^T_{\mu\tau} \\ (a^T_{e\tau})^* & (a^T_{\mu\tau})^* & 0 \end{pmatrix}, \tag{75}$$

and for c coefficients

$$\delta h = \begin{pmatrix} 0 & c^{TT}_{e\mu} & c^{TT}_{e\tau} \\ (c^{TT}_{e\mu})^* & 0 & c^{TT}_{\mu\tau} \\ (c^{TT}_{e\tau})^* & (c^{TT}_{\mu\tau})^* & 0 \end{pmatrix}, \tag{76}$$

where the uppercase T indices indicate an isotropic analysis. For the methods used to diagonalize the full hamiltonian, the simulations, details of the higher-dimensional analysis of IceCube, and the sophisticated plotting of the results, consult references [118,120–122]. These involved analyses yield excellent bounds for a and c.

6. Summary

The above review presented a summary of the Lorentz violation searches within the SME framework in the neutrino sector. These searches can be carried out in Earth-based laboratories, but some are based on neutrino behavior as detected from the atmosphere, from the sun, or from possibly traversing galactic and extragalactic distances. Some of these constraints are the tightest in the field of SME searches and experimental searches, and the phenomenological aspects are further refined to suit the expanding astrophysical observation and analysis. The latest opportunity to see EHE neutrinos of extragalactic origin provides fundamentally important information about space-time symmetries, thereby opening new doors into this sector.

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