Article

Geodesic Circular Orbits Sharing the Same Orbital Frequencies in the Black String Spacetime

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Abstract: We consider isofrequency pairing of geodesic orbits that share the same three orbital frequencies associated with $\Omega^{\hat{r}}$, $\Omega^{\hat{\phi}}$, and $\Omega^{\hat{\omega}}$ in a particular region of parameter space around black string spacetime geometry. We study the effect of a compact extra spatial dimension on the isofrequency pairing of geodesic orbits and show that such orbits would occur in the allowed region when particles move along the black string. We find that the presence of the compact extra dimension leads to an increase in the number of the isofrequency pairing of geodesic orbits. Interestingly we also find that isofrequency pairing of geodesic orbits in the region of parameter space cannot be realized beyond the critical value $\mathcal{J}_{cr} \approx 0.096$ of the conserved quantity of the motion arising from the compact extra spatial dimension.

Keywords: isofrequency; geodesic orbits; black string

1. Introduction

In general relativity, black holes are regarded as one of the most interesting and fascinating objects due to their geometric and remarkable gravitational properties. However, the point to be noted is that their existence had been predicted as a potential tool in explaining observational phenomena until the discovery of the gravitational waves detected by LIGO and Virgo scientific collaborations [1,2] and as well as the first direct shadow of supermassive black hole at the center of galaxy 87 by the Event Horizon Telescope (EHT) collaboration [3,4]. These observations opened a qualitatively new stage to reveal black hole's unknown properties and test remarkable nature of the background geometry around black hole's horizon irrespective of the fact that there are still fundamental problems that general relativity faces, i.e., the occurrence of singularity, spacetime quantization, etc. In this framework, the motion of test particles in the strong gravitational field regime has been a productive field of study for several years [5–19]. On the other hand, there is an extensive body of work devoted to understand the nature of radiative inspirals around black holes as a source of gravitational waves and binary systems [20–28].

It is worth noting that experiments and observations always allow to test the nature of the geometry through alternative theories of modified gravity in the strong field regime. The spactime geometry that can depart from the spherically symmetric Schwarzschild black hole was considered by Grunau and Khamesra [29]. This spacetime metric describes a five dimensional black string due to the compact extra dimension added to the well known Schwarzschild metric. Thus, it is one of reasons why this solution is interesting object to

investigate its properties. The motion of test particles in the different gravity string models has been investigated thoroughly by several authors [30–35]. Furthermore, the geodesic motion of test particles in the black string spacetime for both rotating and non-rotating cases has been studied in detail [29]. The geodesic motion of colliding particles was also studied in the vicinity of black string spacetime [36].

The theoretical investigation of isofrequency pairing of geodesic circular orbits has been considered in the vicinity of Schwarzschild and Kerr black holes, i.e., the region of the parameter space where such geodesic orbits occur has been discussed by providing an intuitive explanation of their occurrence. The above investigation was addressed by taking an external asymptotically uniform magnetic field in the Schwarzschild spacetime [25]. It turns out that the surface of allowed region where isofrequency pairing orbits occur decreases as a consequence of the presence of the external magnetic field. It was later extended to the spinning particles in the Schwarzschild-de Sitter spacetime [26]. In the present paper, we consider isofrequency pairing of geodesic orbits around black string spacetime geometry, as shown by the line element in [29]. We then study the effect of the compact extra spatial dimension ω on the isofrequency pairing of geodesic spiral orbits sharing the same three orbital frequencies.

In this paper, we consider a new frequency $\Omega^{\hat{\omega}}$ arisen from the compact extra spatial dimension associated with the motion in the extra direction along the black string. Generally, the bound orbits are confined to the interior of a compact special spiral given by $h_{z1}(e = 0) \leq h_z \leq h_{z2}(e = 1)$; frequencies, in particular, $\Omega^{\hat{r}}$ and Ω^{θ} are considered as "libration"-type frequencies associated with the radial and longitudinal periods, while $\Omega^{\hat{\varphi}}$ is a "rotation"-type frequency. In the case of black string, new frequency $\Omega^{\hat{\omega}}$ can be defined by a "libration-rotation" type frequency which involves two kinds of type frequencies, as described above. For the sake of clarity, we will further focus on the motion of test particles at the equatorial plane (i.e., $\theta = \pi/2$), and hence we omit $\Omega^{\hat{\theta}}$ as it loses its meaning. Then bound geodesic orbits around black string spacetime are respectively characterized by three orbital frequencies, i.e., $\Omega^{\hat{r}}$, $\Omega^{\hat{\varphi}}$, and $\Omega^{\hat{\omega}}$ associated with the periodic motions. Thus, these orbits are considered as triperiodic orbits around the black string spacetime. In doing so we predict that an infinite number of pairs of such orbits may occur in a region of parameter space around the black string spacetime where physically distinct orbits possessing $\mathcal{E}$, $\mathcal{L}$, and $\mathcal{J}$ share the same orbital frequencies associated with $\Omega^{\hat{\varphi}}$, $\Omega^{\hat{r}}$, and $\Omega^{\hat{\omega}}$, thereby referring to the "isofrequency" bound geodesic orbits synchronized in phase φ while passing periastra at the same time. Isofrequency pairing of geodesic orbits occuring in the vicinity of black string spacetime may in principle play an important role in an astrophysical scenario in understanding the nature of radiative inspirals around black holes.

The paper is organised as follows: In Section 2 we briefly describe the metric for black string and provide the geodesic equations of motion. We further study the effect of the compact extra dimension on the isofrequency pairing of geodesic spiral orbits in Section 3. We end up with conclusion in Section 4. Throughout the paper we use a system of geometric units in which $G = c = 1$.

2. The Metric and the Geodesic Equations of Motion

The metric describing five dimensional black string spacetime with the compact extra spatial dimension ω added to the Schwarzschild metric in the Boyer-Lindquist coordinates $(t, r, \theta, \varphi, \omega)$ is given by [29]

$$
\begin{aligned}
ds^2 &= -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 \\
&\quad + r^2(d\theta^2 + \sin^2 d\varphi^2) + d\omega^2
\end{aligned}
\tag{1}
$$

with M being the total mass of black string and ω corresponding to the compact extra spatial dimension.

The radial motion of geodesic test particles in the equatorial plane of black string spacetime satisfies the following equation

$$\dot{r}^2 = \mathcal{E}^2 - V_{eff}(r; \mathcal{L}, \mathcal{J}),\tag{2}$$

where

$$V_{eff}(r; \mathcal{L}, \mathcal{J}) = \left(1 - \frac{2M}{r}\right)\left(1 + \mathcal{J}^2 + \frac{\mathcal{L}^2}{r^2}\right),\tag{3}$$

with $\mathcal{E} = E/m$, $\mathcal{L} = L/m$, and $\mathcal{J} = J/m$ being the conserved constants per unit mass and related to the particle's specific energy, angular momentum, and conserved quantity arising from ω, respectively.

It is well known that bound orbits exist in the case when $\mathcal{L} > 2\sqrt{3}M$ with $2\sqrt{2}/3 < \mathcal{E} < 1$ for the Schwarzschild black hole. Whereas, in the case of black string, bound orbits only exist when $\mathcal{L} > 2\sqrt{3}(1 + \mathcal{J}^2)^{1/2}M$ with $2\sqrt{2}(1 + \mathcal{J}^2)^{1/2}/3 < \mathcal{E} < (1 + \mathcal{J}^2)^{1/2}$ is satisfied, and the motion of particles is then restricted by the turning points being labeled as the periastron r_{p} and the apastron r_{a}, respectively.

We now consider the innermost stable circular orbit (ISCO) for test particles orbiting around black string spacetime. To find radii of the ISCO one needs to solve the following equation

$$V''_{eff}(r; \mathcal{L}, \mathcal{J}) = 0.\tag{4}$$

In our case r_{ISCO} can be determined implicitly from the condition

$$6\mathcal{L}^2(r - 4M) - 4(1 + \mathcal{J}^2)Mr^2 = 0,\tag{5}$$

with $\mathcal{L}$ obtained on solving $V'_{eff}(r; \mathcal{L}, \mathcal{J}) = 0$. Thus, we have the ISCO radius $r_{ISCO} = 6M$ for black string spacetime. It is worth noting that the ISCO is the same in contrast for the Schwarzschild black hole.

The standard condition $V_{eff}(r_p, \mathcal{L}, \mathcal{J}) = V_{eff}(r_a, \mathcal{L}, \mathcal{J}) = \mathcal{E}^2$ leads to the specific energy $\mathcal{E}$ and angular momentum $\mathcal{L}$ of the particle, given in terms of p and e. Here, $r_p = Mp/(1 + e)$ and $r_a = Mp/(1 - e)$ respectively refer to the periastron and the apastron [25]. In between these two turning points lies bound orbits with corresponding specific energy and angular momentum. Note that parameters p and e respectively measure the size of the orbit and its degree of noncircularity [37]. Then we have

$$\begin{aligned}
\mathcal{E}^2 &= \frac{(p - 2 - 2e)(p - 2 + 2e)}{p(p - 3 - e^2)}(1 + \mathcal{J}^2),\\
\mathcal{L}^2 &= \frac{p^2 M^2}{p - 3 - e^2}(1 + \mathcal{J}^2).
\end{aligned}\tag{6}$$

Following Cutler et al. [38], we turn to the integration of the geodesic equations in terms of $t(r)$, $\varphi(r)$, and $\omega(r)$ coordinates

$$t(r) = \mathcal{E}\int \frac{r\,dr}{(r - 2M)\left[\mathcal{E}^2 - V_{eff}(r; \mathcal{L}, \mathcal{J})\right]^{1/2}},\tag{7}$$

$$\varphi(r) = \mathcal{L}\int \frac{dr}{r^2\left[\mathcal{E}^2 - V_{eff}(r; \mathcal{L}, \mathcal{J})\right]^{1/2}},\tag{8}$$

and

$$\omega(r) \;=\; \mathcal{J}\int \frac{dr}{\left[\mathcal{E}^2 - V_{eff}(r;\mathcal{L},\mathcal{J})\right]^{1/2}}. \tag{9}$$

Using Equations (6)–(9) and substituting $r(\chi) = Mp/(1 + e\cos\chi)$ into Equation (3), we then have the following expressions together with χ related to the coordinates t, φ, and ω as

$$\frac{dt}{d\chi} \;=\; \frac{Mp^2[(p-2)^2 - 4e^2]^{1/2}(p-6-2e\cos\chi)^{-1/2}}{(p-2-2e\cos\chi)(1+e\cos\chi)^2}, \tag{10}$$

$$\frac{d\varphi}{d\chi} \;=\; \frac{p^{1/2}}{(p-6-2e\cos\chi)^{1/2}}, \tag{11}$$

$$\frac{d\omega}{d\chi} \;=\; \frac{\mathcal{J}Mp^{3/2}(p-3-e^2)^{1/2}}{(1+\mathcal{J}^2)^{1/2}(p-6-2e\cos\chi)^{1/2}(1+e\cos\chi)^2}. \tag{12}$$

The radial period and frequency are defined by [24,25,38]

$$T^{\hat{r}} = \int_0^{2\pi} \frac{dt}{d\chi}\,d\chi, \qquad \Omega^{\hat{r}} = \frac{2\pi}{T^{\hat{r}}}. \tag{13}$$

The azimuthal frequency of the orbit will then have the form as

$$\Omega^{\hat{\varphi}} = \frac{1}{T^{\hat{r}}}\int_0^{T^{\hat{r}}} \frac{d\varphi}{dt}\,dt = \frac{\Delta\varphi}{T^{\hat{r}}}, \tag{14}$$

with azimuthal phase $\Delta\varphi$, which is given by

$$\begin{aligned}
\Delta\varphi &= \int_0^{2\pi} \frac{d\varphi}{d\chi}\,d\chi = \\
&= \int_0^{2\pi} \frac{\sqrt{p}}{\sqrt{p-6-2e\cos\chi}}\,d\chi = 4\sqrt{\frac{p}{\epsilon}}\,K\!\left(-\frac{4e}{\epsilon}\right),
\end{aligned} \tag{15}$$

where $\epsilon = p - p_s(e) = p - 6 - 2e$ and $K(x) = \int_0^{\pi/2} d\theta (1 - x\sin^2\theta)^{-1/2}$ is the complete elliptic integral of the first kind.

As a consequence of the presence of the compact extra dimension ω, the new frequency can be defined by

$$\Omega^{\hat{\omega}} = \frac{1}{T^{\hat{r}}}\int_0^{T^{\hat{r}}} \frac{d\omega}{dt}\,dt = \frac{\Delta\omega}{T^{\hat{r}}}, \tag{16}$$

where $\Delta\omega$ is the new extra phase accumulated over time interval $T^{\hat{r}}$.

3. Isofrequency Pairing Orbits in Black String Spectime Geometry

Now we investigate the isofrequency and bound orbits of the particles in the vicinity of the black string spacetime. We consider orbits lying extremely close to the separatrix ϵ in order to estimate the divergent quantities of the azimuthal and the new extra phases, the

radial period, and their ratio in Equations (14) and (16). Considering the near-separatrix analytic expansions we obtain the corresponding expression for $\Delta\omega$ as

$$\Delta\omega = \frac{\mathcal{J}M(3+2e-e^2)^{1/2}}{(1+\mathcal{J}^2)^{1/2}(1+2e)}\sqrt{\frac{(6+2e)^3}{e}}$$
$$\times\left[1+\mathcal{O}\left(\frac{\epsilon}{4e}\right)\right]\ln\left(\frac{64e}{\epsilon}\right). \tag{17}$$

The previous investigations suggest that the occurrence of isofrequency pairing of geodesic orbits is strongly related to the presence of boundary regions referred as *separatrix* and *circular-orbit duals* (COD), and each and every circular orbit in the open range between the separatrix and singular curve and between the singular curve and COD as well has a dual isofrequency pairing. Note that the existence of the *separatrix* plays an important role for occurrence of these dual orbits in this isofrequency scenario. Considering the singular curve and the circular orbit duals in the particular region of the gravitational object is particularly important, and the COD can also play an important role due to the fact that as a boundary region allowing to keep pairs of isofrequency orbits that exist in this region [24–26].

Let us then study the effect of the compact extra dimension on the occurrence of the isofrequency pairing of geodesic orbits in black string spacetime. Based on the above discussions and Equations (13)–(15) we keep the result for which the range of frequencies of any pairs of orbits on $\Omega^{\hat{\phi}} = (M/r_b^3)^{1/2}$ in the black string is given by

$$\tilde{\Omega}^{\hat{\phi}}_{e=0} = 0.062 < \tilde{\Omega}^{\hat{\phi}} < \tilde{\Omega}^{\hat{\phi}}_{e=1} = 0.125, \tag{18}$$

where we have defined $\tilde{\Omega}^{\hat{\omega}} = M\Omega^{\hat{\phi}}$ as a dimensionless parameter.

Further, we determine the values of radial and azimuthal frequencies numerically and tabulate their values in Table 1.

Table 1. Values of $\Omega^{\hat{r}}$ and $\Omega^{\hat{\phi}}$ are tabulated for different values of eccentricity e for isofrequency pairing of geodesic spiral orbits around the black string.

e	0.1	0.3	0.4	0.5	0.6
$\Omega^{\hat{\phi}}$	0.063	0.068	0.073	0.079	0.091
$\Omega^{\hat{r}}$	0.0069	0.0124	0.0152	0.0181	0.0226

e	0.7	0.8	0.89	0.96	1
$\Omega^{\hat{\phi}}$	0.103	0.109	0.115	0.121	0.125
$\Omega^{\hat{r}}$	0.0272	0.0304	0.0334	0.0362	0.0380

In Table 2, we show numerical values of the frequency $\tilde{\Omega}^{\hat{\omega}}$ of the bound orbits around the black string. As can be seen from Table 2 the value of the frequency $\tilde{\Omega}^{\hat{\omega}}$ arisen from the compact extra spatial dimension is increasing with an increase in the value of the forth conserved quantity $\mathcal{J}$, but the height of the horizontal direction along black string is decreasing for given values of eccentricity e. However, numerical analysis leads to the fact that when the value of the fourth conserved quantity attains $\mathcal{J}_{cr} \approx 0.096$ geodesic spiral orbits are not allowed all through to occur around the black string. Thus, the pairs of geodesic circular spiral orbits of particles with the same radial $\tilde{\Omega}^{\hat{r}}$, azimuthal $\tilde{\Omega}^{\hat{\phi}}$, and new $\tilde{\Omega}^{\hat{\omega}}$ frequencies occur in the particular region of the gravitational object under this critical value irrespective of the fact that these orbits are physically distinct with different values of conserved quantities (i.e., $\mathcal{E}$, $\mathcal{L}$, and $\mathcal{J}$). Then the pairs of these spiral orbits of particles with three frequencies associated with ($\tilde{\Omega}^{\hat{r}}$, $\tilde{\Omega}^{\hat{\phi}}$, $\tilde{\Omega}^{\hat{\omega}}$) would move and oscillates in a cylindrical form along the black string.

It turns out that as a consequence of the decrease in the value of $\mathcal{J}$ the above mentioned cylindrical region becomes larger around the black string. It could then play an important role in allowing the particles to move along the black string, thus increasing the amount of

isofrequency pairs of geodesic spiral orbits in the region between the *separatrix, circular-orbit duals* and *horizontal length h_z*, i.e., in the particular region of the black string. Hence, one can keep in mind that there would exist an infinite number of pairs of circular spiral orbits having the same three orbital frequencies around the black string regardless of physically distinct obits with three conserved quantities. As a consequence of the presence of compact extra spatial dimension, the occurrence of isofrequency pairing of spiral orbits would be of particularly importance in understanding and explaining the behavior of black string spacetime and the nature of the radiative inspirals as a source of gravitational waves or binary systems.

Table 2. The values of $\Omega^{\hat{\omega}}$ are tabulated for different values of the fourth conserved parameter $\mathcal{J}$ for isofrequency pairing of geodesic spiral orbits around the black string. The point to be noted here is that h_z corresponds to the horizontal length and will be valid only for the non vanishing compact extra dimension, i.e., $\omega \neq 0$.

	$\mathcal{J}$	**0.01**	**0.03**	**0.05**	**0.07**	**0.09**	**0.096**
$e = 0$	$\tilde{\Omega}^{\hat{\omega}}$	0.007	0.021	0.035	0.049	0.063	-
	r	27.17	13.05	9.29	7.43	6.29	
	h_z	26.51	11.59	7.09	4.38	1.89	
$e = 1$	$\tilde{\Omega}^{\hat{\omega}}$	0.006	0.019	0.033	0.046	0.059	-
	r	28.25	13.57	9.66	7.73	6.55	
	h_z	27.53	11.98	7.25	4.36	1.45	

4. Conclusions

We considered isofrequency pairing of geodesic spiral orbits in the black string spacetime, which recovers the Schwarzschild one in the case of vanishing ω. Thus, in an astrophysical scenario, it is worth investigating these geodesic orbits in the vicinity of black string as it allows to test the effects arising from the five dimensions. Our analysis suggests that the compact extra dimension ω leads to an increase in the number of pairs of circular spiral orbits. These pairs of orbits having the same three orbital frequencies (i.e., $\Omega^{\hat{r}}$, $\Omega^{\hat{\varphi}}$, and $\Omega^{\hat{\omega}}$) occur in the particular region and oscillate in the cylindrical form around the black string in spite of the fact that they are physically distinct orbits possessing different conserved quantities (i.e., $\mathcal{E}$, $\mathcal{L}$, and $\mathcal{J}$). Furthermore, we showed that pairs of geodesic spiral orbits are not allowed all through to occur in the black string vicinity in the case when the fourth conserved quantity attains $\mathcal{J} \approx 0.096$.

The obtained results suggest that an infinite number of pairs of such orbits may occur in the particular region around the black string and it would be particularly important in explaining not only the possibility of occurrence of pairs of geodesic spiral orbits around the black holes, but also the nature of the black string spacetime. These theoretical results and discussions would be useful to the possible interpretation of astrophysical observations, and they could help to provide details of the validity of alternative models to black holes and also explain the nature of the radiative inspirals since isofrequency paring of geodesic spiral orbits can give the same information as such inspiral objects.

In a future work, we would be considering the possible extensions of recent results to the case of rotating black string and studying the effects of the rotation and a compact extra dimension on the specific isofrequency pairing of geodesic orbits. This would be of primary astrophysical importance.

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